# Functional analysis—spectral theory in Hilbert space

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## Contents

# 1 Spectral theory in Hilbert space—the classical approach

**Introduction** In these notes we consider one of the most attractive and important results from the theory of functional analysis – the spectral theorem for self-adjoint operators on Hilbert space. This is an infinite dimensional analogue of the classical result on the diagonalisation of hermitian matrices which, in turn, has its origin one of the climaxes of Greek mathematics – the study of conic sections. In the process of his work on integral equations, Hilbert showed that for operators on $L^2$ which are defined by summetric kernels, an exact analogue of the finite dimensional results can be obtained – more precisely the space is spanned by an orthonormal basis consisting of eigenvectors of the operator and in fact he classical function systems which played such a central role in the mathematics of the nineteenth century arise in this way. Later, partly motivated by the needs of quantum theory (where non bounded self-adjoint operators appear as observables), the theory was generalised to operators on Hilbert space, which are not necessarily bounded. For such operators, the spectrum need no longer be compact and eigenvectors need no longer exist so that the final form of the theorem is necessarily more abstract and complex.

We begin with a direct proof of the spectral theorem for bounded operators in Section 1. In order to prepare for a more abstract approach, we give a brief introduction to Banach algebras in the second section, culminating in the Gelfand-Naimark theorem. Using this theory we give an alternative version of the spectral theorem in §3 and use this to give the final form – that for unbounded operators – in §4.

The elementary theory of Hilbert spaces can be found in the notes "Functional Analusis—Banachspaces".

**The classical approach** We now prove the spectral theorem for self-adjoint operators in Hilbert space. Recall that the classical theorem for finite dimensional spaces can be stated in the following form: if $T$ is a self-adjoint operators in Hilbert space. Recall that the classical theorem for finite dimensional space can be stated in the following form: if $T$ is a self-adjoint operator on a finite dimensional Hilbert space, thenwe can find an orthonormal basis $(x_1, \dots, x_))$ for $H$ and real scalars $\lambda_1, \dots, \lambda_n$ so that

$$T\left(\sum_{j=1}^{n} \xi_j x_j\right) = \sum_{j=1}^{n} \lambda_j \xi_j x_j$$

i.e. $T$ acts as a multiplication operator on the coordinates. $\lambda_1, \dots, \lambda_n$ are just the eigenvalues of $T$ and it is convenient to label the corresponding eigenvectors $x_1, \dots, x_n$ in such a way that $\lambda_1 \leq \dots \leq \lambda_n$.

If we introduce the orthogona projections

$$0 = P_0, P_1, \dots, P_n = I$$

where $P_i$ is the projection onto $[x_1, \dots, x_i]$ then we can restate this result as follows:

$$T = \sum_{i=1}^{n} \lambda_i (P_i - P_{i-1})$$

or

$$T + \int \lambda dE_\lambda$$

where $E_\lambda$ is the (projection)-valued function

$$\lambda \to \begin{cases} 0 & (\lambda < \lambda_1) \\ P_i & \lambda_i \leq \lambda < \lambda_{i+1} \\ I & \lambda \geq \lambda_n \end{cases}$$

and the integral is regarded as a Stieltjes integral in the natural way. (For the sake of simplicity we are assuming that the $\lambda_i$ are distinct.) It is in this form (which is admittedly rather artificial inthe finite dimensional case) that we shall generalise the spectral theorem to operators on infinite dimensional spaces. Here we shall replace the finitely valued step function $E$ with more general increasing projection valued functions.

We begin with a few informal remarks which we hope will clarify the proof of the spectral theorem. First we note that the formula

$$T = \int \lambda dE_\lambda$$

for a finite dimensional operators can be used to define a functional calculus for such operators. That is, if $x$ is a function defined on the spectrum $\{\lambda_1, \dots, \lambda_n\}$ of $T$ we can define an operator $x(T)$ by

$$x(T) = \int x(\lambda) dE_\lambda.$$

In particular, if $x$ is the function

$$e_\lambda = \chi_{]\infty, \lambda]}$$

we see that $E_\lambda = e_\lambda(T)$.

In our proof we shall reverse this process. We begin by developing a functional calculus using elementary methods – firstly for polynomials and then by a suitable limiting process for functions which are pointwise monotone limits of polynomials, in particular for the characteristic functions of intervals. This allow us to define a family $E_\lambda$ of orthogonal projections by the formula

$$E_\lambda = e_\lambda(T).$$

It then follows from some simple propertis of the functional calculus that

$$T = \int \lambda dE_\lambda$$

in a suitable sense.

We begin with some simply properties of operators in Hilbert spaces. Recall that $A \in L(H)$ is **non-negative** (written $A \geq 0$) if $A$ is hermitian and $(Ax|x) \geq 0$ $(x \in H)$. Then the generalised Cauchy-Schwarz inequality holds:

$$(Ax|y)^2 \leq (Ax|x)(Ay|y)$$

(it is proved exactly as for the classical case).

$A \leq B$ means $B - A \geq 0$. We introduce the **bounds** $\lambda(A) = \inf\{(Ax|x) : x \in B_H\}$, $\mu(A) = \sup\{(Ax|x) : x \in B_H\}$ for the hermitian operator $A$.

Then $A \geq 0$ if and only if $\lambda(A) \geq 0$ and we have the equality

$$||A|| = \max\{|\lambda(A), |\mu(A)|\}$$

resp. the inequalities

$$\lambda(A)\mathrm{Id} \leq A \leq \mu(A)\mathrm{Id}.$$

**Lemma 1** *Let $(A_n)$ be a bounded sequence of hermitian operators and suppose that $A_n$ is monotone (i.e. $A_n \leq A_{n+1}$ for each $n$, resp. $A - n \geq A_{n=1}$). Then there is a bounded hermitian operator $A$ so that $A_n \overset{s}{\to} A$.*

PROOF. Without loss of generality we can suppose that $0 \leq A_n \leq \mathrm{Id}$ and $(A_n)$ is increasing. Now if $x \in H$, we have the follwing estimate (if $m > n$).

$$\begin{aligned} ||(A_m - A_n)x||^4 &= ((A_m - A_n)x|(A_m - A_n)x)^2 \\ &\leq ((A_m - A_n)x|x)((A_m - A_n)^2x|(A_m - A_n)x) \\ &\quad \text{(Cauchy-Schwarz inequality)} \\ &\leq ((A_m - A_n)x|x)||x||^2. \end{aligned}$$

(since $||A_m - A_n|| \leq 1$).

The sequence $(A_nx|x)$ ismonotone and bounded and so Cauchy. This imlies that $(A_nx)$ is Cauchy. The result follows easily.

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**Proposition 1** *(The square root lemma:) If $A \geq 0$ in $L(H)$ then there is a $B \geq 0$ so that $B^2 = A$.*

PROOF. The proof is based on an elementary iteration process to obtain the square root of a non-negative number. We can assume without loss of generality that $0 \leq A \leq \mathrm{Id}$. Note that $B^2 = A$ if and only if

$$(I - B) = 1/2((I - A) + (I - B)^2).$$

Hence we define inductively a sequence $C_n$ of operators as follows: $C_0 = 0$, $C_1 = 1/2(I - A)$, $C_{n+1} = 1/2((I - A) + C_n^2)$. Then it is easy to see

1. that each $C_n \geq 0$,
2. that $C_{n=1} - C_n$ is a polynomial in $(I - A)$ with positive coefficients and so is non-negative (for $C_{n+1} - C_n = \frac{1}{2}(C_n - C_{n-1}))C_n + C_{n-1}))$,
3. that $0 \leq C_n \leq \mathrm{Id}$.

Hence by the Lemma above, $C_n$ converges strongly to an operator $C$ and $B = I - C$ is the required operator.

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**Exercises**

A. Show that the $B$ constructed above commutes with $A$ and indeed with every operator which commutes with $A$.

B. Show that $B$ is uniquely determined by the above conditions (so we can write $B = A^{1/2}$).

C. Show that if $A \geq 0$, $B \geq 0$ and $AB$ commute, then $AB \geq 0$ (write $(ABx|x)$ as $(AB^{1/2}B^{1/2}x|x)$).

D. Let $(P_n)$ be a sequence of orthogonal projections. Show that

   a) if it is decreasing then $s - \lim P_n$ exists and is the projection on $\cap P_n H$;

   b) if it is increasing then $s - \lim P_n$ exists. Identigy the latter operator as a suitable projection.

Now recall the following definitions. If $T \in L(H)$, then

$$\begin{aligned} \rho(T) &= \{\lambda \in \mathbf{C} : (\lambda \mathrm{Id} - T) \text{ is invertible}\} \\ \rho(T) &= \mathbf{C} \setminus \rho(T). \end{aligned}$$

If $p$ is a polynomial, $p(T)$ is the operator obtained by formally substituting $T$ in $p$ (i.e. if

$$p(\lambda) \quad = \quad a_0 + a_1\lambda + \cdots + a_n\lambda^n$$
$$p(T) = a_0\mathrm{Id} + a_1 T + \cdots + a_n T^n).$$

Then we have the following simple relationships:

1. $\rho(T^*) = \overline{\rho(T)}$ $(= \{\bar{\lambda} : \lambda \in \rho(T)\})$;
2. $\rho(T^{-1}) = \rho(T)^{-1}$ $(= \{1/\lambda : \lambda \in \rho(T)\})$ if $T$ is an isomorphism;
3. $p(\lambda) \in \rho(p(T))$ if and only if $\lambda \in \sigma(T)$ and so $p(\sigma(T)) = \sigma(p(T))$.

(For if $\lambda \in \sigma(T)$, then

$$p(\lambda) 0 p(\lambda_0) = (\lambda - \lambda_0) q(\lambda)$$

for some polynomial $q$ and so

$$p(T) - p(\lambda_0)\mathrm{Id} = (T - \lambda_0 \mathrm{Id}) q(T)$$

which is not invertible. Conversely, suppose that $\mu$ is in $\sigma(p(T))$. Then we can write

$$p(\lambda) - \mu = (\lambda - \lambda_1) \dots (\lambda - \lambda_n)$$

for suitable $\lambda - 1, \dots, \lambda_1$ (where the $\lambda$'s are the reimages of $\mu$ under $p$). Now

$$p(T) - \mu\mathrm{Id} = (T - \lambda_1\mathrm{Id}) \dots (T - \lambda_n \mathrm{Id}).$$

Since the left-hand side is not invertible, one of the factors on the right side must also be non-invertible i.e. some $\lambda_k \in \sigma(T)$.

Now suppose that $A \in L(H)$ is a hermitian operator and put $\alpha = \lambda(A)$ $\beta = \mu(A)$.

**Lemma 2** *Let $p$ be a real polynomial so that $p(\lambda) \geq 0$ for $\lambda \in [\alpha, \beta]$. Then $p(A) \geq 0$.*

PROOF. $p$ has a factorisation

$$\sigma \prod_{i=1}^{r} (\lambda - a_i) \prod_{j=1}^{s} (b_j - \lambda) \prod_{k=1}^{t} ((\lambda - c_k)^2 + d_k^2)$$

where the $a_i$ are the roots in $]-\infty, \alpha]$, the $b_j$ are the roots in $[\beta, \infty[$, the $(c_k + \mathrm{Id}_k)$ are the complex roots and $\sigma > 0$. Then

$$p(A) = \sigma \prod (A - a_i \mathrm{Id}) \prod (b_j \mathrm{Id} - A) \prod (A - c_k \mathrm{Id})^2 + (d_k \mathrm{Id})^2.$$

Each factor is non-negative and the factors commute – hence $p(A) \geq 0$ (cf. 1.3.C).

