Functional Analysis—Banach spaces

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Contents

1 Normed spaces

We begin with the elementary theory of normed spaces. There are vector spaces with a suitable distance function. With the help of his distance, the usual procedures involving limit operations (approximation of non linear operators by their derivatives, approximation methods for constructuing solutions of equations etc.) can be carried out. Definition 1.1 below, which was explicitly introduced by Banach and Wiener, was already implicit in earlier work on integral equations by Riesz (who employed the concrete normed spaces $C(I)$, $L^p(I)$ which will be introduced below).

The plan of the section is simple. We begin by introducing the two main concepts of the chapter—normed spaces and continuous linear operators. Their more obvious properties are discussed and some concrete examples—mainly the so-called ℓ^p-spaces—are introduced.

Definition 1 *A* **seminorm** *on a vector space E (over $\mathbf{C}$ or $\mathbf{R}$) is a mapping $x \mapsto ||x||$ from E into $\mathbf{R}^+$ with the properties*

1. $||x+y|| \leq ||x|| + ||y|| \quad (x, y \in E)$*;*
2. $||\lambda x|| = ||\lambda|\, ||x|| \quad (x \in E, \lambda \in \mathbf{C}$ *or* $\mathbf{R}$*resp.);*
3. $||x|| = 0$ *implies* $x = 0 \quad (x \in E)$*.*

A **normed space** *is a pair $(E, ||\,||)$ where E is a vector space and $||\,||$ is a norm on E.*

If $||\,||$ is a seminorm (resp. a norm) the mapping

$$d_{||\,||} : (x, y) \to ||x - y||$$

is a semimetric (resp. a metric) on E. We call it the **semimetric** (resp. metric) **induced by** $||\,||$. Thus every normed space $(E||\,||)$ can be regarded in a natural way as a metric space and so as a topological space and we can use, in the context of normed spaces, such notions as continuity of mappings, convergence of sequences or nets, compactness of subsets etc.

If $(E, ||\,||)$ is a normed spaces, we write $B_{||\,||}$ or $B(E)$ for the closed unit ball of E i.e. the set $\{x \in E : ||x|| \leq 1\}$.

EXERCISES.

A. A subset A of a vector space is **absolutely convex** if $\lambda x + \mu y \in A$ whenever $x, y \in A$, $\lambda, \mu \in \mathbf{C}$ (respectively $\mathbf{R}$) and $|\lambda| + |\mu| \leq 1$.

   A is **absorbing** if for each $x \in E$ there is a $\rho > 0$ so that $\lambda x \in A$ when $|\lambda| \leq \rho$. Show that $B_{||\,||}$ is absolutely convex and absorbing and that if A is absolutely convex and absorbing then

   $$||\,||_A : x \mapsto \inf\{\rho > 0 : x \in \rho A\}$$

   is a seminorm on E (it is called the **Minkowski functional** of A).

   Show that it is a norm if and only if A contains no non-trivial subspace of E.

B. Let E be a vector space, A an absolutely convex subset which does not contain a non-trivial subspace.

   Let $E_A = \bigcup_{n \in \mathbf{N}} nA$. Show

   1. that E_A is a vector subspace of E;
   2. that A absorbs E_A;

3. that $(E, || \, ||_A)$ is a normed space.

The usual constructions (products, subspaces, quotiens etc.) can be carried out in the context of normed spaces. For example, if G is a vector subspaces of the normed space $(E, || \, ||)$, the restriction $|| \, ||_G$ of $|| \, ||$ to G is a norm thereon and so we can regard G in a natural way as a normed space (this norm is called **the norm induced on G by $|| \, ||$**).

Similarly, if π_G denotes the natural projection from E onto the quotient space E/G, then the mapping

$$y \mapsto \inf\{||x|| : x \in E \text{ and } \pi_G x = y\}$$

is a seminorm. The question of when it is a norm is examined in an exercise below.

There are several possibilities for defining norms on product spaces and we shall discuss these in some detail later. For our present purposes, the following one on a product of two spaces will suffice: Let $(E, || \, ||_1)$ and $F(|| \, ||_2)$ be normed spaces. The mapping

$$(x, y) \to \max\{||x||_1, ||y||_2\}$$

is a norm on $E \times F$ which (with this norm) is then called the **normed product** of E and F. (Note that the unit ball of $E \times F$ is then just the Cartesian product of the unit balls of E and F.)

Exercises.

1. Show that the topology induced by $|| \, ||_G$ on G coincides with the restriction to G of the topology of E;

2. if $x \in E$, show that $||\pi_G(x)||$ (the norm in E/G) is just the distance from x to G i.e.
$$\inf\{||x - y|| : y \in G\}.$$
Deduce that the seminorm on E/G is a norm if and only if G is closed. Use this to give an example where it is not a norm.

3. Show that the topology induced by the norm on $E \times F$ is the product of the topologies on E and F.

It follows from the very definition of the topology via the norm that it is very closely related to the linear structure of E. In fact, the following properties are valid:

1. the mappings
$$A : (x, y) \mapsto x + y$$
and
$$M : (\lambda, x) \mapsto \lambda x$$
from $E \times E$ into E resp. $\mathbf{C} \times E$ or $\mathbf{R} \times E$ into E are continuous for the topology generated by the norms. For
$$||(x, y) - (x_1 + y_1)|| \leq ||x - x_1|| + ||y - y_1||$$
and
$$\begin{aligned} ||\lambda x - \lambda_1 x_1|| &= ||(\lambda x - \lambda_1 x) + (\lambda_1 x - \lambda_1 x_1)|| \\ &\leq |\lambda - \lambda_1|\,||x|| + |\lambda_1|\,||x - x_1||. \end{aligned}$$

2. Let G be a subspace of $(E, || \, ||)$. Then the closure $\bar{G}$ of G is also a subspace.

Exercises. Let $|| \, ||$ be a seminorm on E. Show that $E_0 := \{x \in E : ||x|| = 0\}$ is a subspace of E. If π_0 denotes the natural projection from E onto E/E_0, show that
$$\pi_0(x) \to ||x||$$
is a well-defined mapping on E/E_0 and is, in fact, a norm. E/E_0, with this norm, is called the **normed space associated** with E. (This simple exercise is often useful on occasions when a natural construction "should" produce a normed space but in fact only produces a seminormed space. We simply factor out the zero subspace.)

As is customary in mathematics, we identify spaces which have the same structure. The appropriate concept is that of an isomorphism. It turns out that there are two natural ones in the context of normed spaces:

Let E, F be normed spaces. E and F are **isomorphic** if there is a bijective linear mapping $T : E \to F$ so that T is a homeomorphism or the norm topologies. T is then called an **isomorphism**. If T is, in addition, norm-preserving (i.e. $||Tx|| = ||x||$ for $x \in E$), T is an **isometry** and E and F are **isometrically isomorphic** (we write $E \sim F$ resp. $E \cong F$ to indicate that E and F are isomorphic— resp. isometrically isomorphic).

Two norms $|| \, ||$ and $|| \, ||_1$ on a vector space E are **equivalent** if Id_E is an isomorphism from $(E, || \, ||)$ onto $(E, || \, ||_1)$ i.e. if $|| \, ||$ and $|| \, ||_1$ induce the same topology on E.

Isomorphisms are characterised by the existence of estimates from above and below—let $T : E \to F$ be a bijective linear mapping. Then T is an isomorphism if and only if there exist M and m (both positive) so that

$$m||x|| \leq ||Tx|| \leq M||x|| \quad (x \in E)$$

(for a proof see the Exercise below).

Thus the norms $|| \, ||$ and $|| \, ||_1$ on E are equivalent if and only if there are $M, m > 0$ so that $m||x|| \leq ||x_1|| \leq M||x|| \quad (x \in E)$.

We now bring a list of some simple examples of normed spaces. in the course of the later chapters we shall extend it considerably.

Examples

A. The following mappings on $\mathbf{C}^n$ (resp. $\mathbf{R}^n$) are norms:

$$\begin{aligned} || \, ||_1 &: (\xi_1, \dots, \xi)n) \mapsto (|\xi_1| + \cdots + |\xi_n|) \\ || \, ||_2 &: (\xi_1, \dots, \xi)n) \mapsto (|\xi_1|^2 + \cdots + |\xi_n|^2)^{1/2} \\ || \, ||_3 &: (\xi_1, \dots, \xi)n) \mapsto \sup(|\xi_i| : i = 1, \dots, n\}. \end{aligned}$$

Each of these norms induces the usual topology on $\mathbf{C}^n$ (resp. $\mathbf{R}^n$). Note that the respective unit balls are (for $n = 3$) the octahedron, the euclidian ball and the cube (or hexahedron).

B. Let K be a compact space. $C(K)$ denotes the space of continous, complex-valued functions on K. This space has a natural vector spaces structure and the mapping

$$|| \, ||_\infty : x \mapsto \sup\{|x(t)| : t \in K\|$$

is a norm. $|| \, ||_\infty$ induces the topology of uniform convergence on K (that is, a sequence or net of functions in $C(K)$ is norm-convergent if and only if it is uniformly convergent on K).

C. Let I be a compact interval in $\mathbf{R}$, n a positive integer.

The space

$$C^n(I) := \{x \in C(I) : x, x; , \dots, x^{(n)} \text{ exists and are continuous}\}$$

has a natural vector space structure and the mapping

$$|| \, ||_\infty^n : x \to \max\{||x||_\infty, \dots, ||x^{(n)}||_\infty\}$$

is a norm on $C^n(I)$. Note that $C^n(I)$ is a vector subspace of $C(I)$, but that $(C^n(I), || \, ||_\infty^n)$ is not a normed subspace of $(C(I), || \, ||_\infty)$—that is $|| \, ||_\infty^n$ is not the norm induced on $C^n(I)$ from $C(I)$—or even equivalent to it.

D. Let $\{(E_k, ||\,||_k) : k = 1, \dots, n\}$ be a family of normed spaces. On $E = \prod_{k=1} E_k$ we define two norms:

$$\begin{aligned} ||\,||_s : (x_1, \dots, x_n) &\rightarrow \sum_{k=1}^{n} ||x_k||_k \\ ||\,||_\infty : (x_1, \dots, x_n) &\rightarrow \max_{k=1,\dots,n} ||x_k||_k. \end{aligned}$$

Then $||\,||_s$ and $||\,||_\infty$ are distinct norms on E (if $n > 1$) which are, however, equivalent. In fact, we have the inequality:

$$||x||_\infty \leq ||x||_s \leq n||x||_\infty.$$

(Note that this means geometrically that the unit ball of $||\,||_s$ is contained in that of $||\,||_\infty$ resp. contains a copy of it reduced by a factor $\frac{1}{n}$).

Exercises.

A. Show that on $C([0,1])$, the mapping $x \mapsto \int_0^1 |x(t)|dt$ is a norm which is not equivalent to $||\,||_\infty$.

B. Show that the mapping $x \mapsto (x, x', \dots, x^{(n)})$ is an isomorphism from $C^n(I)$ onto a subspace of the product space $C(I) \times \cdots \times C(I)$ ($(n+1)$ factors).

An important role in the theory of infinite dimenstional spaces is played by linear operators. In contrast to the finite dimensional case, we impose the following condition, which takes account of the topological resp. norm structure:

Definition 2 *A linear mapping* $T : (E, ||\,||_1) \rightarrow (F, ||\,||_2)$ *is* **bounded** *if there is a* $C > 0$ *so that*

$$||Tx||_2 \leq C||x||_1 \quad (x \in E).$$

In fact, this is equivalent to continuity. Indeed we have equivalence of the following three conditions on a linear operator T between normed spaces E and F:

1. T is continuous;

2. T is continuous at 0;

3. T is bounded.

PROOF. 1. implies 2. is immediate.

2. implies 3.: since T is continuous at 0 and $B_{|| ||_2}$ is a neighbourhood of $0 = T(0)$, there is a $\delta > 0$ so that $Tx \in B_{|| ||_2}$ if $||x||_1 \leq \delta$. Now for each $x \in E$ with x non-zero,

$$\left\| \frac{\delta x}{||x||_1} \right\|_1 \leq \delta$$

and so $\left\| T\left(\frac{\delta x}{||x||_1}\right) \right\|_2 \leq 1$ i.e. $||Tx||_2 \leq ||x||_1/\delta$.

3. implies 1.: we suppose C chosen as above.Then

$$||Tx - Ty||_2 = ||T(x-y)_2 \leq C||x-y||_1$$

and so T is even Lipschitz continuous.

We write $L(E, F)$ for the set of bounded linear mappings from E into F. This space has a natural vector space structure (via pointwise addition and multiplication by scalars).

We define a norm on it as follows:

$$||T|| = \inf\{C > 0 : ||Tx||_2 \leq C||x||_1 \quad (x \in E)\}.$$

If E, F, G are normed spaces and $T \in L(E, F)$, $S \in L(F, G)$, then the composed mapping ST is also bounded and we have the estimate $||ST|| \leq ||S||\,||T||$ for its norm.

■

EXERCISES.

A. Use 1.1.7 to obtain the characterisation of isomorphism given before 1.1.5.

B Show that the formula given above does indeed define a norm on $L(E, F)$ and that

$$\begin{aligned} ||T|| &= \sup\{||Tx||_2/||x||_1 : x \in E, \quad x \neq 0\} \\ &= \sup\{||Tx||_2 : x \in B_E\} = \sup\{||Tx||_2 : ||x||_1 = 1\} \end{aligned}$$

C. A subset of a normed space E is **bounded** if the norm is bounded on it. Show that this is equivalent to the fact that the set C is absorbed by the unit ball (i.e. there is a $K > 0$ so that $C \subseteq KB_E$). Show that a linear mapping T between normed space is bounded if and only if it maps bounded sets into bounded sets and that a subset of $L(E, F)$ is bounded if and only if it is equicontinuous.

Examples We now bring some basic examples of operators:

A. Let $A = [a_{ij}]$ be an $m \times n$ matrix. Then the corresponding linear mapping

$$T_A = (\xi_j) \mapsto \left(\sum_{j=1}^{n} a_{ij}\xi_k \right)_i^m = 1$$

is bounded from $(\mathbf{R}^n|| \,||_p)$ into $(\mathbf{R}^m, || \,||_p)$ for $p = 1, 2$ or ∞.

B. We define a mapping $D : C^1(I) \to C(I)$ by $D : x \to x'$. D is linear and since

$$||Dx||_\infty = ||x'||_\infty \leq \max\{||x||_\infty, ||x'||_\infty\} = ||x||_\infty^1$$

it is bounded and $||D|| \leq 1$. More generally, one can define the operator

$$D^k : C^{n+k}(I) \to C^n(I)$$

$(n, k \in \mathbf{N})$ which maps x onto its k-th derivative.

C. Let a be a function in $C(I)$. Then the mapping

$$M_a : x \to ax$$

is continuous and linear from $C(I)$ into itself and $||M_a|| \leq ||a||_\infty$.

D. Differential operators: Let $a_0, \ldots, a_n$ be elements of $C(I)$. Then we define a **differantial operator**

$$L : C^n(I) \to C(I) \text{ by } L := \sum_{k=0}^{n} M_{a_k} \circ D^k.$$

E. Let K be bounded, continuous complex-valued functin on $I \times J$ (I, J compact intervals in$\mathbf{R}$). We define the **integral operator** I_K **with kernel** K as follows:

$$I_K : C(J) \ni x \to \left(s \to \int_J K(s,t)x(t)dt \right) \in C(I).$$

I_K is a continuous linear mapping from $C(J)$ into $C(I)$.

F. (**Projections.**) An operator $T \in L(E)$ is a **projection** if $T^2 = T$. Then $\mathrm{Id} - T$ is also a projection (since $(\mathrm{Id} - T)^2 = \mathrm{Id} - 2T + T^2 = \mathrm{Id} - T$).

Also $T(E) = \{x : Tx = x\} = \mathrm{Ker}\,(\mathrm{Id} - T)$ and so is closed.

The standard example of a projection is the mapping $(x, y) \to (x,))$ from a product space $E_1 \times E_2$ onto the factor E_1. In a sense this is the only one since if $T \in L(E)$ is a projection then the mapping

$$x \to (Tx, (\mathrm{Id} - T)x)$$

is an algebraic isomorphism form E onto the rpoduct space $E_1 \times E_2$ where $E_1 = T(E)$, $E_2 = (\mathrm{Id} - T)(E)$. In fact it is also an isomorphism for the norm structure on a product. Hence a projection in thes case causes a splitting up of the space into a product of two spaces.

A subspace E is **complemented** if it is the range of a projection $T \in L(E)$. E is then simultaneously a subspace of E and also isomorphic to a quotient space.

Exercises.

A. Let $A = (a_{ij})$ be as in 1.1.9.A and consider T_A as a mapping from $(\mathbf{C}^n, || \, ||_\infty)$ into $(\mathbf{C}^m, || \, ||_\infty)$. Show that $||T_A|| = \sup_{1 \leq i \leq m} \sum_{j=1}^n |a_{ij}|$. What is its norm as an operator for the norms $|| \, ||_1$ and $|| \, ||_2$?

B. Show that $||D|| = 1$ and $||M_a||^2 = ||a||_\infty$. Give an estimate for the norm of L and K_K (notation as above).

C. Let E, F be normed spaces, G a closed subspace of E, T a bounded linear operator from E into F. Show that if $T(G) = \{0\}$ there is a continuous linear operator $\tilde{T}$ from E/G into F so $\tilde{T} \circ \pi = T$.

We shall be interested in the following propertis of mappings $T \in L(E, F)$:

injectivity: This means that $\operatorname{Ker} T = \{0\}$;

surjectivity: i.e. that $T(E) = F$;

bijectivity: i.e. injectivity and surjectivity;

isomorphicity: cf. definition after above;

existence of a right inverse: i.e. an $S \in L(F, E)$ so that $TS = \mathrm{Id}_F$;

existence of a left inverse i.e. an $S \in L(F, E)$ so $ST = \mathrm{Id}_E$.

Note that for a linear operator T on a finite dimensional space E, all of these notions coincide as we know from linear algebra. In the infinite dimensional case, this is no longer true. We shall give some examples here and later.

In connection with these definitions, we can give various generalisations of the notion of an eigenvalue of an operator. We shall begin here with the most useful one. Later we shall consider refinements.

If $T \in L(E)$, the **spectrum** of T is the set of those $\lambda \in \mathbf{C}$ for which $(\lambda I - T)$ is not an isomorphism.

To illustrate these concepts, consider the following examples:

A. The identity mapping from $C([0,1])$ is injective if and only if its set of zeros has empty interior. It is surjective if and only if a has no zeros. In the latter case it is an isomorphism. The spectrum of M_a is the range of a.

A generalisation of the concept of a linear mapping which is often useful is that of a multi-linear mapping whereby a mapping T from a product $\prod_{k=1}^n E_k$ of vector spaces into the vector space F **is multilinear** if for each $i \in \{1, \dots, n\}$, the partial mapping

$$x \mapsto T(x_1, \dots, x_{i-1}, x, x_{i+1}, \dots, x_n).$$

The space of multilinear mappings is denoted by $\mathcal{L}(E_1, \dots, E_n; F)$. It has a natural linear structure. If the E's and F are normed spaces then the following conditions on such a mapping T are easily seen to be equivalent:

a) T is continuous as a mapping from $\prod_{k=1}^n E_k$ (with the product topology) into F;

b) T is bounded i.e. there is $C > 0$ so that

$$||T(x_1, \dots, x_n)|| \leq C ||x_1|| \dots ||x_n||$$

for each $x_1, \dots, x_n$.

(This fact is poved exactly as for linear mappings.)

The space of such multilinear mappings is denoted by $L(E_1, \dots, E_n; F)$. It is linear subspace of $\mathcal{L}(E_1, \dots, E_n; F)$ and the mapping

$$T \to \sup\{||T(x_1, \dots, x_n)|| : x_i \in B_{E_i}\}$$

is a norm theoreon. The case $F = \mathbf{R}$ (resp. $\mathbf{C}$) is particularly important and then we write $\mathcal{L}(E_1, \dots, E_n)$ and $L(E_1, \dots, E_n)$ for $\mathcal{L}(E_1, \dots, E_n; \mathbf{R})$ and $L(E_1, \dots, E_n; \mathbf{R})$ resp. $\mathcal{L}(E_1, \dots, E_n; \mathbf{C})$ etc.

In aplications, we usually have the situation where all of the E_i are equal to given space E. Then we use the notation $L^n(E; F)$ for the space $L(E_1, \dots, E_n; F)$ resp. $L^n(E)$ for $L(E_1, \dots, E_n)$.

Typical examples of such multilinear mappings are tensors (in the finite dimensional case) or mappings of the form

$$(x_1, \dots, x_n) \to x_1(s_1) \dots x_n(s_n) K(s_1, \dots, s_n) ds_1, \dots ds_n$$

on $C([0,1])$ where K is a suitable kernel i.e. a continuous function on $[0,1]^n$.

In the theory of differentiation for functions between Banach space, we shall encounter "nested" spaces of linear operators e.g. spaces like $L(E, L(E, F))$, $L(E, L(E, L(E, F)))$ etc. Fortunately, these can be more conveniently represented by spaces of multilinear meppings as the following Proposition shows:

Proposition 1 *The mapping*

$$T \mapsto (x_1 \mapsto ((x_1, \dots, x_n) \mapsto T(x_1, \dots, x_n)))$$

is a linear isometric isomorphism from $L(E_1, \dots, E_n; F)$ onto $L(E_1, L(E_2, \dots, E_n; F))$.

Proof. We prove this for the case $n = 2$. First we nove that the mapping $S \mapsto ((x_1, x_2) \mapsto (Sx_1)(x_2))$ is an inverse for the one in the statement of the theorem. That both are isometries follows from the equality:

$$\begin{aligned} ||T|| &= \sup\{||T(x_1, x_2)|| : ||x||_2 \leq 1||x_2|| \leq 1\} \\ &= \sup\{||x_1|| \leq 1 \sup\{||T(x_1, x_2)|| : ||x_2|| \leq 1\} \end{aligned}$$

If we apply the result repeadly, we see that the nested space $L(E_1, LL(F_2, \dots, L(E_n, F))$ is isometrically isomorphic to $L(E_1, \dots, E_n; F)$.

We continue this section with some remarks on finite dimensional spaces. These have played an increasingly important role in the theory of normed spaces in recent years as building blocks for infinite dimensional ones. They are also very helpful in providing a geometrical intuition which is useful even in the infinite dimensional case.

The first result shows that, as far es the topological structure is concerned, all finite dimensional spaces (of the same dimension) ar the same. This should not be interpreted as stating that all finite dimensional space are trivial. Their geometry can be very distincts.

Proposition 2 *Every real, finite dimensional normed space E is isomorphic to $(\mathbf{R}^n, || \, ||_\infty)$ where $n = \dim E$. Similarly, every n-dimensional normed space over $\mathbf{C}$ is isomorphic to $(\mathbf{C}^n, || \, ||_\infty)$.*

Proof. Let $(x_1, \ldots, x_n)$ be a bases for E and consider the continuous linear map

$$T : (\lambda_1, \ldots, \lambda_n) \to \sum_{i=1}^{n} \lambda_i x_i$$

from $\mathbf{R}^n$ onto E.

The image of the unit sphere of $\mathbf{R}^n$ under T is a compact subset of E whcih does not contain 0 (since the (x_i) ae linearly independent). Hence there is a $\delta > 0$ so that $||T(\lambda)|| \geq \delta$ if $\lambda \in \mathbf{R}^n$, $||\lambda||_\infty = 1$. From this it follow that $||T^{-1}|| \leq 1/\delta$. The complex case can be probed similarly.

■

Corollar 1 *Any two norms on a finite dimensional normed space are equivalent.*

In view of this fact it is interesting to consider some infinite dimensional normed spaces other than those of 1.1.5.A. In $\mathbf{R}^n$ or $\mathbf{C}^n$ we consider the set

$$A_p = \{x = (\xi_i) : |\xi_1|^p + \cdots + |\xi_n|^p \leq 1\}$$

where $1 \leq p < \infty$. Then A_p is absolutely convex and absorbing (see Exercise 1.15 below) and so its Minkowski functional $||\,||_p : x \to (\sum |\xi_i|^p)^{1/p}$ is then a norm. We denote the space $\mathbf{R}^n$ (resp. $\mathbf{C}^n$) with this norm by ℓ^p_n. (In particular, $||\,||_2$ and $||\,||_1$ coincide with the norms introduced in 1.1.5.A). In line with this notation, we denote $(\mathbf{R}^n, ||\,||_\infty)$ by ℓ^∞_n.

It is a simple exercise to show that A_p increases as p increases. This means that the identity from ℓ^p_n into ℓ^q_n has norm ≤ 1 if $q \geq p$.

Exercises. Prove the following unproven statements from the last paragraph:

1. A_p is absolutely convex (use the fact that the function $f : t \to t^p \quad (t < 0)$ is convex i.e. satisfies the condition

$$f(\lambda s + (1-\lambda)t) \leq \lambda f(s) + (1-\lambda) f(t) \quad (0. \leq s < t, 0 < \lambda < 1).$$

2. A_p incereases with p. Show also that

3. $\bigcap_{p>1} A_p = A_1$

4. $\overline{\bigcup_{1 \leq p < \infty} A_p} = A_\infty$

and interpret the last two statements in terms of the norms.

We conclude this secton with some further examples of normed spaces. Firstly a very important class of spaces which are infinite dimensional versions of the ℓ^p_n spaces discussed above:

A. We define ℓ^p $(1 \leq p \leq \infty)$ to be the set of sequences $x = (\xi_n)$ of real (respectively complex) numbers so that

$$\sum |\xi_n|^p < \infty \quad (1 \leq p < \infty)$$

resp.

$$\sup |\xi_n| < \infty \quad (p = \infty)$$

(i.e. ℓ^∞ consists of the bounded sequences). On these spaces we define the mappings

$$|| \, ||_p : x \to \left(\sum |\xi_n|^p\right)^{1/p}$$

resp.

$$|| \, ||_\infty : x \to \sup |\xi_n|.$$

Now these are vector spaces and the $|| \, ||_p$ are norms. Perhaps the easiest way to see this is to use the following two simple facts to reduce to the finite dimensional cases:

a) $x \in \ell^p \Leftrightarrow ||x_n||_p$

b) $||x||_p = \lim ||x_n||_p$
for $p < \infty$ (where $x_n = (\xi_1, \dots, \xi_n, 0 \dots))$.

(That $|| \, ||_\infty$ is a norm is trivial.) Notice that if

$$A_p = \left\{x : \sum |\xi_n|^p \leq 1\right\}$$

then A_p increases with p and so ℓ^p is continuously embedded in ℓ^q if $p < q$. (For if $p < q$ and $x = (\xi_i) \in A_p$ then $\sum |\xi_i|^p \leq 1$ and so $\xi_i| \leq 1$ for each i. Then $\xi_i|^q \leq |\xi_i|^p$ and so $\sum |\xi_i|^q \leq 1$ i.e. $x \in A_q$.)

In the same way we can define the space $\ell^p(S)$ for an arbitrary set S. For $1 \leq p < \infty$ this consists of those functions $x : S \mapsto \mathbf{R}$ for which the expression

$$\sum_{t \in S} |x(t)|^p$$

is finite. Iyt is a normed space under the norm

$$||x||_p = \left(\sum_{t \in S} |x(t)|^p\right)^{1/p}.$$

The space $\ell^\infty(S)$ consists of the bounded functions on S with the obvious norm.

B **Sequence spaces.** A useful generalisation of the ℓ^p-spaces are the so-called **sequence spaces**, that is, vector subspaces of ℓ^∞ which contain ϖ, the set of those sequences which have only finitely many non zero terms. The following are among the most important excamples of such spaces:

$$\begin{aligned}
c_0 &= \{(\xi_n) \in \ell^\infty : \lim_{n\to\infty} \xi_n = 0\}; \\
c &= \{(\xi_n) \in \ell^\infty : \lim \xi_0 \text{ exists}\} \quad (x \text{ for convegent}); \\
cs &= \left\{(\xi_n) \in \ell^\infty : \sum_{n=1}^{\infty} \xi_n \text{ exists}\right\} \quad (c \text{ convergent sections}); \\
bs &= \left\{(\xi_n \in \ell^\infty : \left(\sum_{n=1}^{\infty} \xi_n\right)_m \text{ is bounded}\right\} \quad (\text{bounded sections}); \\
bv &= \left\{(\xi_n \in \ell^\infty : \sum_{n=1}^{\infty} |\xi_{n+1} - \xi_n| < \infty\right\} \text{ is bounded} \quad (\text{bounded variations}); \\
bv_0 &= \{(\xi_n) \in bv'' \lim_{n\to\infty} \xi_n = 0\}.
\end{aligned}$$

We provide these spaces with the following norms:

c_0 and c are given the norm $|| \, ||_\infty$ induced from ℓ^∞. On bs resp. bv we use the norms

$$||('xi_n)||_{bs} = \sup_m \left(\left|\sum_{n=1}^{m} \xi_n\right|\right\}$$

$$||\xi_n||_{bv} = |\xi_1| + \sum_{n=1}^{\infty} |\xi_{n+1} - \xi_n|$$

cs (resp. bv_0) is provided with the norm induced from bs (resp. bv). We leave to the reader the simple task of verifying that they actually are norms.

EXERCISES. If $E \subseteq \ell^\infty$ is a sequence space we define the projection operator $P_n : E \to E$ by putting $P_n((\xi_k)) = (\xi_n, \dots, \xi_n, 0, \dots)$. E is called a K**-space** if these projections are continuous:

It is called

an AK **space** if $P_n x \to x$ for each $x \in E$;

resp. an AD **spaces** if ϖ is dense in E.

(The letters come from the German "Abschnittskonvergenz" and "abschnittdicht"). Which of these properties holds for $\ell^p, c_0, c, bv_0, bv, cs, bs$?

EXERCISES.

A. Let B be a subset of a normed space. Show that B is bounded if ad only if for each sequence (λ_n) is $\mathbf{R}$ which decreases to zero, and each sequence (x_n) in B, $\lambda_n x_n \to 0$. Show that if $x_n \to 0$ in E there is a sequence $'mu_n$ of positive numbers incrasing to infinity such that $\mu_n x_n \to 0$.

B. A normed space E is separable (i.e. E contains a countable dense subset) if and only if there is an increasing sequence (E_n) of finite dimensional subspaces of E so that $\bigcup E_n$ is dense in E.

C. Let E_1, E_2 be normed spaces, $p \in [1, \infty[$. Show that

1. $(x, y) \to (||x||^p + ||y||^p)^{1/p}$ is a norm on $E_1 \times E_2$;
2. each of these norms is equivalent to the norm $(x, y) \to \max\{||x||, ||y||\}$.

Generalise to finite products.

D. Suppose that $0 < p < 1$. Is the mapping

$$(\xi_1, \xi_2) \to (|\xi_1|^p_| \xi_2|^p)^{1/p}$$

a norm on $\mathbf{R}^2$?

E. Show that the norm

$$|| \, || : x \sum^{n-1} i = 0 |x^{(i)}(0)| + ||x^{(n)}||_\infty$$

on $C^n[0, 1]$ is equivalent to the one defined in 1.1.5.C.

F. Let $T \in L(E, F)$ be bijective and suppose that

$$(1 - \epsilon)||x|| \leq ||Tx|| le (1 + \epsilon)||x|| \quad (x \in E).$$

Show that $||T|| \, ||T^{-1}|| \leq \frac{1+\epsilon}{1-\epsilon}$. Deduce that if (x_i) is a basis for the finite dimensional normed space E and (f_i) is the dual basis for E' (i.e. $f_i(x_j) = 0 (i \neq j)$, $1 (i = j)$) and if $\sum ||f_i|| \, ||x_i - y_i|| < \epsilon < 1$, there is an isomorphism $T : E \to E$ mapping x_i into y_i for each i and so that

$$||T|| \, ||T^{-1}|| \leq \frac{1 + \epsilon}{1 - e}.$$

G. Let E be the normed subspace of $C([-1,1])$ consisting of those functions which are the restriction of polynomials to $[-1,1]$. Shwo that the mapping

$$x \to x'(0)$$

is linear, but not bounded, from E into $\mathbf{C}$.

H. Let E, F be normed spaces, $T \in L(E,F)$ and G, H be closed subspaces of E and F respectively. Shows that if $T(G) \subseteq H$ there is a unique $\tilde{T} \in L(E/G, F/H)$ so that diagram commutes.

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$\tilde{T}$ is injective if and only if $G = T^{-1}(H)$ and $||\tilde{T}|| \leq ||T||$ with equality if $H = \{0\}$.

I. Let (T_n) be a norm-bounded sequence in $L(E,F)$ (E, F normed spaces). Show that if there is a dense subset M of E so that $||T_n x|| \to 0$ for each $x \in M$, then $||T_n x|| \to 0$ for each $x \in E$. Show that (T_n) even tends to zero uniformly on precompact subsets of E.

J. Let E, F be normed spaces over $\mathbf{R}$. A mapping $T : E \to F$ is **additive** if

$$T(x+y) = Tx + Ty \quad (x, y \in E).$$

Show that if T is additive and continuous, then T is linear.

K. Let E be a normed space over $\mathbf{R}$. If $x, y \in E$, define

$$H_1(x,y) := \{z \in E : ||z-y|| = ||z-x|| = ||x-y||/2\}.$$

The subsets $H_n(x,y)$ are then defined inductively as follows:

$$\begin{aligned} H_n(x,y) \;&:=\; \{z \in H_{n-1}(x,y) : \text{ for } z_1 \in H_{n-1}(x,y) \text{ we have } 2||z - z_1|| \\ &\leq\; \operatorname{diam} H_{n-1}(x,y)\} \end{aligned}$$

where $\operatorname{diam} H_{n-1}(x,y) : \sup\{||z_1 - z_2|| : z_1, z_2 \in H_{n-1}(x,y)\}$. Show that $\bigcup_{n \in \mathbf{N}} H_n(x,y) = \{(x,y).2\}$. Deduce that if T is a bijective mapping from E onto F so that $T0 = 0$ and

$$Tx - Ty|| = ||x - y||$$

$(x, y \in E)$, then T is linear (note that the above characterisation of $(x+y)/2$ implies that T maps $(x+y)/2$ into $(Tx+Ty)/2$. This implies that T is additive and so linear).

L. A point x in a normed space E with $||x|| = 1$, is said to be an extreme point of the unit ball of E if whenever $x = \frac{1}{2}(y+z)$ with y and z in B_E, then $y = z = x$. What are the extreme points of the unit balls of ℓ^p_n $(1 \leq p \leq \infty)$? How many extreme points does the unit ball of ℓ^1_n (resp. ℓ^∞_n) have?

M. Let C be a compact convex subset of $\mathbf{R}^n$. Show that every point in C can be expressed as a convex combination of at most $n+1$ extereme points of C (use induction on n, distinguishing between the cases whee x lies on the boundary of C resp. in the interior of C).

