Functional analysis—locally convex spaces

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Contents

1 Locally convex spaces

We shall now develop the generalisation of the concept of Banach spaces which is relevant as a framework for the topological structure of spaces of test functions and distributions. Typically the former are spaces of functions which are submitted to an infinite number of conditions, usually of growth and regularity. The appropriate concept is that of a locally convex topology i.e. one which is defined by a family of seminorms rather than by a single norm.

Recall that a **seminorm** on a vector space E is a mapping p from E into the non-negative reals so that

$$p(x + y) \leq p(x) + p(y) \quad p(\lambda x) = |\lambda| p(x)$$

for each x, y in E and λ in $\mathbf{R}$.

We shall use letters such as p, q to denote seminorms. The family of all seminorms on E is ordered in the natural way i.e. $p \leq q$ if $p(x) \leq q(x)$ for each x in E. If p is a seminorm,

$$U_p = \{x \in E : p(x) \leq 1\}$$

is the **closed unit ball** of p. This is an absolutely convex, absorbing subset of E, whereby a subset A of E is **convex** if for each $x, y \in A$ and $t \in [0, 1], tx + (1 - t)y \in A$

balanced if $\lambda A \subset A$ for λ in $\mathbf{R}$ with $|\lambda| \leq 1$

absolutely convex if it is convex and balanced;

absorbing if for each x in E there is a positive ρ so that $\lambda x \in A$ for each λ with $|\lambda| < \rho$. U_p is in addition algebraically closed i.e. its intersection with each one-dimensional subspace of E is closed, whereby these subspaces carry the natural topologies as copies of the line. On the other hand, if$U$ is an absolutely convex, absorbing subset of E, then its **Minkowski functional** i.e. the mapping

$$p_U : x \mapsto \inf\{\lambda > 0 : x \in \lambda U\}$$

is a seminorm on E. In fact, the mapping $p \mapsto U_p$ is a one-one correspondence between the set of seminorms on E and the set of absolutely convex, absorbing, algebraically closed subsets of E. Also $p \leq q$ if and only if $U_p \subset U_q$.

A family S of seminorms on E is **irreducible** if the following conditions are verified:

a) if $p \in S$ and $\lambda \geq 0$, then $\lambda p \in S$;

b) if $p \in S$ and q is a seminorm on E with $q \leq p$, then $q \in S$;

c) if p_1 and p_2 are in S, then so is $\max(p_1, p_2)$;

d) S separates E i.e. if x is a non-zero element of E, then there is a $p \in S$ with $p(x) \neq 0$.

If S is a family of seminorms which satisfies only d), then there is a smallest irreducible family of seminorms which contains S. It is called the **irreducible hull** of S and denoted by $\tilde{S}$. ($\tilde{S}$ is the intersection of all irreducible families containing S – alternatively it consists of those seminorms which are majorised by one of the form

$$\max(\lambda_1 p_1, \ldots, \lambda_n p_n)$$

where the λ_i's are positive scalars and the p_i's are in S).

A **locally convex space** is a pair (E, S) where E is a vector space and S is an irreducible family of seminorms on the former. If S is a family of seminorms which separates E, then the space $(E, \tilde{S})$ is **the locally convex space generated by** S.

If (E, S) is a locally convex space, we define a topology τ_S on E as follows: a set U is said to be a **neighbourhood** of a in E if $(U - a)$ contains the unit ball of some seminorm in S. The corresponding topology is called the **topology associated with** S. This topology is Hausdorff (since S separates E) and, in fact, completely regular, since it is generated by the uniformity which is defined by the semimetrics $(d_p : p \in S)$ where $d_p(x, y) = p(x - y)$. Hence we can talk of convergence of sequences or nets, continuity and completeness in the context of locally convex spaces.

Recall that a **topological vector space** is a vector space E together with a Hausdorff topology so that the operations of addition and multiplication by scalars are continuous. It is easy to see that a locally convex space is a topological vector space. On the other hand, locally convex spaces are often defined as topological vector spaces in which the set of absolutely convex neighbourhoods of zero forms a neighbourhood basis. This is equivalent to our definition. For if (E, S) is a locally convex space as defined above, then (E, τ_S) is a topological vector space and the family $\{U_p : p \in S\}$ is a basis of convex neighbourhoods of zero. On the other hand, if (E, τ) is a topological vector space satisfying the convexity condition, then the set of all τ-continuous seminorms on E is irreducible and the corresponding topology τ_S coincides with τ.