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Now we introduce the notation $C_1([\alpha,\beta])$ for the class of non-negative upper semi-continuous real valued function on $[\alpha,\beta]$. It is well known that such functions are pointwise limits of decreasing sequences of continuous functions and it follows easily from the classical Weierstraß theorem on the density of polynomials that we can take these functions to be polynomials. Hence if $x \in C_1([\alpha,\beta])$ is the pointwise limit of the sequence $(p_n)$, we define $x(A)$ to be $s - \lim p_n(A)$. Note

a) that the limit exists by 1.1 since the sequence $(p_n(A))$ is decreasing (1.4);

b) that the limit $x(A)$ is independent of the choice of sequence $(p_n)$.

**Proof of b).** Suppose that $p_n$ and $q_n$ decrease to $x$. By replacing $p_n$ resp. $q_n$ by $(p_n + \frac{1}{n})$ resp. $(q_n + \frac{1}{n})$ we can suppose that both are strictly decreasing. Then for each $t$ and $n$ there is an $m$ so that $p_m(t) < q_n(t)$. By continuity, this inequality holds on a neighbourhood of $t$. By compactness, we can cover $[\alpha,\beta]$ by finitely many such neighbourhoods and, taking the supremum over the corresponding $m$, we get an $m_0$ such that $p_{m_0} < q_n$ on $[\alpha,\beta]$. Then

$$p_{m_0}(A) < q_n(A)$$

and so $\lim p_n(A) \leq \lim q_n(A)$.

By symmetry, the inequality holds in the opposite direction and so we have equality.

It is easy to check that this functional calculus has the following properties:

a) if $x \leq y$, then $x(A) \leq y(A)$;

b) if $\lambda \geq 0$, then $(\lambda x)(A) = lx(A)$;

c) $(x+y)(A) = x(A) + y(A)$, $(xy)(A) = x(A)y(A)$.

■

Now we write $C_2([\alpha,\beta])$ for the space of those functions which can be expressed as differences

$$x_1 - x_2$$

of functions $x_1, x_2$ in $C_1([\alpha,\beta])$. (Equivalently, those function $x$ for which $x^+$ and $x^-$ belong to $C_1([\alpha,\beta])$.) For such an $x$ we put

$$x(A) = x^+(A) - x^-(A).$$

Then the mapping $x \to x(A)$ is linear and multiplicative. Now if $\lambda \in \mathbf{R}$, the function

$$e_\lambda = \chi_{]-\infty,\lambda]\cap[\alpha,\beta]}$$

is in $C_1([\alpha, \beta])$ and so we can define operators

$$E_\lambda = e_\lambda(A).$$

Then we have the following properties:

1. $E_\lambda$ is an orthogonal projection;
2. if $\lambda \leq \mu$, then $E_\lambda \leq E_\mu$;
3. $E_\alpha = 0$, $E_\beta = \mathrm{Id}$;
4. $E_\lambda$ commutes with $A$;
5. $s - \lim_{\lambda \to \mu^+} E_\lambda = E_\mu$.

**Proof of 5.** We can find a sequence $(p_n)$ of polynomials so that $p_n$ decreases to $e$ and $p_n \leq e_\lambda + \frac{1}{n}$. Then

$$p_n(A) \geq E_{\mu+1/n} \geq E_m u \text{ and} p_n(A) \to E_\mu.$$

Hence $E_{\mu+1/n} \to E_\mu$.

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**Proposition 2** *(Spectral theorem for hermitian operators):*

$$A = \int_\alpha^\beta \lambda dE_\lambda$$

*in the sense of a Riemann-Stieltjes integral with respect to the uniform topology.*

Proof. If $\mu < \lambda < \nu$

$$\mu(e_\nu - e_\mu) \leq \lambda(e_\nu - e_\mu) \leq (e_\nu - e_\mu)$$

and so $\mu(E_\nu - E_\mu) \leq A(E_\nu - E_\mu) = \nu(E_\nu - E_\mu)$. Hence summing over a suitable partition $\alpha = \mu_0 \leq \mu_1 \leq \cdots \leq \mu_n = \beta$ we get

$$\sum_{k-1}^{n} \mu_{k-1}(E_{\mu_k} - E_{\mu_{k-1}}) \leq A \leq \sum_{k=1}^{n} \mu_k(E_{\mu_k} - E_{\mu_{k-1}}).$$

Also the difference between the left and right hand terms is $\sum_{k=1}^n (\mu_k - \mu_{k-1})(E_{\mu_k} - E_{\mu_{k-1}})$ which lies between 0 and $\delta I$ where $\delta = \max_k(\mu_k - \mu_{k-1})$. Hence these terms, which are appropriate Riemann sums for the integral $\int_\alpha^\beta \lambda dE_\lambda$ converge to $A$.

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**Exercises**

A. We can use the spectral theorem to define a functional calculus for self-adjoint operators on Hilbert space. If $x$ is a continuous function on $[\alpha, \underline{\ }]$ we define an operator $x(T)$ on $H$ by the formula

$$\int_\alpha^\beta x(\lambda) dE(\lambda)$$

where once again the integral is to be understood as a Riemann-Stieltjes integral in the natural way.

We denote this operator by $x(T)$. Show that if $x$ is a polynomial then this definition agrees with the original one.

B. Show that if $A, B$ are commuting hermitian operators, with representations

$$\begin{aligned} A &= \int \lambda dE_\lambda \\ B &= \int \lambda dE_\lambda \end{aligned}$$

then the projection $E(\lambda)$ and $F(\mu)$ commute for all $\lambda, \mu$.

C. Show that if $A \in L(H)$ is normal, then $A$ has a representation

$$A = \int_{\mathbf{C}} z dE_z$$

where $z \to E_z$ is a suitable projection valued funciton on $\mathbf{C}$ and the integral is interpreted as a Riemann integral in a suitable way.

D. Let $A$ be a self-adjoint operators which commute. Show how to define operators $|A|, A^+, A^-, \max(A, B), \min(A, B)$ and describe some of their properties.

E. If $A$ is the operator $M_x$ of multiplication by a bounded measurable function on a measure space, calculate explicitly the partition of unity $(E_\lambda)$ involved in the spectral representation of $A$.

## 2 Banach algebras

In this section, we prepare for a more abstract approach to the spectral theorem. To do this we develop some basic elements of the theory of commutive Banach algebras, that is, Banach spaces whose structure is enriched by the presence of a suitable multiplication. The classical example of such a space is $C(K)$, the algebra of continuous funcitons on a compact set, and our main result will be the famous Gelfand-Naimark theorem which gives a characterization of those Banach algebras which have this form. The method of proof involves constructing a natural mapping from a general Banach algebra into an algebra $C(K)$ of the above form (this construction has a certain formal similarity with the embedding of a Banach space in its second dual). The proof is completed by finding conditions which ensure that it is an isomorphism.

The basic construction of this section – the Gelfand Naimark transformation – is of purely algebraic nature. We therefore begin with a rather concise treatment of the necessary algebraic preliminaries.

**Definition 1** *An* **algebra** *over* **C** *is a vector space $A$ with an associative multiplication so that*

*a)* $x(y+z) = xy + xz$ $(y, x, z \in A)$

*b)* $(z+y)z = xz + yz$ $(x, y, z, \in A)$

*c)* $(\lambda x)y = \lambda(xy) = x(ly)$ $(x \in \mathbf{C}, x, y \in A)$.

*$A$* **unit** *of $A$ is an element $e$ so that $xe = ex = x$ $(x \in A)$. An element $x$ $A$ is invertible if there is a $y \in A$ with $xy = yx = e$ – otherwise it is* **singular**.

Examples. If $S$ is a set, then $\mathbf{C}^S$, the space of mappings from $S$ into $\mathbf{C}$, has a natural algebra structure – that induced by pointwise operations. It is a commutative algebra with unit (the constant function 1). If $E$ is a vector space, $\mathcal{L}(\mathcal{E})$, the space of linear mappings from $E$ into itself, has a natural algebra structure – where multiplication is composition of functions. It is **non-commutative** (if $\dim E \geq 2$) and has an unit $\mathrm{Id}_E$. We write

| | |
|---|---|
| $\mathrm{Inv}(A)$ | for the set of invertible elements of $A$ |
| $\mathrm{Sing}(A)$ | for the set of non-invertible elements. |

A mapping $\phi : A \to A_1$ is an **algebra homomorhism** if it is linear and multiplicative (i.e. $\phi(xy) = \phi(x)\phi(y)$ we say that $\phi$ is **unit preserving** (or **unital**) if $\phi(e) = e_1$.

If $A$ is an algebra, then a subset $A_1$ of $A$ is a **subalgebra** if it is both a vector subspace and is closed under multiplication. A subalgebra $I$ is an **ideal** if, in addition, $xy \in I$ and $yx \in I$ whenever $y \in I, x \in A$. An ideal $I$ is **proper** if it is not equal to $A$. If $I$ is an ideal, the quotient spaces $A/I$ has a natural algebra structure – it is called a **quotient algebra**.

**Exercise** Show that a subset $I$ of an algebra $A$ is an ideal if and only if there is an algebra $A_1$ and an algebra morhism $\varpi : A \to A_1$ so that $I = \mathrm{Ker}(\varpi)$.

The following are examples of subalgebras, resp. ideals.

I. If $S$ is a topological space, then the subsapces listed below are subalgebras:

| | |
|---|---|
| $C(S)$ | – the continuous function in $\mathbf{C}^S$; |
| $C^\infty(S)$ | – the bounded, continuous functions; |
| $C_0(S)$ | – the continuous function which vanish at infinity ($S$ locally compact. |

II. If $E$ is a Banach space, $L(E)$, the space of bounded linear operators on $E$, is a subalgebra of $\mathrm{End}(E)$.

III. If $S$ is a set, $S_1$ a subset, then

$$I_{S_1} := \{x \in \mathbf{C}^S : X|_{S_1} = 0\}$$

is an ideal in $\mathbf{C}^S$.

IV. If $E$ is a Banach space, then $C_f(E)$, the space of bounded operators of finite rank, and $K(E)$, the space of compact operators, are ideals in $L(E)$.

In the context of algebrals we can give an abstract definition of the spectrum (cf. I.7). If $x \in A$ ($A$ an algebra with unit $e$), then

$$\sigma_A(x) := \{\lambda \in \mathbf{C} : \lambda e - x \in \mathrm{Sing}(A)\}$$

is called the **spectrum** of $A$. We write simply $\sigma(x)$ if the algebra $A$ is not in doubt.

For example, if $E$ is a finite dimensional vector space and $T \in \mathrm{End}(E)$ then $\sigma_{\mathrm{End}}(T)$ is the set of eigenvalues of $T$.

If $x$ is an appropriate function on a set (resp. topological space) $S$ then

– in $\mathbf{C}^S$ $\sigma(x) = x(S)$;

– in $C(S)$ $\sigma(x) = x(S)$;
– in $C^\infty$ $\sigma(x) = \overline{x(S)}$, the closure of the range of $x$.

If $T \in L(E)$, $E$ a Banach space, then $\sigma_{L(E)}(T)$ is the classical spectrum of $x$.

**Exercises**

A. Prove the identity

$$(e + yx)^{-1} = e - y(e + xy)^{-1}x$$

(for $x, y \in A$ – provided that either $e + xy$ is invertible). Deduce that $\sigma(xy) \cup \{0\}\sigma(yx) \cup \{0\}$ $(x, y \in A\}$.

B. If $T \in L(E)$ ($E$ a Banach space), show that

$$\sigma_{L(E)}(T) = \sigma_{\mathrm{End}(E)}(T).$$

C. Let $p$ be a non-constant polynomial with complex coefficients, $x \in A$. Show that

$$\sigma(p(x)) = \{p(\lambda) : \lambda \in \sigma(x)\}$$

where $p(x)$ is the element of $A$ obtained by formal substitution of $x$ in $p$ (cf. p. ...).