N. For which sets S do we have that $\ell^1(S)$ and $\ell^\infty(S)$ are isometrically isomorphic (consider the real case only).

O. Show that if $n > 2$, then $\ell^p_n \cong \ell^{p_1}_n$ if and only if $p = p_1$ (again for real spaces).

P. S_n (the set of stoschastic matrices) is the set of those $n \times n$ matrices with non-negative elements so that the row sums are one. Show that his is a closed, bounded compact subsets of $\mathbf{R}^{n^2}$. and that the set of extreme points consists of those matrices with exactly one "1" in each row. What is the corresponding result for the set DS_n of doubly stochastic matrices (i.e. those A for which both A and A^t are stochastic)?

2 The Hahn-Banach Theorem

In this section we shall be cncerned with the infinite dimensional analogue of the duality theory of finite dimensional spaces. The role of the dual V^* of such a vector space is taken by the space of continuous linear mappings fromthe normed spaces E into $\mathbf{R}$ of $\mathbf{C}$ resp. This is called the **dual** of E and denoted by E'.

It is a normed space with the norm

$$||f|| = \sup\{|f(x)| : x \in B_{|| \, ||}\}$$

(i.e. the norm defined on $L(E, \mathbf{R})$—cf. p. 12).

We begin with a characterisation of continuity for linear functionals (i.e. linear mappings with values in the scalar field):

Proposition 3 *Let f be a non-zero linear form on a real normed space E. Then the following are equivalen:*

1. *f is continuous;*

2. *$\mathrm{Ker}\, f$ is closed in E;*

3. *$\mathrm{Ker}\, f$ is not dense.*

Hence a subspace of E is a closed hyperplane (i.e. a closed subspace with codimension 1. of and only if it is the kernel of an $f \in E'$.

PROOF. 1. $\Rightarrow$ 2. and 2. $\Rightarrow$ 3. are trivial.

3. $\Rightarrow$ 1.: Suppose that $\mathrm{Ker}\, f$ is not dense. Then there is a ball $U(x, \epsilon) = \{y \in E : ||y - x|| < \epsilon\}$ so that $f(y) \neq 0$ for $y \in U(x, \epsilon)$. Translating to the origin we get an $\alpha \in \mathbf{R}$ and an $\epsilon > 0$ so that $\alpha \notin f(U(0, \epsilon))$. Now since $U(0, \epsilon)$ is absolutely convex, so is its image in $\mathbf{R}$. Hence $f(U(0, \epsilon) \subseteq] - |\alpha|, |\alpha|[$ i.e. f is bounded on $U(0, \epsilon)$ and so on B_E.

Note that the above proof actually establishes the following quantitative form of the result: if $f \in E'$, then $||f|| \leq 1$ if and only if $H_f^1 \cap U = \varpi$ where H_f^1 is the hyperplane $\{x \in E : f(x) = 1\}$ and U is the open unit ball of E.

Another consequence of the proof of 2.1 is the following: if f is a non-continuous linear functional, then $f(U) = \mathbf{R}$ for every non-trivial open subset U of E. Hence such functionals are highly pathological. We shall later show to construct examples (cf. 3.18.H).

■

We now turn to one of the most basic results on normed spaces—the Hahn-Banach theorem. THis has two forms—an analytical one involving the existence of continuous linear forms with certain propertiesand a geometrical one involving the separation of convex subsets of a normed space by hyperplanes. We shall begin with the analytic approach—the geometric one will be dealt with later. In order to give the result in its most natural form we require the following generalisation of the notion of a seminorm:

Definition 3 *A* **subnorm** *on a vector space E is a function $p : E \to \mathbf{R}^+$ so that*

(a) $p(x + y) \leq p(x) + p(y) \qquad (x, y \in E)$;

(b) $p(\lambda x) = \lambda p(x) \qquad (\lambda \geq 0, x \in E)$.

A subnorm bears the same relationship to convex subsets of E wich contain 0 *as seminorms do to absolutely convex subsets. More precisely, if p is a subnorm then*

$$U_p = \{x : p(x) \leq 1\}$$

is such a set. On the other hand, the Minkowski functional of a convex, absorbing set which contains 0 *is a subnorm.*

The version of the Hahn Banach theorem that we shall bring states that a linear functional on a vector space which satisfies a suitable inequality can ve extended to the whole spaces without losing this property.

The main analytical difficulty lies in extending it to a space of dimension one more. This is taken care of in the following result:

Lemma 1 *Let E be a real vector space, p a subnorm on E, F a subspace of E of codimension 1. If f is a linear mapping from F into* $\mathbf{R}$ *so that $f(x) \leq p(x)$ for all $x \in F$, then there is an extnsion $\tilde{f}(x) \leq p(x)$ $(x \in E)$.*

Proof. Choose $x_0 \in E \setminus F$. Every element $y \in E$ has a unique representation $y = x + \lambda x_0$ $(x \in F, \lambda \in \mathbf{R})$. Now if $x_1, x_2 \in F$ then

$$f(x_2)-f(x_01) = f((x_2+x_0)+(-x_1-x_0)) \leq p((x_2+x_0)+(-x_1-x_0)) \leq p(x_2+x_0)+p(-x_1-x_0).$$

hence

$$-f(x_1) - p(-x_1 - x_0) \leq -f(x_2) + p(x_2 + x_0)$$

and so

$$\sup\{-f(x_1) - p(-x_1 - x_0) : x_1 \in F\} \leq \inf\{-f(x_2) + p(x_2 + x_0) : x_2 \in F\}.$$

Thus we can choose $\xi \in \mathbf{R}$ lying between these two values and define $\tilde{f}$ on E by

$$\tilde{f} : (x + \lambda x_0) \to f(x) + \lambda\xi.$$

$\tilde{f}$ is clearly linear and, if $\lambda : 0$,

$$\xi \leq p\left(\frac{x}{\lambda}x_0\right) - f\left(\frac{x}{\lambda}\right)$$

and so

$$\lambda\xi + f(x) \leq p(x + \lambda x_0) \text{ i.e. } \tilde{f}(y) \leq p(y)$$

if $y = x + \lambda x_0$ $(\lambda \geq 0)$.

The case $\lambda < 0$ is treated similarly.

■

A standard application of Zorn's Lemma now leads to the following version of the Hahn-Banach theorem:

Proposition 4 *Let E be a real vector space, p a subnorm on E, F a subspace of E. Suppose that f is a linear form on F so that $f(x) \leq p(x)$ $(x \in F)$. Then there is a linear extension $\tilde{f}$ of f to a linear form on E so that $\tilde{f}(x) \leq p(x)$ for all $x \in E$.*

PROOF. We consider the set $\mathcal{P}$ of all pairs (M, g) where M is a subspace of E containing F and g is an extension of f to al linear form on M so that $g(x) \leq p(x)$ on M. We order $\mathcal{P}$ by declaring $(M, g) \leq (M_1, g_1)$ if and only if $M \subseteq M_1$ and g_1 is an extension of g. $\mathcal{P}$, under this ordering, satisfies the chain condition and so has a maximal element (M_0, g_0). But it folows easily from 2.3 that $M_0 = E$ and so the Proposition is proved.

In functional analysis we usually require the following version of this result.

■

Proposition 5 *Let E be a vector space over $\mathbf{R}$ (resp. $\mathbf{C}$), $|| \, ||$ a semiorm on E, F a subspace of E, f a linear mapping from F into $\mathbf{R}$ (resp. $\mathbf{C}$) so that $|\tilde{f}| \leq || \, ||$ on F. Then there exists a linear extension $\tilde{f}$ of f to E so that $|\tilde{f}| \leq || \, ||$ on E.*

PROOF. The case $\mathbf{R}$: This follows immediately from 2.4 since the inequalities $f \leq || \, ||$ and $|f| \leq || \, ||$ are equivalent (for $-f(x) = f(-x) \leq ||-x|| = ||x||$).

The case $\mathbf{C}$: We can regard E as a real vector space (by forgetting that one can multiply by complex numbers). If f is a complex linear form on a subspace F, then

$$g : x \Re f(x)$$

is a real linear form on F. We can recover f from g since

$$\Im f(x) = -\Re(i f(x)) = -\Re(f(ix)) = -g(ix).$$

Now we apply the real case to g to obain a suitable extension $\tilde{g}$ of g to E. Then we define

$$\tilde{f} : x \to \tilde{g}(x) - i\tilde{g}(ix).$$

$\tilde{f}$ is linear extension of f and if $x \in E$ we can choose $\alpha \in \mathbf{C}$ with $|\alpha| = 1$ so that $\alpha \tilde{f}(x) \geq 0$. Then

$$|\tilde{f}(x)| = |\tilde{f}(\alpha x)| = |\tilde{g}(\alpha x)| \leq ||\alpha x|| = ||x||.$$

From this result we can deduce a list of corollaries.

■

Proposition 6 *Let E be a subspace of a normed space $(E, || \, ||)$. If f is a continuous linear form on F, then there is an extension $\tilde{f}$ of f to a continuous linear form on E so that $||\tilde{f}|| = ||f||$.*

PROOF. ince $|f(x)| \leq ||f|| \, ||x||$ for $x \in F$, we can apply 2.5 with the seminorm $x \to ||f|| \, ||x||$.

■

Corollar 2 *If $x \in E$, there is a continuous linear form f on E so that $f(x) = ||x||$ and $||f|| = 1$.*

PROOF. Let F be the one-dimensional subspace spanned by x. Apply 2.6 to the linear form

$$\lambda x \to \lambda ||x|| \text{ on } F.$$

■

Corollar 3 *Let F be a subspace of $(E, || \, ||)$ and let $x_0 \in E$ be so that $d_{||\,||}(x_0, F) > 0$. Then there is a continuous linear form $\tilde{f}$ on E so that*

(a) $\tilde{f}(x_0) = 1$

(b) $\tilde{f}(x) = 0$ *if* $x \in F$

(c) $d_{||\,||}(x_0, F) ||\tilde{f}|| = 1$.

PROOF. Let M denote the linear span of $F \cup \{x_0\}$. On M we define a linear form f as follows:

$$f : x + \lambda x_0 \to \lambda \quad (x \in F, \lambda \in \mathbf{C}).$$

Then since

$$||x + \lambda x_0|| = |\lambda| \left\| \frac{x}{\lambda} + x_0 \right\| \geq |\lambda| d_{||\,||}(x_0, F)$$

it follows that f is continuous on M and $d|| \, ||(x_0, F). \; ||f|| \leq 1$. On the other hand, for $0 < \epsilon$, there is an $x \in F$ so that $||x - x_0|| < d_{||\,||}(x_0, F) + \epsilon$ and so

$$1 = |f(x - x_0)| \leq ||x - x_0|| \, ||f|| \text{ i.e. } (d_{||\,||}(x_0, F) + \epsilon)||f|| \geq 1.$$

Hence $d_{||\,||}(x_0, F)||f|| = 1$ and the result follows from an application onf 2.6 to f.

Corollar 4 *Let G be a subset of E, $y_0 \in E$. Then y_0 lies in the closed linear span of G if and only if the following condition holds: for each continuous linear form f on E, if f vanishes on G, then $f(y_0) = 0$. In particular, a sequence (x_n) in E is complete (i.e. such that its closed linear span $[\bar{x}_n] = E$) if and only if whenever $f' in E'$ is such that $f(x_n) = 0$ for each n, then $f = 0$.*

Proposition 7 *Let A be closed, absolutely convex subset of a normed space E, x_0 a point in $E \setminus A$. Then thee is a continuous linear form f on E so that $|f(x) \leq 1$ for $x \in A$ and $f(x_0) > 1$.*

PROOF. Choose $\epsilon > 0$ so that $2\epsilon < d(x_0, A)$. Then the set $U = A + \epsilon B_{|| \, ||}$ is absolutely convex and absorbing and $||x_0||_U > 1$ where $|| \, ||_U$ is the Minkowski functional of U and so by applying 2.5 to $(E, || \, ||_U)$ we get a linear form f with $|f| \leq 1$ on U (and hence on A) and $f(x_0) > 1$. f is continuous since it is bounded on $B_{|| \, ||}$.

■

Corollar 5 *A subset A of a normed space E is closed and absolutely convex if and only if there is a set S of continuous linear form on E so that*

$$A = \{x \in E : |f(x)| \leq 1 \text{ for each } f \in S\}.$$

EXERCISES.

A. Let A be a subset of the normed space $(E, || \, ||)$, f a mapping from A into $\mathbf{C}$. Show that f can be extended to a continuous linear functional on E if and only if the following conditions holds: There exists $K > 0$ so that for each $x - 1, \ldots, x_n$ in A and $\lambda_1, \ldots, \lambda_n$ in $\mathbf{C}$

$$\left| \sum_{k=1}^{n} \lambda_k f(x_k) \right| \leq K \left\| \sum_{k=1}^{n} \lambda_k x_k l \right\|.$$

B. A family $\{x_\alpha\}_{\alpha \in A}$ of elements of a normed space E is **topologically free** if, for each $\alpha \in A$, x_α does ot lie in the closed linear span of $\{x_\beta : \beta \in A, \beta \neq \alpha\}$ (i.e. the smallest closed subspace of E containing $\{x_\beta\}_{\beta \in A} \setminus \{x_\alpha\}$). Show that this is equivalent to the following condition: there exists a family $\{f_\alpha\}\alpha \in A$ of continuous linear forms on E so that for each $\alpha, \beta \in A$, $f_\alpha(x_\beta) = 0$ $(\alpha \neq \beta)$, $f_\alpha(x_\alpha) = 1$.

C. Let $T : E_1 \to F$ be a continuous linear mapping from E_1 into F (E_1 a normed space, F a finite dimensional normed space). If E_1 is a subspace of the normed space E, show that there is a $\tilde{T} \in L(E, F)$ with $\tilde{T}|_{E_1} = T$. Can one always et such an extension with $||\tilde{T}|| = ||T||$?

D. Show that if E_1 is a finite dimensional subspace of E, there is a bounded linear projection P onto E_1. Deduce that there is a closed subspace $F_1 \subseteq E$ with

$$E = E_1 \oplus F_1.$$

E. Let E_1 be a subspace of the normed space E and suppose that $T \in L(E_1, \ell^\infty)$. Show that there is a $\tilde{T} \in L(E, \ell^\infty)$ which extends T and has the same norm. Deduce that ℓ^∞ is complemented in any space into which it can be isometrically embedded.

F. Let $x_1, \dots, x_n$ be lienearly independent vectors in the normed space E. Show that there are elements $f_1, \dots, f_n$ in the dual so that

$$f_i(x_j) = \begin{cases} 0 & \text{if} \quad i \neq j \\ 1 & \text{if} \quad i = j. \end{cases}$$

We now consider in more detail the dual space E' of a normed space. We begin by calculating the duals of some concrete spaces:

I. Finite dimensional spaces: If E is a finite dimensional space, then $E' = E^*$, the algebraic dual, and this has the same dimension as E. Hence the interesting part is the calculation of the geometric form of the unit ball. We caltulate this explicitly for the spaces ℓ^p_n $(1 \leq p \leq \infty)$. To do this we use the following inequality: suppose that $1 < p < \infty$ and q is conjugate to p i.e. is such that $\frac{1}{p} + \frac{1}{q} = 1$. Then $xy \leq \frac{x^p}{p} + \frac{y^p}{q} \quad (x, y \geq 0)$.

EXERCISES. Prove this inequality by examining the minimum of the function

$$y \mapsto \frac{x^p}{p} + \frac{y^p}{q} - xy.$$

We can now state our result: if $1 \leq p \leq \infty$, the dual of ℓ^p_n is ℓ^q_n where p and q are conjugate. (N.B. 1 and ∞ are conjugate.)

More precisely, we mean that if we identify the dual of $\mathbf{R}^n$ with itself in the usual way then the norm of a y as a functional on ℓ^p_n is the same as its ℓ^q-norm. To make the proof a little clearer we introduce the map

$$T : y \to T_y \text{ from } \mathbf{R}^n \text{ into } (\mathbf{R}^n)^*$$

where T_y is the form $(\xi_1, \dots, \xi_n) \to \xi_1 n_1 + \dots + \xi_n \eta_n$ $(y = (\eta_1, \dots, \eta_n))$. We must show that $||T_y|| = ||y||_q$. We assume that $1 < p < \infty$ (the case $p = \infty$ or $p = 1$ is easier).

Firstly $||T_y|| \leq ||y||_q$. For if $x = (\xi_n) \in \mathbf{R}^n$ with $||x||_p \leq 1$ and $||y||_q \leq 1$ then

$$|T_y(x)| = \sum |\xi_i \eta_i| \leq \frac{\sum |\xi_i|^p}{p} + \frac{\sum |\eta_i|^p}{q}$$

by the above inequality: Hence $||T_y|| \leq 1$ if $||y||_q \leq 1$.

On the other hand we have $||T_y|| \geq ||y||_q$. For if $y = (\eta_i)$ with $||y||_q = 1$ we choose $x = (\xi_i)$ where $\xi_i = \pm |\eta_i|^{q-1}$ with $\pm$ sing according to the sign of η_i. THen

$$||x||^p_p = \sum |\eta_i|^{p(q-1)} = \sum |\eta_i|^q = 1 \quad (\text{since } p(q-1) = q).$$

Also

$$||T_y(x)|| = \sum |\eta_i|^q = 1 \text{ and so } ||t_y|| \geq 1.$$

II. Infinite dimensional spaces: Using this result we can calculate the duals of the ℓ^p-spaces $(1 \leq p < infty)$. Firstly, we note that to every $f \in (\ell^p)'$ $(1 \leq p \leq \infty)$, we can associate a sequence $y = (\eta_n)$ by defining

$$(\eta_n) = f(e_n) \text{ where } e_n = (0, \dots, 0, 1, 0, \dots).$$

The proof uses the following steps:

1. if $y = (\eta_n) \in \ell^q$ then $T_y : x \to \sum \xi_i \eta_i$ is a continuous linear form on ℓ^p with $||T_y|| \leq ||y||$ (this is proved as in the finite dimensional case).

2. If $y_n = (\eta_1, \dots, \eta_n, 0, \dots, 0)$ then $||y_n||_q = ||T_{y_n}|| = ||T_{y_n}|| \leq ||y||$. Hence the ℓ^q norms of the sections (y_n) are bounded in ℓ^q. From this it follows that $y \in \ell^q$.

3. f agrees with T_y on the finite dimensional spaces ℓ^p_n and hence on ℓ^p by continuity, if $p < \infty$.

4. The equality of the norms: this follows from the fact that $||y_n||_q = ||T_{y_n}||$ and the facts that $||y_n||_q \to ||y||_q$ and $||T_{y_n}|| \to ||T_y||'$.

Warning: The dual of ℓ^∞ is **not** ℓ^1 and in fact cannot be identified in any natural way with a sequence space. The problem lies in the fact that ℓ^∞ is not an AK space i.e. if $x \in \ell^\infty$ it need not happen that $p_n x \to x$ (in fact, this happens if and only if $x \in c_0$). This means that step 3) above breaks down. However, the same proof shows that the dual of c_0 is ℓ^1.

We return to some more general facts about linear functionals. Using the Hahn-Banach theorem, we can establish the following symmetry between the norms in E and E'.

Proposition 8 *If $x \in E$, then*

$$||x|| = \sup\{|f(x)| : f \in E', ||f|| \leq 1\}.$$

PROOF. This is a restatement of 2.7.

■

As is well known, finite dimensional space are isomorphic (in a natural way) to their second duals, and isomorphic (but not in a natural way) to their duals. This is mirrored in the fact that, for a general infinite dimensional space,E and E' can have completely different linear topological structures. On the other hand there is a natural isometry of E into its second dual, which need not in general be onto.

Notation. We write E'' for $(E')'$, the dual of E'. It is called the **bidual** of E. There is a natural mapping $J_E : E \to E''$ defined by

$$J_E : x \to (f \to f(x)).$$

It is clearly inear and 2.14 can be restated as follows: J_E is an isometric embedding from E onto a subspace of E''. In general, as we shall see later, this mapping need not be surjective. If it is, we say that E is **reflexive**.

Exercises. Every finite dimensional space is reflexive (clear). If$1 < p < \infty$, then ℓ^p is reflexive. This follows from the equalities $(\ell^p)'' \cong (\ell^q)' \cong \ell^p$ with a **little care**. (There are examples of non reflexive spaces for which E'' and E are isometrically isomorphic). To do this, we note that if $f \in (\ell^p)'$, $x \in \ell^p$, then

$$J)\ell^p x(f) = f(x) = T_y(x) = \sum \xi_i \eta_i$$

where $i = Tx$. Since every element of $(\ell^q)'$ is of this form, this shows that J_{ℓ^p} is surjective.

We shall return to the concept of reflexivity in more detail later. Before doing so, we investigate the behaviour of the dyality with respect to the simple operations on Banach spaces discussed in the first paragraph.

Firstly, we identify the dual of a subspace of E with a quotient of E'. Let F be a subspace of a normed space E. If $f \in E'$ we write $\rho_F \circ f$ for the restriction of f to F. Then ρ_F is a linear mapping from E' into F' and its kernel is the set $\{f \in E'' f(x) = 0 \text{ for each } x \in\}$. We call this set the **polar** of F in E' (written F^0). It is a closed subspace of E'. The mapping $\rho_F : E' \to F'$ induces an injective mapping $\tilde{\rho}_F$ from E'/F^0 into F'.

Proposition 9 *$\tilde{\rho}_F$ is an isometry from E'/F^0 onto F'.*

PROOF. It is clear that $\tilde{\rho}_F$ is linear and norm decreasing. Ito follws from 2.6 that for each $g \in F'$ there is an $f \in E'$ with $\rho_F(f) = g$ and $||f|| = ||g||$. This means that ρ_F maps the unit ball of E' onto the unit ball of F' and this clearly implies the result.

■

Corollar 6 *If F is a dense subspace fo E, then F' and E' are isometrically isomorphic.*

Now we discuss quotients—we shall show that the dual of a quotient space is a subspace of the dual space. For suppose that F is a closed subspace of the normed space E and write π_F for the projection from E onto the quotient space E/F. If $f \in (E/F)'$, then f defines in a natural way a continuous linear form on E, namely the composition

$$E \stackrel{\pi_F}{\to} E/F \stackrel{f}{\to} \mathbf{C}.$$

We write $\tilde{\pi}_F(f)$ for the form $f \circ \pi_F$.

Proposition 10 *The mapping $\tilde{\pi}_F : f \to \tilde{\pi}_F(f)$ is a linear isometry from $(E/F)'$ onto F^0.*

PROOF. The image of $(E/F)'$ under π_F consists of those elements of E' which can ve lifted to E/F and this is precisely F^0. If $f \in (E/F)'$, $x \in B_E$, then

$$|\tilde{\pi}_F(f)(x)| = |f(\pi_F(x)) \leq ||f|| \, ||\pi_F(x)|| \leq ||f||$$

and so $||\tilde{\pi}_F(f)|| \leq ||f||$. On the other hand, if $y \in E/F$, $||y|| \leq 1$, then for $\epsilon > 0$, there is an $x \in E$ with $||x|| \leq 1 + \epsilon$ and $y = \pi_F(x)$. Then

$$||f(y)| = |\tilde{\pi}_F(f)(x)| \leq ||\tilde{\pi}_F(f)||(1 + \epsilon)$$

and so $||f|| \leq ||\pi_F(f)||$.

■

Now we turn to products. Just as in the case of finite dimensional vector spaces, the dual of a product is the product of the duals. However we have to be al little more careful with the norms. Recall that

$$|| \, ||_\infty : (x_1, \dots, x_n) \to \max\{||x_1||, \dots, ||x_n||\}$$

is a norm on the linear space $E := E_1 \times \cdots \times E_n$, as is the mapping

$$|| \, ||_s : (x_1, \dots, x_n) \to ||x_1|| + \cdots + ||x_n||.$$

$(E, || \, ||_\infty 0$ is called the **normed product** ot the spaces (E_k), $(E, || \, ||_s)$ the **normed sum** (as we know, these two spaces are isomorhic (1.5.D). If $f = (f_1, \dots, f_n)$ is an element of $E_1' \times \cdots \times E_n'$, then the mapping

$$s_F : (x_1, \dots, x_n) \to f_1(x_1) + \cdots + f_n(x_n)$$

is a continuous linear form on E. Infact this establishes an isomorhism between $(\prod E_k)'$ and $(\prod E_k')$.

Proposition 11 *The mapping $f \to S_f$ is an isometric isomorphism from $(\pi E_k', || \, ||_s)$ onto $(E, || \, ||_\infty)'$ (resp. $(\pi E_k', || \, ||_\infty)$ onto $(E, || \, ||_s)'$).*

Proof. It is clear that $f \to S_f$ is linear. If $g \in E'$, we define the form f_k on E_k as the composition $E_k \to E \stackrel{g}{\to} \mathbf{C}$ where the first mapping is the natura injection from E_k into E. Then if $f := (f_1, \dots, f_n)$, $S_f = g$ and so the mapping is surjective.

We now assume that E has the norm $|| \, ||_\infty$, E' the norm $|| \, ||_1$. If $f \in \prod_{k=1}^n E_k'$, $x \in B_E$, then

$$|S_f(x)| = |f_1(x_1) + \cdots + f_n(x_n)| \leq ||f_1|| \, ||x_1|| + \cdots + ||f_n|| \, ||x_n|| \leq ||f||_s ||x||$$

and so $||S_f|| \leq ||f||_s$.

On the other hand, if $\epsilon > 0$, there is an $x_k \in B_{E_k}$ so that $f_k(x_k) \geq ||f_k|| - \epsilon/n$. If $x := (x_1, \dots, x_n)$, then $||x||_\infty \leq 1$ and $S_f(x) \geq ||f||_s - \epsilon$ and so $||f||_s \leq ||S_f||$.

Exercises. Complete the proof of 2.20.

In orde to relate the concept of duality to linear mappings we bring the following defiition which is completely analogous t the finite dimensional one.

Definition 4 *If $T \in L(E,F)$ $(E, f$ normed spaces), we denote by T' the mapping $f \to f \circ T$ from F' into E'—the* **adjoint** *or* **transported mapping** *of T. Then by definition $T'(f)(x) = f(Tx)$ $(f \in F', x \in E)$.*

For example if $A = (a_{ij})$ is an $m \times n$ matrix with T_A the associated linear mapping from $\mathbf{C}^n$ into $\mathbf{C}^m$ (cf. 1.9.A) and if we identify $(\mathbf{C}^n)'$ with $(\mathbf{C}^n)$ then the adjoint mapping $(T_A)'$ is T_At where A^t is the transposed $n \times m$ matrix (a_{ji}).

If E and F are as in 2.16 then ρ_F is the adjoint of the natural injection i_F from F in E. Similarly, $\tilde{\pi}_F$ is the adjoint of π_F.

It is easy to check that $(\lambda S + \mu T)' = \lambda S' + \mu T'$ and $(S \circ T)' = T' \circ S'$. Hence if $T \in L(E,F)$ is an isomorhism so is T'. For if S is an inverse for T, S' is an inverse for T'. Of course, if $T \in L(E,F)$, then T' is continuous and in fact, $||T||' = ||T'||$.

Proof. If $f \in F'$, then $||T'(f)|| = \sup\{|T'(f)(x)| : x \in B_E\} = \sup\{|f(Tx)|' : x \in B_E\}$.

■

Hence

$$\begin{aligned} ||T'|| &= \sup\{||T'||(f)|| : f \in B_{F'}\} = \sup_{f \in B_{F'}} (\sup\{|f(Tx)| : x \in B_E\}) \\ &= \sup_{x \in B_E} \{(\sup\{|f(Tx| : f \in B_{F'}\})\} = \sup\{||Tx|| : x \in B_E\} = ||T||. \end{aligned}$$

The following remark is often useful. If $T \in L(E,F)$, we denote by T'' the adjoint of T'. Then the following diagram commutes file=bild3a.eps,height=3.0cm,width=8cm

i.e. if we regard E (resp. F) as a subspace of E'' (resp. of F''), then T'' is an extension of T. For we must show that $T''J_E(x) = J_F T(x)$ ($x \in E$). But if $f \in G$, then

$$(T''J_E(x))(f) = J_E(x)(T'f) = (T''f)(x) = f(T(x))$$

while $(J_F T(x))(f) = f(T(x))$.

We use this to show that a closed subspace of a reflexive space is reflexive. Let F be such a subspace of the reflexive space E and consider the following diagramm

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where the horizontal arrows are the natural injections of F into E resp. its second adjoint. Now J_E is surjective and we must show tht J_F is also. Suppose that the latter is not the case i.e. that there is an $x \in F''$ where is not in the image of J_F. Now the second adjoint of the injection $F \to$ identifies F'' with the bipolar F^{00} of F in E'' (i.e. the polar of F^0 in E'') by 2.16 and 2.19.

Suppose then that $x_0 \in F^{00}$, but that $x_0 \notin F$. Then x_0 has the form $J_E(y_0)$ for $y_0 \in E$. Since $x_0 \notin F$, there is an $f \in E'$ which vanishes on F but is such that $f(x_0) = 1$. Then

$$J_E(y_0)(f) = f(x_0) = 1$$

which contradicts the fact that $f \in F^0$ and $x_0 \in F^{00}$.

As stated int eh introductio to this chapter, the Hahn-Banach theorem cn also be regarded as a geometrical result and we shall now discuss this aspect in some detail. This will also provide an alternative, more intuitive proof of the result.

Recall that if f is a linear form on a normed space E, then f is continuous if and only if there is a non-empty open set U on whch it is bounded or, equivalently, there is some hyperplane of the form

$$H_f^\alpha = \{x \in E : f(x) = \alpha\}$$

which does not intersect U.

Also thenorm of a linaer form f can be given the following geometrical interpretation: it is the inverse of the distance of the point 0 from the hyperplane H_f^1 (for the latter distance is the radius of the largest open ball which does not meet H_f^1 and this is the largest open ball on which f is less than 1 in absolute value).

Similarly, if U is an open, convex subset of the normed space E with $0 \in U$ and p_U is the subnorm whch has U as open ball i.e.

$$P_U(x) = \in \{\rho > 0 : x \in \rho U\}$$

then the inequality $f \leq p_U$ is equivalent to the fact that $H^1_f \cap U = \emptyset$. Using these facts we can restate the analytic Hahn-Banach theorem in the following form:

Proposition 12 *Let U be non-empty, convex open subset of the normed space E and suppose that M is an affine subspace which does not intersect U. Then there exists a closed hyperplane $\tilde{M}$ containing M so that $\tilde{M} \cap U = \emptyset$.*

PROOF. Suppose that M is closed (this is possible since the closure $\bar{M}$ of M also has empty intersection with U). Let E_0 be the subspace of E which is spanned by M. If $x_0 \in M$, then $M_0 = M - x_0$ is a subspace of E_0 with codimension 1 since every $z \in E_0$ has a representation

$$z = x + \lambda x_0 \quad (x \in M, \lambda \in \mathbf{R}).$$

Consider the functional

$$f : x + \lambda x_0 \mapsto \lambda$$

on E_0. f is continuous since its kernel M_0 is closed. Also $M = H^1_f$ and so $H^1_f \cap U = \emptyset$. By the Lemma, we have $f \geq P_U$ on E_0 and so there exists an $\tilde{f} \in E'$ for which the same inequality holds i.e. $\tilde{f} \leq p_U$ on E. Then $\tilde{M} = H^1_{\tilde{f}}$ is the required hyperplane.

■

We shall ow bring a geometrical proof of this resul. We begin with the two dimensional case:

Lemma 2 *Let U be an open, convex subset of the normed space E where* $\dim E \geq 2$. *If $0 \notin U$, there exits a line L through 0 with $L \cap U = \phi$.*

PROOF. It is no loss of generality to suppose that $E = \mathbf{R}^2$ since we can work in a two dimensional subspace of E. Consider the mapping

$$\emptyset : (\xi_1, \xi - 2) \to \frac{(\xi - 1, \xi - 2)}{\sqrt{(\xi_1^2, \xi_2^2)}}$$

from $\mathbf{R}^2 \setminus 0$ onto the unit circle

$$S^1 = \{x \in \mathbf{R}^2 : \xi_1^2 + \xi_2^2 = 1\}.$$

Since U is open and convex, its image $\phi((U)$ is an open angular interval in S^1 whose angular length is at most π (otherwise, we would have two points in U which lie on a straight line through 0 and are on opposite sides of 0. Then $0 \in U$ by convexity). Hence we can find an $x \in S^1$ with $x \notin \phi(U)$ and $-x \notin \phi(U)$. This means that the line through x and $-x$ does not cut U.

■

We now prove 2.23 directly:

PROOF. We can suppose without loss of generality that $0 \in M$ (i.e. M is subspace) and that M is closed. Put

$$\mathcal{M} = \{\mathcal{N} \subseteq \mathcal{E} : \mathcal{N} \text{ is closed subspace of } \mathcal{E}, \mathcal{M} \subseteq \mathcal{N} \text{ and } \mathcal{N} \cap \mathcal{U} = \phi\}.$$

The directed set $(\mathcal{M}, \subseteq)$ has a maximal element, say $\tilde{M}$, by Zorn's Lemma. We clain that $\tilde{M}$ is a hyperplane. If not, then the quotient space $E/\tilde{M}$ is a normed space with dimension at least 2 and $\tilde{U} = \pi(U)$ doesn't cntain 0 (where π is the natural quotient map from E onto $E/\tilde{M}$). Then there is a line L through 0 with $L \cap \tilde{U} = \phi$. $\tilde{\tilde{M}} = \pi^{-1}(L)$ is a closed subspace of E which strictly contains M and fails to meet U. This contradicst the maximality of M.

■

EXERCISES. The above proof implicitly uses the following fact: if F is a closed subspace of the normed space E, then π, the natural projection from E onto E/F, is open (i.e. if U is open in E, then $\pi(U)$ is open in E/F. Prove this.

Of course, we can reverse the above process and deduce the analytic version of the Hahn-Banach theorem from 2.23. The readre is invited to carry this out in detail.

EXERCISES.

A. Let A and B be non-empty convex subsets of the normed space E and suppose that they do not intersect. Show that if A has non-empty interior, then there is a closed hyperplane B_f^α which **separates** A and B i.e. is such that

$$A \subseteq \{x : f(x) \leq \alpha\}, \quad B \subseteq \{x : f(x) \geq \alpha\}.$$

Show that if A and B are both open then the above hyperplane **separates A and B strictly** i.e.

$$A \subseteq \{x : f(x) < \alpha\}, \quad B \subseteq \{x : f(x) > \alpha\}.$$

B. Show that if A is a closed, convex set and B is a compact, convex set (both non-empty) with empty intersection then there is a closed hyperplane which strictly separates A and B.