We remark that if a family S of seminorms on the vector space E satisfies conditions a) - c) above, but not necessarily d) and we define

$$N_S = \{x \in E : p(x) = 0, p \in S\}$$

then we can define a natural locally convex structure on the quotient space E/N_S. This is a convenient method for dealing with non-Hausdorff spaces which sometimes arise.

Examples: I. Of course, normed spaces are examples of locally convex space, where we use the single norm to generate a locally convex structure.
II. If E is a normed space and F is a separating subspace of its dual E', then the latter induces a family S of seminorms, namely those of the form $p_x : x \mapsto |f(x)|$ for $f \in F$. This induces a locally convex structure on E which we denote by $S_w(F)$. The corresponding topology $\sigma(E, F)$ is called **the weak topology induced by F.** The important cases are where $F = E'$, respectively where E is the dual G' of a normed space and F is G (regarded as a subspace of $E' = G''$).
III (the fine locally convex structure): If E is a vector space, then the set of **all** seminorms on E defines a locally convex structure on E which we call the **fine structure** for obvious reasons. IV. The space of continuous functions: If S is a completely regular space, we denote by $\mathcal{K}(\mathcal{S})$ or simply by $\mathcal{K}$, the family of all compact subsets of S. If $K \in \mathcal{K}$, then

$$p_K(x) = \sup\{|x(t)| : t \in K\}$$

is a seminorm on $C(S)$, the space of continuous functions from S into $\mathbf{R}$. The family of all such seminorms defines a locally convex structure $S_{\mathcal{K}}$ on $C(S)$ – the corresponding topology is that of compact convergence i.e. uniform convergence on the compacta of S.
V. Differentiable functions. If k is a positive integer, $C^k(\mathbf{R})$ denotes the family of all k-times continuously differentiable functions on $\mathbf{R}$. For each $r \leq k$ and K in $\mathcal{K}(\mathbf{R})$, the mapping

$$p_K^r : x \mapsto \sup\{|x^{(r)}(t)| : t \in K\}$$

is a seminorm on $C^k(\mathbf{R})$. The family of all such seminorms defines a locally convex structure on $C^k(\mathbf{R})$.
VI. Spaces of operators: Let H be a Hilbert space. On the operator space $L(H)$, we consider the following seminorms:

$$p_x : T \mapsto \|Tx\|$$

$$p_x^* : T \mapsto \|T^*x\|$$

$$p_{x,y} : T \mapsto |(Tx|y)|$$

for x and y in H. The family of all seminorms of the first type define the bf strong locally convex structure on $L(H)$, while those of the first two type define the **strong *-structure**. Finally, those of the third type define the **weak operator structure**.
IV. Dual pairs. We have seen that the duality between a normed space and its dual can be used to define weak topologies on E and E'. For our purposes, a

more symmetrical framework for such duality is desirable. Hence we consider two vector spaces E and F, together with a bilinear form $(x, y) \mapsto \langle x, y\rangle$ from $E \times F$ into $\mathbf{R}$, which is **separating** i.e. such that

- if $y \in F$ is such that $\langle x, y\rangle = 0$ for each x in E, then $y = 0$;;
- if $x \in E$ is such that $\langle x, y\rangle = 0$ for each y in F, then $x = 0$.

Then we can regard F as a subspace of E^*, the algebraic dual of E, by associating to each y in F the linear functional

$$x \mapsto \langle x, y\rangle.$$

Similarly, E can be regarded as a subspace of F^*. (E, F) is then said to be a **dual pair**. The typical example is that of a normed space, together with its dual or, more generally, a subspace of its dual which separates E. For each $y \in F$, the mapping $p_y : x \mapsto |\langle x, y\rangle|$ is a seminorm on E and the family of all such seminorms generates a locally convex structure which we denote by $S_w(F)$ – the **weak structure generated by** F.