The following definition is motivated by the involution $T \to T^*$ which we studied for operators on Hilbert spaces. An **involution** on an algebra $A$ is a mapping $x \to x^*$ so that

a) $(x + y)^* = x^* + y^*$ $(x, y \in A)$;

b) $(\lambda x)^* = \bar{\lambda} x^*$ $(\lambda \in \mathbf{C}, x \in A)$;

c) $x^{**} = x$ $(x \in A$;

d) $(xy)^* = y^* x^*$ $(x, y \in A)$.

$x \in A$ is **self-adjoint** if $x = x^*$.

An algebra with involution is called a **$*$-algebra**. An algebra morphism $\phi(x^*) = (\phi(x))^*$ for $x \in A$.

Examples. Prove that if $A$ has a unit $e$ then $e^* = e$.

The mapping $x \to \bar{x}$ (complex-conjugation) is an involution on the algebras $\mathbf{C}^s, C(S), C^\infty(S)$. Of course, if $H$ is a Hilbert space, then $T \to T^*$ (the adjoint operator) is an involution on $L(H)$.

**Proposition 3** *Let $A$ be a $*$-algebra. Then*

1. *the self-adjoint elemets span$A$;*
2. *if $A$ has a unit, then $\sigma(x^*) = \overline{\sigma(x)}$ $(x \in A)$.*

Proof.

1. Put $\Re x = \frac{1}{2}(x + x^*)$, $\Im x = \frac{1}{2i}(x - x^()$. Then $x = \Re x + i\Im x$ and $\Re x$ and $\Im x$ are self-adjoint.
2. It follows from 2.4 and condition d) above that $y$ is an inverse for $x$ if and only if $y^*$ is an inverse for $x^*$. Hence Ł $\in \sigma(x)$ if and ony if $(\lambda e - x)$ is not invertible i.e. if and only if $(\lambda e - x^*) = \bar{\lambda} e - x^*$ is not invertible. The latter just means that $\bar{\lambda} \in \sigma(x^*)$.

■

If $M$ is a subset of an algebra $A$, we put

$$M^C := \{x \in A : xy = yx \text{ for each } y \in M\}$$

($M^C$ is the **commutator** of $M$)
$M^{CC} := (M^C)^C$ is the **double commutator** of $M$. Note that $M \subseteq M^{CC}$.

**Proposition 4** 1. *$M^C$ is a subalgebra of $A$;*

2. *$M$ is commutative if and only if $M \subseteq M^C$;*
3. *$M \subseteq N$ implies $N^C \subseteq M^C$;*
4. *if $M$ is a commutative subset of $A$ then $M^{CC}$ is a commutative subalgebra of $A$.*

Proof. 1., 2. and 3. are clear.

4. If $M$ is commutative, then $M \subseteq M^C$ and so $M^{CC} \subseteq M^C$. Hence if $x, y \in <^{CC}$, then $x \in M^C$ and $y \in M^{CC}$ so that $x$ and $y$ commute.

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**Proposition 5** *Every commutative subset of an algebra $A$ is contained in a maximal commutative subset of $A$ and this is a subalgebra (with unit if $A$ has a unit).*

PROOF. The first statement follows from Zorn's Lemma. If $M$ is maximal commutative subset, then $M = M^{CC}$ and so $M$ is a subalgebra by 2.6.4.

EXAMPLES. If $x \in \text{Inv}(A)$, show that $x^{-1} \in \{x\}^{CC}$. Deduce that if $x \in A$ and $A^1$ is a maximal commutative subalgebra of $A$ containig $x$ then $\sigma_A(x) = \sigma_{A_1}(x)$.

Let $A$ now be an algebra with unit $e$. A **maximal ideal** in $A$ is a **proper** ideal $I$ so that if $I_1$ is an ideal with $I \subseteq I_1$, then $I_1 = I$ or $I_1 = A$.

In this connection note that an ideal is proper if and only if it does not contain the unit element $e$. Hence a proper ideal cannot contain an invertible element – in particular, the only proper ideal in a field $F$ is the trivial one $\{0\}$.

**Proposition 6**
1. *Every proper ideal in $A$ is contained in a maximal ideal;*
2. *if $A$ is commutative, then an ideal is maximal if and only the quotient $A/I$ of $A$ by this idealis a field.*

PROOF. 1. follows from Zorns's lemma which 2. is an immediate consquence of the above remark on ideals in a field.

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A **multiplicative functional** on an algebra $A$ is a non-zero algebra homomorphism $f : A \to \mathbf{C}$. If $A$ has a unit $e$, we demand, in addition, that $f(e) = 1$. For the remainder of this section we shall tacitly assume that all algebras have a unit.

**Proposition 7**
1. *If $f$ is a multiplicative functional on $A$, then $Kerf$ is a maximal ideal;*
2. *A maximal ideal has the form $Kerf$ ($f$ as above) if and only if the quotient space $A/I$ is isomorphic to $\mathbf{C}$.*

PROOF.

1. $f$ induces an isomorphism from $A/\text{Ker}f$ onto $\mathbf{C}$ and so $\text{Ker}f$ is maximal.

2. Define $f$ to be the composition $A \to A/I \cong \mathbf{C}$. Then $I = \text{Ker}f$.

■

Our aim in this chapter is to obtain representations of algebras as spaces of functions. To this end we indtroduce the following definitions. The set of all multiplicative functionals $f$ on $A$ is denoted by $M(A)$ and called the **spectrum** of $A$.

If $x \in A$ then we can define the function

$$\hat{x} : f \to f(x)$$

on $M(A)$. Then $x \to \hat{x}$ is an algebra homomorphism from $A$ into $\mathbf{C}^{M(A)}$. It is called the **Gelfand-Naimark transform** (GN-transform).

Our ultimate aim is a characterization of algebras for which this mapping is an isomorphism onto a suitable subalgebra of $\mathbf{C}^{M(A)}$. We consider first the question of its injectivity. If $A$ is a **commutative** algebra, the **radical** $\mathrm{Rad}(A)$ is defined to be the ideal $\bigcap_{f \in M(A)} \mathrm{Ker} f$. $A$ is **semi-simple** if and only if $\mathrm{Rad}(A) = \{0\}$.

**Proposition 8** *Let $A$ be a commutative algebra. Then*

1. *Rad$(A)$ is the kernel of the GN-transform;*
2. *$A$ is semi-simple if and only if the GN-transform is injective.*

If $A$ is a $*$-algebra and $f \in M(A)$ we define $f^*$ by $f^* : x \to f(x^*)$. Then $f^* \in M(A)$. $f$ is **hermitian** if $f^* = \bar{f}$ (complex-conjugation) (i.e. if $f$ is a $*$-algebra homomorphism from $A$ into $\mathbf{C}$).

Then the following are equivalent:

a) the GN-transform is a $*$-algebra homomorphism;

b) for each $f \in M(A)$, $f^* = \bar{f}$ (i.e. each $f \in M(A)$ is hermitian).

**Exercises**

A. Give an example of elements $x, y$ of an algebra so that $- \in \sigma(xy)$, $0 \notin \sigma(yx)$ (consider the operators

$$\begin{aligned} S^r : (x_1, x_1, x_3, \dots) &\to (0, x_1, x_2, \dots) \\ S^l : (x_1, x_1, x_3, \dots) &\to (x_1, x_2, \dots) \end{aligned}$$

on a Hilbert space $l^2$.

B. Let $A$ be an algebra. A **commutator** in $A$ is an element of the form $xy - yx$ $(x, y \in A)$. Denote by $C$ the ideal generated by the commutators in $A$. Show

1. that $A/C$ is commutative;
2. if $I$ is an ideal in $A$, $A/I$ is commutative if and only if $C \subseteq I$;
3. every algebra homomorphism from $A$ into a commutative algebra $B$ factors over $A/C$ (so that, in particular, the GN-transform factors over $A/C$).

C. Let $\lambda = (\lambda_n)$ be a bounded sequence (i.e. $\lambda \in \ell^\infty$). Denote by $M_\lambda$ the operator

$$(x_1, x_2, x_3, \dots, x_n, \dots) \to (\lambda_1, x - 1, \dots, \lambda_n, x_n, \dots)$$

from $\ell^2$ into itself. Show that $\sigma(M_\lambda) = \overline{\{\lambda_n\}}$.

Deduce that if $K$ is compact subset of $\mathbf{C}$, there is a Hilbert space $H$ and an operator $T \in L(H)$ so that $K = \sigma(T)$.

We now turn to the situation which will particularly interest us – algebras provided with a suitable norm:

**Definition 2** *A* **normed algebra** *is a pair* $\{A, \|| \||)$ *where* $A$ *is an algebra and* $\|| \||$ *is a norm on* $A$ *so that* $||xy|| \leq ||x|| \, ||y||$ $(x, y \in A)$*.* $A$ *is a* **Banach algebra** *if* $(A, \|| \||)$*, as a normed space, is complete.*

*If* $A$ *has a unit* $e$ *we assume that* $||e|| = 1$*. If* $A$ *and* $A_1$ *are normed algebra, a* **normed algebra morphism** *from* $A$ *into* $A_1$ *is a norm-contractive algebra homomorphism.* $A$ *and* $A_1$ *are* **isomorphic** *if there are normed algebra morphism* $\phi : A \to A_1$ *and* $\psi : A_1 \to A$ *which are mutually inverse.*

For example, if $S$ is a set, then $\ell^\infty(S)$, the space of bounded function in $\mathbf{C}^S$, is a Banach algebra under the supremum norm. If $A$ is a topological space, then $C^\infty(S)$ and $C_0(S)$ (if $S$ is locally compact) are Banach algebras.

If $E$ is a normed space, then $L(E)$, with the uniform norm, is a normed algebra. it is a Banach algebra if $E$ $E$ is a Banach space. Similarly, $K(E)$, the space of compact operators in $L(E)$, is a Banach algebra.

It is clear that if $(A, \|| \||)$ is a normed algebra, $A_1$ a subalgebra, $I$ an ideal and $M$ a subset. Then

$$\begin{array}{rl} \bar{A}_1 & \text{is a subalgebra;} \\ \bar{I} & \text{is an ideal;} \\ M^C & \text{is a closed subalgebra.} \end{array}$$

Of course if $A_1$ is a subalgebra of a normed algebra, then it is a normed algebra with the induced norm. If $A$ is a Banach algebra and $A_1$ is closed, then $A_1$ is a Banach algebra.

If $I$ is a closed ideal of $A$, then $A/I$ with the quotient norm is a normed algebra. It is a Banach algebra if $A$ is a Banach algebra. (We show that the quotient norm is submultiplicative: denote by $\pi$ the natural projection from $A$ onto $A/I$. Then

$$\begin{aligned} ||xy|| &= \inf\{||x'y'||''x'y' \in A, \pi(x') = x, \pi(y') = y\} \\ &\leq \inf\{||x'||\,||y'|| : \{i(x') = x, \pi(y') = y\} \\ &\leq \inf\{||x'|| : \pi(x') = x\} \inf\{||y'|| : \pi(y') = y\} \\ &= ||x||\,||y||. \end{aligned}$$

**Proposition 9** *Let $A$ be a normed algebra with a unit $e$. Then $A$ is isomorphic to a subalgebra of an algebra $L(E)$ where $E$ is a Banach space.*

PROOF. If $x \in A$ define $L_x : A \to A$ by

$$L_x : a \to xa.$$

Then $L_x \in L(A)$ and $x \to L_x$ is an algebra homomorphism from $A$ into $L(A)$. Also this mapping is an isometry since

$$||L_x|| := \sup\{||xa|| : a \in A, ||a|| \leq 1\}$$

and the right hand side is bounded below by $||x||$ (take $a = e$) and above by $||x||$ (by submultiplicativity). We finish the proof by noting that $L(A)$ is a normed subalgebra of $L(\hat{A})$ ($\hat{A}$ is the normed space completion of $A$).