C. Show that if K is a closed, convex subset of a normed space E, then K is the intersection of sets of the form

$$L_f^\alpha = \{x : f(x) \leq \alpha\}$$

where $f \in E'$ (these are called closed **half-spaces**). Show also that each point x on the boundary ∂K of K has a **supporting hyperplane** i.e. there is an $f \in E'$ with

$$K \subseteq \{y \in E : f(y) \leq f(x)\}.$$

We close this section with an important characterisation of reflexive spaces. The proof is based on a generalisation of the Hahn-Banach space to so-called locally convex spaces. These will be discussed in more detail in Chapter IV. In the meantime, we bring some preliminary material:

Definition 5 *A* **topological vector space** *is a vector space E, together with a Hausdorff topology so that the mappings*

$$(x, y) \mapsto x + y \text{ and } (\lambda, x) \mapsto \lambda x$$

of addition and scalar multiplication are continuous.

We can then define, in the obvious way, the concept of continuous linear mappings between topological vector spaces and hence that of a continuous linear functional on such a space.

The most common way of defining a topological vector space is by means of seminorms as follows: let E be a vector space, S a family of seminorms on E which separates the latter i.e. is such that if $x \in E$ is nonzero then there exists $p \in S$ so that if $x \in E$ is non-zero then there exists $p \in S$ so that $p(x) \neq 0$. Then if we define

$$V_{p,\epsilon} = \{x \in E : p(x) < \epsilon\}$$

the sets of the form $\{V_{p,\epsilon} : p \in S, \epsilon > 0\}$ are a subbasis of neighbourhoods of zero for a linear topology on E.

At present we are only interested in the following types of locally convex spaces:

Let E be a normed space, F a separating subset of the dual (i.e. F is such that if $x \in E$ and $f(x) = 0$ for each $f \in F$, then $x = 0$). We define the so-called **weak topology** $\sigma(E, F)$ on E to be that linear topology which is defined by the family $\{p_f : f \in F\}$ of seminorms ofn E where

$$p_f : x \mapsto |f(x)|.$$

Thus this topology has a subbasis consisting of the sests

$$U_{x_0, f\epsilon} = \{x \in E : |f(x) - f(x_0)| < \epsilon\}$$

as x_0 runs through E, f through F and $\epsilon > 0$. (Note that if F is a subspace, which it usually will be in our applications, we can replace the ϵ be a "1").

We shall be almost exclusively concerned with two cases:

a) F is the dual E' of E. The resulting topology $\sigma(E, E;)$ is simply referred to as the **weak topology** of E;

b) E is the dual F' of a normed space and we regard F as a subspace of $E' = F''$. The resulting topology $\sigma(F', F)$ is called the **weak $*$-topology** of f'.

We note the following properties of weak topologies:

As a first step towards the characterisation of reflexive spaces we prove an important result which is generally knows as ALAOGLU's theorem:

Proposition 13 *The unit ball $B_{F'}$ of the dual of a normed space F is $\sigma(F', F)$-compact.*

PROOF. The proof is an application of Tychonov's theorem that the product of campact spaces is compact. We consider the mapping

$$f \to (f(x))_{x \in E}$$

from $B_{F'}$ into the product space $\prod_{x \in E} I_x$ where I_x is the closed interval $[-||x||, ||x||]$ in $\mathbf{R}$.

Now by the very definition of the weak topology, this is a homeomorphism of $B_{F'}$, onto its range and so, since the product is compact, it will suffice to show that the latter is closed. However, regarding the elements of the product as functions on E, the elements of the range are characterised by their linearity (boundedness is automatically ensured since $|f(x) \leq ||x||$ for any function in the product). But linearity means that the functional is in the intersection of the sest of the form

$$U_{\lambda, x} = \{f : f(-x, x) = \lambda f(x)\}$$

resp.

$$V_{x,y} = \{f : f(x+y) = f(x) + f(y)\}$$

as λ, x, y range through $\mathbf{R}$, resp. E and each of these sets is clearly closed in the product.

When E is reflexive, we can identify E and E'' and then the weak topology $\sigma(E, E')$ on E and the weak topology $\sigma(E'', E')$ on E'' coincide. Hence the following corollary follows immediately from the above:

■

Corollar 7 *If E is reflexive, then its unit ball B_E is $\sigma(E, E')$-compact.*

We shall now proceed to prove the converse. In order to do this we require a version of the Hahn-Banach theorem for locally convex spaces. This will be discussed in detail in Chapter IV.2. For our purposes, the following version which can be proved almost ecactly as 2.10 will suffice.

Proposition 14 *Let U be an open, non-empty convex subset of a locally convex space E, x a point in $E \backslash U$. Then there is a contiuous linear functional f in E' so that $f(x) \leq 1$ on U and $f(x) > 1$.*

In order to apply this Lemma to spaces with weak topologies, we must identify their duals. We use a simple algebraic Lemma:

Lemma 3 *Let $f_1, \dots, f_n$ be linear functionals on a vector space E. Then if $f \in E^*$ is such that $\bigcap Ker(f_i) \subseteq Ker f$, f is a linear combination on the f_i.*

PROOF. We can assume that the f_i are linearly independent (if they are not, we replace them by a linearly independent family with the same span). Then the mapping

$$\pi : x \mapsto (f_1(x), \dots, f_n(x))$$

is a surjection from E onto $\mathbf{R}^n$. The hypothesis states that f vanishes on the kernel of this mapping. Hence it can be written in the form $g \circ \pi$ where g is in the dual of the finite dimensional space $\mathbf{R}^n$. If we take into account the form of linear functionals on $\mathbf{R}^n$, we see that this is just the required result.

■

From this we can deduce the following description of the duals of spaces with weak topologies:

Proposition 15 *Let F be a subspace of the dual E' of the normed space E which separates the points of E. Then the dual of E with the weak topology $\sigma(E, F)$ is F (more precisely, each $f \in F$ is $\sigma(E, F)$-continuous and each $\sigma(E, F)$-continuous linear form is in F). In particular, the dual of $(E, \sigma(E, E'))$ is E' and the dual of $(E', \sigma(E', E))$ is E.*

Proof. It is clear that $f \in F$ defines a $\sigma(E,F)$-continuous form. On the other hand, if $f \in E$ is $\sigma(E,F)$-continuous then it is bounded on some neighbourhood of 0 which we can take of the form $\{x : |f_1(x)| \leq 1, \dots, |f_n(x)| \leq 1\}$ for elements $f-1, \dots, f_n$ of F. But then $\bigcup_i \operatorname{Ker} f_i \subseteq \operatorname{Ker} f$ and so f is a linear combination of the f_i i.e. in F.

■

These results can be conveniently restated using **polar sets** which are defined as follows: if $A \subseteq E$, then the **polar** A^0 in E' is the set $\{f \in E' : |f(x)| \leq 1\}$ for $x \in A$. Similarly, if $b \subseteq E'$, its polar B_0 in E is the set $\{x \in E : |f(x)| \leq 1 \text{ for } f \in B\}$. Then if $A \subseteq E$ and $B \subseteq E'$ we can define the bipolars $(A^0)_0$ resp. $(B_0)^0$ which are subsets of E and E' resp. (note that this notation coincides with the polar introduced before 2.16 in the special case of subspaces).

Proposition 16 *If B is a subset of a normed space E, then the bipolar $(B^0)_0$ of B is the $\sigma(E,E')$-closed absolutely convex hull of B.*

Proof. Since a polar set A_0 $(A \subseteq E')$ is the intersection of $\sigma(E,E')$-closed, absolutely convex sets, it automatically has these properties itself. Hence the bipolar is closed and absolutely convex. Note also that the polar (and hence bipolar) of B and $\overline{\Gamma(B)}$, the closed, absolutely convex hull of B, are identical. Hence it suffices to show that if B is closed and absolutely convex, then $B = (B^0)_0$. B is clearly a subset of thi set. Suppose that it is a proper subset i.e. that there exists $x \in (B^0)_0 \setminus B$. Then by 2.31 we can find an $f \in E'$ with $f(x_0) > 1$ and $|f(x)| \leq 1$ $(x \in B)$. But then $f \in B^0$ and so $x_0 \in (B^0)_0$ which is a contradiction.

■

In a similar way we can prove:

Proposition 17 *If B is a subset of E, then $(B^0)_0$ (bipolar in E'') is the $\sigma(E'',E')$-closed, absolutely convex hull of B.*

Exercises. Prove this result.

Corollar 8 *Let E be a normed space. Then the unit ball B_E of E is $\sigma(E'',E')$-dense in $B_{E''}$.*

Proof. Notice that by its very definition, the unit ball of E'' is the bipolar $((B_E)^0)^0$ of B_E (as a subset of E'').

■

Another way of stating this result is as follows: $B_{E''}$ is the completion of B_E with respect to the uniformity induced by the topology $\sigma(E, E')$.

We can now summaris our result as follows:

Proposition 18 *A Banach space E is reflexive if and only if its unit ball B_E is $\sigma(E, E')$-compact.*

PROOF. The only remaining point to be proved is the fact that if the unit ball is compact, then E is reflexive. But this follows immediately from 2.37. ∎

EXERCISES.

A. Show that a convex subset A of a normed space E is norm-closed if and only if it is closed for the weak topology $\sigma(E, E')$.

B. Show that if f is an element in the dual E' of a reflexive normed space, then there exists an $x \in B_E$ so that $f(x) = ||f||$ (we say then that f **attains its norm**).

C. Show that the dual of a reflexive Banach space is also reflexive.

EXERCISES.

A. Calculate the norms of the following elements of the dual of $C([0, 1])$:

$$\delta_t : x \mapsto x(t)$$

$$\sum_{i=1}^{n} \lambda_i \delta_{t_i} \text{ where } t_1, \dots, t_n \text{ are distinct points of } [0, 1];$$

$$x \mapsto \int_0^1 xy \text{ where } y \in C([0, 1]).$$

B. Let F be a subspace of a normed space E. Show that if $f \in E'$ then the distance from f to F^0 is given by the formula

$$\sup\{|f(x)| : x \in F, ||x|| \leq 1\}.$$

C. Show that if a normed space E is reflexive, then so is the quotient E/F by a closed subspace. Show that on the other hand if E has a closed subspace F so that both F and E/F are reflexive, then E is reflexive.

D. Suppose that E and F are normed space and that $x_1, \dots, x_n$ (resp. $y_1, \dots, y_n$) are vectors in E resp. F whereby the x_i are linearly independent. Show that there is a $T \in L(E, F)$ so that $Tx_i = y_i$ for each i.

E. Show that if E is separable, then $(E', \sigma(E', E))$ satisfies the first axiom of countability. Show that if the dual of a normed space E is separable, then so is E but that the converse is false.

K. Let (x_n) be a sequence in ℓ^1 which converges weakly to zero. Show that it also converges in norm to zero.

(This is proved using the so-called **gliding hump method** as follows: show that if (x_n) is a sequence in ℓ^1 so that $x_n \to 0$ weakly but $||x_n|| = 1$, then one can construct a subsequence (x_{k_n}) and a strictly increasing sequence (N_k) of integers so that for each k

$$\sum_{r<N_k} |\xi_r^k| + \sum_{r \geq N_{k+1}} |\xi_r^k| \leq \epsilon \quad \text{(where } x_{n_k} = (\xi_r^k))$$

for some small ϵ. Use this to construct an element f of the dual of ℓ^1 so that $f(x_{n_k}) \not\to 0$.)

3 Banach spaces

As mentioned in the introduction, we require a suitable completeness condition on our normed space in order to obtain more substantial results. The natural one is that of Cauchy comleteness with respect to the induced metric.

Definition 6 *A normed space $(E, || \, ||)$ is called a* **Banach space** *if it is complete under the associated metric i.e. if each Cauchy sequence converges.*

A useful property of Banach spaces is the following: if (x_n) is a sequence so that $\sum ||x_n|| < \infty$, then $\sum x_n$ converges i.e. the partial sums $\sum_{k=1}^n x_k$ converge to a point x in the space. For it follows from the triangle inequality, that the sequence of partial sums is Cauchy. Series with the above property are called **absolutely convergent**.

The following normed space are Banach spaces:

$C(K)$ (K a compact space);

$C^n(I)$ (I a compact interval, $n \in \mathbf{N}$);

$L(E, F)$ (E a normed space, F a Banach space).

In particular, the dual E' of a normed space is always a Banach space.

PROOF. The completeness of $C(K)$: this is essentially a restatement of the fact that the uniform limit of continuous functions is continuous. Similarly, the completeness of $C^n(I)$ follows from that of $C(I)$ and the fact that if a sequence of functions in $C^n(I)$ converges together with their derivatives up to order n, then the limit function is in $C^n(I)$.

■

EXERCISES. Prove that if F is a Banach space, so is $L(E, F)$. Show that if E, F are normed spaces with E non-trivial (i.e. $E \neq \{0\}$) then F is isomorphic to a closed subspace of $L(E, F)$. Deduce that F is complete if $L(E, F)$ is (choose a non-zero continuous linear form f on E and consider the mapping $y \to (x \to f(x)y)$ from F into $L(E, F)$.

EXERCISES. Show that the space $C([0, 1])$ with norm $||x|| = \int_0^1 |x(t)|dt$ is not a Banach space.

It is clear that a closed subspace of a Banach space is a Banach space with the induced norm. Conversely if a normed subspace of a normed space is a Banach space, then it is closed. Also a product of a finite family of Banach spaces is a Banach space.

Just as we can embed non complete metric spaces in complete spaces, so every non complete normed space can be embedded in a Banach space as we show below (3.5).

Proposition 19 *Let L be a dense subspace of a normed space $(E, || \, ||)$, T a continuous linear mapping from L (with the induced norm) into a Banach space $(F, || \, ||_2)$. Then there is a unique continuous linear mapping $\hat{T}$ from E into F which extends T. The norms of $\hat{T}$ and T coincide.*

PROOF. If $x \in E$, there is a sequence (x_n) in L so that $x_n \to x$. (x_n) is Cauchy in L and so (Tx_n) is Cauchy in F. Let $y := \lim Tx_n$. y is independent of the choice of the approximating sequence (x_n) (the reader should check this) and so the correspondence $x \to y$ can be used to define a mapping $\hat{T}$ from E into F. We omit the details required to show that T has the required properties.

■

EXERCISES. Show that when E is separable, 2.5 can be proved without using Zorn's lemma (let $\{x_n\}$ be a dense sequence in E and put $F_n = \operatorname{span} F \cup \{x_1, \dots, x_n\}$. Apply 2.2. sucessively to $F_1, F_2, \dots$. The apply 3.4 to the functional obtained on the dense subspace $\cup F_n$).

Proposition 20 *Let E be a normed space. Then E can be embedded as a dense subspace of a Banach space E_1 so that the following property holds:*

every continuous linear operator T from E into a Banach space F has a unique extension to a continuous linear operator $\hat{T}$ from E_1 into F. The norms of $\hat{T}$ and T coincide.

PROOF. We identify E with $J_E(E) \subseteq E''$ and let E_1 be the closure of $J_E(E)$ in E''. The rest follows from 3.4.

■

The space E_1 is unique in the following sense. Suppose that we can embed E in a second space E_2 with the same properties. Then we can extend both of these embeddings to get linear contractions between E_1 and E_2 which are the identity of E and so are mutually inverses. Hence E_1 and E_2 are in this natural way isometrically isomorphic. This justifies the name **completion** for the above space which we denote by $\hat{E}$.

■

EXERCISES.

A. Show that $B_{\hat{E}}$ is the closure of B_E in $\hat{E}$.

B. Let F be a dense subspace of E. Show that if $x \in E$ there is a sequence (x_n) in F which is such that $\sum ||x_n|| < \infty$ and $\lim_n \sum_{k=1}^n x_k = x$. Deduce that a normed space E is a Banach space if and only if for every sequence (x_n) in E which is such that $\sum ||x_n|| < \infty$, the series $\sum x_n$ converges in E.

Proposition 21 *Let F be a closed subspace of a Banach space E. Then the quotient E/F is also a Banach space.*

PROOF. We use 3.7.B. Let (y_n) be sequence in E/F so that $\sum ||y_n|| < \infty$. Then there is a sequence (x_n) in E so that for each n $\pi_F(x_n) = y_n$ and $||x_n|| \leq ||y_n|| + 2^{-n}$. Then $\sum ||x_n|| < \infty$ and so $\sum x_n$ converges in E, say to x. Clearly $\sum y_n$ converges in E/F to $\pi_F(x)$.

■

We now give a useful criterium for demonstration the completeness of conrete Banach sequence or function spaces.

Proposition 22 *Let S be a set, A is closed, absolutely convex subset of $\mathbf{C}^S$ (with the product topology) which is bounded i.e. such that for each $t \in S$ there is a $K > 0$ so that $|x(t)| \leq K$ whenever $x \in A$. Then if $E_A = \bigcup_n nA$, $(E_A, || \, ||_A)$ is a Banach space.*

PROOF. Of course E_A is a normed space—we show that it is complete. If (x_n) is an $|| \, ||$-Cauchy sequence, each $(x_n(t))$ is Cauchy (this follows from the fact that A is bounded—the reader should check this). Hence by the completeness of $\mathbf{C}$ there is a function x so that $x_n(t) \to x(t)$ for each $t \in S$. We show that $x \in E_A$ and $x_n \to x$ in E_A. For each $\epsilon > 0$ there is an $N \in \mathbf{N}$ so that $x_m - x_n \in \epsilon A$ if $m, n \geq N$. Letting $n \to \infty$ and using the fact that A is pointwise closed we see that $x_m - x \in \epsilon A$ for $m \geq N$. From this it follows easily that $x \in E_A$ and $||x_m - x||_a \to 0$.

■

As an application of this result consider the sets

$$A_p = \left\{ x \in \mathbf{C}^S : \sum_{s \in S} |x(s)|^p \leq 1 \right\}$$

for $1 \leq p < \infty$. We know (1.16) that A_p is close and absolutely convex in $S^{\mathbf{C}}$. Hence the corresponding normed spaces (i.e. $(\ell^p(S), || \, ||_p)$) are complete. In the special case where $S = \mathbf{N}$ we obtain the sequence space ℓ^p and so we see that these are Banach space. Similarly,

$$A_\infty = \{x \in S^{\mathbf{C}} : \sup_{s \in S} |x(s)| \leq 1\}$$

generates

$$\ell^\infty(S) = \{x \in \ell^\infty(S) : \text{ for each } \epsilon < 0 \ \{t \in S : |x(t) \geq \epsilon\} \text{ is finite}\}$$

of $\ell^\infty(S)$.

$c_0(S)$ is closed in $\ell^\infty(S)$ and so is a Banach space. (For if $x_n \to x$ in $\ell^\infty(S)$ and each $x_n \in c_0(S)$, then for $\epsilon > 0$ we choose an x_N with $||x_N - x|| \leq \epsilon/3$. There is a finite $A_1 \subseteq S$ so that the values of x_N outside of S_1 have absolute value at most $\epsilon/3$. Then the values of x outside of S_1 have absolute values at most ϵ. This method of proof is known as an $\epsilon/3$ argument for obvious reasons.)

One of the most important motivations for the study of Banach spaces and their linear operators was the classical theory of integral equations and we conclude this section with some remarks on that subject. We begin with an elementary criterium for the invertibility of an operator which is the basis for many perturbation results:

Proposition 23 *Let E be a Banach space, T an operator in $L(E)$ with $||T|| < 1$. Then $(I - T)$ is an isomorphism.*

PROOF. If $||T|| = \lambda < 1$ then for each n, $||T^n|| \leq \lambda^n$ and so the series $\sum_{n=0} T^n$ is absolutely convergent in $L(E)$. Suppose that it converges to S. Then if $S_n = \sum_{k=0}^n T^k$, $(I-T)S_n = S_n(I-T) = I - T^{n+1}$ and if we let n go to infinity we see that

$$(I-T)S = S(I-T) = I.$$

■

Essentially the same proof gives the following sharper form:

Corollar 9 *If* $T \in L(E)$ *is such that* $\limsup ||T^n||^{1/n} < 1$*, then* $(I-T)$ *is an isomorphism.*

As an application we discuss briefly intergral equations of **Volterra type**. We consider an operator of the form I_K where $K : I \times I \to \mathbf{R}$ is a kernel on a compact interval $I = [a, b]$. We suppose that K is a Volterra kernel, that is

1. $K(s,t) = 0$ if $s < t$.
2. K is continuous on $\{(s,t) \in R^2 : a \leq t \leq s \leq b\}$.

Then I_K is a continuous linear operator from $C(I)$ into itself and $||I_K|| \leq M(b-a)$ where

$$M = \sup\{|K(s,t)| : (s,t) \in I^2\}.$$

We study the integral equation $x(s) = y(s) + \int_a^s K(s,t)y(t)dt$ where x is given and y is sought. In our notation this takes the form

$$x = y + I_K y.$$

I.e. $x = (\text{Id} + I_K)y$.

hence if we can show that $(\text{Id} + I_K)$ is invertible we can write the solution in the form $y = (\text{Id} + I_K)^{-1}x$ and this suggests using the above results.

Before carrying this out, we remark that integral equations of this type arise, for example, in the solution of differential equations of the form

$$y' = pt + q$$

where p, q are given continuous functions on I and a suitable y, say with initial conditions $y(a) = y_0$, is sought. Then, integrating, we can transform this into an integral equation

$$y(s) + \int_a^s (-p(t)y(t)dt = y_0 + \int_a^s q(t)dt.$$

In order to be able to employ 3.11 to solve the Volterra equation $x = y + I_K y$ we analyse the form of the iterates I_K^n of I_K. By elementary results on the double integrals of continuous functions (the so-called "Fubinito"), I_K^2 is the operator I_{K_2} where K_2 is the kernel

$$(s,t) \to \int_s^t K(s,u)K(u,t)du$$

and, more generally, I_K^n is I_{K_n} where the kernel K_n satisfy the recursion formula:

$$K_{n+1} : (s,t) \to \int_s^t K(s,u)K_n(u,t)du.$$

Now it is easy to prove by induction that

$$\sup_{s,t \in I} |K_n(s,t)| \leq \frac{M^n}{(n-1)!}(b-a)^{n-1}$$

and so

$$||I_K^n|| \leq \frac{M^n}{(n-1)!}(b-a)^{n-1}.$$

hence $\limsup ||I_K^n||^{1/n} < 1$ (in fact $= 0$) and so $(I + I_K)$ is invertible. It follows that the given equation always has a unique solution.

(In fact, the formula for the inverse of $(I + I_K)$ gives a little more, namely that the solution is given by the integral formula

$$y = I_{\tilde{K}}(x)$$

where $\tilde{K}$ is the kernel

$$(s,t) \to \sum_{n=0}^{\infty} (-1)^n K_n(s,t)$$

(the right hand side converges uniformly because of the above estimate for K_n). This series representation of the inverse is called the **von Neumann series**.

EXERCISES.

A. Show that the solution of the Fredholm equation

$$x(s) \to \lambda \int_0^1 stx(t)dt = y(s)$$

is

$$x(s) = y(s) + \frac{3\lambda s}{3-\lambda} \int_0^1 ty(t)dt$$

for $|\lambda| < 3$. Show that this solution is also valid for $\lambda \neq 3$.

B. Extend the result of the above paragraphs to systems of Volterra equations:

$$x_i(s) = y_i(s) + \sum_{j=1}^{n} \int_a^s K_{ij}(s,t)y_j(t)dt$$

where $\{K_{ij} : i, j = 1, \dots, n\}$ are Volterra kernels on I. (Work in the Banach space $C(I, \mathbf{R}^n) = C(I)^n$.)

C. Consider the differential equation

$$y' = py + q \text{ with initial value } y(a) = y_0,$$

where p and q are continuous functions on $[a, b]$.

Use the von Neumann series to obtain the classical solution:

$$y(s) = e^p(s)\left[y + 0 + \int_a^s e^{-P(u)}q(u)du\right]$$

where P is the primitive of p which vanishes at a.

The spaces ℓ^1, ℓ^∞ and $C([0,1])$ play an important role in the theory of Banach spaces because of some properties which we now consider. First we need a Lemma.

Lemma 4 *Let E be a separable Banach space. Then the unit ball $B_{E'}$ is a compact metric space (and hence separable) for the weak star topology $\sigma(E', E)$.*

Proof. We know that $B_{E'}$ is compact. Suppose that (x_n) is a dense sequence in B_E. And $\epsilon/3$ argument (cf. Exercise 1.18.I) shows that on $B_{E'}$ $\sigma(E', E)$ coincides with the topology defined by the seminorms

$$f \to |f(x_n)|$$

$(n \in \mathbf{N})$. But as in the proof 2.30 this implies that $B_{E'}$ is homeomorphic to a subspace of a countable product of copies of the unit interval and so is metrisable.

■

Proposition 24 *Let E be a separable Banach space. Then E is isomometrically isomorphic*

a) to a quotient space of ℓ^1;

b) to a subspace of ℓ^∞;

c) to a subspace of $C(K)$ where K is a compact metric space.

PROOF. We choose a dense sequence (x_n) in B_E and a weak $*$ dense sequence (f_n) is B'_E and consider the mappings

a) $(\xi_n) \to \sum \xi_n x_n$ from ℓ^1 into E;

b) $x \to (f_n(x))$ from E into ℓ^∞;

c) $x \to (f \to f(x))$ from E into $C(K)$ where K is $(B_{E'}, \sigma(E', E))$.

■

EXERCISES. Check that these three mappings have the required propertis.

If we are prepared to assume a little descriptive topology we can strengthen part c) above as follows:

Proposition 25 *Let E be a separable Banach space. Then E is isometrically isomorphic to a subspace of $C(p0, 1])$.*

PROOF. The result that we require is the following: for every compact metric space K there is a continuous surjection

$$\pi : \mathbf{Can} \to K$$

where **Can** is the Cantor set (see below). Then

$$x \to x\pi$$

embeds $C(K)$ isometrically as a subspace of $C(\mathbf{Can})$. Now if we realise the Cantor set as a closed subset of the unit interval I in the usual way (by excluded middles), then $C(\mathbf{Can})$ can be regarded as a subspace of $C(I)$ by mapping $x \in C(\mathbf{Can})$ into $\tilde{x}$ where $\tilde{x}$ is obtainded from x by extending the latter linearly on the excluded intervals.

■

EXERCISES. The Cantor space **Can** is one of the most important topological spaces in analysis and we review some of its properties. For many purposes, the most convenient definition is as the product $2^{\mathbf{N}}$ of countably many copies of a 2-point set. The relationship to the more geometric definition as $\cap F_n$ where F_n is the closed subset of $[0, 1]$ indicated in the diagram

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is established as follows. F_n consists of these points in $[0, 1]$ which have a tryadic development of the form

$$x = \sum_{n=1}^{\infty} a_n 3^{-n}$$

where $a_n = 0$ or 2 (together with the point "1"). We map this x onto the sequence $(\tilde{a}_n)$ where $\tilde{a}_n = 0$ if $a_n = 0$ and $\tilde{a}_n = 1$ if $a_n = 2$. This establishes a homeomorphism between $\cap F_n$ and $\{0,1\}^{\mathbf{N}}$.

The importance of the Cantor space lies in the fact which we used above, namely that every compact, metrisable space is the continuous image of **Can** . This is proved by means of the following steps which we now sketch:

1. It is true for the unit interval I (we map a sequence (a_n) in $\{0,1\}^{\mathbf{N}}$ onto $\sum a_n 2^{-n}$).

2. It is true for $I^{\mathbf{N}}$ (for gy 1) there is a continuous surjection from $(\mathbf{Can}\,)^{\mathbf{N}}$ onto $I^{\mathbf{N}}$. But
$$(\mathbf{Can}\,)^{\mathbf{N}} \cong (2^{\mathbf{N}})^{\mathbf{N}} \cong 2^{\mathbf{N}} \cong \mathbf{Can}$$
(the symbol $\cong$ denotes "is homeomorphic to" in this context).

3. Every closed subset of **Can** is a retract i.e. there is a continuous mapping π from **Can** onto the closed subset K_1 so that $\pi(x) = x$ for $x \in K_1$. (To prove this consider the metric
$$d(x,y) = \sum 2^{-n}|a_n - b_n|$$
where $x = (a_n)$, $y = (b_n)$ are in $2^{\mathbf{N}}$. Then if $x \in 2^{\mathbf{N}}$, there is a point $y \in K_1$ so that
$$d(x,y) = d(x,K_1)$$
(this follows from the special nature of the metric that if y_1 is such that $d(x,y) = d(x,y_1)$, then $y = y_1$. Hence the point y is uniquely defined. Then $\pi : x \mapsto y$ is the required mapping.

4. Every compact metric space is homeomorphic to a closed subspace of $I^{\mathbf{N}}$. This is a standard result of elementary topology. It is proved rather similarly to 3.13 by constructing a sequence (x_n) of continuous functions from the space K into $[0,1]$ which separates K (i.e. is such that if $s \neq t$, then $x_n(s) \neq x_n(t)$ for some n) and using the mapping $s \mapsto (x_n(s))$ which is a homeomorphism from K onto a suitable subset of $[0,1]^{\mathbf{N}}$.

Using these four facts we can establish the main result as follows: let K be compact and metrisable. We can assume, without loss of generality, that K is a subspace of $I^{\mathbf{N}}$. Let $K_1 = f^{-1}(K)$ where f is a continuous surjection from **Can** onto $I^{\mathbf{N}}$ and let π be a retraction from **Can** onto K_1. Then $f|_{K_1} \circ \pi$ is the required surjection from **Can** onto K.

EXERCISES.

A. Let $(E_k)_{k=1}^n$ be non-trivial normed spaces. Show that $\prod_{k=1}^n E_k$ is a Banach space if and only if each E_k is.

B. With the notation of 1.11 show that $L(E_1, \dots, E_n : F)$ is a Banach space if F is.

C. Let F be a closed subspace of a normed space E. Show that if both F and E/F are Banach spaces, then E is also Banach.

D. Let T be a continuous linear mapping from $(E, || \, ||_1)$ into $(F, || \, ||_2)$. Let G be a subspace of F with a norm $|| \, ||_3$ so that the injection from G into F is continuous. Show that the mapping

$$|| \, ||_T : x \to ||x||_1 + ||Tx||_3$$

is a norm on $E_T := \{x \in E : Tx \in G\}$ and that $(E, || \, ||_T)$ is a Banach space if E and G are both Banach spaces.

E. With the notation of 1.4 show that E/E_0 is a Banach space if E is complete under the semimetric induced on E by $|| \, ||$.

F. (For readres familiar with the concept of uniform spaces.) Prove the following more general and elegant form of 3.9: Let M be a complete uniform space, $(E, || \, ||)$ a normed space which, as a set, is a subset of M.

Suppose

1. that the uniformity induced by $|| \, ||$ on E is finer than that of M.
2. $|| \, ||$ is lower semicontinuous i.e. for each $\epsilon > 0, y \in E$, $\{y : ||x - y|| \leq \epsilon\}$ is closed in M. Show that E is a Banach space.

G. Let (w_n) be a decreasing sequence of positive numbers which theds to zero but whose sum $\sum w_n$ diverges. We define a sequence space $d(w_n; p)$ $(p \in [1, \infty[)$ as follows:

$$d(w_n; p) := \left\{ x(\xi_n) : ||x|| := \sup \left(\sum_{n=1}^{\infty} |\xi_{\pi(n)}|^{p w_n} \right)^{\frac{1}{p}} < \infty \right\}$$

(the supremum being taken over all permutations of $\mathbf{N}$). Show that this space is a Banach space (the so-called **Lorentz space**).

H. Let E be a Banach space, (x_α) a Hamel basis for E, (f_α) the dual elements in the algebraic dual E^* (i.e. $f_\alpha(x_\beta) = 1$ (if $\alpha = \beta$), 0 (if $\alpha \neq \beta$)). Show that at least one f_α is not continuous if E is infinite dimensional. (Consider an element of the form $\sum \frac{1}{2^n} \frac{x-n}{||x_n||}$ for some sequence of distinct elements of $\{x_\alpha\}$.)

(Note that this result shows that discontinuous linear functionals exist on infinite dimensional Banach spaces. The proof uses implicity the axiom of choise. It has in fact been shown recently that it is consistent with the usual axioms of set theory—without the axiom of choice—to assume the axiom that every linear operator between Banach spaces is continuous. This implies that it is impossible to **construct** a discontinuous linear functional on a Banach space.)

I. Prove that the following sequence spaces are complete: bv, cs, bv_0, bs.

J. Let $A = [a_{ij}]$ be an infinite matrix with $\sup_j \sum_{j=1}^\infty |a_{ij}| < 1$. Show that for each sequence y in ℓ^∞ there is a sequence x in ℓ^∞ so that

$$\xi_i = \sum_{j=1}^\infty a_{ij}\xi_j + \eta_i \quad (i = 1, 2, \dots).$$

K. Let $i : E_1 \to E_2$ be a continuous linear injection and suppose that $i(B_{E_1})$ is closed in E_2. Show that the extension $\hat{i}$ of i to a continuous linear mapping from $\hat{E}_1$ into $\hat{E}_2$ is also an injection.

L. Let E be a Banach space. Show that E contains a subspace isomorphic to ℓ^1 (resp. c_0) if and only if there is an $M > 0$ and a sequence (x_n) of unit vectors in E so that for each finite sequence $\lambda_1, \dots, \lambda_m$ of scalars

$$\sum_{k=1}^m |\lambda_k| \leq M \left\| \sum_{k=1}^m \lambda_k x_k \right\|$$

resp.

$$1/M \sup\{|\lambda_k|\} \leq \left\| \sum_{k=1}^m \lambda_k x_k \right\| \leq M \sup\{|\lambda_k| : k = 1, \dots, m\}.$$

M. Show that if E is a Banach space and $B \subseteq E$ is compact then so is its closed convex hull (i.e. the closure of the convex hull of B). Let E be the normed space ϖ (the space of sequence in ℓ^2 with finite support), provided with the ℓ^2-norm. Show that the set $\{e_n/n\} \cup \{0\}$ is compact, but its closed convex hull is not.

N. Show that the space E of all polynomials is a non-complete normed space under the norm

$$x \to |a_0| + |a_1| + \cdots + |a_n|$$

where $x(t) = a_0 + a_1 t + \cdots + a_n t^n$. Give a concrete representation of its completion as a space of functions.

O. Let E, F be Banach space, with F a subspace of E, $T : \ell^1 \to E/F$ a linear contraction. Then for each $\epsilon > 0$ there is continuous linear operator $T : \ell^1 \to E$ so that

$$\pi T = \tilde{T} \text{ and } ||\tilde{T}|| \leq ||T|| + \epsilon$$

where $\pi : E \to E/F$ is the natural mapping. (Consider the sequence (Te_n) in E/F where (e_n) is the natural basis for ℓ^1.)

P. Let F be a closed subspace of the Banach space E and suppose that E/F is isomorphic to ℓ^1. Show that there is a closed subspace F_1 of E so that $E = F \oplus F_1$ (i.e. F is complemented in E).

