Commutative Algebra
Exercises for March 11, 2014

(1) (Ideals) Let R be a commutative ring with unit, and let I, J be ideals of R. Show that

$$I + J = \{i + j \,|\, i \in I, j \in J\}$$

is an ideal of R.

(2) (Residue class rings) Let R be a commutative ring with unit, and let I be an ideal of R. Let r_1, r_2, s_1, s_2 be elements of R such that $r_1 + I = r_2 + I$ and $s_1 + I = s_2 + I$. Prove $(r_1 s_1) + I = (r_2 s_2) + I$.

(3) (ACC) Let $\mathcal{I}(\mathbb{Z})$ be the set of ideals of the ring of integers $\mathbb{Z}$. Show that $(\mathcal{I}(\mathbb{Z}), \subseteq)$ satisfies the (ACC).

(4) (Ideal generation) Determine for which of the sets S the ideal $\langle S\rangle_R$ of the ring $R = \mathbb{R}[x, y]$ satisfies $\langle S\rangle_R = R$.
   (a) $S = \{xy, 2x^3y + 3\}$.
   (b) $S = \{x^2y, xy^2 + 1\}$.
   (c) $S = \{xy + x, 1 + y^2\}$.

(5) (Homomorphisms and ideals) Find generators for the kernels of each of these ring homomorphisms.
   (a) $f : \mathbb{Q}[x, y] \to \mathbb{Q}$, $p \mapsto \hat{p}(\frac{1}{2}, -\frac{1}{3})$.
   (b) $h : \mathbb{Q}[x, y, z] \to \mathbb{Q}[x]$, $p \mapsto p(x, 0, 0)$.