■

**Corollar 1** *Let $A$ be a normed algebra with unit. Then the normed space complection $\hat{A}$ of $A$ has a natural Banach algebra structure so that $A$ is a normed subalgebra of $\hat{A}$.*

PROOF. We can identify $A$ with a subalgebra of $L(A)$. Now the completion of $A$ is identifiable with its closure in $L(\hat{A})$ and this is a Banach algebra.

■

## Exercises

A. Let $A$ be an algebra with unit $e$ and let $|||\,||$ be a norm on $A$ so that multiplication is continuous (i.e. there exists a $K > 0$ so that $||xy|| \leq K||x||\,||y||$ $(x, y \in A)$.

   Show that there exists an equivalent norm $|||\,||y$ on $A$ so that $(A, ||||\,|||)$ is a normed algebra with unit. (Embed $A$ into $L(A)$ as in 2.14 and consider the norm induced from $L(A)$.)

B. Consider the space $C^n([0,1])$ of $n$-times continuously differentiable complex-valued functions on $[0,1]$ with norm

$$||| \, ||_n : x ||x||_\infty + ||x'||_\infty + \cdots + ||x^{(n)}||_\infty .$$

Show that

a) $(C^n[0,1], ||| \, ||_n)$ is not a Banach algebra (for $n \geq 2$);

b) there is an equivalent norm $|||| \, |||_n$ on $C^n([0,1])$ under which it is a Banach algebra.

C. Consider the space $\ell^1(Z)$. If $x = (\zeta_n), y = (\eta_n) \in \ell^1(Z)$, define

$$x * y := \left( \sum_{k \in Z} \zeta_{n-k} \eta_k \right)_{n \in Z} .$$

Show that $\ell^1(Z)$ is a commutative Banach algebra with this multiplication. Does $\ell^1(Z)$ have a unit?

D. Let $\alpha = (\alpha_n)_{n \in \mathbf{N}}$ be a sequence of positive numbers so that $\alpha_{m+n} \leq \alpha_m \alpha_n$ $(m, n \in \mathbf{N})$. Show that there exists a Banach algebra $A$ and an element $x \in A$ so that $||x^n|| = \alpha_n$ $(n \in \mathbf{N})$.

(Let $\ell^1(\alpha)$ be the space $\{(\zeta_n) \in \mathbf{R}^{\mathbf{N}} : \sum_{n \in \mathbf{N}} \alpha_n |\zeta_n| < \infty\}$. Show that this is a Banach algebra under the multiplication

$$x * y := \left( \sum_{r+s=n} \zeta_r \eta_s \right)_n$$

and then take $x = e_1$).

E. let $\{A_\alpha\}_{\alpha \in A}$ be a family of Banach algebras. Show that

$$\left\{ x = (x_\alpha) \in \prod_{\alpha \in A} A_\alpha : ||x||_\infty \sup_{\alpha \in A} ||x_\alpha|| \right\}$$

(i.e. $B \prod_{\alpha \in A} A_\alpha$ – cf. I.9.) is a Banach algebra under this norm (with componentwise multiplication).

We have already used the next result in the context of operators:

**Lemma 3** *Let $x$ be an element of a Banach algebra with unit $e$ such that $||e - x|| < 1$. Then $x \in Inv(A)$ and*

$$x^{-1} = \sum_{n=0}^{\infty} (e - x)^n .$$

PROOF. Let $y := e - x$. Then $\sum_{n=0}^{\infty} ||y^n|| < \infty$ and so $\sum_{n=0}^{\infty} y^n$ is convergent since $A$ is complete. Let $s_n$ denote the partial sum $\sum_{k=0}^{n} y^k$. Then $s_n x = x s_n = e - y^{n=1}$ and so

$$sy = \lim s_n y = \lim(e - y^{n+1}) = e = ys$$

where $s : \sum_{k=0}^{\infty} y^k$.

■

**Corollar 2** *Let $I$ be a proper ideal in $A$. Then $\bar{I}$ is also an proper ideal.*

PROOF. Let $U := \{x \in A : ||e - x|| < 1\}$. Then $U$ is open and $U \subseteq \mathcal{CI}$. Hence $U \subseteq \mathcal{C}\bar{\mathcal{I}}$ and so $\bar{I} \neq A$.

■

**Proposition 10** *Let $A$ be a Banach algebra with unit. Then*

1. *Inv$(A)$ is open;*
2. *Inv$(A)$ is a topological group.*

PROOF.

1. Suppose that $x \in \mathrm{Inv}(A)$ and $y \in A$ with $||y|| < ||x^{-1}||^{-1}$. Then $||x^{-1}y|| \leq ||x^{-1}|| \, ||y|| < 1$ and so $(e + x^{-1}y)$ is invertible. Hence $x + y$ is invertible with inverse $(e + x^{-1}y)^{-1}x^{-1}$.
2. We need only show that inversion is continuous (in the topology induced by the norm). But the series representation shows that inversion is continuous in an open neighbourhood of $e$ and this is sufficient.

■

The next result, which characterises one dimensional Banach algebras, will allow us to identify the maximal ideals of a commutative Banach algebra with the space of multiplicative forms.

**Theorem 1 (Gelfand-Mazur.)** *Let a Banach algebra $A$ with unit be a (skew) field. Then $A = \mathbf{C}$.*

PROOF. We show that for each $x \in A$, there is a $\lambda \in \mathbf{C}$ with $x = \lambda e$. If not, there is an $x$ so that $(\lambda e - x)^{-1}$ exists for each $\lambda \in \mathbf{C}$. Choose $f \in A'$ so that $f(x^{-1}) \neq 0$. Then

$$\varpi : \lambda \to f((\lambda e - x)^{-1})$$

is entire since

$$\begin{aligned} \frac{\varpi(\lambda+\mu) - \varpi(\mu)}{\mu} &= \frac{f((\lambda+\mu)e - x)^{-1} - (\lambda e - x)^{-1})}{\mu} \\ &= -f((\lambda+\mu)e - x)^{-1}(\lambda e - x)^{-1}) \end{aligned}$$

and this converges to $-f((\lambda e - x)^{-2})$ as $\mu \to 0$ (2.19).

Also $|\varpi(\lambda)| = |\frac{1}{\lambda} f([e - \frac{x}{\lambda})]^{-1} \leq \frac{||f||}{|\lambda| - ||x||}$ for large $\lambda$ and so $|\varpi(\lambda)| \to 0$ as $|\lambda| \to \infty$.

Hence $\varpi = 0$ (by Liouville's theorem) – condtradiction (since $\phi(0) + 0$).

■

**Proposition 11** *Let $I$ be a maximal ideal in a commutative Banach algebra with unit. Then*

*a) $I$ is closed;*

*b) $A/I \cong \mathbf{C}$;*

*c) $I$ is the kernel of an $f \in M(A)$.*

PROOF.

a) follows from 2.18;

b) follows from 2.20;

c) follows from 2.10.2.

■

**Proposition 12** *If $f \in M(A)$ ($A$ a Banach algebra with unit), then $f$ is continuous and $||f|| = 1$.*

PROOF. $f$ is continuous since its kernel is closed (2.21). $||f|| \geq 1$ since $||f(e)|| = 1$. If $||f|| > 1$ there is an $x \in A$ with $||x|| \leq 1$ and $f(x) =: \lambda > 1$. Then $(x/\lambda)^n \to 0$ but $f[(x.\lambda)^n] = 1$ – contradiction.

■

Examples. Give an example of a normed algebra with a non-continuous multiplicative functional. For the remainder of this section we will assume all algebras to be commutative with unit.

By 2.21 the sepectrum $M(A)$ is a subset of the unit ball of $A'$, the normed space sual of $A$. It is weakly closed since we can represent it is the intersection

$$\left(\bigcap_{x,y\in A} V_{x,y}\right) \cap W_e$$

where

$$\begin{aligned} V_{x,y} &:= \{f \in A' : f(xy) = f(x)f(y)\} \\ W_e &:= \{f \in A' : f(e) = 1\} \end{aligned}$$

and each of these sets is weakly closed.

Hence $M(A)$ is compact for the weak topology $\sigma(A', A)$, and we regard it as a compact space with this topology.

**Proposition 13** *The GN-transform is linear, multiplicative contraction from A into $C(M(A))$.*

**Proposition 14** $\sigma_A(x) = \sigma_{C(M(A))}\,(\hat{x})$. *In particular, $x \in Inv(A)$ if and only if $\hat{x} \in Inv(C(M(A)))$ (i.e. $f(x) \neq 0$ for each $f \in M(A)$).*

Proof. It is sufficient to prove the second statement. Now $x$ is invertible if and only if it lies in no maximal ideal i.e. if and only if for each $f \in M(A)$, $f(x) \neq 0$. This is equivalent to the fact that for each $f \in M(A)$, $\hat{x}(f) \neq 0$ i.e. that $\hat{x}$ is invertible in $C(M(A))$.

■

If $xA$ we define the **spectral norm** of $x$ to be the express

$$\rho_A(x) := ||hatx||_{C(M)}.$$

We habe the following internal description:

**Proposition 15** $\varpi_A(x) = \lim_{n\to\infty} ||x^n||^{1/n}$.

PROOF. Note that $||\hat{x}|| = ||\hat{x}^n||^{1/n}$ for each $n$ and so

$$||\hat{x}|| = ||\hat{x}^n||^{1/n} \leq ||x^n||^{1/n}.$$

Hence $||\hat{x}|| \leq' liminf ||x^n||^{1/n} \leq \limsup ||x^n||^{1/n}$ and so it suffices to show that the above lim sup is bounded from above by $\rho_A(x)$.

let $U$ denote the open subset

$$\{\lambda \in \mathbf{C} : |\lambda| \geq ||\hat{x}||\}$$

of $\mathbf{C}$. $(\lambda e - x)^{-1}$ exists for each $\lambda \in U$ and the Laurent expansion

$$\phi(\lambda) = \sum_{n=0}^{\infty} x^n / \lambda^{n+1}$$

shows that the function

$$\phi : \lambda \mapsto (\lambda e - x)^{-1}$$

from $U$ into $A$ is analytic. It follows from the natural analogue of the bounds on the Taylor cefficients for Banach space valued functions (cf. II.) that

$$\limsup ||x^n||^{1/n} \leq ||\hat{x}||$$

and the result follows.

■

Using this result we can characterise those Banach algebras for which the GNT is an isometry.

**Proposition 16** *Then GN-transform on $A$ is an isometry if and only if $||x^2|| = ||x||^2$ for each $x \in A$.*

PROOF. This condition is necessary since it is satisfied in $C(K)$ ($K$ is compact).

It is sufficient since if it holds then $||x^{2^n}|| = ||x||^{2^n}$ for each $n \in \mathbf{N}$ and so

$$\rho_A(x) = \lim_{n \to \infty} ||x^{2^n}||^{2^{-n}} = ||x||.$$

■

As an example we calculate the maximal ideal space of an algebra of the form $C(K)$ ($K$ compact). If $s \in S$

$$I_s := \{x \in C(K) x(s) = 0\}$$

is a maximal ideal. We claim that every maximal ideal is of this form. For if $I$ is a maximal ideal in $C(K)$ than

$$Y := \bigcap_{x \in I} \{t \in K : x(t) = 0\}$$

is closed and non-empty in $K$ (by the finite intersection property).

if $s \in Y$ then $I \subseteq I_s$ and so $I = I_s$ by maximality.

Hence there is a natural bijection between $K$ and $M(C(K))$ and it is not difficult to see that this is a homeomorphism.