Q. Let $x_1, \dots, x_n$ be a partition of unity in a space $C(K)$ i.e. they are non-negative functions with norm one whose sum is the constant function 1. Show that their linear span $[x_1, \dots, x_n]$ is isometrically isomorphic to ℓ^∞_n. Deduce that $C(K)$ has the following property: for each finite dimensional subspace E and each $\epsilon > 0$ there is a finite dimensional subspace F containing E (of dimension m say) and antiisomorphism T from F onto ℓ^∞_m so that

$$(1 - \epsilon)||x|| \leq ||Tx|| \leq (1 + \epsilon)||x|| \quad (x \in F).$$

4 The Banach Steinhaus theorem and the closed graph theorem

We now consider three general results on Banach spaces which are closely related to each other. They all use the Baire category theorem in their proof. The first result is the uniform boundedness theorem which states that a pointwise bounded family of continuous **linear** mappings on a Banach space is bounded in norm. This is the basis for a whole spectrum of results in functional analysis which state roughly that various concepts (for example, analyticity of functions) remains unaffected by (not too violent) changes in the topology.

We first give a simple version of the result which does not use linearity. In this setting we can only obtain "local boundedness" i.e. boundedness in some neighbourhood. We then apply the general principle "local and linear implies global".

Lemma 5 *Let (X,d) be a complete metric space, M a family of continuous mappings from X into $\mathbf{C}$ which is pointwise bounded (that is, for each $s \in X, \sup\{|x(s)| : x \in M\} < \infty$). Then there is a ball $U(x_0,\epsilon)$ in X so that M is uniformly bounded on $U(x_0,\epsilon)$. (Recall that $U(x_0,\epsilon) = \{x : d(x,x_0) < \epsilon\}$.)*

PROOF. Let $A_n := \{s \in X : |x(s)| \leq n \text{ for each } x \in M\}$. Then A_n is closed and $x = \bigcup_{n\in\mathbf{N}} A_n$. Hence, by Baire's category theorem, there is an $n_0 \in \mathbf{N}$ so that A_{n_0} contains a ball. M is clearly bounded on this ball.

Proposition 26 (Principle of uniform boundedness.) *Let E be a Banach space, D a normed space, M a set of continuous, linear mappings from E into F so tha M is pointwise bounded, that is for each $x \in E$ there is a $K > 0$ so that $||Tx||leK(T \in M)$. Then M is norm-bounded in $L(E,F)$.*

PROOF. Then set $\{x \to ||Tx|| : T \in M\}$ satisfies the conditions of 4.1 and so there is a ball $U(x_0,\epsilon)$ in E and a $K_1 > 0$ with $||Tx|| \leq K_1$ for each $x \in U(x_0,\epsilon)$. Then if $x \in E, ||x|| \leq 1$, we have that $x_0 + \epsilon x \in U(x_0,\epsilon)$ and so $||T(x_0 + \epsilon x)|| \leq K_1$. Hence

$$\begin{aligned} ||Tx|| &= 1/\epsilon||T(x_0+\epsilon x) - Tx_0|| \\ &\leq 1/\epsilon(||T(x_0+\epsilon x)|| + ||Tx_0|| \leq K \end{aligned}$$

where $K := 1/\epsilon[K-1+\sup\{||Tx_0|| : T \in M\}]$.

■

This result is most often used in the form of the following corollary for which we introduce some notation: A subset B of a normed space E is **weakly bounded** if for each $f \in E'$, there is a $K > 0$ so that $|f(x)| \leq K(x \in B)$.

Corollar 10 *A subset B of a normed space E is normbounded if it is weakly.*

PROOF. The weak-boundedness of B is equivalent to the fact that $J_E(B)$ is pointwise bounded as a subset of $L(E',\mathbf{C})$. Hence $J_B(B)$ is norm-bounded in E'' and so B is bounded in E (J_E is an isometry).

■

Exercises. If E is a Banach space, M a family of continuous, linear mappings from E into F show that M is uniformly bounded if the following condition is satisfied: for each $f \in F', x \in E$ there is a $K > 0$ so that $|f(Tx)| \leq K(T \in M)$.

The principle uniform boundedness implies easily the following famous result—pointwise limits of sequences of continuous linear mappings are continuous. The decisive fact is that such a sequence is bounded (and so equicontinuous) by 4.2 and of course pointwise limits of equicontinuous families are continuous.

Proposition 27 (The Banach-Steinhaus theorem.) *Let (T_n) be a sequence of continuous linear mappings from a Banach space E into a normed space F so that* $\lim(T_n x)$ *exists for each $x \in$. Then the mapping*

$$T : x \to \Im T_n x$$

is continuous and (of course) linear.

Proof. By using the fact that a Cauchy sequence in a normed space is bounded we see that $\{T_n\}$ is pointwise bounded and so by 4.2 there is a $K > 0$ with $||T_n x|| \leq K||x||$ $(x \in E)$. Since $T_n x \to TX$, $||Tx|| \leq K||x||$ i.e. T is bounded.

■

Another useful consequence is the following theorem which is perhaps the simplest among a whole range of results about special situations where separate continuity of functions of two variables implies joint continuity.

Proposition 28 *Let E_1, E_2, F be normed spaces with E_2 Banach. Let $T : E_1 \times E_2 \to F$ be bilinear and separately continuous (i.e. the partial maps $x \to T(x, y_1)$, $y \to T(x_1, y)$ are continuous for each x_1 and y_1). Then T is continuous.*

Proof. Consider the map

$$\tilde{T} : x_1 \to (x_2 \to T(x_1, x_2)).$$

By the continuity with respect to the second variable, this maps E_1 into $L(E_2; F)$. By the continuity in x_1, the family $\{\tilde{T}(x_1) : x_1 \in B_{E_1}\}$ is pointwise bounded in $L(E_2, F)$ and so (4.2) there is a $K > 0$ such that for each $x - 1 \in B_{E_1}$, $||\tilde{T}(x_1)|| \leq K$.

Hence if $x_1 \in B_{E_1}, x_2 \in B_{E_2}$, then $||T(x_{,1}, x_2)|| \leq K$ i.e. T is continuous.

■

Exercises.

A. Strengthen the results of 4.2 and 4.5 as follows: Show that in 4.2 it is sufficient to assume that M is pointwise bouded on a set of second category in E and in 4.5 it is sufficient to assume that (T_n) is convergent on a set of second category in E.

B. Let E be a Banach space whose elements are functions on an interval $I \in \mathbf{R}$. Show that if

(a) for each $t \in I$, the form $x \to x(t)$ is continuous on E;

(b) each function x in E is differentiable at a given point $t_0 \in I$, then the mapping $x \mapsto x;(t_0)$ is continuous on E.

C. Let E denote the normed subspace of $C([-1,1])$ consisting of the restriction of the polynomial to $[-1,1]$. Consider the mappings $f_n : E \to \mathbf{C}$ where

$$f_n : x \to n(x(n^{-1}) - x(0)).$$

Show that (f_n) is pointwise bounded but not norm bounded.

D. Let E be a normed space. A sequence (x_n) is a **basis** for E if for each $x \in E$ there is a unique family (ξ_n) of scalars so that $x \in \sum_{k=1}^{\infty} \xi_k x_k$. Then one can define mappings $f_n : E \to \mathbf{C}$, $S_n : E \to E$ by

$$f_n : x \to \xi_n, S_n : x \to \sum_{k=1}^{n} \xi_k x_k$$

(x_k) is a **Schauder basis** if and only if the (S_n) (or equivalently, the (f_n)) are continuous. Show that

(a) a basis is a Schauder basis if and only if it is topologically free (cf. 2.12.B);

(b) if E is a Banach space and (x_n) is a Schauder basis then (S_n) is norm-bounded.

Let (x_n) be a linearly independent sequence in a Banach space E so that $E_0 = [x_n]$ is dense. If S_n denotes the linear mapping from E_0 into itself defined by the conditions

$$S_n(x_k) = x_k (k \leq n), S_n(x_k) = 0 \quad (k > 0).$$

Show that (x_n) is a Schauder basis for E if and only if the (S_n) are continuous and uniformly bounded.

E. The Schauder basis (x_n) and (y_n) in a Banach space E are **equivalent** if and only if for every sequence (ξ_n) of scalars, $\sum \xi_k x_k$ converges if and only if $\sum \xi_k v_k$ converges. Use the Banach-Steinhaus theorem to show that (x_n) and (y_n) $T : E \to E$ so that $Tx_n = y_n$ $(n \in \mathbf{N})$.

F. (Principle of condensation of singularities.)

Let $T_{m,n}$ be a double sequence in $L(E, F)$. Suppose that for each m, there is an $x_m \in E$ so that $\{T_{m,n}(x_m)\}^\infty$ is unbounded. Show that there is an $x \in E$ so that for each m $\{T_{m,n}(x)\}_{n=1}^\infty$ is unbounded.

G. Let $(x_m$ be an orthogonal system in $C[0,1]$ i.e. such that $\sin x_m x_n = \delta_{m,n}$. (Classical example—the trigonometric functions.) Suppose in addition that $|\bar{x}_n] = C[0,1]$ i.e. the linear span of $\{x_n\}$ is dense. If $x \in C[0,1]$, $\sum_{m=1}^\infty (\int x_m x) x_m$ is called the **Fourier series** of x with respect to (x_m). By applying the principle of condensation of singularities to the functionals

$$S_{m,n} = x \to \sum_{k=1}^{m} \left(\int x_k x \right) x_k(t_n)$$

show that if for a sequence (t_n) in $[0,1]$ there is for each n a function g_n whose Fourier series diverges at t_n, then there is a g whose Fourier series diverges at each t_n.

We now consider three essentially equivalent theorems: The closed graph theorem, the open mapping theorem and the epimorphism theorem. Roughly speaking, they state respectively:

1. that any linear mapping which can be constructed is continuous;
2. if an equation $Tx = y$ is always solvable, then it is well-posed;
3. a given vector space can have at most one Banach space structure.

These statements will be made precise in the following text:

Definition 7 *A continuous linear mapping T from a normed space E into a normed space F is* **open** *if the image of the unit ball in E is a neighbourhood of zero in F. Then T is surjective.*

The condition that T be open can be expressed quantitatively as follows: There is an $M > 0$ so that for each $y \in F$ there is an $x \in E$ with $||x|| \leq M||y||$ and $Tx = y$.

(In the language of the theory of equations this means that the problem $Tx = y$ s **well-posed**—small perturbations of the right hand side can be corrected by small perturbations of the solution.)

Note that T is open in the above sense if and only if it maps open sets onto open sets (sinse open sets are unions of balls).

EXERCISES.

A. Show that $T \in L(E, F)$ is open if and only if the induced mapping $\tilde{T} : E/\mathbf{Ker}\,(T) \to F$ is an isomorphism (cf. 1.10.C for the definition of $\tilde{T}$).

B. Show that if T is an open mapping in $L(E, F)$ and $y_n \to 0$ in F then there is a sequence (x_n) in E so that $x_n \to 0$ in E and $y_n = Tx_n$ $(n \in \mathbf{N})$.

Our next main result states that a surjective linear mapping between Banach spaces is always open. The burden of the proof is contained in the next lemma.

Lemma 6 *Let $T \in L(E, F)$ $(E, F$ Banach spaces) and suppose that $\overline{T(B_E)}$ is a neighbourhood of zero in F. Then T is open.*

PROOF. Suppose that $\overline{T(B_E)} \subseteq U(0, \epsilon)$ where $\epsilon > 0$. Then we shall show that $T(2B_E) \subseteq U(0, \epsilon)$ i.e. $T(B_E) \subseteq U(0, \epsilon/2)$. We have $\overline{T(2^{-n_{B_E}})} \subseteq U(0, 2^{-n}\epsilon)$ for each n. If $y \in U(0, \epsilon)$, we construct a sequence (x_n) in E so that $x_1 \in B_E$ and for each $n \geq 1$.

$$||x_n x_{n+1}|| \leq 2^{-n-1} \text{ and } ||Tx_n - y|| \leq \epsilon \cdot 2^{-n-1}.$$

Then (x_n) is a Cauchy sequence in E and so has a limit x with

$$||x|| \leq ||x_1|| + \left\| \sum_{n=1}^{\infty} x_n - x_{n+1} \right\| \leq 2 \text{ and } Tx = \lim Tx_n = y.$$

We construct (x_n) inductively. Firstly we pick $x_1 \in B_E$ so that $||Tx_1 - y|| \leq \epsilon/4$ $(T(B_E)$ is dense in $U(0, \epsilon))$. Now suppose $x_1, \dots, x_k$ chosen so that the first of the above conditions holds for $n = 1, \dots, k-1$ and the second for $n \leq k$. Then $y \in U(Tx_k, 2^{-k-1}\epsilon)$ and $T(U(x_k, 2^{-k-1}))$ is dense in $U(Tx_k, 2^{-k-1}\epsilon)$ and so we can find an $x_{k+1} \in U(x_k, 2^{-2-1})$ so that $||Tx_{k+1}^{-y}|| \leq 2^{-k-2}\epsilon$. Then the above conditions hold for $n = k$ (resp. $n = k + 1$).

∎

Proposition 29 (The open mapping theorem.) *Let $T \in L(E, F)$ (E, F Banach spaces). Then T is open if and only if it is surjective.*

PROOF. We show that if T is surjective, then it is open. We have $F = \bigcup_{n=1}^{\infty} \overline{nT(B_E)}$. Hence, by Baire, there is an $n \in \mathbf{N}$, an $x \in F$ and an $\epsilon > 0$ so that $\overline{nT(B_E)} \subseteq U(x, \epsilon)$. Then $\overline{T(B_E)} \subseteq U(x', \epsilon')$ where $x' := x/n, \epsilon' := \epsilon/n$. It follows easily that $\overline{T(B_E)} \subseteq U(0, \epsilon'/2)$ and so we can apply 4.10.

■

Corollar 11 (Banach's isomorphism theorem.) *Let T be a continuous bijective linear mapping from E into F (E, F Banach spaces). Then T is an isomorphism.*

If T is a linear mapping from E into F (E, F Banach spaces) the graph of T is defined to be the subset $\Gamma(T) := \{(x, Tx) : x \in E\}$ of $E \times F$. Then $\Gamma(T)$ is a subspace of $E \times F$. If $\gamma(T)$ is closed in the normed space $E \times F$ we say tht T **has a closed graph**.

EXERCISES. Show that the following condition is equivalent to the above definiton: if $x_n \to x$ in E and $Tx_n \to y$ in F then $y = Tx$. Deduce that a continuous operator has a closed graph. Show that if T has a closed graph, then **Ker** (T) is closed.

y

Proposition 30 (Closed graph theorem.) *Let T be a linear mapping from E into F (E, F Banach spaces). Then if T has a closed graph, it is continuous.*

PROOF. $\Gamma(T)$, being a closed subspace of the Banach space $E \times F$, is itself a Banach space with the induced norm. The mapping $(x, Tx) \to x$ from $\Gamma(T)$ into E is clearly a continuous, linear bijection. Hence by 4.12 its inverse, the mapping $x \to (x, Tx)$ from E into $E \times F$ is continuous. This imlies the continuity of T.

The open mapping theorem is often used in proofs of existence theorems for partial differential equations. Perhaps its most useful from is 4.16 below which involves duality. In applications it reduces proofs of such existence theorems to the establishment of **a priori** estimates for the solutions.

■

Part of the proof lies in the following characterisation of maps with dense range:

Lemma 7 *T' is injective if and only if $T(E)$ is dense in F.*

Proof. Suppose that $T(E)$ is dense in F. The if $f \in F'$, $T'(f) = 0$ implies that f vanishes on $T(E)$ and so $f = 0$. On the other hand, if $T(E)$ is not dense in F, there is an $f \in F'$ so that $f \neq 0$ but f vanishes on $T(E)$ (2.10). Then $T'(f) = 0$ and so T' is not injective.

■

Proposition 31 Proposition (the epimorphism theorem). *If $T \in L(E, F)$ (E, F Banach space), then the following are equivalent:*

(a) T is surjective;

(b) T' is a injection and $T'(F')$ is closed in E';

(c) $T(E)$ is dense in F and $T'(F')$ is closed in E';

(d) if C is bounded inE', then $(T')^{-1}(C)$ is bounded in F'.

If this is the case, then $T'(F') = (\mathbf{Ker}\, T)^0$.

Proof. (a) $\Rightarrow$ (b): T lifts to a continuous linear bijection $\tilde{T}$ from E/N onto $F (N := \mathbf{Ker}\, T)$. By the isomorphism theorem (4.12), $\tilde{T}$ is an isomorphism.

Consider the diagrams

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Then $\tilde{T}'$ is an isomoprhism and $\tilde{\pi}_N$ is an injection from $(E/N)'$ onto N^0 and this implies (b).

(b) and (c) are equivalent by 4.15.

(b) implies (d): by (4.12) T' has a continuous inverse from $T'(F')$ onto F' and if C is a bounded set in E', then $(T')^{-1}(C)$ is the image of the bounded set $C \cap T'(F')$ under this mapping.

(d) implies (a): we show that $\overline{T(B_E)}$ is a neighbourhood of zero in F and the result then follows from 4.12. There is a $K > 0$ so that $||f|| \leq K$ if $T'f \in B_{E'}$. Then $\overline{T(B_E)} \subseteq U(0, K^{-1})$. For if this were not the case, we could find a $y \in F$ with $||y|| \leq K^{-1}$ and $y \notin \overline{T(B_E)}$. Then by 2.13, there is an $f \in F'$ so that $f(y) > 1$ and $||f(Tx)|| \leq 1$ if $x \in B_E$. Then $||f|| > K$ and $||T'(f)|| \leq 1$—contradiction.

■

In the above proof we used implicitly the following useful fact: If $T : E \to F$ is a continuous linear operator, we can decompose it as follows:

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where π is the canonical projection onto $E/\mathbf{Ker}\,T$. Then $\tilde{T}$ is bijection from $E/\mathbf{Ker}\,T$ onto a dense subspace of $\overline{T(E)}$. We see that the following are equivalent:

1. T is an open mapping onto $T(E)$;
2. $T(E)$ is closed in F;
3. $\tilde{T}$ is an isomoprhism.

EXERCISES.

A. Let (x_n) be a basis fro a Banach space E (c.f. 4.7.D). Define the norm $|| \, ||$ on E by
$$||| \, ||| : x \to \sup\{||S_n(x)|| : n \in \mathbf{N}\}.$$
Show that
   (a) $(E, || \, ||_n)$ is a Banach space;
   (b) $|| \, ||_n$ and $|| \, ||$ are equivalent;
   (c) (x_n) is a Schauder basis.

   (For (a) let $(\sum_{k=1}^{\infty} \xi_k^n x_k)_{n=1}^{\infty}$ be a $|| \, ||_n$-Cauchy sequence in E. Then it has a $|| \, ||$-limit $\sum_{k=1}^{\infty} \xi_k x_k$ is E. Show that $\xi_k^n \to \xi_k$ $(k \in \mathbf{N})$ and deduce that
$$\left(\sum_{k=1}^{\infty} \xi_k^n x_k \right) \to \left(\sum_{k=1}^{\infty} \xi_k x_k \right) \text{ in } (E, || \, ||_n).$$

B. Let E, F be Banach spaces of functions on an interval I in $\mathbf{R}$ so that the linear forms $x \to x(t)$ $(t \in I)$ are continuous on E and F. Show that if the derivative x' exists for each x in E and lies in F then the mapping $x \to x'$ from E into F is continuous (use 4.7.B to show that this mapping has a closed graph).

C. If E, F, G are Banach spaces and $S \in L(E, G)$, $T \in L(F, G)$ are such that for each $x \in E$, there is a unique $y \in F$ so that $Sx = Ty$, then the mapping $x \to y$ is a continuous linear mapping from E into F. Deduce that if $S \in L(F, G)$ is injective and T is a linear mapping from E into F, then ST is continuous if and only if T is.

D. Let X be a linear space with a Hausdorff topology τ. A **Banach subspace** of X is a subspace E with a norm $|| \, ||_n$ so that $(E, || \, ||_n)$ is a Banach space and τ induces on E a topology coarser than that induced by $d_{|| \, ||_n}$. Show that if $(X-1, \tau_1)$ and (X_2, τ_2) are two such spaces and E (resp. F) is a Banach subspace of X_1 (resp. X_2), then a linear mapping T from X_1 into X_2 which is $\tau_1 - \tau_2$ continuous and maps E into F is continuous as a mapping between the normed space E and F. Deduce that a vector subspace of X can have at most one Banach subspace structure (up to equivalence of norms).

E. Let (f_n) be a sequence of continuous functions from the complete metric space M into $\mathbf{R}$. Show that if f_n converges pointwise to f then f is continuous except on a set of first category.

F. Let M be a family of mappings in $L(E_1, \dots, E_n; F)$ where the E_i and F are Banach spaces. Show that if M is pointwise bounded, then it is bounded.

G. $T \in L(E, F)$ is a **monomorphism** if and only if whenever $S, S_1 \in L(E_1, E)$ are such that $T \circ S = T \circ S_1$, then $S = S_1$. T is an **epimorhism** if whenever $S, S-1 \in L(F, F_1)$ and $S \circ T = S_1 \circ T$ then $S = S_1$.

Show that T is a monomorphism if and only if it is injective and an epimorhism if and only if $T(E)$ is dense in F. Give an example of a T which is both an epimorhism and a monomorphism but not an isomoprhism.

H. If $S : F \to G$, $T : E \to G$ are linear contractions between Banach spaces, a pullback for S, T is a Banach space H_0 with linear contractions $S_0 \in L(H_0, F)$, $T_0 \in L(H_0, E)$ so that $S \circ S_0 = T \circ T_0$ and the following property holds:

for every Banach space H_1 and operators $S_1 \in L(H_1, F)$ $T_1 \in L(H_1, E)$ so that $S \circ S_1 = T \circ T_1$ there is $U \in L(H_0, H_1)$ so that

$$T_0 = T_1 \circ U, \quad S_0 = S_1 \circ U.$$

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Show that such a triple (H_0, S_0, T_0) always exists. (Put $H_0 := \{(x, y, x) \in E \times F \times G : x = Tx = Sy\}$.)

If F is a subspace of a Banach space E show that the diagram file=bild9.eps,height=4cm,width=7cm

is a publlback in this sense.

I. $T : E \to F$ is called a **strict epimorhism** if the corresponding mapping $\tilde{}E/\mathbf{Ker}\,\tilde{T} \to F$ is an isometric isomoprhism onto F.

Show that this is equivalence to the fact that $T(B_E)$ is a dense subset of B_F.

J. (Quadrature formulae): Suppose that $T = (t_0, \dots, t_n)$ is a partition of $[a, b]$ and that $\rho = (\rho_1, \dots, \rho_n)$ is sequence of numbers. We define a linear form

$$f_\rho^T(x) = \sum_{r=1}^{n} \rho_r x(t_r)$$

on $C([a,])$. Given a sequence (T^k) of partitions, resp. a sequence (ρ^k) of coefficients, we are interested in the question of when the corresponding functionals

$$f_k = f_{\rho_k}^{T_k}$$

converge to the integration functional $f : x \to \int_a^b x$. Prove the following facts:

a) the norm of f_ρ^T is $\sum_r |\rho_r|$;

b) if f_k is exact for polynomials of degree at most k (i.e. if $f_k(p) = f(p)$ for such polynomials p), then f_k converges pointwise to f if and only if there is a constant M so that $\sum_r |\rho_r^k| \leq M$ for each k.

K. Recall that a Banach sequence space E is an AK space if $p_n x \to x$ where if $x = (\xi_n)$, $p_n x = (\xi_1, \dots, \xi_n, 0, \dots)$. Show that if this is the case then $E' = E^\beta$ where

$$E^\beta = \left\{ y : \sum \xi_k \eta_k \text{ converges for each } x \in E \right\}.$$

Show that $(bv_0)' = bs$, $(cs)' = bv$.

L. Let E be the space of entire function with the norm $||x|| = \sup\{|x(\lambda)| : |\lambda| < 1\}$. Show that his is a normed space and that the mapping

$$T : x \to (\lambda \to x(\lambda/2))$$

is a vector space isomoprhism from E onto itself which is continuous but whose inverse is not continuous.

M. Prove the following stronger form of the open mapping theorem. Let $TL(E,F)$ (E and F Banach spaces) be such that $T(E)$ is of second category in F. Then $|T|$ is open (and surjective).

5 Hilbert space

In this section we consider every special but important class of Banach spaces—namely those spaces whose metric structure is induced by an inner product. They are thus the infinite dimensional analogues of Euclidean spaces and we shall find that many of the concepts and results for finite dimensional spaces (such as self-adjointness) can be carried over in suitable forms to the infinite dimensional case.

In fact, the existence of a scalar product is an extremely strong condition and just as in the finite dimensional case where every n dimensional Euclidean space is isometric to $\mathbf{R}^n$ (resp. $\mathbf{C}^n$ in the complex case) we shall find that there is essentially only one Hilbert space (up to dimension).

It will be convenient to assume in this section that all vector spaces are complex (the results are valid for the real case with the simplificaitons which arise from the fact that the scalar product is bilinear, not sesquilinear).

Definition 8 *An* **inner product space** *(over* $\mathbf{C}$*) is a vector space H with a positive-definite sesquilinear form i.e. a mapping* $(|) : H \times H \to \mathbf{C}$ *satisfying*

1. *for each* $x \in H, (x|x) \geq 0$ *and* $(x|x) = 0 \Leftrightarrow x = 0$*;*
2. *for each* $y \in H$*, the mapping* $x \to (x|y)$ *is linear;*
3. $(x|y) = (\overline{y|x})^{1/2}$ $(y, x \in H)$*.*

The inner product $(x|y)$ *can be used to define a norm*

$$|| \, || : x \to (x|x)^{1/2}$$

on H*. Thus every inner product space can be regarded in a natural way as a normed space. If* H *under this norm is complete (i.e. is a Banach space) we call* H *a* **Hilbert space***.*

EXERCISES.

A. Show that $x \to (x|x)^{1/2}$ is a norm (first verify the **Cauchy-Schwartz inequality**

$$|(x|y)| \leq ||x|| \, ||y|| \quad (x, y \in H) \,).$$

B. If H is an inner product space ober $\mathbf{C}$ show that the inner product can be recovered from the norm as follows:

$$4(x|y) = ||x+y||^2 - ||x-y||^2 + i||x+iy||^2 - i||x-iy||^2.$$

Show that a norm $||\ ||$ on a linear space H is defined by an inner product if and only if the following identity (the parallelogram law) holds for $x, y \in H$:

$$||x+y||^2 + ||x-y||^2 = 2(||x||^2 + ||y||^2).$$

Proposition 32 *If H is an inner product space, then so is its Banach space completion.*

PROOF. This folows immediately from 5.2. Alternatively it can be verified directly by noting that the inner product on H extends in a unique manner to a continuous function on $\hat{H} \times \hat{H}$ and this function is an inner product which defines **the norm of** $\hat{H}$.

For reasons which will soon be apparent we shall be content with the following examples of Hilbert spaces. The Banach space $(\ell^2(S), ||\ ||_2)$ can be regarded as a Hilbert space with the inner product

$$(x|y) := \sum_{s \in S} x(s)\overline{y(s)}.$$

Similarly, $L^2(\mu)$ is a Hilbert space with the inner product

$$(x|y) := \int d\bar{y} d\mu.$$

Just as in the case of finite dimensional spaces, a central role is played by orthonormal bases i.e. bases whose elements are mutually perpendicular. Hence we define a subset A of an inner product space H to be **orthonormal** if each $x \in A$ has norm 1 and if $x \neq y$ in A implies that $(x|y) = 0$.

■

Lemma 8 (Bessel's inequality.) *If A is orthonormal, then for each $x \in H$, $\sum_{y \in A} |(x|y)|^2 \leq ||x||^2$.*

PROOF. Choose $J \in J(A)$, the family of finite subsets of A, and put $u := \sum_{y \in J}(x|y)y$. Then $(x-u|y) = 0$ for each $y \in J$ and so $(x-u|u) = 0$. Hence

$$0 \leq ||x-u||^2 = (x-u|x-u) = (x-u|x) = ||x||^2 - (u|x)$$

and

$$(u|x) = \sum_{y \in J}(x|y)(y|x) = \sum_{y \in J}|(x|y)|^2.$$

Hence

$$\sum_{y \in J}|(x|y)|^2 \leq ||x||^2 \text{ and so } \sum_{y \in A}|(x|y)|^2 \leq ||x||^2.$$

■

Lemma 9 *If A is an orthonormal set in a Hilbert space and $x \in H$, then $\sum_{y \in A}(x|y)y$ is convergent to an element of H.*

PROOF. Let $u_j = \sum_{y \in J}(x|y)y$ for $J \in \mathcal{J}(\mathcal{A})$. We show that the net $\{u_J : J \in \mathcal{J}(\mathcal{A})\}$ is Cauchy. This follows form the equality

$$||u_J - u_{J'}||^2 = \sum_{y \in K}|(x|y)|^2$$

(where $K = (J \setminus J') \cup (J' \setminus J)$) since $\sum_{u \in A}|(x|y)|^2$ converges by 5.4.

■

An orthonormal set A in H is **maximal** if it is not properly contained in a larger orthonormal set. It is **fundamental** if its linear span is dense.

Proposition 33 *Let A be an orthonormal set in a Hilbert space H. Then the following are equivalent:*

1. *A is maximal;*
2. *A is fundamental;*
3. *if $x \in H$, then $x = \sum_{y \in A}(x|y)y$;*
4. *if $x \in H$, then $||x||^2 = \sum_{y \in A}|(x|y)|^2$;*
5. *if $x, z \in H$, then $(x|z) = \sum_{y \in A}(x|y)\overline{(y|x)}$.*

PROOF. 1. $\Rightarrow$ 3.: By 5.5 $\sum_{y \in A}(x|y)y$ converges say to x'. If $x \in A$, then $(x'|z) = (x|z)$. Hence $(x' - x|z) = 0$ for each $z \in A$ and so $x' - x = 0$ since otherwise $A \cup \{x' - x)/||x' - x||\}$ would be an orthonormal set, properly containing A.

3. $\Rightarrow$ 2. is clear.

2. $\Rightarrow$ 1.: Suppose that A is not maximal. Let $A \cup \{x_0\}$ be an orthonormal system, properly containing A. Then since x_0 is approximable by linear combinations of elements of A to each of which it is orthonormal, $(x_0|x_0) = 0$ which is impossible.

3. $\Rightarrow$ 5. and 5. $\Rightarrow$ 4. are routine calculations.

4. $\Rightarrow$ 3.: Let $x' := \sum_{y \in A}(x|y)y$. Then it follows from 4. that $||x' - x||^2 = 0$ i.e. $x' = x$.

A subset A of a Hilbert space H which satisfies any of the above conditions is called a (**Hilbert**) basis for H. Note that 3. implies that it is then a Schauder basis (cf. 4.7.D) if it is countable.

∎

Proposition 34 *Every orthonormal set in H is contained in a complete orthonormal set (i.e. a basis). In particular, every Hilbert space has a basis.*

PROOF. Apply Zorn's Lemma to the family of orthonormal sets containing the given one.

∎

Proposition 35 *Let H be a Hilbert space. Then the following are equivalent:*

1. *H is separable;*
2. *H has a countable basis;*
3. *every basis of H is countable.*

PROOF. 3. $\Rightarrow$ 2. and 2. $\Rightarrow$ 1. are trivial.

1. $\Rightarrow$ 2.: If H is separable, it is easy to find a linearly independent sequence (x_n) whose linear span is dense in H. We show how to replace (x_n) by an orthonormal sequence (y_n) with the property that $\{y_1, \dots, y_m\}$ spans the same subspace as $\{x_1, \dots, x_m\}$ for each m. Then (y_n) is clearly a basis. To do this we use a process known as Gram-Schmidt orthogonalisation.

We define

$$\begin{aligned} \tilde{y}_1 &:= x_1, y_1 := \tilde{y}_1/||\tilde{y}_1||, \tilde{y}_2 := x_2 - (y_1|x_2)y_1, y_2 := \tilde{y}_2/||y_2||, \dots \\ \tilde{y}_n &:= x_n - \sum_{k=1}^{n-1}(y_k|x_n)y_k, y_n := \tilde{y}_n|||\tilde{y}_n||, \dots . \end{aligned}$$

As an example of the above construction, consider the linearly independent sequence (x_n) in $L^2([-1,1])$ where $x_n(t) = t^n$ $(n = 0, 1, 2, \dots)$. The linear span of this sequence is the space of polynomials which is dense in $C([-1,1])$ for the supremum norm. Since $C([-1,1])$ is dense in $L^2([-1,1])$, the sequence is complete in the latter space.

If we apply the Gram-Schmidt process, we obtain an orthonormal basis (p_n), where p_n is a polynomial of degree n. We can give the following explicit formula for p_n. Put

$$\begin{aligned} P_n(t) &= \frac{1}{2^n n!}\frac{d^n(t^2-1)^n}{dt^n} \\ &= \sum_{r=[\frac{n}{2}]+1}^{n} (-1)^{n-r}\frac{(2r)!}{(2r-n)!}\binom{n}{r} t^{2r-n} \end{aligned}$$

(the first three terms are $1, t, \frac{3}{2}t^2 - \frac{1}{2}$).

We show that (P_n) is orthogonal. To do this it suffices to show that

$$\int_{-1}^{1} P_n(t) t^m dt = 0 \quad (m < n)$$

and this follows from integrating by parts.

In the same way one shows that the L^2-norm of P_n is $\frac{2}{2n+1}$. Hence the sequence (p_n) where

$$p_n(t) = \sqrt{\frac{2n = 1}{2}} P_n(t)$$

is an orthonormal sequence.

■

The following are further examples of orthonormal sequences:

I. The classical example is the system (x_n) of trigonometric functions i.e. the sequence

$$\left(\frac{1}{\sqrt{2\pi}}, \frac{2}{\sqrt{\pi}}\cos t, \frac{1}{\sqrt{\pi}}\sin t, \frac{1}{\sqrt{\pi}}\cos 2t, \frac{1}{\sqrt{\pi}}\sin 2t, \dots\right).$$

It is orthonormal in $L^2([0, 2\pi])$.

II. The Haar functions: Put

$$\chi_0(t) = 1 \quad (t \in [0,1])$$

$$\chi_{2^k+1}(t) = \begin{cases} 1 & t \in \left[\frac{21-2}{2^{k+1}}, \frac{21-1}{2^{k+1}}\right[\\ -1 & t \in \left[\frac{21-1}{2^{k+1}}, \frac{21}{2^{k+1}}\right[\\ 0 & \text{(otherwise)} \end{cases}$$

The n-th Haar function is then

$$h_n = 2^{-k/2}\chi_n \text{ (where } n = 2^k + 1).$$

(h_n) is an orthonormal system in $L^2([0,1])$.

III. The Rademacher functions: There are defined as follows:

$$r_0(t) = 1 \quad (t \in [0,1])$$

$$r_n(t) = \begin{cases} 1 & \text{for} \quad t \in \left[\frac{21}{n}, \frac{21+2}{2^n}\right[\\ -1 & \text{for} \quad t \in \left[\frac{21+1}{2^n}, \frac{21+2}{2^n}\right[\end{cases}$$

$(1 = 0, \dots, 2^{n-1} - 1)$.

(Note that the Rademacher functions are formed by taking the sums of those Haar functions which correspond to the same dyadic partition of $[0,1]$.

The Haar functions are complete in $L^1([0,1])$ whereas the Rademacher functions are not (as is clear from the above description).)

Exercises.

A. Complete the proof of 5.8 by showing that if H has an uncountable basis, then it is not separable.

B. Let A and B be complete orthonormal sets in a Hilbert space H. Show that A and B have the same cardinality. Show that if S is a set with the same cardinality as an orthonormal basis for H, then H and $\ell^2(S)$ are isometrically isomorphic. Deduce that two Hilbert spaces are isomorphic if and only if they have basis with the same cardinality.

We now discuss orthonormal decompositions of Hilbert spaces. For infinite dimensional spaces exactly the same result holds as for euclidian spaces.

Proposition 36 *Let K be a closed subspace of a Hilbert space H and put*

$$K^\perp = \{x \in H : (x|y) = 0 \textit{ for } y \in K\}.$$

Then $K^\perp$ is a closed subspace of H and $H = K \oplus K^\perp$, i.e. each $x \in H$ has a unique decomposition $x = y + z$ $(y \in K, z \in K^\perp)$.

PROOF. We choose an orthonormal basis A for K and extend to an orthonormal basis B for H. Then it is clear that $K^\perp$ is the closed linear span of (B_A) and if $x \in H$ we

$$x = \sum_{y \in B} (x|y)y = \sum_{y \in A} (x|y)y + \sum_{y \in B \setminus A} (x|y)y.$$

The mapping $x \mapsto y$ where x and y are as 5.10 is a projection from H onto K, called the **orthogonal** projection from H onto K and denoted by P_K. it is a continuous linear mapping and in fact has norm 1 (except for the trivial case where $K = \{0\}$).

■

We have seen earlier that the spaces ℓ^2 and $L^2(\mu)$ coincide with their dual spaces. This is due to the fact that they are Hilbert spaces as the next result shows:

Proposition 37 (Riesz.) *For every continuous linear functional f on H, there exists a unique vector $y \in H$ so that f is the form $x \mapsto (x|y)$. More precisely, if we denote this form by Y_y, then the mapping $T_H : y \mapsto T_y$ is an antilinear isometric isomorphism from H onto H' (anti+linear means that the mappings is additive and satisfies the condition $T_H(\lambda y) = \bar{\lambda} T_H(y)$.) In this real case T_H is linear.*

PROOF. Let $f \in H'$. We shall verify the existence of a y with the stated property. We can assume that $f \neq 0$. Let $K = \{x \in H : f(x) = 0\}$ and choose $z \in K^\perp$ with $f(z) = 1$. Then $x - f(x)z \in K$ and so $(x - f(x)z|z) = 0$ from $x \in H$. Hence $(x|z) = f(x)||z||^2$ and $y = \frac{z}{||z||^2}$ is the required element.

■

EXERCISES. Complete the proof of 5.11 (i.e. show that the mapping $y \mapsto T_y$ is an antilinear isometry).

In view of 5.11, it is natural to define an inner product on H', by defining $(T_H x | T_H y)$ to be $(y|x)$ for $x, y \in H$. Then the norm defined on H' by this scalar product coincides with the norm on H' as the dual of H.

Proposition 38 *A Hilbert space H is reflexive.*

PROOF. We shall show that $J_H = T_{H'} \circ T_H$ which implies that J_H is surjective since both of the T's are. For if $f \in H'$, $x \in H$, then

$$\begin{aligned}(T_{H'} \circ T_H(x))(f) &= (f|T_H(x)) = (T_H(T_H^{-1}(f))|T_H(x)) \\ &= (x|T_H^{-1}f) = f(x) = (J_H(x))(f).\end{aligned}$$

(Note that this result follows immediately from the fact that H is isometrically isomorphic to an $\ell^2(S)$ space which we know to be reflexive. However, the above coordinate free proof is instructive.)

■

Another look at duality for L^p-spaces. In ? we established the duality between L^p and L^q spaces. The essential tool was the Radon-Nikodym theorem. In this section we show how this duality can be deduced from the L^2-case (i.e. from Hilbert space duality). For simplicity we work in the context of a probability measure μ on a measure space (Ω, A): We begin with a proof of part of ?? which does not use the Radon-Nikodym theorem.

Proposition 39 *Them mapping*

$$T : y \to (x \to \int xyd\mu)$$

is an isometry from $L^\infty(\mu)$ onto $(L^1(\mu))'$.

PROOF. The only difficult part of the proof is to show that T is onto, which we now do. First note that if we apply the Riesz representation theorem in this context, we get that T is an isometry from $L^2(\mu)$ onto $L^2(\mu)$. Hence if $f \in (L^1(\mu))'$, then, since its restriction to $L^2(\mu)$ is a continuous form on $L^2(\mu)$, it can be represented by a $y \in L^2(\mu)$ i.e.

$$f(x) = \int xyd\mu \quad (x \in L^2(\mu))$$

for each $x \in L^2(\mu)$. Then we claim that $y \in L^\infty(\mu)$ and so $f(x) = \int xyd\mu$ for $x \in L^1(\mu)$ by continuity.

If y were not in L^∞ we could find disjoint measurable sets (A_n) is Ω so that $\mu(A_n) > 0$ and $|y| \geq n$ on A_n. In fact, by reducing to the real-valued case and assuming (as we may without loss of generality) that y is not bounded above, we can assume that $y \geq n$ on A_n. Then if (α_n) is a sequence of positive numbers so that $\sum \alpha_i \mu(A_i) = 1$ and $\sum i\alpha_i\mu(A_i) = \infty$. Then

$$\left\| \sum_{i=1}^n \alpha_i \chi_{A_i} \right\|_1 = \sum_{i=1}^n \alpha_\mu(A_i) \geq 1$$

but

$$f\left(\sum_{i=1}^n \chi_{A_i}\right) \geq \sum_{i=1}^n |i\alpha_i\mu(A_i) \text{—contradiction.}$$

■

EXERCISES. We can now reverse the reasoning of ? and use the duality $(L^1)' = L^\infty$ to prove the Radon Nikodym theorem. We sketch the proof—the exercise consists in filling out the details:

1. Let ν be a measure on Ω which is absolutely continuous with respect to μ (μ and ν both non-negative). Then $x \to \int x d\nu$ is a continuous linear form on $L^1(\nu + \mu)$ and so is represented by $y \in L^\infty(\nu + \mu)$.

2. $0 \leq y \leq 1$ and $\mu\{y = 1\} = 0$.

3. $z = \frac{y}{1-y}$ is in $L^1(\mu)$ and $\nu = z\mu$.

We now consider the space $L(H_1, H_2)$ of all continuous linear mappings from the Hilbert space H_1 into the Hilbert space H_2. The norm on $L(H_1, H_2)$ can be determined by the following formula:

$$||T|| = \sup\{|(Tx|y)| = ||x|| \leq 1, ||y|| \leq 1\}$$

which follows from the equality:

$$||x|| = \sup\{|(x|y)| : ||y|| \leq 1\}.$$

If $T \in L(H_1, H_2)$ then its adjoint T' is an element of $L(H_1', H_2')$. Consider the diagramm

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where the mapping T^* is defined in such a way as to make it commutative i.e. $T^* = T_{H_1}^{-1} \circ T' \circ T_{H_1}$. In other words, if $y \in H_2$, T^*y is defined to be that element of H_1 which satisfies the equation

$$(Tx|y) = (x|T^*y) \quad (x \in H_1).$$

T^* is a continuous linear mapping from H_2 into H_1 and the correspondence $T \to T^*$ satisfies the following conditions:

a) $(T_1 + T_2)^* = T_1^* + T_2^*$;

b) $(\lambda T)^* = \bar{\lambda} T^*$;

c) $(T_1 T_2)^* = T_2^* T_1^*$;

d) $||T^*|| = ||T||$;

e) $(T^*) = T$.

If H_1 and H_2 coincide (in which case we write $L(H)$ instead of $L(H, H)$), then $L(H)$ is a Banach algebra and the map $T \to T^*$ is what we call an **involution**.

Just as in the finite dimensional case, we can use the structure of Hilbert space to isolate certain special classes of operators which are easier to analyse.

Definition 9 *An operator* $T \in L(H)$ *is called* **hermitian** *if* $T = T^*$*;*
normal *if* $TT^* = T^*T$ *'*
isometric *if* $(Tx|Ty) = (x|y)$ *i.e.* $T^*T = I$*;*
unitary *if it is isometric and onto i.e.* $T^*T = I = TT^*$*;*
an (orthogonal) projection *if it is hermitian and* $T^2 = T$*;*
positive *if* $(Tx|x) \geq 0$ $(x \in H)$ *(written* $T \geq 0$*).*

Exercises. Prove the equivalence of the two conditions used in the definition of isometric resp. unitary. Prove that T is isometric in the above sense if and only if $||T(x)|| = ||x||$ for $x \in H$.

Consider the following examplex in connection with 5.17:

I. Shift operators: we define the so-called shift operators S^l and S^r, firstly on $L^2(\mathbf{R})$, as follows:

$$\begin{aligned} S^l : x &\mapsto (s \mapsto x(s+1)) \\ S^r : x &\mapsto (s \mapsto x(s-1)). \end{aligned}$$

On $L^2(\mathbf{R}_+)$ they are defined as follows

$$S^l : x \mapsto (s \mapsto x(s+1))$$

$$S^r : x \mapsto s \mapsto \begin{cases} 0 & \text{if} \quad 0 \leq s < 1 \\ x(s-1) & \text{if} \quad 1 \leq s < \infty \end{cases}$$

Similarly, we can define S^l and S^r on $\ell^2(\mathbf{N})$ and $\ell^2(\mathbf{Z})$. Then in all these space we have $(S^l)^* = S^r$ and $(S^r)^* = S^l$. However wheras these operators are unitary and mutually adjoint resp. inverses on $L^2(\mathbf{R})$ and $\ell^2(\mathbf{Z})$ this is not the case on $L^2(\mathbf{R}_+)$ or $\ell^2(\mathbf{N})$. Here S^r is isometric but not unitary and so not normal (note that an isometric operator is unitary if and only if it is normal). S^l has none of the properties of 5.17.

II. Multiplication operators: Let (Ω, μ) be a measure space, $x \in L^\infty(\mu)$. Then $M_x : y \mapsto xy$ is a continuous linear operator on $H = L^2(\mu)$. The mapping $x \mapsto M_x$ from $L^\infty(\mu)$ into $L(H)$ is linear and preserves multiplication i.e. $M_{xy} = M_x \circ M_y$. In addition,

1. the adjoint of M_x is $M_{\bar{x}}$ where x is the complex conjugate of x);
2. $x \mapsto M_x$ is an isometry i.e. $||M_x|| = ||x||_\infty$;
3. M_x is hermitian if and only if x is real-valued;
4. M_x is normal;
5. M_x is positive if and only if $x \geq 0$ (μ-almost everywhere);
6. M_x is an orthogonal projection if and only if x has the form χ_A for some measurable function;
7. M_x is an isometry if and only if $|x| \leq 1$ a.e. and then M_x is unitary.

We prove 2. The others are easier. If $y \in H$, we have the inequality

$$||M_x y||_2^2 = \int |xy|^2 d\mu \leq ||x||_\infty^2 \int |y|^2 d\mu = ||x||^2\infty||y||_2^2.$$

Now if $\epsilon > 0$ there is a measurable set A of positive, but finite, measure so that

$$|x(s)| \geq ||x|| - \epsilon$$

on A. If $y = \chi_A/\sqrt{\mu(A)}$, then $||y||_2 = 1$ and $||xy||_2 \geq ||x||_\infty - \epsilon$ and so $||M_x|| \geq ||x||_\infty$.

Exercises. Verify the unproven statemets of the above list.

III. Integral operators: Let (Ω, μ) be a finite measure space, K a bounded, measurable (for the product measure $\mu \oplus \mu$) complex-valued function on $\Omega \times \Omega$. Then the mapping

$$I_K : x \to (s \to \int K(s,t)x(t)d\mu)$$

is a continuous linear operator from $L^2(\mu)$ into itself. using the Fubini theorem, one can show that the adjoint of I_K is I_{K^*} where $K^* : (s,t) \to \overline{K(s,t)}$.

Proposition 40 *1. The hermitian operators in $L(H)$ form a real subspace;*

2. the product of two hermitian operators is hermitian if and only if they commute;

3. $A \in L(H)$ is hermitian if and only if $(Ax|x)$ is real for each x in H. Then $||A|| = \sup\{|(Ax|x)| : x \in H, ||x|| \leq 1\}$;

4. if $T \in L(H)$ then T^*T and TT^* are hermitian;

5. if $T \in L(H)$ then $||T^*T|| = ||T||^2$.

PROOF. 3. if A is hermitian, then

$$(Ax|x) = (x|A^*x) = (x|Ax) = (\overline{Ax|x}).$$

On the other hand, the following equalities show that the given assumption implies that $(Ax|y) = (x|Ay)$ for each $x, y \in H$:

$$\begin{aligned} 4(Ax|y) &= (A(x+y)|x+y) - (A(x-y)|x-y) + i(A(x+iy)|x+iy) - i(A(x-iy)|x-iy) \\ 4(x|Ay) &= (x+y|A(x+y)) - (x-y|A(x-y)) + i(x+iy|A(x+iy)) - i(x-iy|A(x-iy)). \end{aligned}$$

Now let $k := \sup\{|(Ax|x)| : ||x|| \leq 1\}$. Then $k \leq ||A||$. Suppose that $x \in H$ with $||x|| \leq 1$, $x \neq 0$, $Ax \neq 0$. Then for any $\lambda \in \mathbf{R} \neq \{0\}$, we have

$$\begin{aligned} ||Ax||^2 &= (Ax|Ax) = (A^2x|x) \\ &= \frac{1}{4}\{A(\lambda Ax + \lambda^{-1}x)|\lambda Ax + \lambda^{-1}x) - (A(\lambda Ax - \lambda^{-1}x)|\lambda Ax - \lambda^{-1}x)\} \\ &\leq (\frac{1}{4})k\{||\lambda Ax + \lambda^{-1}x||^2 + ||\lambda Ax - \lambda^{-1}x||^2 \\ &= \frac{1}{2}k\{\lambda^2||Ax||^2 + \lambda^{-2}||x||^2\}. \end{aligned}$$

Substituting $||x||/||Ax||$ for λ^2 gives the following inequality which is obviously valid also for x with $x = 0$ or $Ax = 0$.

$$||Ax||^2 \leq k||Ax||\,||x|| \text{ and so } ||Ax|| \leq k||x||.$$

5. It is clear that $||T^*T|| \leq ||T||^2$. On the other hand, we have

$$\begin{aligned} ||T||^2 &= \sup\{||Tx||^2 : ||x|| \leq 1\} = \sup\{(Tx|Tx) : ||x|| \leq 1\} \\ &= \sup\{|T^*Tx|x)|L||x|| \leq 1\} = ||T^*T|| \text{ by 3.} \end{aligned}$$

■

We now list some simple properties of orthogonal projections:

Proposition 41 *1. If K is a closed subspace of H, then P_K is an orthogonal projection;*

2. if P is an orthogonal projection and $K := \{x \in H : Px = x\}$ then $P = P_K$ (we then write $R(P)$ for $P(H)$);

3. if P and Q are orthogonal projections; then PQ is an orthogonal projection if and only if P and Q commute and then $R(PQ) = R(P) \cap R(Q)$;

4. if P is the orthogonal projection on K, then $(I - P)$ is the orthogonal projection on $K^\perp$;

5. *the sum of two orthogonal projections is an orthogonal projection if and only if their product is zero;*

6. *if K and L are closed subspaces of H, the following are equivalent: $K \subseteq L$; $P_K P_K = P_K$; $P_K P_L = P_K$; $P_L - P_K$ is positive. Then $P_L - P_K$ is the orthogonal projection onto the orthogonal complement of K in L.*

PROOF.

1. Let $x, x' \in H$ with $x = y + z$, $x' = y' + z'$ where $y, y' \in K$, $z, z' \in K$. Then $P_K x = y$ and $P_K x' = y'$. Hence

$$(P_K x|x') = (y|x') = (y|y') = (x|y') = (x|P_K x')$$

and so P_K is hermitian.

2. K is a closed subspace (as the kernel of the operator $(I-P)$). We show that if $x \in H$ then $Px \in K$ and $(I-P)x \in K^\perp$ so that $x = Px+(I-p)x$ is the representation of x in $K \oplus K^i$. But $P(Px) = P^2x = Px$ and so $Px \in K$. On the other hand, if $y \in K$,

$$((I-P)x|y) = (x|y) - (Px|y) = (x|y) - (x|Py) = (x|y) - (x|y) = 0.$$

3. By 5.20.2 PQ is hermitian if and only if P and Q commute. If this is the case, then

$$(PQ)(PQ) = P(QP)Q = P(PQ)Q = P^2Q^2 = PQ.$$

Then $x \in R(PQ) \Rightarrow x = P(Qx) \in R(P) \cap R(Q)$. On the other hand, if $x \in R(P) \cap R(Q)$, then

$$x = Px = P(Qx) = (PQ)x.$$

4. if $(P + Q)$ is a projection, $(P + Q)^2 = P + Q$ and so $QP + PQ = 0$. Hence $PQP + PQ = 0 = QP + PQP$ and so $PQ = 0$. Conversely if $PQ = 0$, then $QP = (PQ)^* = 0$ and so $(P + Q)^2 = P + Q$.

■

EXERCISES. Identify the range $R(M_{\chi_A})$ of the projection M_{χ_A} on $L^2(\mu)$. Interpret the relation $M_{\chi_A} \leq M_{\chi_B}$.

Proposition 42 *1. $T \in L(H)$ is normal if and only if $||Tx|| = ||T^*x||$ for each $x \in H$;*

2. *if T is normal, then $||T|| = \sup\{|(Tx|x)| : ||x|| \leq 1\}$.*

PROOF. 1. If T is normal, then

$$||Tx||^2 = (Tx|Tx) = (T^*Tx|x) = (TT^*x|x) - (T^*Tx|x) = ||T^*x||^2 - ||Tx||^2 = 0 \text{ if } ||Tx|| = ||T^*x||$$

and so $TT^* - T^*T = 0$ by 5.20.3.

■

EXERCISES. Prove 5.23.2.

Now we discuss briefly various forms of convergence in H and $L(H)$.

Definition 10 *A sequence* (x_n) *in* H **converges strongly** *to* x *in* H *if* $||x_n - x|| \to 0$ *(written* $x_n \stackrel{s}{\to} x$*);*

converges weakly *to* x *in* H *if* $(x_n - x|y) \to 0$ *for each* $y \in H$ *(written* $x_n \stackrel{w}{\to} x$*) (of course these coincide with norm resp. weak convergence in the Banach space* H*;*

a sequence (S_n) **converges uniformly** *to* S *in* $L(H)$ *if* $||S_n - S|| \to 0$ *(written* $S_n \stackrel{u}{\to} S$*);*

converges strongly *to* S *in* $L(H)$ *if* $||S_nx - Sx|| \to 0$ *for each* $x \in H$ *(written* $S_n \stackrel{s}{\to} S$*);*

converges weakly *to* S *in* $L(H)$ *if* $((S_n - S)x|y) \to 0$ *for each* $x, y \in H$ *(written* $S_N \stackrel{w}{\to} S$*).*

Proposition 43 *1. if* $x_n \stackrel{s}{\to} x$ *in* H*, then* $x_n \stackrel{w}{\to} x$*;*

2. *if* $S_n \stackrel{u}{\to} S$ *in* $L(H)$*, then* $S_n \stackrel{s}{\to} S$ *and if* $S_n \stackrel{s}{\to} S$ *then* $S_n \stackrel{w}{\to} S$*;*
3. *if* $S_n \stackrel{u}{\to} S$, $T_n \stackrel{u}{\to} T$ *in* $L(H)$ *and* $\lambda_n \to \lambda$ *in* $\mathbf{C}$*, then* $S_n + T_n \stackrel{u}{\to} S + T$ *and* $\lambda_n S_n \to \lambda S$*—the corresponding result holds for strong convergence;*
4. *if* $S_n \stackrel{s}{\to} S$ *(resp.* $S_n \stackrel{w}{\to} S$*) and* $T_n \stackrel{s}{\to} T$*, then* $S_nT_n \stackrel{s}{\to} ST$ *(resp.* $S_nT_n \stackrel{w}{\to} ST$*;*
5. *if* $S_n \stackrel{u}{\to} S$*, then* $S_n^* \stackrel{u}{\to} S^*$*—the corresponding result holds for weak convergence.*

PROOF. 4. for strong convergence: Take $x \in H$. Since (S_n) is strongly convergent, it is pointwise bounded and so there is a $K > 0$ so that $||S_n|| \leq K$ for each n (principle of uniform boundedness). Then

$$\begin{aligned} ||S_nT_nx - STx|| &= ||S_n(T_n - T)x + (S_n - S)Tx|| \\ &\leq K||(T_n - T)x|| + ||(S_n - S)Tx|| \to 0. \end{aligned}$$

■

EXERCISES. Let (e_n) be the standard basis for $\ell^2(\mathbf{N})$. Show that (e_n) converges weakly but not strongly to zero.

The following exercises give counter-examples to the conjectures:
if $S_n \xrightarrow{s} S$, $T_n \xrightarrow{w} T$ does it follow that $S_n T_n \xrightarrow{w} ST$?
if $S_n \xrightarrow{s} S$, does it follow that $S_n^* \xrightarrow{s} S^*$?

EXERCISES. Show that $(S^l)^n \xrightarrow{S} 0$ and $(S^r)^n \xrightarrow{w} 0$ on ℓ^2 but that $(S^r)^n$ does not converge strongly to zero and $(S^r)^n(S^l)^n$ does not converge weakly to zero.

Proposition 44 *If $T_n \xrightarrow{w} T$ and each T_n is hermitian, then T is hermitian. If $T_n \xrightarrow{s} T$ and each T_n is an orthogonal projection, then so is T.*

EXERCISES. Show that $S_n^* S_n \xrightarrow{w} 0$ if and only if $S_n \xrightarrow{s} 0$. Show that if U_n, U are unitary in $L(H)$ and $U_n \xrightarrow{w} U$, then $U_n \xrightarrow{s} U$.

EXERCISES. Show that if (P_n) is an increasing sequence of projections on H, then there is a projection P so that $P_n \xrightarrow{s} P$.

If K is a closed subspace of a Hilbert space H and $x \in H$ then $P_K x$, the orthogonal projection of x onto K is characterised by the geometrical property that it is the nearest point to x in K i.e.

$$||x - P_K x|| < ||x - z|| \quad (z \in K, z \neq P_K x).$$

(For $x - z = (x - P_K x) + (P_K x - z)$ and so $||x - z||^2 = ||x - P_K x||^2 + ||P_K x - z||^2$).

The existence of such a point is a property which is also possessed by closed convex subsets of H:

Proposition 45 *Let K be a closed convex subset of a Hilbert space H, $x \in H$. Then there is a unique point $P_K x$ in K with*

$$||x - P_K x|| = d(x, K)$$

where $d(x, K) = \inf\{||x - y|| : y \in K\}$ is the distance from x to K.

In addition, the mapping $x \to P_K x$ from H onto K is continuous.

PROOF. We choose a sequence (x_n) in K with

$$||x - x_n|| \leq d + (1/n)$$

where $d = d(x, K)$.

From the inequality

$$\begin{aligned}
||x_n - x_m||^2 &= ||(x_n - x) + (x_m - x)||^2 \\
&= 2||x_n - x||^2 + 2||x_m - x||^2 - || - 2x + x_n + x_m||^2 \\
&= 2||x_n - x||^2 + 2||x_m - x||^2 - 4||x - (x_n + x_m)/2||^2 \\
&\leq 2(d + 1/n))^2 + 2(d + (1/m)) - 4d^2 \\
&= 2d(\frac{1}{m} + \frac{1}{n}) + 2(\frac{1}{n^2} + \frac{1}{m^2})
\end{aligned}$$

it follows that (x_n) is Cauchy. If x_0 is a limit point, then $d(x, K) = ||x - x_0||$.

The uniqueness follows from the fact that if y_0 is a second point with the same property then by exactly the same calculation,

$$||x_0 - y_0||^2 \leq 2d^2 + 2d^2 - 4d^2 = 0.$$

To prove continuity, it is sufficient to note that

$$||P_K x - P_K y|| \leq ||x - y||.$$

This is proved by contradiction as follows. Suppose that this is not the case. Then here are points $x, y \in H$ with $||P_K x - P_K y|| > ||x - y||$. If we conside the three dimensional affine space spanned by $x, y P_K x$ and $P_K y$ then we see that we have a skew rectangle $ABCD$ where A and B represent x and and y, resp. C and D $P_K x$ and $P_K y$ with $|AB| < |CD|$.

file=bild11.eps,height=7cm,width=10cm

Now by elementary geometry there is a point on the segment CD (which lies in K) which is nearer to A than C or nearer to B than D. This is a contradiction.

■

EXERCISES. Let K be a convex, closed subset of a real Hilbert space H. Show that

$$(x - P_K x | y - P_K x) \quad 0$$

for each $x \in H$, $y \in K$ (i.e. the angle between $x - P_K x$ and $y - P_K x$ is obtuse).

We shall round off this section with a proof of the spectral theorem for self-adjoint, compact operators on Hilbert space. This bears a striking similarity to the finite dimensional one and will serve as a motivation for the more general version which will involve our attention in Chapter III.

Definition 11 *A linear operator $T \in L(E, F)$ where E and F are Banach space is* **compact** *if the image $T(B_E)$ of the unit ball of E is relatively compact in F (i.e. if every bounded sequence (x_n) in E has a subsequence (x_{n_k}) for which (Tx_{n_k}) is convergent).*

In order to check that a concrete operator is compact, the following simple facts are useful.

I. *If $T \in L(E, F)$ has finite rank (i.e. if the image space $T(E)$ is finite dimensional), then T is compact.*

II. *If $T \in L(E, F)$ is the uniform limit of a sequence of compact operators, then T is itself compact.*

PROOF. Put $M = T(B_E)$. We must show that for each $\epsilon > 0$ there is an ϵ-net for M. Now choose $N \in \mathbf{N}$ so that

$$||(T_N - T)x|| \leq \epsilon/2 \text{ if } x \in B_E.$$

There is an $\epsilon/2$-net for $T_N(B_E)$ i.e. $x_1, \dots, x_m$ so that

$$T_N(B_E) \subseteq \bigcup_i U(x_i, \epsilon/2).$$

Then $T(B_E) \subseteq \bigcup_i U(x_i, \epsilon)$.

■

Using this fact, we can give the following examples of compact linear mappings:

I. Suppose that $y = (\eta_n)$ is in c_0 and consider the mapping

$$M_y : (\xi_n) \to (\xi_n \eta_n)$$

on ℓ^2. Then this mapping is compact as the uniform limit of the mappings M_{y_n} where
$$y_n = (\eta_1, \dots, \eta_n, 0, 0, \dots)$$
and so M_{y_n} has finite rank.

II. If $KL[0,1]^2 \to \mathbf{R}$ is continuous, then the associated kernel mapping I_K is compact. For there is a sequence (K_n) of continuous functions of the form
$$K_n(s,t) = \sum_{i=1}^{r(n)} x_i^n(s) y_i^n(t)$$

so that $K_n \to K$ uniformly on $[0,1]^2$ (this is a consequence, for example, of the Weierstraß theorem which states that the polynomials in two variables are dense in $C([0,1]^2)$ (see 3.2 for an abstract version of the Weierstraß theorem which implies this result). Then I_{K_n} has finite rank and $I_{K_n} \to I_K$ uniformly.

We can now state and prove the spectral theorem:

Proposition 46 *Let S be a compact, self-adjoint operator on a separable Hilbert space H. Then there exists an orthonormal basis (x_n) and a sequence (λ_n) in $\mathbf{R}$ which converges to zero, so that*

$$Sx_n = \lambda_n x_n \text{ for each } n.$$

REMARK. It follows that if x has the expansion $\sum \xi_n x_n$ with respect to the basis, then

$$Sx = \sum \lambda_n \xi_n x_n$$

i.e. if we identify H and ℓ^2 by means of the isometry

$$U : (\xi_n) \to \sum \xi_n x_n$$

then the mapping S has the representation M_y where $y = (\lambda_n) \in c_0$.

The (x_n) are then eigenvectors of S and we can express the result briefly in the form: a compact self-adjoint operator on a Hilbert space possesses a sequence (x_n) of eigenvectors which form an orthonormal basis for H.

Just as in the finite dimensional case, the decisive step in the proof is that of the existence of **one** eigenvector. This is done in the following Lemma:

Lemma 10 *Let $S \in L(H)$ be compact and self-adjoint. Then there is a unit vector $x_1 \in H$ with $||x_1|| = 1$ so that $Sx_1 = \lambda_1 x_1$ where $|\lambda_1| = ||S|| = \sup\{|(Sx|x)| : x \in B_H\}$.*

PROOF. Without loss of generality we can assume that

$$||S|| = \sup\{(Sx|x) : x \in B_H\}$$

(otherwise we replace S by $-S$).

We choose a sequence (x_n) in B_H so that

$$(Sx_n|x_n) \geq \lambda_1 - \frac{1}{n},$$

where $\lambda_1 = \sup\{(Sx|x) : x \in B_H\}$.

Then

$$||Sx_n - \lambda_1 x_n||^2 = (Sx_n|Sx_n) - 2\lambda_1(Sx_n|x_n) + \lambda_1^2(x_n|x_0)$$
$$\leq \lambda_1^2 - 2\lambda_1(\lambda_1 - \frac{1}{n} + \lambda_1^2) = 2\lambda_1/n \to$$

Since S is compact, there is a subsequence so that (Sx_{n_k}) converges, say to y. In order to keep the notation simple, we shall suppose that (x_n) *is* this subsequence. Then, since $||Sx_n - \lambda_1 x_n|| \to 0$, (x_n) is also a Cauchy sequence, with a limit x_1. Then $y = Sx_1$. It is clear that $Sx_1 = \lambda_1 x_1$ and the result follows.

■

We now use this to complete the proof of 5.35.

PROOF. Let x_1 be as above and consider the restriction of S to $H_1 = [x_1]^\perp$. This space is invariant under S (why?) and so, applying the Lemma again, we get a unit vector x_2, perpendicular to x_1 so that

$$Sx_2 = \lambda_2 x_2 \text{ where } |\lambda_2| = \sup\{|(Sx|x)| : x \in B_{H_1}\}.$$

Continuing, we obtain an orthonormal sequence (x_n) in H and a sequence (λ_n) of real numbers so that

$$Sx_n = \lambda_n x_n \text{ and } |\lambda_{n+1}| \leq |\lambda_n|.$$

Then (λ_n) is in c_0. For if this were not the case, there would be an $\epsilon > 0$ so that $|\lambda_n| \geq \epsilon$ for each n. Now the sequence x_n/λ_n is bounded in H but its image under S is the orthonormal system (x_n) which certainly does not contain a convergent subsequence and this contradicts the compactness of S. Now put $H_\infty = [\bar{x}_n], H_0 = H_\infty^\perp$. The restriction of S to H_0 vanishes (for otherwise we could use the Lemma to obtain an eigenvalue which is smaller (in absolute value) than each of the λ_n which is impossible). We obtain the required basis by combining (x_n) with an orthonormal basis for H_0.

■

REMARK. We have deliberately stated the spectral theorem above for operators on separable Hilbert space to avoid a rather cumbersome notation. The proof shows that if S is a compact, self-adjoint operator on a non-separable Hilbert space H, then we can split H as $H_1 \oplus H_2$ where both parts are S-invariant, H_1 is separable and S vanishes on H_2.

It is often convenient to estate 5.35 as follows:

Proposition 47 *Let S be a compact self-adjoint operator in $L(H)$ where H is an infinite dimensional Hilbert space. Then $\sigma(S)$ contains 0 and $\sigma(S)\setminus\{0\}$ is either finite or consists of a sequence (λ_n) which converges to zero. In the latter case, there exists a sequence $(H_n)_{n=0}$ of closed subspace of H so that* *a)* $H = \oplus_n H_n$*;*

b) each H_n *is* S*-invariant;*

c) $S|_{H_0} = 0$*;*

d) H_n *is finite dimensional* $(n \geq 1)$ *and* $Sx = \lambda_n x$ *for* $x \in H_n$*.*

Corollar 12 *(the Fredholm alternative:) If* $\lambda \notin \sigma(S)$*, then the equation*

$$(\lambda Id - S)x = y$$

has a unique solution x *for every* $y \in H$*. If on the other hand,* $\lambda \in \sigma(S)$ $(\lambda \neq 0)$*, the equation has a solution if and only if* $y \in N(\lambda)^\perp$ *where* $N(\lambda) = \{x : Sx = \lambda x\}$*.*

We now apply the above to Fredholm's theory of integral equations. We suppose that K is a continuous function from $I \times I$ into $\mathbf{C}$ where I is compact subinterval of $\mathbf{R}$ and that K satisfies the condition

$$K(s,t) = \overline{K(t,s)} \quad ((s,t) \in I \times I).$$

Then the operator I_K is hermitian and compact on $H = L^2(I)$. Thus we can apply the above theory to the associated integral equation

$$(\lambda \mathrm{Id} - I_K)x = y$$

to obtain the following information on the solutions:

a) the spectrum of I_K consists of a sequence (λ_n) which converges to 0 and if H_n is the eigenspace of $\lambda_n (\neq 0)$, then H_n is a finite dimensional subspace of $C(I)$;

b) if $x = I_K y$, then
$$x = \sum \lambda_n P_n y$$
where P_n is the orthogonal projection onto H_n and the series converges uniformly on I;

c) if $\lambda \notin \sigma(I_K)$, the equation
$$(\lambda \mathrm{Id} - I_K)x = y$$
has a solution for each $y \in L^2(I)$ given by the series
$$x = \left(\frac{1}{\lambda}\right) y + \sum_{\eta=1} \frac{\lambda_n}{\lambda(\lambda - \lambda_n)} P_n y$$
which converges uniformly on I. Hence if y is continuous, then so is x;

d) if λ is a non-zero eigenvalue, say $\lambda = \lambda_k$, then the above equation has a solution if and only if $P_k y = 0$ and the solution is given by the same formula (where the k-th term is omitted).