Up to this identification, the GN-transform is the identity. The above proof can be used to demonstrate the following result:

**Lemma 4** *Let $A$ be a subalgebra of $C(K)$ containing the constant functions ($K$ a compact space) and suppose that $A$ is a Banach algebra under a norm $||| \, || \geq ||| \, ||_\infty$ so that*

1. *$A$ separates the points of $K$;*
2. *if $x \in A$ is invertible in $C(K)$ then $\frac{1}{x} \in A$;*
3. *if $x \in a$ then $\bar{x} \in A$.*

*Then $M(A)$ is nturally identifiable with $K$.*

**Exercises**

A. Show that $M(C^n[0,1]) = [0,1]$.

B. If $A$ is generated (as a Banach algebra) by $x \in A$ show that $M(A)$ and $\sigma(x)$ are homeomorpic under the mapping

$$f \to f(x)$$

from $M(S)$ into $\sigma(x)$.

As a slightly less trivial example we calculate the spectrum of the algebra $\ell^1(Z)$. If $\alpha \in T$ (the circle group) then

$$f_\alpha : x \to \sum_{n=-\infty}^{\infty} \xi_n \alpha^n \quad (x = (\xi_n)($$

is in $M(\ell^1(Z))$ and every element has this form (for if $f \in M(\ell^1(Z))$ put $\alpha := f(e-1)$. Then $\alpha \neq 0$ since $e_1$ is invertible. We show that $|\alpha| = 1$. If $|\alpha| > 1$ let $x := \sum_{n=1}^{\infty} \alpha^{-n} e_n$. Then $f(x) = \sum_{n \in \mathbf{N}} = \infty$ – contradiction. A similar contradiction can be obtainded by considering $x := \sum_{n=-1}^{-\infty} \alpha^{-n} e_n$ for $|\alpha| < 1$.

hence $M(\ell^1(Z)) = T$ and $\hat{\ell}^1(Z)$ is the spa e of continuous functins on $T$ which have $'ell^1$-Fourier coefficients.

It is perhaps appropriate to remark here that if $A$ is a not necessarily commutative Banach algebra with unit, then one can define the spectrum of $A$ and the GN-transform as for commutative algebras. However, the latter will factorise over the associated commutative algebra (2.12.B) so that one loses no essential generality by considering commutative algebras from the beginning. Certatin results can be extended to non-commutative algebras by using the fact that every Banach algebra is "locally commutative" (that is, every element $x$ lies in a closed commutative subalgebra of $A$ – e.g. the closed algebra generated by $\{e, x\}$).

**Exercises**

A. Denote by $I$ the operator

$$x \to \left(t \to \int_0^t x(x)ds\right)$$

from $C([0,1])$ into itself. Show that $\rho(I) = 0$. (Here we mean $\rho(I)$ in the commutative subalgebra of $L(C[0,1])$ generated by $I$).

B. Let $A_1, A_2$ be commutative, semi-simple Banach algebras with unit, $T$ a unital algebra homomorphism from $A_1$ into $A_2$. Show that $T$ is norm-continuous (consider $A_1$ and $A_2$ as subalgebras of $C(M(A_1))$ and $C(M(A_2))$ resp. and show that $T$ is continuous for the topology of pointwise convergence. Deduce that $T$ has a closed graph).

   Deduce that there is at most one norm (up to equivalence on a commutative, semi-simple algebra $A$ with unit under which $A$ is a Banach algebra.

C. Let $A$ be a commutative Banach algebra with unit. Show that $x \in \mathrm{Rad}(A)$ if and only if $(e + xy) \in \mathrm{Inv}(A)$ for each $y \in A$.

D. Let $A$ be a not-necessarily commutative Banach algebra with unit. Show that if $x \in A$ then $\lim_{n\to\infty} ||x^n||^{1/n}$ exists. Denote it by $\rho_A(x)$.

Show that if $x, y \in A$ commute then

$$\rho_A(x+y) \leq \rho_A(x) + \rho_A(y); \rho_A(xy) \leq \rho_A(x)\rho_A(y).$$

We now characterise those commutative Banach algebras for which the GN-transform is an isomorphism. Of course, such an algebra must possess all the properties of $C(K)$ – in particular it must have an involution.

**Definition 3** *A Banach $*$-algebra is a Banach algebra with an involution so that* $||x^*|| = ||x||$ $(x \in A)$.

Examples of Banach $*$-algebras are the algebras $C^\infty(S)$ and $L(H)$. The following algebras are also Banach $*$-algebras.
PROOF. 1. $\Rightarrow$ 2.: if $f \in M(A)$ then

$$f(e + x^*x) = f(e) + \hat{x^*x}(f) = 1 + |\hat{x}|^2(f) > 0.$$

2. $\Rightarrow$ 3.: Suppose that $h$ is self-adjoint and $\lambda \in \mathbf{C}$ with $\lambda \in \mathbf{C}$ with $\lambda = \alpha + i\beta$ $(\alpha, \beta$ real). Then a simple calculation shows that

$$(h - \lambda e)(h - \bar{\lambda}e) = \beta^2(e + z^*z)$$

where $z = h - \alpha e/\beta$. Hence if $\beta \neq 0$ then $\lambda \notin \sigma(h)$.

3. $\Rightarrow$ 1.: if $x \in A$, we canwrite

$$x = \Re x + i\Im x; x^* = \Re x - i\Im x.$$

Then for $f \in M(A)$, since $f(\Re x)$ and $f(\Im x)$ are real,

$$\begin{aligned} \hat{x^*}(f) &= (\Re x)^\wedge(f) - i(\Im x)^\wedge(f) \\ &= \overline{(\Re x + i\Im x)^\wedge(f)} = \overline{\hat{x}(f)}. \end{aligned}$$

■

We now come to the definition which will characterise those algebras with the property that we are seeking. A Banach $*$-algebra $A$ is called a $B^*$**-algebra** if $||x^*x|| = ||x||^2$ $(x \in A)$ (or equivalently, $||x^*x|| = ||x^*||\,||x||$ $(x \in A)$).

For example the algebras $C(K)$ ($K$ compact) and $L(H)$ ($H$ a Hilbert space) are $B^*$-algebras. Hence so are any closed, self-adjoint subalgebras of such algebras.

**Exercises**

A. Show that a Banach algebra with involution in which $||x^*x|| = ||x||^2$ $(x \in A)$ is a Banach $*$-algebra and so a $B^*$-algebra.

B. Show that a Banach $*$-algebra $A$ in which $||x^*x|| \geq ||x^2||$ $(x \in A)$ is a $B^*$-algebra.

$$\begin{array}{rll} C[0,1] & \text{with the involution} & x^* : t \to \overline{x(1-t)}; \\ H^\infty(U) & \text{with the involution} & x^* : \lambda \to \overline{x(\lambda)}; \\ \ell^1(Z) & \text{with the involution} & x^*(\xi_n^*) \text{ where } \xi_n^* = \overline{x_{-n}}. \end{array}$$

**Exercises**

A. If $(A, \|| \||)$ is a Banach algebra with involution so that $x \to x^*$ is continuous, then there is an equivalent Banach algebra norm $\||| \|||$ on $A$ so that $(A, \||| \|||)$ is a Banach $*$-algebra.

B. Show that if $(A, \|| \||)$ is a commutative, semi-simple Banach algebra with involution, then involution is continuous.

In general the Gelfand-Naimark transformation will not preserve involution. Hence a commutative Banach $*$-algebra is called **symmetric** if the GN-transform is a $*$-algebra morphism.

For example, $C^\infty(S)$, with its standard involution, and $\ell^1(Z)$ are symmetric. $C[0,1]$, with the involution $x^*(t) = \overline{x(1-t)}$, and $H^\infty(U)$ are not.

Once again, it is not difficult to give an invernal characterisation of this concept: Let $A$ be a commutative Banach $*$-algebra with unit. Then the following are equivalent:

1. $A$ is symmetric;

2. for each $x$ in $A$, $(e + x^*x) \in \mathrm{Inv}(A)$;

3. if $h$ is self-adjoint, then $\sigma(h) \subseteq \mathbf{R}$.

**Exercise** The proof of our main result uses one of the central results of approximation theory—the Stone-Weierstraß theorem, which can be stated as follows: let $A$ be a norm-closed subalgebra of $C(K)$ ($K$ compact) so that

1. $A$ separates the points of $K$ (i.e. if $s \neq t$ in $K$ there is an $x \in A$ so that $x(s) \neq x(t)$).

2. $A$ contains the constants.

3. $A$ is self-adjoint (i.e. $x \in A$ implies $\bar{x} \in A$). Then $A = CK$.

Prove this result, using the following steps:

a) use 3. to reduce to the case of real functions;

b) show that $A$ is a lattice i.e. if $x \in A$, then $|x| \in A$ (use the classical result that the function $|t|$ is uniformly approximable by polynomialso on $[-1, 1]$)

c) deduce that ir $x, y \in A$ then so do $\max(x, y)$ and $\min(x, y)$;

d) show that for every $x \in C(K)$, $\epsilon > 0$, there is a $y \in A$ with $y \leq x + \epsilon$ (use 1. and 2.) to find, for each $t \in K$, $ay_t \in A$ so that $y_t(t) < x(t) + \epsilon$. Now use c) and a compactness argument);

e) complete the proof by approximating $x$ from below as in d).

We are now in position to state the main result of this section:

**Proposition 17** *If $A$ is a commutative $B^*$-algebra with identity, then the GN-transform is an isometry from $A$ onto $C(M(A))$.*

PROOF. 1. $x \to \hat{x}$ is an isometry: if $x \in A$, then

$$||x^2||^2 = ||(x^2)^* x^2|| = ||(x^* x)|| = ||x^* x||^2 = ||x||^4$$

and so $||x^2|| = ||x||^2$ – now use 2.27.

■

**Exercises**

A. An element $x$ of a $B^*$-algebra $A$ is **normal** if $x^* x = x x^*$. Show that this is equivalent to the existence of a commutative $B^*$-algebra $A_0$ of $A$ containing $x$.

B. Consider the space $C([0, 1]^2)$ of continuous functions on the square $[0, 1]^2$. Show that this space is a Banach algebra under either of the following two multiplications:

1. $$k_1 \circ k_2 : (s, t) \to \int_s^t k_1(s, u) k_2(u, t) du;$$

2. $$k_1 \circ k_2 : (s, t) \to \int_0^1 k_1(s, u) k_2(u, t) du.$$

Show that both of these algebras are subalgebras of the space $L(C([0, 1])$ of linear operators on $C([0, 1])$ in a natural way.

C. If $A$ is a commutative Banach algebra over $\mathbf{C}$ with unit $e$ then the mapping $\mathrm{Inv} : x \to x^{-1}$ from $\mathrm{Inv}(A)$ into $A$ is analytic. Calculate its derivatives. Do the same for the mapping $\exp : A \to A$.

D. Show that if $x$ is an dement of a Banach algebra $A$ and, for $\lambda \in \mathbf{C} \backslash \sigma(x)$, $d(\lambda)$ is the distance from to $\sigma(x)$ (i.e. $d(\lambda) = \inf\{|\lambda - \mu| : \mu \in \sigma(x)\}$, then

$$||\lambda e - x^{-1}|| \geq 1/d(\lambda).$$

E. Show that the space $BV([0,1])$ of real-valued functions on $[0,1]$ of bounded-variation with $f(0) = 0$ is a Banach algebra under pointwise multiplication and the norm $||f||_{bv} = V(f)$, the variation of $f$.

F. Show that the Gelfand-Naimark transformation ona Banach algebra is an isomorphism onto its range if and ony if there is a $c > 0$ so that $||x^2|| \geq ||x||^2$ for each $x \in A$.

G. Show that the sequence space $\ell^p (1 \leq p \leq \infty)$ is a Banach algebra under pointwise multiplication (i.e. under the multiplication

$$xy = (\xi_n \eta_n) \quad (x = (\xi)n), y = (\eta_n)).$$

H. A linear form on a commutative Banach algebra $A$ is a derivative at $f \in M(A)$ if and only if

$$L(xy) = f(x)L(y) + L(x)f(y)$$

$(x, y \in A)$. Describe all such derivatives for the algebra $C^1([0,1])$.