PROOF. a) and b) follow from the fact that I_K maps $L^2(I)$ continuously into $C(I)$.

c) if $(\lambda I - I_K)x = y$, then

$$\begin{aligned} x &= (\frac{1}{\lambda})(y + I_K x) \\ &= (\frac{1}{\lambda})y + (\frac{1}{\lambda})I_K(\sum_{n=1}^{\infty} \frac{1}{\lambda - \lambda_n} P_n y + \frac{1}{\lambda} P_0(y)) \\ &= (\frac{1}{\lambda})y + \sum_{n=1}^{\infty} \frac{\lambda_n}{\lambda(\lambda - \lambda_n)} P_n y. \end{aligned}$$

d) is similar.

■

EXERCISES.

A. Let T be compact, self-adjoint operator on the Hilbert space H. Show that T is positive if and only if its eigenvalues are non-negative. Show that in general H can be expressed as a direct sum $H_1 \oplus H_2$ where each H_i is invariant under T and T is positive on H_1 resp. negative on H_2. Is this decomposition unique?

B. Let T be a positive, compact operator on the Hilbert space H. Show that if $\{\lambda_n\}$ is the set of non-zero eigenvalues of T arranged in decreasing order, then

$$\lambda_n = \sup\{(Tx|x) : ||x|| \leq 1, x \in (H_1 \oplus \cdots \oplus H_{n-1})^{\perp}\}$$

where the H_i's are as in 5.38.

We shall now show that compact, convex subsets in Hilbert space have the fixed point property i.e. that if K is such a set, then every continuous mapping from K into itself has a fixed point. This will enable us to give a spectacularly simple proof (due to Lomonossov) of a recent result in operator theory.

We recall the famous classical result that every continuous mapping from a compact convex subset K of $\mathbf{R}^n$ into itself has a fixed point (BROUWER's fixed point theorem). From this it is easy to deduce that the same result holds for the Hilbert cube i.e. the compact metric space

$$C := \{(\xi_n) \in \emptyset : |\xi_n| \leq 1/n \text{ for each } n\}$$

provided with the product topology i.e. the topology of coordinatewise convergence. We simply apply the above result to the partial mappings $f_n : p_n \circ f$ where p_n is the fixed point (x_n) and if x is a limit point of the sequence (x_n) then x is clearly a fixed point for f.

From this we can deduce that every compact, convex subset K of a separable Hilbert space has the fixed point property. For we can assume without loss of generality that K is contained in the Hilbert cube. For if $\alpha_n = \sup\{|\xi_n| : x \in K\}$, then K is affinely homeomorphic to a subset of C under the mapping

$$x \mapsto \left(\frac{1}{n\alpha_n}\xi_n\right).$$

Then, since C and K are norm compact, the nom topology and the topology of coordinatewise convergence coincide on them.

Now we show that if

$$f : K \to K$$

is continuous, then f has a fixed point. For then $f \circ P_{K|CX}$ maps C into itself and so has a fixed point x i.e. $f(P_K x) = x$.

But then $x \in K$ and so $P_K x = x$ i.e. x is fixed point of f.

From this result it follows that if B is a closed convex bounded subset of H and $f : B \to B$ is such that $f(B)$ is relatively compact in B, then f has a fixed point.

A Corollary to this result is the following Proposition of Lomonossov concerning the longstanding problem: does every continuous linear operator T on a Banach space E have a non trivial invariant subspace—that is, is there a closed subspace E_1 with $E_1 \neq \{0\}$, $E_1 \neq E$, so that $T(E_1) \subseteq E_1$?

Lomonossov's result shows that this is indeed the case if T commutes with a non-trivial compact operator (in particular, if T itself or a polynomial in T is compact). It has been shown recently that the result is false without some condition on the operator T.

The sake of simplicity we shall consider only operators on separable Hilbert spaces although similar methods apply also to Banach spaces (see Exercise 5.42.N).

Note that only the separable case is interesting since any operator on a non-separable space has a non-trivial invariant subspace (for if $x \neq 0$, then $\overline{[x, Tx, T^2x, \ldots]}$, is an invariant subspace).

Proposition 48 *Let E be an infinite dimensional, separable Hilbert space, $T \in L(E)$ an operator which commutes with some non-trivial compact operator S. Then T has a non-trivial invariant subspace.*

Proof. We can suppose that $||S|| = 1$ and we choose $x_0 \in E$ with $||Sx_0|| > 1$. Then $||x_0|| > 1$. Let

$$B = \{x \in E : ||x - x_0|| \epsilon 1\}.$$

Then $0 \notin B$ and $0 \notin \overline{S(B)}$. Let $\mathcal{A}$ be the set of operators of the form $p(T)$ where P is a polynomial. We can suppose that

$$\mathcal{A}\mathcal{E} = \{\mathcal{A}(\mathcal{E}) : \mathcal{A} \in \mathcal{A}\}$$

is dense in E (otherwise its closure would be an invariant subspace for T). In particular, the sets $\{y \in E : ||Ay - x_0|| < 1\}$ cover $\overline{S(B)}$ as A ranges over $\mathcal{A}$. By compactness, we can find a finite subset $\{A_1, \dots, A_n\}$ so that the sets $\{y : ||A_i y - x_0|| < 1\}$ $(i = 1, \dots, n\}$ cover $S(B)$. Consider now the functions

$$a_j : x \to (1 - ||A_j x - x_0||)^+$$

$$b_j : y \to \frac{a_j(y)}{\sum_{i=1}^n a_i(y)}$$

(note that the denominator is positive on $S(B)$);

$$\psi : x \to \sum_{j=1}^n b_j(Sx) A_j Sx.$$

Then ψ maps $\{x : ||x - x_0|| \leq 1\}$ continuously into itself and its range is relatively compact. Hence it has a fixed point, say u. Put

$$A_0 := \sum_{j=1}^n b_j(Su) A_j S$$

$$E_1 := \{x : A_0 x = x\}.$$

Then E_1 is T-invariant since T commutes with A_0. Also E_1 is neither equal to E (since A_0 is cmpact)not to $\{0\}$ since it contains u.

■

Exercises.

A. Calculate the angle between the chords $e_{t_2} - e_{t_1}$and $e_{t_4} - e_{t_3}$ on the curve

$$t \mapsto e_t = \chi_{[0,t]}$$

in $L^2([0, 1])$ $(t_1 < t_2 < t_3 < t_4)$.

B. Let K be a subspace of the Hilbert space H, E a Banach space, T a continuous linear operator from K into E. Show that there is a $\tilde{T}$ $L(H, E)$ which extends T and has the same norm.

C. If $(H_\alpha)_{\alpha \in A}$ is a family of Hilbert spaces, we define a new space—**the Hilbert direct sum**—as follows: we consider the subspace H of the Cartesion product $\prod_{\alpha \in A} H_\alpha$ consisting of those vectors (x_α) which are such that $\sum_\alpha ||x_\alpha||^2 < \infty$. Show that H is a Hilbert space with the norm

$$||x|| = \left(\sum_\alpha ||x_\alpha||^2 \right)^{1/2} \qquad x = (x_\alpha)$$

and scalar product

$$(x|y) = \sum_\alpha (x_\alpha | y_\alpha).$$

We denote this space by $\oplus H_\alpha$. If $H_\alpha = L^2(\mu_\alpha)$ for a measure space.

D. Show that if (H_α), (K_α) are families of Hilbert spaces (indexed by the same set) and $T_\alpha \in L(H_\alpha, K_\alpha)$, then thee is an operator $T : \oplus H_\alpha \to \oplus K_\alpha$ extending each T_α (i.e. so that $T((x_\alpha) = (T x_\alpha))$ if and only if

$$\sup ||T_\alpha|| < \infty.$$

If $H_\alpha = K_\alpha$ for each α and each T_α is self-adjoint (resp. normal, unitary) what can be said atout T?

E. Let $T \in L(H, H_1)$ be an operator with closed range (i.e. $T(H) = \overline{T(H)}$. Show that there is an $S \in L(H_1)$ so that $TST = T$, $STS = S$ and ST resp. TS are symmetric. Show that if the equation $Tx = y$ is solvable, then the solution is $x = Sy$. For arbitrary $y \in H_1, x = Sy$ is a "least squares solution" of the equation $Tx = y$ (i.e. $||y = Tx|| \leq ||y - Tz||$ for each $z \in H$. (Consider the splittings $H = \operatorname{Ker} T \oplus (\operatorname{Ker} T)^\perp)$, $H_1 = T(H) \oplus T(H)^+)$.

F. Show that the unit ball B_H of a Hilbert space if $\sigma(H, H')$ metrisable if and only if H is separable. Show that the whole space is $\sigma(H, H')$-metrisable if and only if it is finite dimensional.

G. A sesquilinear form on a Hilbert space H is a mapping

$$S : H \times H \to \mathbf{C}$$

so that for each $y \in H$ (resp. $x \in H$) the mapping $S(x, y)$ is linear (resp. $y \to S(x, y)$ is antilinear). Show that every bounded sesquilinear form is induced by an operator $A \in L(H)$ in the sense that

$$S(x, y) = (Ax|y) \quad (x, y \in H).$$

H. Let S be a continuous sesquilinear form on a Hilbert space and suppose that there is a $K > 0$ so that

$$S(x, x) > K||x||^1 \quad (x \in H).$$

Show that there is a linear isomorphism A of H so that

$$S(x, y) = (Ax|y).$$

I. Show that if (U_n) is a sequence of unitary operators so that $U_n \to S$, $U_n^* \to S^*$ (strongly) then S is unitary. Let U_n be the unitary operator

$$(\xi_1, \dots, \xi_n, \dots) \to (\xi_n, \dots, \xi_{n-1}, \xi_{n=1}, \dots)$$

on ℓ^2.

Show that U_n converges strongly to a non unitary operator.

J. Let (Ω, μ) be a finite measure space. Show that $M_{x_n} \stackrel{s}{\to} M_x$ if and only if (x_n) is $|| \, ||_\infty$-bounded and $x_n \to x$ in $L^1(M, \mu)$;

$M_{x_n} \stackrel{w}{\to} M_x$ if and only if $\{x_n\}$ is $|| \, ||_\infty$-bounded and

$$\int_A x_n \to \int_A x \text{ for each } A \in \mathcal{A}.$$

K. Let H be a Hilbert space, M a dense subset, (x_n) a sequence in H, (T_n) a sequence in $L(H)$. Show that

$$\begin{aligned} x_n \stackrel{s}{\to} x \quad &\Leftrightarrow \quad x_n \stackrel{w}{\to} x \text{ and } ||x_n|| \to ||x||; \\ T_n \stackrel{w}{\to} o \quad &\Leftrightarrow \quad (T_n x|y) \to 0 \text{ for each } x, y \in M \end{aligned}$$

provided that (T_n) is bounded.

L. Let T be a continuous linear operator from H_1 into H_2 where H_1 and H_2 are separable Hilbert spaces. Show that

1. the expression
$$||T||_{HS} = \sum_{n=1}^{\infty} ||Tx_n||^2$$
is independent of the orthonormal basis (x_n);
2. $||T||_{HS} = ||T^*||_{HS}$;
3. the space $L_{HS}(H_1, H_2)$ of operators for which $||T||_{HS} < \infty$ is a Hilbert space under the scalar product
$$(S|T)_{HS} := \sum_{n=1}^{\infty} (Sx_n|Tx_n)$$
(the operators in this space are called **Hilbert-Schmidt** operators);
4. every Hilbert-Schmidt operator is compact;
5. in ℓ^2, the multiplication operator M_y is **Hilbert-Schmidt** if and only if $y \in \ell^2$.

M. Show that if H is an infinite dimensional Hilbert space, then $\{x : ||x|| = 1\}$ is $\sigma(H, H')$-dense in the unit ball of H. Consider the mapping
$$(x, y) \to (x|y)$$
on $\ell^2 \times \ell^2$, where ℓ^2 is provided with the weak topology.

a) for which points is it jointly continuous?
b) for which points is it continuous on $B_{\ell^2} \times \ell^2$?
c) for which points is it continuous on $B_{\ell^2} \times B_{\ell^2}$?

N. Show that in a Banach space E a compact, convex subset cube (show that one can reduce to the case where E is separable and so the norm topology coincides with that defined by a sequence of linear functionals). Deduce that K has the fixed point property and use this to extend 5.34 to operators between Banach spaces.

O. Consider the following sequences in $H = \ell^2$.
$$\begin{aligned} x_n &= \{0, \dots, 0, 1, 0, \dots) \\ y_n &= (0, \dots, 0, 1, \frac{1}{n+1}, 0, \dots) \end{aligned}$$
and put $E_1 = [\overline{x_n}], E_2 = [\overline{y}_n]$. Show that $E_1 \cap E_2 = \{0\}$ and that $E_1 + E_2$ is dense but not closed in H.

P. Let $T : C \to C$ be a completely continuous mapping on a bounded, closed, convex subset of a Hilbert space. Show directly that T has a fixed point. (Define a sequence (x_n) by means of the recursion formula

$$x_{n+1} = aTx_n + (1-a)x_n$$

where $0 < a < 1$ and show that it converges to a fixed point.)

Q. Let T be a compact operator from H_1 into H_2. Show that there are orthonormal sequences (e_i) and (f_i) in H_1 and H_2 respectively and a sequence (λ_n) in c_0 so that

$$Tx = \sum_{i=1}^{\infty} \lambda_i (e_i|x) f_i.$$

6 Bases in Banach spaces

For infinite dimensional Banach spaces, to concept which corresponds to that of a basis for finite dimensional spaces is that of a Schauder basis (see 4.7.D). In fact it is one of the most fundamental and useful ones of the theory and we shall use it to prov some results on the structure of subspaces of ℓ^p spaces.

We begin with a list of definitions (with some repetitions of earlier ones):

Definition 12 *Let (x_n) be a sequence in a Banach space. Then (x_n) is* **complete** *if $[\overline{x_n}]$, the closed linear span of $\{x_n\}$, is E i.e. if every element of E can be approximated by linear combinations of the x_n. (x_n) is* **strongly linearly independent** *if $x_n \notin \overline{lin}\{x_m, m \neq n\}$ $(n \in \mathbf{N})$. Recall that this is equivalent to the existence of a sequence (f_n)* **biorthogonal** *to (x_n) (i.e. such that $f_m(x_n) = \delta_{mn}$). A sequence (f_n) is E' is* **total** *if for each $x \in E$, $f_n(x) = 0$ for each n implies that $x = 0$.*

(x_n) is a **Markuševič basis** *if it is fundamental and strongly linearly independent and the (uniquely determined) biorthogonal system (f_n) is total. (x_n) is a* **(Schauder) basis** *if for each $x \in E$ there is a unique sequence (λ_n) is a Markuševič scalars so that $x = \sum_{n=1}^{\infty} \lambda_n x_n$. Then (x_n) is a Markuševič basis with biorthogonal sequence (f_n) in E', we define continuous linear operator S_n on E by*

$$S_n : x \to \sum_{k=1}^{n} f_k(x) x_k.$$

Of course if a Banach space has a basis then it must be separable. The first question which arises naturally is—does every separable Banach space have a basis? This problem was raised by Banach and remained one of the most *famous problems in functional analysis until it was solved in the negative in 1973, in fact by the example of Enflo mentioned on p. 184 since it is not difficult to see that if E is a Banach space with a Schauder basis then every compact $T \in L(E)$ is uniformly approximable by finite dimensional operators. We begin by showing that a weaker result holds—namely every separable Banach space has a Markuševič basis.*

Proposition 49 *Let E be a separable Banach space, $F \subseteq E'$ separable and total. Then there is a Markuševič basis $(y_n : g_n)$ in E so that each g_n lies in F.*

Proof. The proof employs a Gram-Schmidt type procedure. We commence with a fundamental sequence (x_n) in E and a total sequence (f_n) in F (why does the latter exists?). We choose a biorthogonal sequence (y_n, g_n) by an induction process as follows: if the (y_n, g_n) are chosen up to (y_{2k}, g_{2k}) we choose y_{2k+1} to be the first x_ℓ not in $[y_1, \dots, y_k]$ and define

$$g_{2k+1} = \frac{f_m - f_m(x_1)g_1 - \cdots - f_m(x_{2k})g_{2k}}{f_m(x_{2k+1})}$$

where f_m is any element of $[f_n]$ which does not vanish on $x_{2k=1}$. We now choose $(y_{2k+2}; g_{2k+2})$ by a dual process i.e. g_{2k+2} is the first f_ℓ not in $[g_1, \dots, g_k]$ and put

$$y_{2k+1} = \frac{x_m - g_1(x_m)y_1 - \cdots - g_{2k+1}(x_m)y_{2k+1}}{g_{2k+2}(x_m)}$$

where x_m is chosen so that $g_{2k+2}(x_m) \neq 0$. Then (x_n, g_n) is biorthogonal and $\operatorname{lin}\{x_n\} = \operatorname{lin}\{y_n\}$, $\operatorname{lin}\{f_n\} = \operatorname{lin}\{g_n\}$ from which it follows that $(y_n; g_n)$ has the desired properties.

■

Corollar 13 *Every separale Banach space contains a Markuševič basis $(x'_n f_n)$ with the property that $\overline{[f_n]}$ is norming i.e.*

$$||x|| = \sup\{|f(x)| : f \in \overline{[f_n]}, ||f|| \leq 1\}.$$

Exercises. Let E_1 be a closed subspace of a separale Banach space E. Let (x_n, f_n) be a Markuševič basis for E_1. Show that there is a Markuševič basis (y_m, g_m) for E with the property that (x_n) is a subsequence of (y_m). Deduce that every subspace E_1 of a separable Banach space has a **quasi-complement** i.e. a closed subspace E_2 so that $E_1 \cap E_2 = \{0\}$, $E_1 + E_2$ is dense in E.

(Warning: the coresponding result is, in general, false for non-separable E).

The difference between Markuševič bases and Schauder bases, is made explicit in the next result.

Proposition 50 *Let (x_n) be a Markuševič basis. Then (x_n) is a Schauder basis if and only if the associated projection operators (S_n) are uniformly bounded.*

PROOF. Firstly suppose that (x_n) is a basis. Then $S_n x \to x$ for each $x \in E$ and so $\{S_n x\}$ is bounded. Hence $\{S_n\}$ is uniformly bounded (cf. 4.2). Now suppose that $\{S_n\}$ is uniformly bounded. It suffices to show that if $x \in E$ then $S_n x \to x$. But this holds for each x in the linear span of $\{x_n\}$ which is dense in E. Hence the result follows from 1.18.I.

■

EXERCISES.

A. Let (x_n) be a sequence in a Banach space E so that for each $x \in E$ there is a unique sequence (λ_n) of scalars with $f(\sum_{n=1}^m \lambda_n x_n) \to f(x)$ for each $f \in E'$. Show that (x_n) is a Schauder basis.

   (Define a new norm $|| \, ||_n : x \to \sup_m || \sum_{n=1}^m \lambda_n x_n||$ and ape the proof of 4.17.A to show that $|| \, ||_n$ is equivalent to $|| \, ||$).

B. Let (x_n) be a complete sequence of non-zero elements of E. Show that (x_n) is a basis if and only if the following condition holds: There is a constant $M \geq 0$ so that for each $m \leq n$ in $\mathbf{N}$ and each sequence $\{\lambda_1, \dots, \lambda_n\}$ of scalars,

$$\left\| \sum_{k=1}^m \lambda_k x_k \right\| \leq M \left\| \sum_{k=1}^n \lambda_k x_k \right\|.$$

   Show that this can be reformulated geometrically as follows:

$$\text{dist}\,(V_n, x_n) \geq \alpha \text{ for some } \alpha > 0$$

   where V_n is the unit ball in $\text{lin}\,\{x_1, \dots, x_1\}$ and x_n is $[\overline{x_{n+1}, \dots}]$.

Examples of Schauder bases abound. The classical function systems (trigonometric functions, Legendre and Hermite polynomials etc.) are bases for suitable function spaces. We bring a few simple examples:

I. The vectors (e_n) form a basis for c_0, ℓ^p $(1 \leq p < \infty)$ but not for ℓ^∞ (why not?).

II. The space $C[0,1]$ has a basis (x_n) defined as follows: x_0 is the constant function 1 and x_1 is the identity function. If n has the form $2^k + j$ $(k \geq 1, j = 0, \ldots, 2^k - 1)$ then x_n is the polygonal function

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The sequence (x_n) is complete in $C[0,1]$ (why?) and for any $n \in \mathbf{N}$ and any sequence $(\lambda_0, \ldots, \lambda_{n+1})$ of complex numbers,

$$\left\| \sum_0^n \lambda_i x_i \right\| \leq \left\| \sum_0^{n+1} \lambda_i x_i \right\|.$$

Hence by 6.6.B, (x_n) is a basis.

III. The Haar system defined below is a basis for each of the spaces $L^p[0,1]$ $(1 \leq p < \infty)$. x_0 is as in II and if $n = 2^k + j$ then x_n is the function.

(Note that the function of II are indefinite ingegrals of the above functions.)

This follows from the following facts. Firstly the span of the Haar functions is the space of step functions which constant on dyadic intervals and these functions are dense in $L^p([\theta, 1])$. Secondly, a simple calculation shows that

$$\left\| \sum_{i=0}^n \lambda_i x_i \right\| \leq \left\| \sum_{i=0}^{n+1} \lambda_i x_i \right\|$$

for any sequence $\lambda_0, \ldots, \lambda_n, \lambda_{n+1}$ of real numbers.

IV. If $1 < p < \infty$, then the trigonometric functions $\{\exp(2n\pi it)n \in \mathbf{Z})\}$ form a basis for $L^p([0,1])$. This follows from the following result of M. Riesz: there is a K_p so that

$$\left\| \sum_{k=0}^n \exp(2k\pi it) \right\|_p \leq K_p \left\| \sum_{k=-n}^n \xi_k \exp(2k\pi it) \right\|_p$$

for finite sequences $(\xi_{-n}, \ldots, \xi_1, \xi_0, \ldots, \xi_n)$ of scalars (see, for example, A.Z. Zygmund, Trigonometric series ...).

Although a separable Banach space need not have a basis it does possess a plentiful supply of sequences which are bases for their closed linear span and these play an important role in the theory.

Definition 13 *A sequence* (x_n) *in a Banach space* E *is a* **basic sequence** *if it is a basis for* $[\overline{x_n}]$. *If the* x_n *are non-zero this is (by ...) equivalent to the existence of a* $K > 0$ *so that*

$$\left\|\sum_{k=1}^{m} \lambda_k x_k\right\| \leq K \left\|\sum_{k=1}^{n} \lambda_k x_k\right\|$$

for each $n \geq m$ *in* $\mathbf{N}$ *and each sequence* $\{\lambda_1, \dots, \lambda_n\}$ *of scalars. We now show that every Banach space has a basic sequence.*

Lemma 11 *Let* E_1 *be a finite dimensional subspace of an infinite dimensional Banach space* E, ϵ *a positive number. Then there is an* $x \in E$ *so that* $||x|| = 1$ *and*

$$||y|| \leq (1+\epsilon)||y + \lambda x||$$

$(y \in E_1, \lambda \in \mathbf{C})$.

PROOF. Let $\{y_i\}_{i=1}^{m}$ be an $\epsilon/2$ net for the unit sphere of E_1 i.e. such that every point therein lies within a distance of $\epsilon/2$ from one of the y_i. (The existence of such a net follows from the precompactness of B_{E_1}.) Now let $(f_1, \dots, f_n)$ be a sequence in E' with $||f_i|| = 1$ and $f_i(y_i) = ||y_i||$ for each i. Choose $x \in E$ with $||x|| = 1$ and $f_i(x) = 0$ for each i. Then x satisfies the conditions of the Lemma. For if $y \in E_1$ with $||y|| = 1$ and i is chosen so that $||y - y_i|| \leq \epsilon/2$ we have, for each λ,

$$\begin{aligned} ||y + \lambda x|| &\geq ||y_i + \lambda x|| - \epsilon/2 \\ &\geq f_i(y_i + \lambda x) - \epsilon/2 \\ &= 1 - \epsilon/2 \geq ||y||/(1+\epsilon). \end{aligned}$$

■

Proposition 51 *Every infinite-dimensional Banach space* E *has a basic sequence.*

PROOF. Choose $\epsilon > 0$ and $\epsilon_n > 0$ so that $\prod(1+\epsilon_n) \leq 1+\epsilon$. For x_1 we take an arbitrary element of E with norm one. We then use the Lemma to construct inductively a sequence (x_n) so that for each n

$$||y|| \leq (1+\epsilon_n)||y + \lambda x_{n+1}||$$

for each λ and $y \in [x_1, \dots, x_n]$. Then a simple calculation shows that the inequality of 6.7 holds with $k = 1+\epsilon$.

■

EXERCISES. Refine the above proof to show that if (x_n) is a sequence in E with $||x_n|| = 1$ and $f(x_n) \to 0$ for each $f \in E'$ (i.e. $x_n \to 0$ in $\sigma(E, E')$) then there is a subsequence of (x_n) which is basic. (Note that in the construction of x in the Lemma, it suffices to demand that $|f_i(x)| < \epsilon/4$ for each i).

One property of bases (respectively basic sequences) which is used constantly in applications is their stability under small perturbations. There exists a host of results of this type (so called PALEY-WIENER theorems) and we present a few of the most useful.

Let (x_n) and (y_n) be sequences in E, F resp. we say that (x_n) **dominates** (y_n) (in symbols $(x_n) > (y_n)$) if one of the two following equivalent conditions holds:

1. there is a continuous linear operator $T : [\overline{x_n}] \to [\overline{y_n}]$ so that $Tx_k = y_k$ for each k,

2. there is a $K > 0$ so that

$$\left\|\sum_{k=1}^{n} \lambda_k y_k\right\| \epsilon K \left\|\sum_{k=1}^{n} \lambda_k x_k\right\|$$

for each finite sequence $\{\lambda_1, \dots, \lambda_n\}$ of scalars. If, in addition, (y_n) dominates (x_n) then (x_n) and (y_n) are **equivalent**. In this case the T in 1. is an isomorphism.

Note that if (x_n) and (y_n) are equivalent then (x_n) has one of the following properties if and only if (y_n) also does: being strongly linearly independent, being a basic sequence. In both are fundamental, then (x_n) is a (Markuševič) basis if and only if (y_n) is.

EXERCISES. Show that if (x_n) is a basis then $(y_n) < (x_n)$ if and only if for each sequence (λ_n) of scalars.

$$\sum \lambda_n x_n \text{ converges } \Leftrightarrow \sum \lambda_n y_n \text{ converges.}$$

Prove the unproven statements in the above paragraph.

Proposition 52 *Let (x_n) be a strongly linearly pendent sequence and suppose that $(y_n) < (x_n)$ with constant K (i.e. $||\sum \lambda_k y_k|| \leq K||\sum \lambda_k x_k||$ for scalars $\{\lambda_1, \dots, \lambda_m\}$). Then if $\epsilon < 1/K$, the sequence $(x_n + \epsilon y_n)$ is equivalent to x_n.*

PROOF. The operator defined formally by the equation

$$\sum \lambda_n x_n \to \epsilon \sum \lambda_n y_n$$

has a continuous linear extension to an operator T from $[\overline{x_n}]$ into $[\overline{y_n}]$ and $||T|| < 1$. Hence $I + T$ is an isomorphism (3.10) and $(I + T)x_n = x_n \epsilon y_n$.

■

Definition 14 *If (ϵ_n) is a sequence of non negative numbers, a sequence (x_n) is (ϵ_n)-***stable** *if it is equivalent to each sequence (y_n) which is such that $||x_n - y_n|| \leq \epsilon_n$.*

Proposition 53 *Let (x_n) be a strongly linearly independent sequence of unit vectors in E with biorthogonal system (f_n). Then if (ϵ_n) is a sequence of non negative numbers with $\sum \epsilon_n ||f_n|| < 1$, (x_n) is (ϵ_n)-stable.*

PROOF. Let (u_n) be a sequence with $||u_n|| \leq \epsilon_n$. We show that $(x_n) - (x_n + u_n)$ by applying the above Proposition with $\epsilon = 1$.

We can estimate: for $x = \sum \lambda_k x_k = \sum f_k(x) x_k$,

$$\begin{aligned} \left\| \sum \lambda_k u_k \right\| &= \left\| \sum f_k(x) u_k \right\| \\ &\leq \sum |f_k(x)| \epsilon_k \\ &\leq k||x|| \text{ where } k = \sum ||f_k|| \epsilon_k < 1 \end{aligned}$$

and so the condition of 6.12 are satisfied.

■

The next result is of a similar nature.

Proposition 54 *Let (x_n) be a Schauder basis for the Banach space E with $||x_n|| = 1$ and suppose that (x_n) has basis constant K (i.e. $K = \sup\{||S_n|| : n \in \mathbf{N}\}$). Then if (y_n) is a sequence in E which is near to (x_n) in the sense that $\sum ||x_n - y_n|| < 1/2K$, (y_n) is a basis for E, equivalent to (x_n).*

PROOF. This is a corollary of 6.12 but we bring a direct proof. We define a linear operator $T : E \to F$ by

$$\sum_{n=1}^{\infty} \lambda_n x_n \to \sum_{n=1}^{\infty} \lambda_n y_n.$$

Note that T is well-defined since if $\sum_{n=1}^{\infty} \lambda_n x_n$ converges (say to x) then $|\lambda_n| \leq 2K||x||$ and

$$\sum_{k=1}^{n} \lambda_k y_k = \sum_{k=1}^{n} \lambda_k (y_k - x_k) + \sum_{k=1}^{n} \lambda_k x_k$$

and the first series on the right hand side converges absolutely. Also we have $||I - T|| < 1$ and so T is an isomorphism from which everything follows. For if $x = \sum_{n=1}^{\infty} \lambda_n x_n$, we have

$$\begin{aligned} ||(I-T)|| &\leq \left\| \sum_{n=1}^{\infty} \lambda_n x_n - \lambda_n y_n \right\| \\ &\leq 2K \sum_{n=1}^{\infty} ||x_n - y_n|| \, ||x||. \end{aligned}$$

■

we now use the techniques of basis theory to prove that the spaces $\ell^p (1 \leq p < \infty)$ and c_0 have the following property: every infinite dimensional subspace contains a subspace isomorphic to the original space. (Note that this result does not involve the concept of a basis). In the next section we shall use it to show that every complemented infinite dimensional subspace of one of these spaces is isomorphic to the original space.

To do this we indroduce the important concept of **block sequences**.

If (x_n) is a basic sequence in a Banach space then a **block sequence** of (x_n) is a sequence (y_k) constructed as follows: let (n_k) be a strictly increasing sequence in $\mathbf{N}$ and let (λ_n) be a sequence of scalars. Define

$$y_k = \sum_{n=n_k+1}^{n_{k+1}} \lambda_n x_n.$$

Then (y_k) is clearly a basic sequence with basic constant at most that of (x_n). In the next result we show that c_0 and ℓ^p have the property that every block sequence of the standard basis (e_n) is equivalent to (e_n) in rather strong sense.

Proposition 55 *Let (y_k) be a block sequence of the standard basis of $E = \ell^p$ $(1 \leq p < \infty$, with $||y_k|| = 1$ $(k \in \mathbf{N})$. Then (y_n) is equivalent to (e_k) and infact if $E_1 = [\overline{y_k}]$ then E_1 is isometric to E and there is a projection of norm one from E onto E_1. The same holds for $E = c_0$.*

PROOF. We suppose that

$$y_k = \sum_{n=n_k+1}^{n_{k+1}} \lambda_n e_n$$

with $\sum_{n=n_k+1}^{n_{k+1}} |\lambda_n|^p = 1$. Now if (ξ_k) is a sequence in ℓ^p we have

$$\left\| \sum_{k=1}^{\infty} \xi_k y_k \right\|^p = \left\| \sum_{k=1}^{\infty} \left(\sum_{n=n_k+1}^{n_{k+1}} \lambda_n e_n \right) \right\|^p = \sum \sum |\xi_k|^p |\lambda_n|^p - \sum_{k=1}^{\infty} |\xi_k|^p$$

and so (y_k) is equivalent to (e_n) and E_1 and E are isometric. We construct the projection P as follows: for each k choose $f_k \in \text{lin}\,\{e_n\}_{n=n_k+1}^{n_{k+1}}$ in $(\ell^p)' = \ell^q$ with $||f_k|| = 1$ and $f_k(y_k) = 1$. Then

$$P : x \to \sum_{k=1}^{\infty} f_k(x) y_k$$

is the required projection. We can estimate the norm of P as follows:

$$\begin{aligned} \left\| P\left(\sum_{n=1}^{\infty} \xi_n e_n \right) \right\|^p &= \sum_{k=1}^{\infty} |f_k(x)|^p \\ &\leq \sum_{k=1}^{\infty} \sum_{n=n_k+1}^{n_{k+1}} |\xi_k|^p = \left\| \sum \xi_n e_n \right\|^p \end{aligned}$$

and so $||P|| = 1$.

■

EXERCISES. Complete the proof of 6.17 by considering the case $E = c_0$.

Proposition 56 *Let E be a Banach space with basis (x_n), K a closed infinite dimensional subspace. Then F contains a subspace G which has a basis equivalent to a block sequence of (x_n).*

PROOF. Since F is infinite dimensional there is, for every $p \in \mathbf{N}$, a $y \in F$ with $||y|| = 1$, $y \in [\overline{x_{p+1}, x_{p+2}}]$. (Otherwise F would be in span $\{x_1, \dots, x_p\}$ for some $p \in \mathbf{N}$). Now we construct inductively a sequence (y_k) as follows:

1. Choose any $y_1 = \sum_{n=1}^{\infty} \xi_{1,n} x_n$ with $||y_1|| = 1$. Take $k_1 \in \mathbf{N}$ so that $|| \sum_{n=k_1+1} \xi_{1,n} x_n || < 1/4K$ (K the basis constant of (x_n)).

2. Choose $y_2 \sum_{n=k_1+1}^{\infty} \xi_{2,n} x_n$ in F with $||y_2|| = 1$ and $k_2 \in \mathbf{N}$ so that $|| \sum_{n=k_2+1}^{\infty} \xi_{2,n} x_n|| < \frac{1}{16}K$.

   Continuing, at the r-th step we get a $y_r = \sum_{n=k_r+1}^{\infty} \xi_{r,n} x_n$ and a $k_r \in \mathbf{N}$ so that
$$\left\| \sum_{n=k_{r+1}}^{\infty} \xi_{r,n} x_n \right\| \leq \frac{1}{4^r K}.$$
   Then if $z_r = \sum_{n=k_{r-1}} \xi_{r,n} x_n$, (z_r) is a block sequence of (x_n) and $||y_r - z_r|| \leq \frac{1}{4^r K}$ so that (y_r) and (z_r) are equivalent by 6.15.

■

Proposition 57 *If $E = \ell^p$ or c_0 then every infinite dimensional closed subspace of E contains a complemented subspace isomorphic to E.*

PROOF. The subspace contains a subspace G with a basis equivalent to a block sequence of (x_n) (6.18) and this space satisfies the given condition.

■

Most of the standard bases have additional symmetry properties which we now discuss. We begin with some remarks on unconditional convergence in Banach spaces. A series $\sum x_n$ in a Banach space is **unconditionally convergent** if for each permutatin π of $\mathbf{N}$, $\sum x_{\pi(n)}$ converges. Then, of course, the sum is independent of the permutation (this can be proved directly of by using the Hahn-Banach theorem to reduce to the case where $E = \mathbf{C}$). Fortunately this notion of convergence coincides with several other natural ones.

■

Proposition 58 *Let $\sum x_n$ be a series in a Banach space. Then the following are equivalent:*

1. *$\sum x_n$ is unconditionally convergent;*
2. *every subseries $\sum x_{n_k}$ is convergent;*
3. *if $(\epsilon_k) \in \{-1, 1\}^{\mathbf{N}}$ then $\sum \epsilon_k x_k$ converges;*
4. *$\sum x_n$ is summable i.e. the net $\{s_J : J \in J(\mathbf{N})\}$ converges where $s_J = \sum_{n \in J} x_n$ and J ranges over the set $J(\mathbf{N})$ of the finite subsets of $\mathbf{N}$, directed by inclusion.*

PROOF. We prove that 2. implies 4. (the other parts are similar). Suppose that 4. does not hold—then we can construct a sequence (J_n) is $\mathcal{J}(\mathbf{N})$ with J_n to the right of J_{n-1} for each n (i.e. the largest element in J_{n-1} is smaller than the smallest one in J_n) so that $||\sum_{i\in J_n} x_i|| \geq \epsilon$ for some fixed $\epsilon > 0$. Then the subseries $\sum_{i\in J} x_i$ diverges where $J = \bigcup_n J_n$.

■

EXERCISES.

A. Show that if $\sum x_n$ is unconditionally convergent then the map

$$(\epsilon_n) \to \sum \epsilon_n x_n$$

from the Cantor set $\{-1,1\}^{\mathbf{N}}$ into E is continuous. Deduce that $\{\sum \epsilon_n x_n : (\epsilon_n) \in \{-1,1\}^{\mathbf{N}}\}$ is compact in E.

B. Let $\sum x_n$ be an unconditionally convergent series in a Banach space over $\mathbf{R}$. Show that if (λ_n) is bounded sequence of scalars, $\sum \lambda_n x_n$ converges and

$$\left\|\sum \lambda_n x_n\right\| \leq ||(\lambda_n||_\infty \sup\left\{\left\|\sum \epsilon_k x_k\right\| : (\epsilon_k) \in \{-1,1\}^{\mathbf{N}}\right\}.$$

Definition 15 *A basis (x_n) in E is* **unconditional** *if for each $x \in E$, its series expansion $\sum \lambda_n x_n$ converges unconditional. Then by 6.20 and 6.21.B*

a) every permutation $(x_{\pi(n)})$ of (x_n) is a basis;

b) every subsequence (x_{n_k}) of (x_n) is a basic sequence;

c) if $\sum \lambda_n x_n \in E$ and (ξ_n) is a bounded sequence then $\sum \xi_n \lambda_n x_n$ converges in E.

EXERCISES. If (x_n) is an unconditional basis, $\sigma = (x_{n_k})$ is a subsequence and $\epsilon = (\epsilon_k) \in \{-1,1\}^{\mathbf{N}}$ we can define linear operators

$$\begin{aligned} P_\sigma : \sum \lambda_n x_n &\mapsto \sum \lambda_{\bar{n}_k} x_{n_k} \\ M_\epsilon : \sum \lambda_n x_n &\to \sum \epsilon_n \lambda_n x_n. \end{aligned}$$

Show that $\{P_\sigma\}$ and $\{M_\epsilon\}$ are uniformly bounded.

Natural examples of unconditional bases are the canonical basis for the ℓ^p-spaces $(1 \leq p < \infty)$ resp. c_0. The Haar basis for $L^p([0,1])$ is unconditional for $p > 1$ but this is difficult to prove (see, ...).

If we define f_n to be the sequence $(0, \dots, 0, 1, 1, 1, \dots)$ where the first "1" is in the n-th place, then (f_n) is a conditional basis for c (i.e. it is not unconditional).

EXERCISES.

A. A Schauder basis (x_n) for E is **symmetric** if (x_n) is equivalent to each permutation $(x_{\pi(n)})$. Then there is an isomorphism V_π of E so that $V_\pi x_n = x_{\pi(n)}$. Show that $\{||V_\pi|| : \pi \in S(\mathbf{N})\}$ is bounded. Deduce that there is an equivalent norm on E so that V_π is an isometry. Show that one can even find an equivalent norm on E so that V_π is an isometry. Show that one can even find an equivalent norm with the property that

$$\left\|\sum \lambda_n \epsilon_n x_{\pi(n)}\right\| = \left\|\sum \lambda_n x_n\right\|$$

for each $x = \sum \lambda_n x_n$, each $\pi \in S(\mathbf{N})$ and each sequence $(\xi - n)$ in $\{-1, 1\}^{\mathbf{N}}$.

B. Show that if (x_n) is symmetric then (x_n) is equivalent to each subsequence (x_{n_k}) of itself.

Show that the coordinate vectors (e_n) form a basis for the sequence space

$$E = \left\{ (\xi_n) : ||(\xi_n)|| := \sup \sum_{k=1}^{\infty} |\xi_{n_k}| k^{-1/2} < \infty \right\}$$

(where the supremum is taken over all subsequences (ξ_{n_k})) which is not symmetric but which satisfies the above condition. (Compare the norm of $(1, \frac{1}{\sqrt{2}}, \dots, \frac{1}{\sqrt{k}}, 0, \dots)$ and its permutation $(\frac{1}{\sqrt{k}}, \dots, \frac{1}{\sqrt{2}}, 1, 0, \dots)$.)

EXERCISES.

A. A basis (x_n) for E is **uniform** if $\sum f_n(x) x_n$ converges to x uniformly on the unit ball. Show that E has a uniform basis if and only if it is finite dimensional.

B. A basis (x_n) for E is **absolute** if for every $x \in E$ the series $\sum f_n(x) x_n$ converges absolutely. Show that E has an absolute basis if and only if it is isomorphic to ℓ^1.

C. Let E be a finite dimensional space. show that there are basis (x_k) of E, resp. (f_k) of E' so that

1. $f_i(x_k) = \delta_{ik}$

2. $||f_k|| = ||x_k|| = 1$ for each k'.

(One can suppose that $E = \mathbf{R}^n$. For each n-tuple (y_k) in $\mathbf{R}^n$ let $D(y_1, \dots, y_n)$ denote the determinant of the matrix with the y_k's as colums. Choose a point $(x_1, \dots, x_n)$ in the unit sphere where $|D|$ attains its maximum and define

$$f_i : y \to \frac{D(x_1, \dots, x_{i-1}, y, x_{n=1}, \dots, x_n)}{D(x_1, \dots, x_n)}.$$

D. Show that if E_0 is an n-dimensional subspace of a normed space E, then there is a projection $P : E \to E_0$ with $||P|| \leq n$.

E. Consider the sequence

$$x_1 = (1/2, 0, \dots), x_2 = (-1, 1/2, 0, \dots), x_3 = (0, -1, 1/2, 0, \dots)$$

in c_0. Show that it is linearly independent but that $\sum 2^{-n} x_n = 0$.

F. Let E be a Banach space, (x_n) a basis for E, $T : E \to F$ a continuous surjective linear mapping, $y_n = Tx_n$. Show that (y_n) is a basis for F if and only if $T' : F' \to E'$ is surjective. If (y_n) is a sequence in a Banach space, put

$$E = \left\{ (\xi_n) \in \emptyset : \sum \xi_n y_n \text{ converges} \right\}.$$

By equipping E with the norm $||(\xi_n)|| = \sup || \sum \xi_n y_n||$ deduce a criterium for (y_n) to be a basis.

G. Let E and F be Banach spaces, (x_n) a basis for F with basis constant 1. Then if $T : E \to F$ is a continuous linear operator, T has the form

$$Tx = \sum_{n=1}^{\infty} f_n(x) x_n$$

where (f_n) is a sequence in E' so that the series converges everywhere. Also we have

$$||T|| = \limsup_{n \to \infty} \sup \left\{ \left\| \sum_{i=1}^{n} f_i(x) x_i \right\| = ||x|| \leq 1 \right\}.$$

H. Show that if E and F are Banach spaces and F has a basis, then an operator $T \in L(E, F)$ is compact if and only if it is uniformly approximable by finite dimensional operators.

I. A basis is **monotone** if $||S_n|| \leq 1$ for each n. Show that if (x_n) is a basis for Hilbert space, then (x_n) is monotone if and only if it is orthogonal.

J. Let (x_n) be a basis in E, (f_n) the corresponding biorthogonal sequence. Show that

1. $\{x_n\}$ is bounded if and only if $\{f_n\}$ is bounded away from zero (i.e. $\inf ||f_n|| \geq 0$).
2. $\{x_n\}$ is bounded away from zero if and only if $\{f_n\}$ is bounded.

K. Let (x_n) be a normalized basis for E (i.e. $||x_n|| = 1$ for each n). Let (λ_n) be a sequence of non zero scalars. Show that if

$$y_n = \sum_{i=1}^{n} \lambda_i x_i$$

then (y_n) is a basis if and only if $\{y_n/\lambda_{n+1})$ is bounded.

L. If (x_n) is an unconditional basic sequence, then so is each block sequence of (x_n).

M. Show that a complete sequence (x_n) of non-zero vectors n a Banach space E is an unconditional basis if and only if there is a $\delta > 0$ so that $\delta(V_{N_1}, E_{N_2}) > \delta$ for each partition of $\mathbf{N}$ into disjoint sets N_1 is the unit sphere of E_{N_1}.

N. Consider the sequence (x_n) in ℓ^1 where $x_1 = e_1$ and

$$x_n = e_n - e_{n-1}$$

for $n > 1$. Show that

a) (x_n) is a basis for ℓ^{1}'
b) $\sum \lambda_n x_n$ converges if and only if $\sum |\lambda_n - \lambda_{n=1}| < \infty$.

Deduce that (x_n) is conditional.

O. Let (x_n) be a normalised unconditional basis for the Hilbert space H. Show that $\sum \lambda_n x_n$ converges if and only if $\sum |\lambda_n|^2 < \infty$ i.e. that (x_n) is equivalent to an orthogonal basis.

P. Let (x_n) be an unconditional basis for E. Show that for any $p \in [1, \infty[$, then norm

$$||x||_p = \left(\int_0^1 \left\| \sum r_n(t) f_n(x) x_n \right\|^p dt \right)^{1/p}$$

on E is equivalent to the original one and has the property that

$$\left\| \sum \lambda_n x_n \right\|_p = \left\| \sum \epsilon_n \lambda_n x_n \right\|_p$$

for all sequences (ϵ_n) in $\{-1, 1\}^{\mathbf{N}}$. ((r_n) denotes the sequence of Rademacher functions, (f_n) the sequence biorthogonal to (x_n).)

Q. Show that if a sequence (x_n) is completein $L^1([0, 1])$ and X_n is a primitive of x_n, then the sequence(X_n), together with the constant function 1, is complete in $C([0, 1])$.

R. Let K be a compact metric space and (s_n) a sequence of distinct points. Suppose that there is a sequence (x_n) in $C(K)$ so that

a) if $x \in C(K)$ there is a sequence (λ_n) of scalars with $x = \sum \lambda_i x_i$;

b) $x_n(s_m) = \delta_{nm}$ for each n, m.

Show that (x_n) is then a basis for $C(K)$ and that the sequence (s_n) is necessarily dense in K.

S. Let K be a compact metrix space, (s_n) a sequence in K. Suppose that there is a sequence (x_n) in $C(K)$ so that

1. if $x \in C(K)$ there is a sequence (λ_n) of scalars with $x = \sum \lambda_i x_i$;
2. $x_n(s_n) \neq 0$ and $x_m(s_n) = 0$ if $m > n$.

Show that (x_n) is then a basis for $C(K)$ and that the sequence (s_n) is necessarily dense in K.

7 Construction on Banach spaces

The Banach spaces which arise in practice belong either to one of the two classical types (namely the $C(K)$or L^p-spaces) or can be constructed in a suitable manner from spaces of these types. We have already met such methods of construction in the earlier chapters and we shall now study some more sophisticate ones in some detail. In 1.1 we defined sums and products of finite families of Banach spaces. We shall begin by extending this to infinite families:

Let $\{(E_\alpha, ||\,||_\alpha)\}_{\alpha\in A}$ be a familiy of Banach spaces. We denote by E_0 the Cartesian product $\prod_{\alpha\in A} E$ of the underlying vector spaces. Then put

$$\begin{aligned} E_1 &:= \left\{x = (x_\alpha) \in E_0 : ||x||_1 := \sum_\alpha ||x_\alpha||_\alpha < \infty\right\} \\ E_\infty &:= \{x = (x_\alpha) \in E_0 : ||x||_\infty := \sup_\alpha ||x_\alpha||_\alpha < \infty\}. \end{aligned}$$

$(E, ||\,||_1)$ and $(E_\infty, ||\,||_\infty)$ are normed spaces. In fact, as can easily be deduced from 3.18.F, they are Banach spaces. $(E_1, ||\,||_1)$ is called the **Banach space sum** of the family (written $B\sum E_\alpha$) and $(E_\infty, ||\,||_\infty)$ is called the **Banach space product** (written $B\prod E_\alpha$). Note that each E_α is naturally isometric to a closed subspace and quotient space of both $B\sum E_\alpha$ and $B\prod E_\alpha$ so tht if either of the latter spaces possesses any property which is inherited by closed subspaces or quotients, then each E_α must have this property (for example, – separability, reflexivity).

We start with some examples:

I. If $\{E_1, \dots, E - n\}$ is a finite family of Banach spaces then $B\sum_{k\in\mathbf{N}} E_k$ and $B\prod_{k\in\mathbf{N}} E_k$ are just the spaces $(\prod E_k, ||\,||_s)$, $(\prod E_k, ||\,||_\infty)$ introduced in 1.5.D.

II. If each of the spaces E_α is the one-dimensional Banach space $\mathbf{C}$, then $B\sum_{a\in A} E_\alpha$ is just $\ell^1(A)$ and $B\prod_{\alpha\in A} E_\alpha$ is $\ell^\infty(A)$.

III. Let $\{S_\alpha\}$ be a family of completely regular spaces and denote by $S = \coprod S_\alpha$ the topological direct sum i.e. as a set S is the disjoint union of the S_α and the topology on S consists of those subsets whose intersection with each S_α is open. Then if $x \in C^\infty(S)$ into $B\prod C^\infty(S_\alpha)$. It is easy to check that $(\Omega_\alpha, \mu_\alpha)$ is a family of measure spaces. If Ω is the disjoint union of the Ω_α then we can define a σ-algebra $\sum$ on Ω by defining $A \subseteq \Omega$ to be measurable if its intersection with each Ω_α is measurable. We define a measure μ on Ω by putting $\mu(A) = \sum \mu_\alpha(A \cap \Omega_\alpha)$ (possibly infinite).

   Then if $x \in L^1(\mu)$ and we define $x_\alpha = x|_{\Omega_\alpha}$ it is a routine matter to check that the mapping $x \to (x_\alpha)$ is an isometry from $L^1(\mu)$ onto $B\sum_\alpha L^1(\mu_\alpha)$.

Exercises.

A. Show that if each E_α is non trivial then $B_\alpha \sum_{\alpha\in A} E_\alpha$ contains an isometric copy of $\ell^1(A)$ and $B\prod_{\alpha\in A} E_\alpha$ contains an isometric copy of $\ell^\infty(A)$. Deduce that

1. $B\sum_{\alpha\in A} E$ and $B\prod_{\alpha\in A} E$ are reflexive if and only if A is finite and each E_α is reflexive;
2. $B\sum_{\alpha\in A} E$ is separable if and only if A is countable and each E_α is separable;
3. $B\prod_{\alpha\in A} E$ is separable if and only if A is finite and each E_α is separable.

(All under the assumption, that each E_α is non-trivial.)

B. Show that if each E_α is isometric to ℓ^∞ then so is $B\prod_{\alpha\in A} E_\alpha$ provided that A is countable. What happens if A is uncountable? What are the corresponding results for ℓ^1?

We now discuss duality for sums and products the next result can be proved almost word for word as in the proof of the duality between ℓ^1 and ℓ^∞.

Proposition 59 *If (E_α) is a family of Banach space sthen the bilinear form*

$$((x_\alpha),(f_\alpha)) \to \sum_{\alpha\in A} f_\alpha(x_\alpha)$$

induces an isometry from $B_{\alpha\in A}\prod E'_\alpha$ onto $(B\sum_{\alpha\in A} E_\alpha)'$. The same mapping imbeds $B\sum_{\alpha\in A} E'$ into $(B\prod_{\alpha\in A} E_\alpha)'$ but not onto (provided infinitely many of the E_α are non trivial).

There is a generalisation of the above construction which often proves useful: If (E_α) is a family of Banach spaces and $1 \leq p < \infty$ put

$$\ell^p_{\alpha\in A} - \sum E_\alpha = \left\{ x \in \prod E_\alpha : ||x||_p = \left(\sum ||x_\alpha||^p\right)^{1/p} < \infty \right\}.$$

Then $\ell^p - \sum_{\alpha\in A} E_\alpha$ is a Banach space. Of course $\ell^1 - \sum_{\alpha\in A} E_\alpha = B\sum_{a\in A} E_\alpha$ and its is sometimes convenient to write $\ell^\infty - \sum_{\alpha\in A} E_\alpha$ instead of $B\prod_{\alpha\in A} E_\alpha$.

An important special case is that of the ℓ^2-sum of Hilbert spaces. Then $H = \ell^2 - H_\alpha$ is itself a Hilbert space under the scalar product

$$((x_\alpha)|(y_\alpha)) = \sum (x_\alpha|y_\alpha)$$

(cf. 6.41.C). Later we shall use such sums where the H_α's are L^2-spaces and we shall require the following concrete representation of the sum. Let $H_\alpha = L^2(\mu_\alpha)$ where μ_α is a non-negative Radon measure on the compact space K_α. Then the sum $\ell^2 - H\alpha$ is naturally isomorphic to the space $L^2(\mu)$ where μ is the corresponding measure on the topological direct sum $M = \prod_{\alpha\in A} K_\alpha$ (cf. 5.52).

EXERCISES.

A. Show that the bilinear form defined in 7.3 induces a duality between $\ell^p - \sum_{\alpha \in A} E_\alpha$ and $\ell^q - \sum_{\alpha \in A} E'_\alpha$ $(1 < p < \infty, \frac{1}{p} + \frac{1}{q} = 1)$. Deduce that $\ell^p - \sum_{\alpha \in A} E_\alpha$ is reflexive if and only if each E_α is for these values of p.

B. $\ell^p - \sum_{\alpha \in A} L^p(\mu_\alpha) \cong L^p(\mu)$ (notation as in example IV above). Show that $\ell^p \sum E_n \cong \ell^p$ if each $E_n \cong \ell^p$.

These constructions can be used as the basis of useful methods for proving that various spaces are isomorphic. As a concrete example consider the following result:

Proposition 60 (The Pelczyński decomposition method.) *Suppose that E and F are Banach spaces so that*

1. *E is isomorphic to a complemented subspace of F;*
2. *F is isomorphic to a complemented subspace of E;*
3. *$E \cong \ell^p - \sum E_n$ for some p where each E_n is isomorphic to E (we then write simply $E = \ell^p(E)$). Then $E \cong F$.*

Proof. We can suppose that $E = F \times G$ for some space G. Then

$$\begin{aligned} E &\cong \ell^p(E) \cong \ell^p(F \times G) \cong \ell^p(F) \times \ell^p(G) \\ &\cong \ell^p(F) \times F \ell^p(G) \cong E \times F. \end{aligned}$$

■

Exercises. Justify carefully the steps $\ell^p(F \times G) \cong \ell^p(F) \times \ell^p(G)$ resp. $\ell^p(F) \cong \ell^p(F) \times F$ used in the above proof.

With this method we can prove the following result on the structure of ℓ^p spaces.

Proposition 61 *Let E be one of the spaces $\ell^p (1 \leq p < \infty)$. Then every infinite dimensional complemented subspace F of E is isomorphic to E.*

Proof. For we have shown in 1.8 that F contains an isomorphic copy of E which is complemented in E and so in F. Hence we can apply 7.5 to deduce that $E \cong F$.

■

Exercises. Apply similar methods to show that the space $E = c_0$ has the same property (i.e. every infinite dimensional, complemented subspace of E is isomorphic to E).

We now return to two of the most ubiquitous construction in mathemtics – inductive and projective limits. Before giving a formal definition we begin with a simple example. Let S be a σ-compact, locally compact space i.e. S is the union $\cup K_n$ of an increasing family of compact subsets where the interior of K_{n+1} contains K_n for each n. We consider the space $C^\infty(S)$ of bounded continuous functions on S.

Each function x in $C^\infty(S)$ defines a socalled "thread" (x_n) of functions where each x_n is the restriction of X to K_n. The functions (x_n) are of course bounded and further $\sup_n\{||x_n||\} < \infty$ where the norm of x_n is taken in the Banach space $C(K_n)$. On the other hand if (x_n) is a sequence where

1. $x_n \in C(K_n)$
2. the x_n are compatible in the sense that $x_{n+1}|_{K_n} = x_n$ for each n
3. $\sup\{||x_n|| : n \in \mathbf{N}\} < \infty$.

Then there is a uniquely determined $x \in C^\infty(S)$ so that x_n is the restriction of x to K_n. Also

$$||x|| = \sup_n\{||x_n||\}.$$

Hence in a certain sense the information contained in the Banach space $C^\infty(S)$ is already contained in the sequence $(C(K_n))$ and the linking maps π_n where π_n is the restriction mapping from $C(K_{n+1}))$ onto $(C(K_n))$.

Hence in a certain sense the information contained in the Banach space $C^\infty(S)$ is already contained in the sequence $(C(K_n))$ and the linking maps π_n where π_n is the restriction mapping from $(C(K_{n=1}))$ onto $(C(K_n))$.

We formalize this situation in the following definition (where we also treat the dual construction).

Let (E_n) be a family of Banach spaces and suppose that for each $n \in \mathbf{N}$ there is a linear contraction

$$\pi_n : E_{n+1} \to E_n.$$

We call the sequence $((E_n), (\pi_n))$ a **projective spectrum** of Banach spaces and define its projective limit $E = B \overset{\lim}{\leftarrow} E_n$ as follows:

$$E = \{(x_n) \in B \prod E_n : \pi_n(x_{n+1}) = x_n \text{ for each } n\}.$$

E is clearly a closed subspace of $B \prod E_n$ and we regard it as a Banach space with the induced norm. The dual notion is that of an **inductive spectrum** i.e. a sequence (F_n) of Banach space and, for each n, a linear contraction $i_n : F_n \to F_{n=1}$. The inductive limit of this spectrum is defined to be the quotient of the space $B \sum F_n$ by the closed subspace N generated by elements of the form $x - i_{n,m}(x)$ ($x \in F_n$ where $i_{n,m}$ is the mapping $i_{m-1} \circ i_{m-2} \cdots \circ i_n$ from F_n into F_m) (i.e. roughly speaking we identify x with its successive images under the imbeddings).

The above definition of the inductive limit may seem rather artificial. In practice, inductive spectra often have a particularly simple form which allows a much more transparent construction of the inductive limit. Suppose that F is a Banach space and that (F_n) is a sequence of closed subspaces indexed by $\mathbf{N}$ so that $F_n \subseteq F_{n+1}$. Then if i_n denotes the natural injection from E_n into F_{n+1}, then (F_n, i_n) forms an inductive spectrum of Banach spaces and the inductive limit is the closure of the linear subspace $\cup F_n$ in E.

Exercises. We illustrate these concepts with some examples:

I. Every separable Bananch space is the inductive limit of a sequence of finite dimensional spaces (cf. Exercise 1.18.B).

II. If S is a σ-compact, locally compact space, say $S \cup K_n$ with the K_n as above, we denote by $C_{K_n}(S)$ the space of those x in $C^\infty(S)$ with support in K_n. Then there is a natural injection from $C_{K_n}(S)$ into $C_{K_{n+1}}(S)$ and so they form an inductive spectrum.

 The unionof the $C_{K_n}(S)$ is the space of continuous functions on S with support in some K_n and the closure of this space is $C - 0(S)$, the space of those continuous functions on S which vanish at infinity i.e.

$$c_0(S) = B \overset{\lim}{\to} C_{K_n}(S).$$

III. Now suppose that $\mathcal{A}$ s a countably generated σ-algebra on Ω, generated by the sets (A_n). We denote by $\mathcal{A}_\backslash$ the σ-algebra generated by $\{A_1, \dots, A_n\}$.

 We can find a finite partition $\mathcal{B}_\backslash = \{\mathcal{B}_{\backslash\rangle} : \rangle = \infty, \dots, \|(\backslash)\}$ which generates $\mathcal{A}_\backslash$.

 Note that $\mathcal{B}_{\backslash=\infty}$ is a refinement of $\mathcal{B}_\backslash$. Now suppose that μ is a probability measure on A and denote by $L^p(\mu_n)$ the L^p space associated with $\mu_n = \mu|_{\mathcal{B}_\backslash}$. THen $L^p(\mu_n)$ consists of the function of the form

$$x := \sum_{i=1}^{k(n)} \alpha_i \chi_{B_{n,i}}$$

and the norm is

$$||x||_p = \left(\sum |\alpha_i|^p \mu(B_{n,i})\right)^{1/p}.$$

Now $L^p(\mu_n)$ is isometrically imbedded in $L^p(\mu_{n+1})$ so we have an inductive sequence of Banach spaces:

$$L^p(\mu)$$

is the inductive limit of this sequence ($1 \leq p < \infty$).

IV. We retain the notation of the above example. We now note that we can define projection operators

$$P_n : x \to \sum^{k(n)} i = 1 \frac{\int_{B_{n,i}} x d\mu}{\mu(B_{n,i}) \chi_{B_{n,i}}}$$

from $L^p(\mu_{n=1})$ onto $L^p(\mu_n)$ (i.e. the operator of conditional expectation). (We can assume that $\mu(B_{ni}) \neq 0$ for each $(L^p(\mu_n), P_n)$.

Now we can embed $L^p(\mu)$ into the projective imit

$$B \overset{\lim}{\leftarrow} L^p(\mu_n)$$

of this spectrum. (We associate to $x \in L^p$ the sequence $(P_n x)$ where $P_n x$ is defined exactly as above.) Then

$$L^p(\mu) = B \overset{\lim}{\leftarrow} L^p(\mu_n)$$

if $1 < p \leq \infty$ i.e. the above injection is surjective. (NB. this is not the case when $p = 1$).

For if (x_n) is a thread i.e. $(x_n) \in L^p(\mu_n)$ and $\sup ||x_n|| < \infty$ then, regarding (x_n) as a sequence in $L^p(\mu)$, it is bounded ther and so (by the reflexivity of $L^p(\mu)$) contains a weak limit point x. It is clear that x is associated with the original thread (note that this argument works because we have here the rather special situation that our spectrum is simultaneously a projective and inductive one.

V. Let E be a Banach space with basis (x_k). We define

$$E_n := [x_1, \dots, x_n].$$

Then we can regard (E_n):

a) as an inductive spectrum with the natural injections

b) as a projective separable with the natural projections $P_n : E_{n+1} \to E_n$.

Then E is naturally equal to $B \overset{\lim}{\leftarrow} (E_n)$. The basis is said to be **boundedly complete** if this subspace is the whole space i.e. if $E = \overset{\lim}{\leftarrow} E_n$.

VI. Consider the family of all probability measures on a measure space $(\Omega, \mathcal{A})$. We order this set by putting $\mu \ll \nu$ if and only if μ is absolutely continuous with respect to ν. Then there is a natural isometry

$$i_{\mu,\nu} : x \to xy^{1/2}$$

from $L^2(\mu)$ into $L^2(\nu)$ where y is the Radon-Nikodym derivatives of μ with respect to ν.

$$\{i_{\mu,\nu} : L^2(\mu) \to L^2(\nu), \mu \ll \nu\}$$

is an inductive spectrum of Hilbert spaces. Its inductive limit (which is a Hilbert space) is important in spectral multiplicity theory.

One of the important properties of inductive respectively projective limits lies in the following description of operators between such spaces.

Proposition 62 *Suppose we are given a bananch space G and spaces*

$$\begin{aligned} E &= B \overset{\lim}{\leftarrow} E_n \\ F &= B \overset{\lim}{\rightarrow} F_n. \end{aligned}$$

Then if (S_n) and (T_n) are sequences where

1. *$S_n \in L(G, E_n)$ (resp. $T_n \in L(F_n, G)$);*

2. *$\pi_n \circ S_{n=1} = S_n$ (resp. $T_{n+1} \circ i_n = T_n$) for each n;*

3. *$\sum_n \{||S_n||\} < \infty$ (resp. $\sup_n \{||T_n||\} < \infty$), there exists precisely one $S \in L(E, G)$ (resp. $T \in L(F, G)$) so that $\pi_n \circ S = S_n$ (resp. $T \circ i_n = T_n$) for each n.*

PROOF. If $x \in G$ we define Sx to be the thread $(S_n x)$. That this is compatrible follows from condition 2.—that it is bounded follows from 3.

To define T we first define an operator $\tilde{T}$ from $B \sum F_n$ to G by putting

$$\tilde{T}((x_n)) = \sum T_n x_n$$

we that note that $\tilde{T}$ vanishes on the space N by condition 2. and so lifts to a mapping from G into G.

■

EXERCISES. A. In many applications, the condition of countability on the indexing set is unnecessarily restrictive. The reader is invited to remove it i.e. to define inductive respectively projective limits for spectra

$$\begin{aligned} \{i_{\alpha\beta} : E_\alpha &\rightarrow E_\beta, \alpha, \beta \in A, \alpha \leq \beta\} \\ \{\pi_{\beta\alpha} : F_\beta &\rightarrow F_\alpha, \alpha\beta \in A, \alpha \leq \beta\} \end{aligned}$$

where the indexing set A is a directed set i.e. a set with a partial ordering $\leq$ so that if $\alpha, \beta \in A$ there is a $\gamma \in A$ with $\alpha \leq \gamma, \beta \leq \gamma$ and $\pi_{\beta\gamma} \circ \pi_{\alpha\beta} - \pi_{\alpha\gamma}$ resp. $i_{\alpha\gamma} = i_{\beta\gamma} \circ i_{\alpha\beta}$ if $\alpha \leq \beta \leq \gamma$.
B. Let H be a Hilbert space with orthogonal basis (x_k) and put $E_n = L(H_n)$ where $H_n = [x_1, \dots, x_n]$. Show that (E_n) forms both an inductive and a projective spectrum in a natural way and identify its projective and inductive limit.

We now turn to tensor products of Banach spaces. Recall that if E and F are vector spaces, a tensor product for E and F is a vector space M together with a bilinear mapping

$$u : E \times F \rightarrow M$$

so that

1. $u(E \times F)$ spans M;

2. for any bilinear mapping $f : E \times F \rightarrow G$ (G a vector space) there is a (unique) linear mapping $T : M \rightarrow G$ so that $f = T \circ u$

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It is a standard result of linear algebra that such a space exists and that it is unique in the sense that if (M_1, μ_1) is a second pair with the same properties then there is an isomorphism $T : M \rightarrow M_1$ so that $u_1 = T \circ u$. Perhaps the easiest way to construct it is to consider the bilinear mapping

$$u : (x, y) \rightarrow x \circ y$$

from $E \times F$ into $B(E^*, F^*)$ where

$$x \otimes y(f, g) = f(x)g(y).$$

We then take M to be the linear span of $u(E \times F)$ in $B(E^*, F^*)$.

Due to the uniqueness we may denote this space by $E \otimes F$, the **tensor product** of E and F. We shall now examine this construction in the contect of Banach spaces. Before doing so we recall some of its algebraic properties:

I. If $S : E_1 \to F_1, T : E_2 \to F_2$ are linear mappings there is a unique linear mapping $S \otimes T : E_1 \otimes E_2 \to F_1 \otimes F_2$ so that $S \otimes T(x \otimes y)S(x) \otimes T(y)$ $(x \in E_1, y \in E_2)$.

II. If $(x_\alpha : \alpha \in A)$ resp. $(y_\beta : \beta \in B)$ ae Hamel bases for E resp. F, then $(x_\alpha \otimes y_\beta : (\alpha, \beta) \in A \times B)$ is a Hamel basis for $E \otimes F$.

Two important and illuminanting examples are obtained by consdering tensor products of continuous or integrable functions. If K and L are compact topological spaces, $x \in C(K)$ and $y \in C(L)$, the we define the function

$$\begin{array}{rcl} x \otimes y : K \times L & \to & \mathbf{C} \\ (s, t) & \to & x(s)y(t). \end{array}$$

$y \otimes y \in C(K \times L)$ and the mapping $(x, y) \to x \otimes y$ is bilinear mapping from $C(K) \times C(L)$, into $C(K \times L)$. The linear span $\{x \otimes y : x \in C(K), y \in C(L)\}$ is a tensor product of $C(K)$ and $C(L)$. hence we can identify the tensor product of $C(K)$ and $C(L)$ with a subspace of $C(K \times L)$. (We say that the tensor product "is" a subspace of $C(K \times L)$.)

EXERCISES. Show that $C(K) \otimes C(L)$ is a dense subset of $C(K \times L)$ (use the Stone-Weierstraß theorem cf. 3.2) and that it is, in general, a proper subspace. Give an example of spaces K, L so that $C(K) \otimes C(L) = C(K \times L)$.

If (Ω, μ), $(\Omega; \nu)$ are σ-finite measure space, $x \in L^1(\mu)$ $y \in L^1(\nu)$, one can define $x \otimes y$ in $L^1(\mu \otimes \nu)$ in a similar way to the above and one can identify $L^1(\mu) \otimes L^1(\nu)$ with a subspace of $L^1(\mu \otimes \nu)$ $L^1(\mu) \otimes L^1(\nu)$ is a dense subspace of $L^1(\mu \otimes \nu)$.

We now assume that E and F are normed spaces and define two norms on their tensor product. Put

$$\begin{array}{rcl} || \, ||_{\text{proj}} : z & \to & \inf \left\{ \sum_{i=1}^n ||x_i|| \, ||y_i|| : z \sum_{i=1}^n x_i \otimes y_i \right\} \\ || \, ||_{\text{ind}} : z & \to & \sup\{|(f \otimes g)(z)| : f \in E', ||f|| \leq 1 g \in F', ||g|| \leq 1\}. \end{array}$$

Proposition 63 *(i) The mappings* $|| \, ||_{proj}$ *and* $|| \, ||_{ind}$ *are norms on* $E \otimes F$ *and* $||z||_{ind} \leq ||z||_{proj}$ *for each* $z \in E \otimes F$*;*

(ii) $||x \otimes y||_{ind} = ||x \otimes y||_{proj} = ||x|| \, ||y||$ *for* $x \in E, y \in F$*'*

(iii) if $S \in L(E, E_1)$*,* $T \in L(F, F_1)$*,* $S \otimes T$ *is continuous for the norms* $|| \, ||_{proj}$ *(resp. for the norm* $|| \, ||_{ind}$*) and in both cases* $||S \otimes T|| \leq ||S|| \, ||T||$*.*

PROOF.