Show that the defining condition is equivalent to the fact that $L(e) = 0$ and $L(A_f^2) = 0$ where $A_f$ is the kernel of $f$ and deduce that a derivative exists if and only if $A_f^2 \neq A_F$ and that it can be chosen to be continuous if $A_f^2$ is not dense in $A)f$. (Here $A_f^2$ is the ideal generated by the elements of the form $xy$ where $x, y \in A_f$).

I. Let $E$ be a Banach space and define

$$E_0 = \mathbf{C}, E_1 = E, E_2 = E \hat{\oplus} E, E_2 = E \hat{\oplus} E \hat{\oplus} E$$

etc. Show that $A = B \sum_{n=0} E_n$ is a Banach algebra with unit under the natural multiplication (induced by the natural bilinear form of tensor product from $E_m \hat{\oplus} E_n$ into $E_{m+n}$) and that is posseses the following universal property: if $T : E \to A_0$ is a linear contraction from $E$ into a Banach algebra with unit, then there is a unique extension $\tilde{T}$ of $T$ to a unit proserving homomorphism from $A$ into $A_0$.

Show how to modify the construction to obtain a commutative Banach algebra $B$ with an analogous universal property.

## 3 The spectral theorem—abstract approach

We now proceed to give amore abstract form of the spectral theorem, based on the representation theorem for commutative $B^*$-algebras. Every self-adjoint operator generates such a subalgebra of $L(H)$ and this is aturally isomorphic to an algebra $C(K)$. We show how to define a measure $\mu$ on $K$ so that the algebra (and, in particular, the origin operator) is represented by a multiplication operator on $L^2(\mu)$.

in the spirit of this approach we conseder simultaneously sets of operators (which are usually $B^*$-algebras i.e. subsets of $L(H)$ whch are norm-closed self-adjoint subalgebras) rather than individual operators.

The construction of the measure is particularly simple when the algebra of operators has a socalled cyclic vector.

**Definition 4** *Let $A$ be a set of operators in $L(H)$. A vector $x \in H$ is* **cyclic** *for $A$ if $Ax := \{Tx : T \in A\}$ is dense in $H$.*

**Exercise** Characterise those $x \in L^2(\mu)$ which ae cyclic for $L^\infty(\mu)$ (regarded as an algebra of operators on $L^2(\mu)$.

The abstract form of the spectral theorem states that up to unitary equivalence a self-adjoint operator "is" a multiplication operator. The precise definition we require is as follows:

**Definition 5** *If $H_1, H_2$ are Hilbert spaces, $U$ a unitary linear mapping from $H_1$ onto $H_2$, then the mapping*

$$T \to UTU^{-1}$$

*is a $B^*$-isomorphism from $L(H_1)$ onto $L(H_2)$. If $A_1$ (resp. $A_2$) is a $B^*$-algebra of continuous linear operators on $H_1$ (resp. $H_2$), we say that $A_1$ and $A_2$ are* **unitarily equivalent** *(or* **spatially isomorphic***) if the above mapping induces an isomorphism from $A_1$ onto $A_2$.*

We are now in position to state and prove the spectral theorem for a set of operators with a cyclic vector.

**Proposition 18** *Let $A$ be a commutative $B^*$-algebra of operators on a Hilbert space $H$ with a cyclic vector $x$. Then there is a positive Radon measure $\mu$ on $M := M(A)$ so that $A$ is unitarily equivalent to a subalgebra of $L^\infty(\mu)$ (regarded as a $B^*$-algebra of operators on $L^2(\mu)$).*

PROOF. The Gelfand-Naimark transform

$$T \to \hat{T}$$

is an isomorphism from $A$ onto $C(M)$. The mapping

$$\hat{T} \to (Tx|x)$$

is a positive linear form on $C(M)$ and so there is a positive Radon measure $\mu$ on $M$ so that

$$(Tx|x) = \int_M \hat{T} d\mu.$$

If $S, T \in E$, then

$$\begin{aligned}(Sx|Tx) &= (T^*Sx|x) = \int_M (T^*S) d\mu^\wedge \\ &= \int_M \hat{S}(\hat{\bar{T}}) d\mu = (\hat{S}|\hat{T})_{L^2}.\end{aligned}$$

Hence we can define the mapping

$$U : Tx \to \hat{T}$$

from $Ax$ onto $C(M)$. Since this mapping is scalar-product preserving from the dense subset $Ax$ of $H$ onto the dense subspace $C(M)$ of $L^2(\mu)$, we can extend it in a unique manner to a unitary operator from $H$ onto $L^2(\mu)$. If $S, T \in A$, then

$$U(S(Tx)) = (ST)^\wedge = \hat{S}\hat{T} = M_{\hat{S}}(\hat{T})$$

and so $US - M_{\hat{S}}U$ on the dense subspace $Ax$ of $H$. Since these are both continuous mappings, they agree on $H$ and so $USU^{-1} = M_{\hat{S}}$. Hence $A$ is unitarily equivalent to the subalgebra $C(M)$ of $L^\infty(\mu)$.

■

To cover the general case (i.e. where the existence of a cyclic vector is not ensured) we use a standard procedure – we apply a maximality argument to split the Hilbert space into a direct sum of invarian subspaces on which the algebra does have a cyclic vector. To do this we use the results of the following simple exercises:

EXAMPLES. Let $K$ be a closed subspace of $H$. Show that the following are equivalent:

(i) $T$ maps $K$ into $K$;

(ii) $T^*$ maps $K^\perp$ into $K^\perp$;

(iii) $T \circ P_K = P_K T P_K$ ($P_K$ the orthogonal projection onto $K$).

Deduce that $T$ maps $K$ into $K$ and $K^\perp$ into $K^\perp$ if and only if $T$ commutes with $P_K$.

**Lemma 5** *Let $A$ be a $B^*$-algebra of continuous linear operators on the Hilbert space $H$. Then $H$ can be expressed as the direct sum $\oplus_{i\in I} H_i$ so that each $H_i$ is invariant under $A$ and $A|_{H_i}$ has a cyclic vector.*

Proof. This is a simple application of Zorn's Lemma, applied to the collection $\mathcal{P}$ of families of orthogonal $A$-invariant subspaces of $H$ with cyclic vectors, ordered in the natural way.

■

**Proposition 19** *Let $A$ be a $B^*$-algebra of operators on a Hilbert space. Then there is a locally compact space $M$ and a Radon measure $\mu$ on $M$ so that $A$ is unitarily equivalent to a subalgebra of $L^\infty(\mu)$.*

Proof. Let $H = \oplus_{i\in I} H_i$ be a representation of $H$ as in 3.6. We can apply 3.3 to find a unitary $U_i$ mapping $H_i$ onto a Hilbert space of the type $L^2(\mu_i)$ where $M_i$ is the spectrum of the algebra $A|_{H_i}$ and $\mu_i$ is a positive Radon measure on $M_i$ and $U_i$ establishes an equivalence between $A|_{H_i}$ and a subalgebra of $L^\infty(\mu_i)$. If we let $M$ denotes the direct sum of the topological spaces $M_i$ and $\mu$ the sum of the family of measures $\{\mu_i\}$. If we let $M$ denotes the direct sum of the topological spaces $M_i$ and $\mu$ the sum of the family of measures $\{\mu_i\}$, then there is a natural unitary mapping $U$ from the Hilbert sum $\oplus_{i\in I} L^2(\mu_i)$ onto $L^2(\mu)$. Then the composition

$$H = \oplus_{i\in I} H_i \; \oplus \vec{U}_I \; \oplus_{i\in I} L^2(\mu_i) \; \vec{U} \; L^2(\mu)$$

is a unitary mapping from $H$ onto $L^2(\mu)$ which establishes the desired equivalence between $A$ and a subalgebra of $L^\infty(\mu)$ (cf. I.9).

■

**Proposition 20** *Let $S$ be a commuting, self-adjoint set of operators in $L(H)$. Then the set $S$ is unitarily equivalent to a subset of a space $L^\infty(\mu)$ for some measure space $(M, \mu)$.*

Proof. We apply 3.7 to the $B^*$-subalgebra of $L(H)$ generated by $S$.

■

**Corollar 3** *Let $T$ be a normal operator in a Hilbert space $H$. Then $T$ is unitarily equivalent to an operator $M_x$ on a Hilbert space of the form $L^2(\mu)$ (where $x \in L^\infty(\mu)$). If $T$ is self-adjoint, $x$ is real-valued.*

There is one case when the existence of a cyclic vector for the wohole space is automatically satisfied – that of a maximal commutative $B^*$-algebra of operators on a separable Hilbert i.e. a commutative $B^*$-subalgebra of $L(H)$ which is not contained in a larger commutative subalgebra. Note firstly that every self-adjoint set $S$ of commuting operators in $L(H)$ is contained in a maximal commutative $B^*$-algebra of operators on $H$.
PROOF. We let $A_0$ be the closed algebra generated by $S \cup \{I\} - A_0$ is a commutative $B^*$-algebra of operators. Let $\mathcal{P}$ denote the set of all commutative $B^*$-subalgebras of $L(H)$ which contain $A$. We order $\mathcal{P}$ by set-theoretical inclusion. Every chain in $\mathcal{P}$ has an upper bound and se we can apply the Lemma of Zorn to deduce the existence of a maximal element $A$ of $\mathcal{P}$. Then $A$ is a maximal commutative subset of $L(H)$. For if $T \in L(H)$ commutes with all elements of $A$, then so do the hermitian operators $(T + T^*)/2$ and $(T - T^*)/2i$ and so both lie in $A$ – hence $T \in A$.

■

**Proposition 21** *Let $A$ be maximal commutative $B^*$-subalgebra of operators on a separable Hilbert space $H$. Then $A$ possesses a cyclic vector. hence $E$ is unitarily equivalent to an algebra $L^\infty(\mu)$ of multiplication operators on $L^2(\mu)$ for some Radon measure on its spectrum.*

PROOF. We split $H$ into the Hilbert direct sum

$$H = \oplus H_\alpha$$

as in 3.6. Since $H$ is separable there are at most countably many $H_\alpha$ so we can write the decomposition in the form

$$H = \oplus H_n.$$

If $x_n$ is a cyclic vector for $A|_{H_n}$ with $||x_n|| = 1$. Then

$$x = \sum \frac{x_n}{n}$$

is in $H$. Then since $x_n = nP_n(x)$ where $P_n$ is the orthogonal projection onto $H$ and $P_n \in A$ by the meaximalities it is not difficult to check that $x$ is a cyclic vector for $A$.

■

We now show how to recover the classical spectral theorem from this result. We take the opportunity to express it in a more modern form (cf. the formulation of 1.5):

**Definition 6** *A* **spectral measure** *on a Hilbert space $H$ is a mapping $\mathcal{E}$ from the $\sigma$-algebra of Borel measurable subsets of* $\mathbf{C}$ *into $L(H)$ so that*

a) *$\mathcal{E}$ is a finitely additive, multiplicative measure with values in $L(H)$ i.e. $\mathcal{E}(\mathcal{A} \cup \mathcal{B}) = \mathcal{E}(\mathcal{B})$ if $A \circ B = \phi$ and $\mathcal{E}(\mathcal{A} \cup \mathcal{B}) = \mathcal{E}(\mathcal{A})\mathcal{E}(\mathcal{B})$ for Borel sets $A, B$;*

b) *$\mathcal{E}(\mathcal{A})$ is an orthogonal projection for each Borel set $A$;*

c) *$\mathcal{E}$ is strongly $\sigma$-additive i.e. for each $y \in H$, the measure*

$$\mathcal{E}_{\mathcal{A}}(\dagger) := \mathcal{A} \to \mathcal{E}(\mathcal{A})(\dagger)$$

*with values in $H$ is $\sigma$-additive. (I.e. if $A$ is the disjoint union of the measurable sets $A_n$ then $\mathcal{E}(\cup \mathcal{A}_\backslash)\dagger = \sum \mathcal{E}(\mathcal{A}_\backslash)(\mathcal{Y})$).*

For example, let $T$ be a normal operator in $L(H)$, $U$ a unitary mapping from $H$ onto $L^2(\mu)$ so that

$$T = UM_xU^{-1} \quad (x \in L^\infty(\mu)).$$

If $A$ is a Borel subset of the plane, we define

$$\begin{aligned} C_A &:= \{x \in A\} \\ \tilde{}E(A) &:= M_{\chi_{C_A}}, \mathcal{E}(\mathcal{A}) := \mathcal{U}\tilde{\mathcal{E}}(\mathcal{A})\mathcal{U}^{-\infty}. \end{aligned}$$

Then $\tilde{}E$ is a spectral measure in $L^2(\mu)$ and if $y \in L^2(\mu)$, then a simple calculation shows that

$$M_x(y) = \int_C i d\tilde{E}(y)$$

where $i$ is the identity function $t \to t$ in $\mathbf{C}$.

Since all these properties are invariant under unitary equivalence, we deduce the existence of a spectral measure $\mathcal{E}$ in $H$ so that

$$Ty = \int \mathrm{Id} d\mathcal{E}(\dagger).$$

We note further that $\mathcal{E}$ has its support in $\sigma(T)$. In particular, if $T$ is hermitian, $\mathcal{E}$ is supported by the real axis. (By definition, $\mathcal{E}$ has its support in a closed subset $C$ if $\mathcal{E}(\mathcal{U}) = \prime$ whenever $U$ is an open set disjoint from $C$). We can summarise our result as follows:

**Proposition 22** *Let $T$ is an operator $M_y$ ($y \in L^\infty(\mu)$), we put*

$$x(M_y) := M_{x \circ y}.$$

*Then this mapping satisfies the above properties. For general $T$, we choose a unitary $U$ from $H$ onto $L^2(\mu)$ so that $T = UM_yU^{-1}$ and define $x(T) := UM_{x\circ y}U^{-1}$.*

This concrete representation as a multiplication operator can be used to give easy proofs of some less trivial facts about operators – this fact is expsoited in the following exercises.

**Exercises**

A. Use 3.11 to prove the following statement: A normal operator $T$ in $L(H)$is hermitian if and only if $\sigma(T) \subseteq \mathbf{R}$;

   positive if and only if $\sigma(T) \subseteq \mathbf{R}_+$;

   unitary if and only if $\sigma(T) \subseteq \{\lambda \in C : |\lambda| = 1\}$.

B. Show that if $T$ is a normal operator in a Hilbert space $T$ can be represented in the form $T = UP$ where $U$ is unitary, $P$ is positive and $U, P$ commute with each other and with all operators in $L(H)$ which commute with $T$.

C. Let $T$ be an operator in $L(H)$. We define $\sigma_p(T)$ (

   **the point spectrum of T**) to be the set of those $\lambda$ in $\mathbf{C}$ so that $\lambda I - T$ is not injective;

   $\sigma_c(T)$ (**the continuous spectrum**) to be the set of those $\lambda$ in $C$ so that $\lambda I - T$ is injective and $(\lambda I - T)H$ is dense in $H$ but not equal to $H$;

   $\sigma_r(T)$ (**the residual spectrum**) to be the set of those $\lambda$ in $\mathbf{C}$ so that $\lambda I - T$ is injective and $(\lambda I - T)H$ is dense in $H$ but not equal to $H$.

   Show that $\sigma(T)$ is the disjoint union of $\sigma_p(T), \sigma_c(T)$ and $\sigma_r(T)$. If $T$ is an operator of the form $M_x$ ($x \in L^\infty(\mu)$) on $L^2(\mu)$, show that

   > $\sigma_p(T)$ if and only if $\{t \in M : x(t) = \lambda\}$ is of positive measure;
   > $\sigma_c(T)$ if and only if $\{t \in M : x(t) = \lambda\}$ has zero measure but for each $\epsilon > 0$, $\{t \in M : x(t) = \lambda\}$ has zero measure but for each $\epsilon > 0$, $\{t \in M : |x(t) - \lambda| \leq \epsilon\}$ has positive measure;
   > $\sigma_r(T)$ is empty.

Deduce that if $T$ is a normal operator

$\sigma_p(T) = \{\lambda \in C : \mathcal{E}\{\lambda\} \neq 0\}$ ($\mathcal{E}$ as in 3.13)
$\sigma_r(T)$ is empty.

Show that if $T$ is a normal operator in $H$ and $\sigma(T)$ is countable, then $H$ has an orthonormal basis consisting of eigenvectors of $T$.

D. Let $(x_n)$ be a sequence in $L^\infty(\mu)$ ($\mu$ is positive bounded). Show that

$M_{x_n} \vec{w} M_x (x \in L^\infty(\mu))$ if and only if $\{x_n\}$ is uniformly bounded and $\int_A (x_n - x) d\mu \to 0$ for each measurable set $A$;

$M_{x_n} \vec{s} M_x (x \in L^\infty(\mu))$ if and only if $\{x_n\}$ is bounded in $L^\infty(\mu)$ and $\int_\Omega |x_n - x| d\mu$ converges to zero.

E. Let $T$ be a self-adjoint operator so that $T^m = T^n$ for some $m \neq n$ in $\mathbf{N}$. Show that $T$ is an orthogonal projection.

F. If $A \in L(H)$ show that $A$ is a projection if and only if $(\mathrm{Id} - 2A)$ is self-adjoint and unitary (what is the geometrical meaning of $(\mathrm{Id} - 2A)$).

G. If $A \in L(H)$, put $\lambda(A) = \inf\{(Ax|x) : ||x|| = 1\}$. Show

1. if $A$ is hermitian, $\lambda(A) > 0$ implies that $A$ is invertible;
2. if $T \in L(H)$, $T$ is invertible if and only if $\lambda(T^*T) > 0$ and $\lambda(TT^*) > 0$.

H. Show that if $f$ is a continuous function from $\mathbf{R}$ into $\mathbf{C}$, then the induced mapping

$$T \to x(T)$$

is continuous from $L(H)_{sa}$, the set of self-adjoint operators, into $L(H)$ in the sense that if $T_n \vec{s} T$ then $x(T_n) \vec{s} x(T)$.

I. Let $T \in L(H)$ be normal. Show that $T$ is compact if and only if $T^r$ is for some $r \in \mathbf{N}$.

J. If $T \in L(H)$ is normal, then

$$\begin{aligned} \mathrm{Ker} T &= T(H)^\perp \\ \overline{T(H)} &= (\mathrm{Ker} T)^\perp \\ \overline{T(H)} &= \overline{T^* T(H)}. \end{aligned}$$

K. Show that every $T \in L(H)$ is a linear combination of unitary operators.

L. Let $E$ be a subset of $L(H)$. A vector $x_0 \in H$ is separating for $E$ if whenever $T \in L(H)$ is such that $Tx_0 = 0$, then $T = 0$.

Show

a) that if $x$ is cyclic and $E$ is commutative then $x$ is separating;

b) if $E$ is a maximal commutative $B^*$-algebra and $x$ is separating, then $x$ is cyclic.

# 4 The spectral theory – unbounded operators

We now turn our attention to under operators on Hilbert space. These play a central role in the formaism of quantum theory. Typical examples are the classical differential operators. Since the unbounded operators which we shall consider have closed graphs, they cannot be defined on the whole of the Hilbert space (otherwise they would be continuous by the closed graph theorem - cf. I.4). This fact leads to complicatons in their treatment and the whole subject is much more delicate that that of bounded operators. We shall confine ourselves to proving the spectral theorem for such operators in the form that they can be represented as multiplication operators (by unbounded functions, of course) on $L^2$-places. The method of proof is rather natural. We show that an unbounded self-adjoint operators is representable in the form $ST^{-1}$ where $S$ and $T$ (but not necessarily $T^{-1}$) are bounded, self-adjoint operators with commute. We then represent $S$ and $T$ simultaneously by multiplication operators and this leads to the required representation for the original operator.

**Definition 7** *A* **partial linear operator** *(abbreviated p.l.o.) on a Hilbert space $H$ is a pair $(D(T), T$ where $D(T)$ is a subspace of $H$ and $T$ is a linear mapping from $D(T)$ into $H$. We shall often write simply $T$ for a p.l.o., its domain of definition $D(T)$ being understood. A p.l.o. $(D(T)T)$ is*

**densely defined** *if $D(T)$ is dense in $H$;*

**closed** *if its graph $\Gamma(T) := \{(x, Tx) : x \in D(T)\}$ is a closed subspace of $H$.*

**Exercises**

A. Show that the partial operator $T$ is closed if and only if the following condition is satisfied: If $(x_n)$ is a sequence in $D(T)$ so that $x_n \to x$ in $H$ and $Tx_n \to y$ in $H$, then $x \in D(T)$ and $Tx = y$.

B. Show that a p.l.o. $T$ is continuous if and only if $D(T)$ is closed in $H$.

C. Show that if $T$ is a p.l.o. in a Hilbert space $H$, then the mapping

$$(x, y) \to (x|y)_{D(T)} := (x|y) + (Tx|Ty)$$

is a scalar product on $D(T)$ and that $D(T)$, with this scalar product, is a Hilbert space if and only if $T$ is closed. Deduce that if $T$ is a closed p.l.o. in a Hilbert space $H$, then there exists a Hilbert space $H_1$ and a continuous injection $j$ from $H_1$ into $H$ that $T \circ j$ is continuous.

If $S, T$ are p.l.o.'s in $H$, we define $S+T$ to be the p.l.o. with $D(S+T) := D(S) \cap D(T)$ and

$$S + T : D(S) \cap D(T) \ni x \to Sx + Tx$$

$\lambda S$ ($\lambda$ a scalar) is the p.l.o. so that $D(\lambda S) = D(S)$ and

$$\lambda S : D(S) \ni x \to \lambda Sx.$$

$S$ is the p.l.o. so that $D(ST) := \{x \in D(T) : Tx \in D(S)\}$ and

$$ST : D(ST) \ni x \to S(Tx).$$

$T^{-1}$ (if $T$ is injective) is the p.l.o. with $D(T^{-1}) = T(D(T))$ and $T^{-1}y = x$ if and only if $y = Tx$ ($y \in D(T^{-1})$).

If $S, T$ are p.l.o.'s in $H$ we say that $T$ is an **extension** of $S$ (written $S \subseteq T$) if $D(S) \subseteq D(T)$ and $Sx = Tx$ for each $x \in D(S)$.

If $T$ is a densely defined, closed p.l.o. in $H$, we define $T^*$, the **adjoint** of $T$, as follows:

$$D(T^*) := \{y \in H \text{ for some } y' \in H, (Tx|y) = (x|y')(x \in H)$$

and if $y \in D(T^*), T^*y$ is defined to be the $y'$ of this equation (this is unique since $D(T)$ is dense).

Owing to the complication introduced by the fact that p.l.o.'s are not defined everywhere, care must be taken when manipulation sums and products of such operators as the following list shows.