(i) If $z \in E \otimes F$ and $z = \sum_{i=1}^n x_i \otimes y_i$ is a representation of z, then for any $f \in E'$, $g \in F'$ with $||f|| \leq 1, ||g|| \leq !, |f \otimes g(z)| = |\sum_{i=1}^n f(x_i) g(y_i)| \leq \sum_{i=1}^n ||x_i|| \, ||y_i|| and so ||z||_{\text{ind}} \leq ||z||_{\text{proj}}$.

$|| \, ||_{\text{ind}}$ is obviously a seminorm. If $z \in E \otimes F$ is non-zero, it can be deduced from Property II above that z has representation $\sum_{i=1}^n x_i \otimes y_i$ where $\{x_1, \ldots, x_n\}$ is linearly independent and at least one y_i, say y_1, is non-zero. Then there is an $f \in E'$ so that $f(x_1) \neq 0$ and $f(x_i) = 0$ $(i = 2, \ldots, n)$ and a $g \in F'$ so that $g9y_1) \neq 0$. We can further assume that $||f|| \leq 1$ and $||g|| \leq 1$. Then

$$f \otimes g(z) = f(x_1) g(y_1) \neq 0$$

and so $||z||_{\text{ind}} \neq 0$.

The homogenity of $|| \, ||_{\text{proj}}$ is clear. Suppose that z_1 and z_2 are in $E \otimes F$. Then for any $\epsilon > 0$, there are representations $z_1 = \sum_{i=1}^m x_i \otimes y_i$ and $z_2 = \sum_{j=1}^n x'_j \otimes y'_j$ so that

$$\sum_{i=1}^m ||x_i|| \, ||y_i|| \leq ||z_1||_{\text{proj}} + \epsilon/2 \text{ and } \sum_{j=1}^n ||x'_j|| \, ||y'_j|| \leq ||z_2||_{\text{proj}}^{\epsilon/2}.$$

Then $\sum_{i=1}^m x_i \otimes y_i + \sum_{j=1}^n x'_j \otimes y'_j$ is a representation for $z_1 + z_2$ and so

$$\begin{aligned} ||z_1 + z_2||_{\text{proj}} &\leq \sum_{i=1}^m ||x_i|| \, ||y_i|| + \sum_{j=1}^n ||x'_j|| \, ||y'_j|| \\ &\leq ||z_1||_{\text{proj}} + ||z_2||_{\text{proj}} + \epsilon. \end{aligned}$$

(ii) Clearly $||x \otimes y||_{\text{proj}} \leq ||x|| \, ||y||$. Also, there is an $f \in E'$ (resp. a $g \in F'$) so that $||f|| = ||g|| = 1$ and

$$f(x) = ||x|| \text{ and } g(y) = ||y||.$$

hence

$$||x \otimes y||_{\text{ind}} \geq |(f \otimes g)(x \otimes y)| = ||x||\,||y||.$$

Thus we have the chain of inequalities

$$||x||\,||y|| \leq ||x \otimes y||_{\text{ind}} \leq ||x \otimes y||_{\text{proj}} \leq ||x||\,||y||.$$

(iii) We consider the continuity of $S \otimes T$ under the projective norm. If $z \in E \otimes F$ and $||z||_{\text{proj}} \leq 1$, then z_n has a representation $\sum_{j=1}^n x_i \otimes y_i$ where $\sum_{i=1}^m ||x_i||\,||y_i|| \leq 1 + \epsilon(\epsilon > 0$. Then

$$\begin{aligned}(S \otimes T)(z) &= \sum_{i=1}^n (S \otimes T)(x_i \otimes y_i) \\ &= \sum_{i=1}^n Sx_i \otimes Ty_i\end{aligned}$$

and

$$\begin{aligned}\sum_{i=1}^n ||Sx_i||\,||Ty_i|| &\leq ||S||\,||T|| \sum_{i=1}^n ||x_i||\,||y_i|| \\ &\leq ||S||\,||T||(1+\epsilon).\end{aligned}$$

hence $S \otimes T$ is continuous and $||S \otimes T|| \leq ||S||\,||T||$. The continuity of $S \otimes T$ for the indurctive norm is proved similarly.

■

Exercises. Show that if $z = \sum_{i=1}^n x_i \otimes y_i \in E \otimes F$, then

$$\begin{aligned}||z||_{\text{ind}} &= \sup\left\{\left\|\sum_{i=1}^n f(x_i)y_i\right\| : f \in E', ||f|| \leq 1\right\} \\ &= \sup\left\{\left\|\sum_{i=1}^n g(y_i)x_i\right\| : g \in E', ||g|| \leq 1\right\}.\end{aligned}$$

We now identify these norms on the tensor product $C(K) \otimes C(L)$ resp. $L^1(\mu) \otimes L^1(\nu)$.

Firstly we show that the norm $||\,||_{\text{ind}}$ on $C(K) \otimes C(L)$ coincides with that induced by the norm on $C(K \times L)$ (when we regard $C(K) \otimes C(L)$ as a subspace of $C(K \times L)$).

If $z = \sum_{i=1}^n x_i \otimes y_i$ where $x_i \in C(K)$, $y_i \in C(L)$, then

$$\begin{aligned}
||z||_{\text{ind}} &= \sup \left\{ \left\| \sum_{i=1}^n f(x_i) y_i \right\| : f \in C(K)', ||f|| \leq \right\} \\
&= \sup_{f \in C(K)'} \left(\sup \left\{ \left| \sum_{i=1}^n f(x_i) y_i(t) \right| : t \in L \right\} \right) \\
&= \sup_{t \in L} \left(\sup \left\{ \left| f\left(\sum_{i=1}^n y_i(t) x_i \right| : f \in C(K)', ||f|| \leq 1 \right\} \right\} \right) \\
&= \sup \left\{ \left\| \sum y_i(t) x_i \right\|_\infty : t \in L \right\} \\
&= \sup_{t \in L} \left(\sup \left\{ \left| \sum_{i=1}^n x_i(s) y_i(t) \right| : s \in K \right\} \right) \\
&= ||z||_\infty.
\end{aligned}$$

Now consider the norm $|| \, ||_{\text{proj}}$ on $L^1(\mu) \otimes L^1(\nu)$. It coincides with the norm induced from that of $L^1(\mu \otimes \nu)$. For if $z = \sum_{i=1}^n x_i \otimes y_i$ is an element of $L^1(\mu) \otimes L^1(\nu)$, then

$$\begin{aligned}
||z||_{L^1} &= \int_{\Omega_1 \times \Omega_2} \left| \sum_{i=1}^n x_i \otimes y_i \right| d(\mu \otimes \nu) \\
&= \sum_{i=1}^n \int_{\emptyset_1 \times \Omega_2} |x_i \otimes y_i| d(\mu \otimes \nu) \\
&\leq \sum_{i=1}^n ||x_i|| \, ||y_i||
\end{aligned}$$

and so $||z||_{L^1} ||z||_{\text{proj}}$.

For the inequality in the other direction, it is sufficient ot assume that z is a simple function whose sets of constancy are rectangles (that is, sets of the form $A \times B$ where $A \subseteq \Omega_1$ and $B \subseteq \Omega_2$ are measurable).

Then z has a representation $\sum_{i=1}^n \chi_{A_i} \otimes y_i$ where the A_i's are disjoint, measurable subsets of Ω_1 and then $||z||_{\text{proj}} \leq \sum_{i=1}^n ||\chi_{A_i}|| \, ||y_i|| = ||x||_{L^1}$.

If E and F are Banach spaces, we denote by

$$\begin{aligned}
E \ &\tilde{\otimes} \ F \text{ the completion of } (E \otimes F, || \, ||_{\text{proj}}) \\
E \ &\approx\!\!\otimes \ F \text{ the completion of } (E \otimes F, || \, ||_{\text{ind}})
\end{aligned}$$

$E \tilde{\otimes} F$ (resp.$E \overset{\approx}{\otimes} F$) is called the **projective tensor product** of E and F (resp. the **injective tensor product**). For example it follows from the above that $C(K)\overset{\approx}{\otimes}C(L)$ can be identified with $C(K\times L)$ and $L^1(\mu)\tilde{\otimes}L^1(\nu)$ with $L^1(\mu\otimes\nu)$.

If $S\in L(E,E_1)$, $T\in L(F,F_1)$, then we can extend $S\otimes T$ to linear continuous mappings from $E\tilde{\otimes}F$ into $E_1\tilde{\otimes}F_1$ and from $E\overset{\approx}{\otimes}F$ into $E_1\overset{\approx}{\otimes}F_1$. We denote these extensions by $S\tilde{\otimes}T$ and $S\overset{\approx}{\otimes}T$. Since the identity mapping from $(E\otimes T$ and $S\overset{\approx}{\otimes}T$. Since the identity mapping from $(E\otimes F,||\,||_{\text{proj}})$ into $(E\otimes F,||\,||_{\text{ind}})$ is continuous, it extends to a continuous, linear mapping from $E\tilde{\otimes}F$ into $E\overset{\approx}{\otimes}F$. In most cases encountered in applications, this mapping is an injection (so that we can regard $E\tilde{\otimes}F$ as a subspace of $E\overset{\approx}{\otimes}F$) but this is not true in general.

The defining property of the algebraic tensor product has the following counterpart:

Let E,F,G be Banach spaces. Recall that a bilinear mapping u from $E\times F$ into G is continuous if there is a $K>0$ so that

$$||u(x,y)||\leq K||x||\,||y|| \quad (x\in E, y\in F)$$

and that $L(E,F;G)$ denotes the space of continuous, bilinear mappings from $E\times F$ into G.

Proposition 64 *Let E,F,G be Banach spaces. Then for every continuous bilinear mapping v from $E\times F$ into G there is a continuous linear mapping T from $E\tilde{\otimes}F$ into G so that the diagram*

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commutes. Conversely the bilinear mapping $\otimes: E\times F\to E\tilde{\otimes}F$ is continuous and so the above correspondence is a linear isomorphism from $L(E,F;G)$ onto $L(E\tilde{\otimes}F;G)$.

PROOF. If $z\in E\otimes F$, $z=\sum_{i=1}^n x_i\otimes y_i$, then $T(z)=\sum_{i=1}^n v(x_i,y_i)$. Hence if $||z||_{\text{proj}}\leq 1$, there is a representation $\sum_{i=1}^n x_i\otimes y_i$ so that $\sum_{i=1}^n ||x_i||\,||y_i||\leq 1+\epsilon$.

Then

$$\begin{aligned} ||Tx|| &= \left\|\sum_{i=1}^n v(x_i,y_i)\right\| \leq \sum_{i=1}^n ||v(x_i,y_i)|| \\ &\leq K\sum_{i=1}^n ||x_i||\,||y_i|| \leq K(1+\epsilon). \end{aligned}$$

Hence T is continuous and $||T||\leq K$. On the other hand, it follows from 7.13(ii) that $\otimes$ is continuous.

∎

Corollar 14 *The dual space of* $E\tilde{\otimes}F$ *is naturally isomorphic to* $L(E,F;\mathbf{C})$.

Proposition 65 *If* E, F *are Banach spaces,* $x \in E \tilde{\otimes} F$*, then there are bounded sequences* (x_n) *in* E *and* (y_n) *in* F *and a sequences* (λ_n) *in* $\mathbf{C}$ *with* $\sum_{i=1}^{\infty} |\lambda_n| < \infty$ *so that*

$$x = \sum_{i=1}^{\infty} \lambda_n x_n \otimes y_n.$$

PROOF. There is a sequence (z_i) in $E \otimes F$ and a sequence (μ_i) in $\mathbf{C}$ so that $\sum_{i=1}^{\infty} |\mu_i < \infty$, $||z_i|| \leq 1$ for each i and $x = \sum_{i=1}^{\infty} \mu_i z_i$ (cf. Exercise 3.7.B).

We can find representation

$$z_i = \sum_{k=1}^{r_i} \nu_{ik} x_{ik} \otimes y_{ij}$$

where $||x_{ik}|| = 1$, $||y_{ik}|| = 1$ and $\sum_{k=1}^{r_i} |\nu_{ik}|| \leq 2$. Then $x = \sum_{i=1}^{\infty} \sum_{k=1}^{r_i} \nu_{ik} x_{ik} \otimes y_{ik}$ and this can be rearranged to the required form.

■

We now consider the spaces $\ell^1 stE$, $\ell^1 \stackrel{\approx}{\otimes} E$ for a given Banach space. If E is a Banach space, a sequence (x_n) in E is **weakly summable** if for each $f \in E'$

$$\sum |f(x_n)| < \infty.$$

We write $\ell^1[E]$ for the family of all such sequences. It is clearly a vector subspace of $E^{\mathbf{N}}$. If $(x_n) \in \ell 61[E]$ then

$$A := \left\{ \sum_{i=1}^{n} \alpha_i x_i : |\alpha_i| \leq 1, \quad n \in \mathbf{N} \right\}$$

is weakly bounded subset of E (since

$$\left| f\left(\sum_{i=1}^{n} \alpha_i x_i \right) \right| \leq \sum_{n=1}^{\infty} |f(x)n)| \text{ for each } f)$$

and so is norm-bounded. Suppose that $||x|| \leq K$ $(x \in A)$. Then if $f \in E'$ with $||f|| \leq 1$ we have

$$\sum_{n=1}^{\infty} |f(x_n)| = \sum_{n=1}^{\infty} \lambda_n f(x_n) \leq K$$

where λ_n is such that $\lambda_n f(x_n) = |f(x_n)|$. Hence we can define a norm $|| \, ||_{ws}$ on $\ell^1[E]$ as follows

$$|| \, ||_{ws} : (x_n) \to \sup \left\{ \sum |f(x_n)| : f \in B(E') \right\}.$$

The sequence (x_n) is **absolutely summable** if $\sum ||x_n|| < \infty$. The space $\ell^1\{E\}$ of all absolutely summable sequences in E is a vector subspace of $\ell^1[E]$. We regard it as a normed space with the norm

$$|| \, ||_{as} : (x_n) \to \sum ||x_n||.$$

Note that $\ell^1\{E\}$ and $\ell^1[E]$ are Banach spaces. This can be deduced from 3.18.F. ($\ell^1[E]$ coincides with the space $\ell^1 - \sum E_n$ where each E_n is an isometric copy of E.)

Proposition 66 *There are natural isomeorphisms*

$$\ell^1 \tilde{\tilde{\otimes}} E \cong \ell^1[E], \ell^1 \tilde{\otimes} E \cong \ell^1\{E\}.$$

PROOF. First note that we can identify the algebraic tensor product $\ell^1 \otimes E$ with the spaces of summable (absolutely or weakly) sequences which take values in finite dimensional subspace of E. It now suffices to verify that on $\ell^1 \otimes E$ the norm $|| \, ||_{\text{ind}}$ agrees with $|| \, ||_{ws}$ and $|| \, ||_{\text{proj}}$ with $|| \, ||_{as}$. This is a calculation similar to that carried out on $C(K) \otimes C(L)$ resp. $L^1(\mu) \otimes L^1(\nu)$. ∎

EXERCISES.

A. Show that if $\epsilon > 0$, then $(x_n), (y_n), (\lambda_n)$ can be chosen as in 7.17 with the further property that

$$||x_n|| = ||y_n|| = 1 \text{ and } \sum |\lambda_n| \leq ||z|| + \epsilon.$$

B. Let E, F be Banach spaces with basis (x_m) and (y_n) respectively. Show that the sequence obtained by ordering the double sequence

$$\begin{array}{ccccccc}
x_1 \otimes y_1 & \rightarrow & x_1 \otimes y_2 & \dots & \rightarrow & x_1 \otimes y_n \\
 & & & \downarrow & & \\
x_2 \otimes y_1 & \leftarrow & x_2 \otimes y_2 & \dots & & x_2 \otimes y_n \\
\downarrow & & & & & \\
\dots & & & & & \\
\downarrow & & & & & \\
x_m \otimes y_1 & \rightarrow & x_m \otimes y_2 & \dots & \rightarrow & x_m \otimes y_n
\end{array}$$

as shown in the diagram is a basis for $E \tilde{\otimes} F$ and $E \tilde{\tilde{\otimes}} F$. (Show that the new sequence is linearly idependent and express the associated projection operators in terms of the projections associated with (x_m) and (y_n). Deduce that they are uniformly bounded.)

C. Show that if E and F are Banach spaces with bases, then the natural mapping from $E \tilde{\otimes} F$ into $E \tilde{\tilde{\otimes}} F$ is an injection.

D. If K is compact topological space, F a Banach space, show that $C(K; F)$, the space of continuous mappings from K into F has a natural Banach space structure and that $C(K; F)$ is isomorphic (as a Banach space) to $C(K) \tilde{\tilde{\otimes}} F$.

E. A norm $|| \, ||_\alpha$ on $E \otimes F$ is called **reasonable** if

   1. $||x \otimes y||_\alpha \leq ||x|| \, ||y|| \quad (x \in E, y \in F)$;
   2. $||f \otimes g||'_\alpha \leq ||f|| \, ||g|| \quad (f \in E', g \in F')$;
      (where $|| \, ||'_\alpha$ is a dual norm to $|| \, ||_\alpha$ on $E' \otimes F'$ regarded as a subspace of $(E \otimes F, || \, ||_\alpha)'$). Show
   3. that $||x \otimes y||_\alpha = ||x|| \, ||y||$;
   4. $||f \otimes g||'_\alpha = ||f|| \, ||g||$;
   5. $|| \, ||_{\text{ind}} \leq || \, ||_\alpha \leq || \, ||_{\text{proj}}$ i.e. $|| \, ||_{\text{ind}}$ (Resp. $|| \, ||_{\text{proj}}$)

   is the greatest (resp. least) reasonable crossnorm on $E \otimes F$.

We now turn to a construction which is a little more subtle—that of ultrapowers. As we shall see, they provide a useful tool for investigating the structure of the finite dimensional suspaces of a Banach space. The construction is based on the conscepts of ultrafilters resp. ultraproducts. Recall that a **filter** on a set A is a non-empty collections $\mathcal{F}$ of subsets of A so that

1. $F \in \mathcal{F}$ implies $F \neq \varpi$;
2. $F, G \in \mathcal{F}$ implies $F \cap G \in \mathcal{F}$;
3. $F \in \mathcal{F}$ implies that each set G containing F is also in $\mathcal{F}$.

The set of filters on a set is ordered by inclusion and the maximal objects with respect to this ordering are called **ultrafilters**. It follows from Zorn's lemma that each filter $\mathcal{F}$ is containded in an ultrafilter. Ultrafilters can be cahracterised as those filters $\mathcal{F}$ so that if $F \subseteq A$ then either F or $A \setminus F$ is in $\mathcal{F}$. There exists a very simple class of ultrafilters, namely those of the form

$$\mathcal{F}_\alpha = \{\mathcal{F} \subseteq \mathcal{A} : \dashv \in \mathcal{F}\}$$

($a \in A$). For our purposes these are not very interesting and we will require ultrafilters whose existence is ensured by Zorn's lemma.

The classical example is one obtained by applying it to the **Frechet filter** on $\mathbf{N}$ i.e. the set of those subsets of $\mathbf{N}$ which contain a set of the form $\{n \in \mathbf{N} : n \geq m\}$ for some $m \in \mathbf{N}$.

We begin with the algebraic setting for our construction.

Let $\mathcal{U}$ be a n ultrafilter on a set A. If $\{B_\alpha\}$ is a family of sets indexed by A, their **ultraproducts** $\prod_\alpha^{\mathcal{U}} B_\alpha$ is the set constructed as follows: on the cartesian product $\prod_{\alpha \in A}^{\mathcal{U}} B_\alpha$ we introduce an equivalence relation $\sim$ as follows:

$$(x_\alpha) \sim (y_\alpha) \Leftrightarrow \{\alpha : x_\alpha = y_\alpha\} \in \mathcal{U}.$$

Then $\prod_\alpha^{\mathcal{U}} B_\alpha := \prod_\alpha B_\alpha|_\sim$.

We have the following obvious facts: if C_α, D_α are subsets of B_α for each α then $\prod_\alpha^{\mathcal{U}} C_\alpha$ and $\prod_\alpha^{\mathcal{U}} D_\alpha$ can be regarded as subsets of $\prod_\alpha^{\mathcal{U}} B_\alpha$ and

$$(\prod_\alpha^{\mathcal{U}} C - \alpha) \cap (\prod_\alpha^{\mathcal{U}} D_\alpha) = \prod_\alpha^{\mathcal{U}} (C_\alpha \cap D_\alpha)$$

$$(\prod_\alpha^{\mathcal{U}} C_\alpha) \cup (\prod_\alpha^{\mathcal{U}} D_\alpha) = \prod_\alpha^{\mathcal{U}} (C_\alpha \cup D_\alpha)$$

$$\prod_\alpha^{\mathcal{U}} (C_\alpha \setminus D_\alpha) = (\prod_\alpha^{\mathcal{U}} C_\alpha) \setminus (\prod_\alpha^{\mathcal{U}} D_\alpha).$$

If each B_α is equal to a given set B we write $B^{\mathcal{U}}$ for the corresponding ultraproduct.

Now if the B_α have some algebraic structure then this can usually be carried over in the natural way to $\prod_\alpha^{\mathcal{U}} B_\alpha$ (as a quotient of a product). In particular the ultraproduct of a family of groups (rings, vector spaces, algebras) is itself in a naturalway a group (ring, vector space, algebra).

Now suppose that we have a family $\{(E_\alpha, ||\,||_\alpha)\}$ of Banach spaces indexed by A. We form the ultraproduct $B\prod_\alpha^{\mathcal{U}} E_\alpha$ as follows. First we consider the space $B\prod_\alpha E_\alpha$. Now

$$N_{\mathcal{U}} := \{(x_\alpha) : \lim_{\mathcal{U}} ||x_\alpha|| = 0\}$$

is a closed subspace of $B\prod E_\alpha$.

Note that the limit in the above formula exists for each $x \in B_{a\in A}E_\alpha$. This follows from the fact that an ultrafilter on a compact space always converges.

We define the **ultraproduct** to be the quotient space $B\prod_\alpha E_\alpha/N_{\mathcal{U}}$- denoted by $B\prod_{a\in A}^{u} E)\alpha$. $T \in L(E_\alpha, F_\alpha)$ and $\sup ||T_\alpha|| < \infty$ are defined in the obvious way i.e. we first combine the T_α's to an operator from $B\prod_\alpha E_\alpha$ into $B\prod_\alpha F_a$ and notice that this operator preserves the corresponding $N_{\mathcal{U}}$-spaces.

Once again we wwite $E^{\mathcal{U}}$ for the special ultraproduct obtained by taking each factor to be E.

Proposition 67 *If $x \in B\prod_\alpha E_\alpha$, then*

$$||[x]|| = \lim_u ||x_\alpha||$$

where $[x]$ is the image of x in $B\prod_{\alpha\in A}^{u} E$.

PROOF. For any $y \in B\prod_\alpha E_\alpha$ with $[y] = [x]$ we have

$$\lim_u ||y_\alpha|| = \lim_u ||x_\alpha||.$$

Hence

$$\lim_u ||x_\alpha|| \leq \inf_{y\in[x]} \lim ||y_\alpha|| \leq \inf_{y\in[x]} ||(y_\alpha)|| = ||[x]||.$$

On the other hand, if $\epsilon > 0$ there is an $A \in \mathcal{U}$ so that $||x_\alpha|| \leq \lim_u ||x_\alpha|| + \epsilon$ for $\alpha \in A$. Then if y is the element (y_α) where $(y_\alpha) = (x_\alpha)$ on A and $y_\alpha = 0$ otherwise, we have $||y|| \leq \lim_u ||x_\alpha|| + \epsilon$ and so, since $[y] = [x]$, $||x|| \leq \lim_u ||x_\alpha||$.

■

For the next result, we introduce the following notation: two Banach spaces E and F are λ-isomorphic if there is an isomorphism $T : E \to F$ with $\max(||T||, ||T^{-1}||) \leq \lambda$ (i.e. if $d(E, F) \leq \lambda$ in the notation of 1). Of course λ is then at least 1. The next result states that every finite dimensional subspace of an ultraproduct $B\prod_\alpha E_\alpha$ is "almost isomeric" to a subspace of "almost every" E. Hence the finite dimensional structure of an ultraproduct cannot be more complicated than that of its factors.

Proposition 68 *If F is)isometric to) a finite dimensional subspace of $B\prod_\alpha^{\mathcal{U}} E_\alpha$ then for each $\lambda > 1$ there is an $A_0 \in \mathcal{U}$ so that for each $\alpha \in A_0$, F is λ-isomorphic to a subspace of E_α.*

PROOF. Let $\{x_1, \dots, x_n\}$ be a basis for F, with representation say

$$x_i = [(x^i_\alpha)]$$

in the ultraproduct. Put $F_\alpha := [(x^i_\alpha) : i = 1, \dots, n]$ in E_α.

Now if T_α is the linear mapping from F into F_α which maps each x_i into x^i_α then the family $\{T_\alpha\}$ is bounded (by the way the norm is defined in ultraproducts). Also if $x \in F$ then $\lim ||T_\alpha x|| = ||x||$. Hence we can find a set $A_x \in \mathcal{U}$ so that

$$1/\lambda' ||x|| < ||T_\alpha x|| < \lambda' ||x||$$

for $\alpha \in A_x$ (where $1 < \lambda' < \lambda$).

Now choose a δ-set $\{y_1, \dots, y_m\}$ for the unit ball of F. Then the proof is finished by defining $A = \cap_i A_{y_i}$ and using Exercise 1.1. for suitably small values of δ^i.

■

On the other hand, if F is a Bananch space which has the property that every finite dimensional subspace is almost isometric to a subspace of one of a given class of Banach spaces, then F is isometric to a subspace of a suitable ultraproduct of such spaces as the following result shows:

Proposition 69 *Let $\mathcal{B}$ be a class of Banach spaces and supposethat a Banach space E has the property that for each finite dimensional subspace F and each $\lambda > 1$, there is an $E_1 \setminus \mathcal{B}$ so that F is λ-isomorphic to a subspace of E_1. Then E is isometric to a subspace of an ultraproduct of spaces of $\mathcal{B}$.*

PROOF. Put $A = \{(F, \lambda) : F$ is a finite dimensional subspace of E and $\lambda \geq 1\}$. For $\mathcal{U}$ we choose an ultrafilter containing the filter base consisting of all sets of the form $\{(F, \lambda) : F \subset F_0, \lambda \leq \lambda_0\}$ for a fixed pair (F_0, λ_0) (i.e. the tails in the ordering

$$(F, \lambda) \leq (F_1, \lambda_1) \Leftrightarrow F \subset F_1, \lambda \geq \lambda_1).$$

For each $\alpha = (F, \lambda)$ there is a λ-isomorphism $T_\alpha : F \to E_\alpha$ for some $E_\alpha \in \mathcal{B}$. We show that E is isometric to a subspace of $B_\alpha \pi^{\mathcal{U}} E_\alpha$ under the mapping

$$J : x \to [(y_\alpha)] \text{ where } y_\alpha = \begin{cases} T_\alpha x & \text{if } x \in F \\ 0 & \text{otherwise.} \end{cases}$$

We shall show that if $\lambda_0 > 1$ then

$$1/\lambda_0 ||x|| \leq ||Jx|| \leq \lambda_0 ||x||$$

which will finish the proof.

For if $A_0 = \{(F, \lambda) : x \in F$ and $\lambda \leq \lambda_0\}$ then $A_0 \neq \mathcal{U}$ and

$$\lambda_0^{-1}||x|| \leq ||y|| \leq \lambda_0||x||$$

which proves the result.

An important consequence is the following:

Proposition 70 *A Bananch space F is finitely represented in a second space E if and only if it is isometric to a subspace of some ultraproduct of E.*

Here the fact hat F is finitely represented in E means t hat each finite dimensional subspace F_0 of F is almost isometric to a subspace of E i.e. for each $\lambda > 1$ there is a subspace of E which is λ-isomorphic to F_0.

EXERCISES.

A. Show that if the separable space F is finitely representable in E then it is isometric to a subspace of $E^{\mathcal{U}}$ where $\mathcal{U}$ is the natural ultrafilter on **N**.

B. A Bananch space E is defined to be **super-reflexive** if and only if every Banach space which is finitely representable in it is reflexive. Show that this is equivalent to the fact that $E^{\mathcal{U}}$ is reflexive for each $\mathcal{U}$ and use this to give some non-trivial examples of super-reflexive spaces.

EXERCISES. Let

$$beaBanachspacesothat$$

E'$andhencealso$E$isseparable.Let$(F$_n$) be a sequence of finite dimensional subspace of E' so that F_n is dense in E'. Show

1. that $|| \, || : f \to ||f|| + \sum_n 2^{-n} d(f, F_n)$ is an equivalent norm on E';
2. if $f_n(x) \to f(x)$ for each $x \in E$ and $|||f_n||| \to |||f|||$ then $|||f_n - F - n||| \to 0$.

Let E be a Banach space with basis (x_n) and corresponding projections (S_n). Show that

a) $|||x|||x| := \sup_{m<n} ||S - nx - S_m x||$ is an equivalent norm on E;

b) $|||x||| := |||x||| + \sum_n 2^{-n}|||x - S_n x|||$ is an equivalent norm no E;

c) if $x_n \to x$ in the weak topology, and $|||x_n||| = |||x|||$ for each n then $x_n \to x$ in E.

let K be a compact space. A **partition of unity** on K is a family $\{\varpi_\alpha\}_{\alpha\in A}$ in $C(K)$ so that

1. $0 \leq \varpi_\alpha \leq 1$;
2. $\{\varpi_\alpha\}$ is locally finite i.e. each $t \in K$ has a heighbourhood U so that $\{\alpha : \text{supp}\varpi_\alpha \cap U \neq \varpi\}$ is finite;
3. $\sum_\alpha \varpi_\alpha = 1$.

Then the mapping
$$(\lambda_\alpha) \to \sum \lambda_\alpha \varpi_\alpha$$
is an isometry from $\ell^\infty(A)$ into $C(K)$ (and hence $CK(K)$ contains a complemented copy of $\ell^\infty(A)$).

If E is a Banach space and $T \in L(\ell^1, E)$ then (T_{e_n}) is in $\ell^\infty(E)$. Show that this correspondence induce an isometry between $L(\ell^1, E)$ and $\ell^\infty(E)$.

Show that c_0 and $\ell^p (p > 1)$ are not isomorphic to subspaces of ℓ^1.

Show that the closed graph theorem implies the principle of uniform boundedness. (If $(T_\alpha)_{\alpha\in A}$ is a pointwise family in $L(E, F)$ consider the mapping $x \to (T_\alpha(x))$ from E into $\ell^\infty(S, F)$.)

Let E be a Banach space, S a locally compact space. Show that $C^b(S) \overset{\approx}{\otimes} E$ can be identified with the space of continuous functions $x : S \to E$ whose ranges are relatively compact in E.

Let E be a Banach space so that E contains an increasing sequence (E_n) of subspaces where

1. $\dim E_n = n$;
2. $d(E)n, \ell^2_n) \leq M$ for some constant M;
3. $\cup E_n$ is dense in E.

Then E is isomorphic to Hilbert space. (Choose an inner product norm $|| \, ||_n$ on E_n with
$$||x||_n \leq ||x|| \leq M||x||_n$$
$(x \in E_n)$. Show that there is an increasing sequence (n_k) of integers and a dense set $F \subseteq E$ so that $\lim ||x||_{n_k}$ exists for $x \in F$. Use this limit to define a suitable norm on E.)

(Spaces defined by linear mappings): Let E, F be normed linear spaces. A partial linear mapping from E into F is a linear mapping T from a subspace $\mathcal{V}_\mathcal{T}$ of E into F. We define the graph of T by

$$\Gamma(T) := \{(x.Tx) : x \in \mathcal{D}_\mathcal{T}\}.$$

Then T has a closed graph if $\Gamma(T)$ is closed in $E \times F$ (i.e. whenever (x_n) is a sequence in E so that $\lim Tx_n$ exist, then $x \in \mathcal{D}_\mathcal{T}$ and $\lim T_{T_{x_n}} = T(\lim x_n)$).

Now let E be a normed space, X a linear space with a Hausdorff topology, $(F, || \, ||)$ a Banach subspace of X (cf. 5.9.D). Then if T is a linear mapping from E into X which is $|| \, || - \tau$ continuous, we define the space

$$T^F = \{x \in E : Tx \in F\}$$

with norm

$$T^{|} | \, || : x \to ||x|| + ||Tx||.$$

Show

i) $\mathcal{D}_\mathcal{T}$ is a Banach space if E and F are and T is closed

ii) T^F is a Banach space if E is Banach.

iii) Suppose that $\mathcal{D}_\mathcal{T}$ is dense in E. Then the dual space of $(\mathcal{D}_\mathcal{T}, || \, ||_\mathcal{T})$ is $E' \times F'/N$ where

$$N = \{(f_1, f_2) \in E' \times F' : f_1 + T' f_2 = 0.)$$

(Interpolation spaces): Consider the following situation

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where E_1, E_2, F_1, F_2 are Banach spaces and T_1, T_2, S, S' are continuous linear operators so that the diagram commutes. Then we know that

$$G_1 := T_1(E_1) \text{ and } G_2 := T_2(E_2)$$

are Banach spaces with the norms:

$$||y_1|| := \inf\{||x|| : x \in E_1 \text{ and } T_1 x = y\}$$

resp.

$$||y_2|| := \inf\{||x|| : x \in E_2 \text{ and } T_2 x =\}.$$

Show that S induces a linear operator $\tilde{S}$ from G_1 into G_2 with $||\tilde{S}|| \leq ||S||$.

(Tensor products of Hilbert spaces.) Let H_1, H_2 be Hilbert spaces. On the algebraic tensor product $H_1 \otimes H_2$ we define an inner product by putting

$$(x_1 \otimes y_1 | x_2 \otimes y_2) = (x_1|x_2)(y_1|y_2)(x_1, x_2 \in H_1, y_1, y_2, \in H_2)$$

and extending linearly. Show that this is in fact an inner product. We denote the competion by $H_1 \tilde{\otimes}_2 H_2$.

a) Show that if (e_α) bzw. (f_β) is an orthonormal basis for H_1 resp. $H_2 (e_\alpha \otimes f_\beta)$ is an orthonormal basis for $H_1 \otimes_2 H_2$.

b) Give concrete representations of $L^2(\mu_1) \tilde{\otimes}_2 L^2(\mu_2)$, resp. $\ell^2(S_1) \tilde{\otimes}_2 \ell^2(S_2), L^2(\mu) \tilde{\otimes} H$.