**Proposition 23** *Let $S, T, U$ be p.l.o.'s in $H$¿ Then*

*(i)* $S + T = T + S$;

*(ii)* $(S + T)U = S + (T + U)$;

*(iii)* $OS \subseteq S$;

*(iv)* $S(TU) = (ST)U$;

*(v)* $(S+T)U = SU + TU$;

*(vi)* $S(T+U) \subset ST + SU$;

*(vii)* $(ST)^{-1} = T^{-1}S^{-1}$ *when* $S$ *and* $T$ *are injective;*

*(viii)* *if* $S \subseteq T$ *and* $S$ *and* $T$ *are closed and densely defined, then* $T^* \subset S^*$;

*(ix)* $(\lambda T)^* = \lambda T^*$ $(\lambda \neq 0)$ *if* $T$ *is closed an densely defined;*

*(x)* $(S+T)^* \subset S^* + T^*$ *if* $S, T$, $S+T$ *are closed and densely defined;*

*(xi)* $(ST)^* \subset T^*S^*$ *if* $S, T$, $ST$ *are closed and densely defined.*

There is equality in (x) and (xi) if $S$ is in $L(H)$.

The proofs are mostly very simple. Those which are not are dealt with in the next exercises.

EXAMPLES. If $S, T$ are p.l.o.'s, $\lambda \in \mathbf{C}$, show that:

1. $\Gamma(\lambda S) = U_\lambda \Gamma(S)$ where $U_\lambda$ is the linear mapping
$$(x, y) \to (x, \lambda y)$$
from $H \times H$ into $H \times H$;

2. $\Gamma(ST) = \Gamma(S) \circ \Gamma(T)$ where if $A, B$ are subsets of the product space $H \times H$, $A \circ B$ is defined as follows:
$$A \circ B := \{(x, y) : (x, y) \in B \text{ and } (y, z) \in A \text{ for some } y \in H\}$$

3. if $T$ is closed and densely defined, $\Gamma(T^*) = U(\Gamma(T)^\perp) = (U\Gamma(T))^\perp$ where $U$ is the unitary mapping
$$(x, y) \to (-y, x)$$
from $H \times H$ into $H \times H$;

4. if $T$ is injective, then $\Gamma(T^{-1}) = V\Gamma(T)$ where $V$ is the unitary mapping
$$(x, y) \to (y, x)$$
from $H \times H$ onto $H \times H$.

Use the results to prove 4.3.

**Proposition 24** *If* $T$ *is densely defined and closed, then so is* $T^*$ *and* $T = (T^{()})^*$.

PROOF. $T^*$ is closed since $\Gamma(T^*) = (U\Gamma(T))^\perp$ and this is a closed subspace of $H \times H$. Suppose that $x_0 \in (D(T^*))^\perp$. Then if $x \in D(T^*)$,

$$O = (x|x_0) = ((-T^*x, x)|(O, x_0))$$

and so

$$\begin{aligned}(O, x_0) \in (U\Gamma(T^*))^\perp &= (UU(\Gamma(T)^\perp))^\perp \\ &= \Gamma(T)^{\perp\perp} \text{ since } UU = -I \\ &= \Gamma(T) \text{ since } \Gamma(T) \text{ is closed.}\end{aligned}$$

Thus $x_0 = TO = O$.

Similarly, $\Gamma(T^{**}) = (\Gamma(T))^{\perp\perp} = \Gamma(T)$ and so $T = T^{**}$.

The typical example of a p.l.o. is a follows: Let $(\Omega; \mu)$ be a measure space, $x$ a complex-valued measurable funciton on $\Omega$. Consider the p.l.o. $M_x$ defined as follows:

$$D(M_x) = \{y \in L^2(\mu) : xy \in L^2(\mu(\}$$

and

$$M_x : y \to xy.$$

$D(M_x)$ is dense in $H$ for if

$$A_n := \{t \in \Omega : |x(t)| \leq n\}$$

then $\chi_{A_n} y \in D(M_x)$ for all $y$ in $L^2(\mu)$ and $\chi_{A_n} y \to y$ in $L^2(\mu)$ as $n$ tends to infinity.

$M_x$ is closed for if $(y_n)$ is a sequence in $D(M_x)$ so that $y_n \to y$ and $xy_n \to y'$ in $L^2(\mu)$, then we can choose a subsequence $(y_{n_k})$ so that

$$y_{n_k} \to y \text{ and } xy_{n_k} \to y' \text{ almost everywhere}$$

and so $y' = xy$.

Hence $M_x$ is closed and densely defined. We show that $M_x^* = M_{\bar{x}}$.

If $y_1 \in D(M_x)$, $y_2 \in D(M_{\bar{x}}) = D(M_x)$, then

$$\begin{aligned}(M_x y_1 | y_2) &= \int_M x y_1 \bar{y}_2 d\mu = \int y_1 (\bar{x} y_2)^= d\mu \\ &= (y_1 | M_{\bar{x}} y_2)\end{aligned}$$

and so $M_{\bar{x}} \subseteq M_x^*$.

On the other hand, suppose that $y_2 \in D(M_x^*)$. Then there is a $y_2'$ so that

$$\int_M x y_1 \bar{y}_2 d\mu = \int_M y_1 \bar{y}_2' d\mu$$

for each $y_1 \in D(M_x)$. Hence, for any $y_1$ in $L^2(\mu)$

$$\int_M x \chi_{A_n} y_1 \bar{y}_2 d\mu = \int_M \chi_{A_n} y_1 \bar{y}_2' d\mu$$

that is,

$$\int_{A_n} x y_1 \bar{y}_2 d\mu = \int_{A_n} y_1 \bar{y}_2' d\mu.$$

Thus $(I + T^*T)$ is injective and since $(I + T^*T)B = I$, $(I + T^*T)$ is surjective. Hence $B = (I + T^*T)^{-1}$. If $x, y \in H$,

$$\begin{aligned} (Bx|y) &= (Bx|(I + T^*T)By) = (Bx|By) + (Bx|T^*TBy) \\ &= (Bx|By) + (T^*TBx|By) = (x|By) \end{aligned}$$

and so $B$ is hermitian. It is positive since if $x \in H$,

$$(Bx|x) = (Bx|(T + T^*T)Bx) = (Bx|Bx) + (TBx|TBx) \geq 0.$$

■

**Definition 8** *Let $T$ be a densely defined, closed p.l.o. ina Hilbert space. Then $T$ is*

> **self-adjoint** *if $T = T^*$ '*
>
> **normal** *if $TT^* = T^*T$.*

*It follows from the example after 4.5 that $M_x$ is normal and that it is self-adjoint if and only if $x$ is real-valued. Note also that $(I + B_x^* M_x)^{-1}$ and $M_x(I + M_x^* M_x)^{-1}$ are the operators of multiplication by the functions $(1 + |x|^2)^{-1}$ and $x(1 + |x|^2)^{-1}$.*

**Exercises**

A. Show that $T$ is self-adjoint (resp. normal) if and only if $C(= T(I + T^*T)^{-1})$ is.

B. Show that if $T$ is a self-adjoint p.l.o., $B$ hermitian in $L(H)$, then $T + B$ is self-adjoint. Show that if $T$ is a injective hermitian operator in $L(H)$, then $T^{-1}$ is self-adjoint.

Thus $x\bar{y}_2 = y_2'$ almost everywhere on each $A_n$ and so $\bar{x}y_2 = y_2'$ almost everywhere on $M$, that is,$y_2 \in D(M_{\bar{x}})$ and $M_{\bar{x}}y_2 = y_2' = M_x^* y_2$.

**Proposition 25** *Let $T$ be a closed, densely defined p.l.o. in $H$. Then$I+T^*T$ is injective and the operators*

$$B := (I + T^*T)^{-1}; \quad C := T(I + T^*T)^{-1}$$

*are in $L(H)$ and $||B|| \leq 1$, $||C|| \leq 1$. In addition, $B$ is positive.*

Proof. By 4.4, $H \times H = \Gamma(T) \oplus U\Gamma(T^*)$. Hence if $z \in H$, $z, 0)$ has a unique representation

$$(z, 0) = (x, Tx) + (T^*y, -y)(x \in D(T), y \in D(T^*))$$

i.e. the equation

$$x + T^*y = z; \quad Tx - y = 0$$

have unique solutions $x \in D(T)$, $y \in S(T^*)$.

We write $x =: Bz$, $y =: CzB, C$ are linear operators defined on $H$ and

$$C = TB, I = (I + T^*T)B$$

(for $(I + T^*T)x = x + T^*y = z$).

Now $||x||^2 = ||x||^2 + ||Tx||^2 + ||T^*y||^2 + ||y||^2$ and so $||Bz||^2 + ||Cz||^2 \leq ||z||^2$. Hence $||B|| \leq 1$ and $||C|| \leq 1$. If $x \in D(I + T^*T)$,

$$((I + T^*T)x|x) = (x|x) + (T^*Tx|x) = ||x||^2 + ||Tx||^2 \geq ||x||^2.$$

■

**Proposition 26** *If $T$ is a densely defined, closed p.l.o., then $T^*T$ is self-adjoint.*

Proof. By 4.5 $(I + T^*T)^{-1}$ is hermitian and so $I + T^*T$ and hence and $T^*T$ are self-adjoint (4.8.B).

■

**Exercises**

A. Show that if $T_1, T_2$ are self-adjoint, $T_1 \subseteq T_2$, then $T_1 = T_2$.

B. Show that a closed, densely defined p.l.o. $T$ is normal if and only if the following condititons are satisfied:

$$D(T) = D(T^*) \text{ and } ||Tx|| \quad (x \in D(T)).$$

We can now state and prove the spectral theorem for unbounded operators.

**Proposition 27** *Let $T$ be a normal p.l.o. in $H$. Then there exists a unitary mapping from $H$ onto a space $L^2(\mu)$ and a measurable function $x$ on $\Omega$ so that*

$$T = UM_xU^{-1}$$

*$T$ is self-adjoint if and only if $x$ is real-valued.*

PROOF. We apply 3.8 to the operators $C(= T(I + T^*T)^{-1})$ and $B(= I + T^*T)^{-1})$. Then $\{C, B\}$ is unitarily equivalent to a pair $\{M_y, M - z\}$ on $L^2(\mu)$ where $z \geq 0$ and

$$A := \{t \in M : z(t) = 0\}$$

is negligible. We define the function $x$ on $M$ as follows:

$$x(t) = \begin{cases} 0 & \text{if} \quad t \in A, \\ y(t)/z(t) & \text{if} \quad t \notin A. \end{cases}$$

Then $x$ is measurable and if $U$ is the unitary mapping from $H$ onto $L^2(\mu)$ establishing the equivalence of $\{C, B\}$ to $\{M_y, M_z\}$

$$UTU^{-1} = UCB^{-1}U^{-1} = UCU^{-1}UB^{-1} = M_y(M_z)^{-1} = M_x.$$

■

**Exercises**

A. Show that a p.l.o. $T$ is normal if and only if it has a representation $T = UT_1$ where $U$ is unitary, $T_1$ is self-adjoint and $U$ and $T_1$ commute.

B. Show that if $T$ is a self-adjoint p.l.o. and $\lambda \in \mathbf{C} \setminus \mathbf{R}$, then $(\lambda I - T)^{-1}$ exists and is in $L(H)$, with

$$||(\lambda I - T)^{-1}|| \leq |\Im\lambda|^{-1}.$$

C. Let $T$ be a self-adjoint p.l.o. on $H$. Show that thee is a sequience $(H_n)$ of closed subspaces of $H$ so that

a) $H = \oplus_{n=1}^{\infty} H)1;$

b) $P_{H_n}$ commutes with $T$ for each $n$ and

$$T_n = P_{H_n} \circ T|_{H_n} \text{ is bounded;}$$

c) $D(T) = \{x \in H : \sum_n ||T_n P_{H_n} x||^2 < \infty\}$ and then

$$Tx = \sum_n T_n P_{H_n} x \quad (x \in D(T)).$$

D. A self-adjoint p.l.o. $T$ on $H$ is **bounded from below** if there is an $m > 0$ so that

$$(Tx|x) > m(x|x) \quad (xD(T)).$$

Show that then $T^{-1}$ exists and is bounded.