A subset B of F is said to be **bounded** for the duality if for each x in E,

$$\sup\{|\langle x, y\rangle| : y \in B\} < \infty.$$

In this case, the mapping

$$p_B : x \mapsto \sup\{p_y(x) : y \in B\}$$

is a seminorm on E. Let $\mathcal{B}$ denote a family of bounded subsets of F whose union is the whole of F. Then the family $\{p_B : B \in \mathcal{B}\}$ generates a locally convex structure $S_{\mathcal{B}}$ on E, that of **uniform convergence** on the subsets of $\mathcal{B}$.

Thus if $\mathcal{B}$ consists of the singletons of F, we rediscover the weak structure. If $\mathcal{B}$ is taken to be the family of those absolutely convex subsets of F which are compact for the topology defined by $S_w(E)$ on F, then $S_{\mathcal{B}}$ is called the **Mackey structure** and the corresponding topology (which is denoted by $\tau(E, F)$) is called the **Mackey topology**. Finally, if we take for $\mathcal{B}$ the family of all bounded subsets of F, then we have the **strong structure**–the corresponding topology is called the **strong topology**.

A rich source of dual pairs is provided by the so-called **sequence spaces**. These are, by definition, subspaces of the space $\omega = \mathbf{R}^{\mathbf{N}}$ i.e. the family of all real-valued sequences which contain ϕ, the spaces of those sequences with finite support (i.e. $\phi = \{x \in \omega : \xi_n = 0$ except for finitely many $n\}$.

ϕ and ω are regarded as locally convex spaces, ω with the structure defined by the seminorms $p_n : x \mapsto |\xi_n|$ and ϕ with the fine structure.

Further examples of sequence spaces are the ℓ^p-spaces.

If E is a sequence space, we define its **α-dual** E^α as follows:

$$E^\alpha = \{y = (\eta_n) : \sum |\xi_n \eta_n| < \infty \text{ for each } x \in E\}.$$

Then (E, E^α) is a dual pair under the bilinear form

$$\langle x, y \rangle = \sum_{n=1}^{\infty} \xi_n \eta_n.$$

Of course, if $E \subset F$, then $E^\alpha \supset F^\alpha$. Also E is clearly a subspace of $(E^\alpha)^\alpha$). A sequence space E is **perfect** if $E = E\alpha\alpha$. Thus

$$\emptyset_\alpha = \phi \quad \phi^\alpha = \emptyset \quad (\ell^p)^\alpha = \ell^q$$

the latter for *all* values of p and q. hence all of these spaces are perfect.

As in the case of normed spaces, we are interested in linear mappings between locally convex spaces which preserve their topological structures. Corresponding to the fact that boundedness and continuity are equivalent for linear mappings between normed spaces, we have the following equivalences: for a linear mapping $T : E \to F$ whereby (E, S) and (F, S_1) are locally convex spaces, the following are equivalent:

- T is $\tau_S - \tau_{S_1}$-continuous;
- T is $\tau_S - \tau_{S_1}$-continuous at zero;
- for each $p \in S_1$, $p \circ T \in S$;
- for each $p \in S_1$, there are finite sequences $q_1, \dots, q_n$ in S and $l_1, \dots, l_n$ of positive numbers so that

$$p \circ T \leq l_1 q_1 + \cdots + l_n q_n.$$

We remark that the last characterisation is also valid (i.e. equivalent to the continuity of T) in the case were S and S_1 are merely separating families of seminorms which generate the corresponding locally convex structures.

The most important examples of such mappings are differential operators i.e. mappings of the form

$$L : x \mapsto \sum_{i=0}^{n} a_i x^{(i)}$$

where a_0, $\dots$,a_n are smooth functions, say on $\mathbf{R}$. This operator can be regarded as a continuous linear mapping from $C^k(\mathbf{R})$ into $C^{k-n}(\mathbf{R})$.

2 Constructions on locally convex spaces:

Subspaces and quotients: If E is a locally convex space and F is a subspace, then F can be regarded as a locally convex spaces in its own right, simply by using the restrictions of the seminorms on E. For example, if U is an open subset of the complex plane, we can consider $H(U)$, the space of holomorphic functions on U, as a locally convex space as a subspace of $C(U)$. (We are tacitly using the space of *complex-valued* continuous functions on U). If F is, in addition, closed in E, then we can regard the quotient space E/F as a locally convex space by using the method which is employed to provide a quotient of a normed space with a norm. In other words, if p is a seminorm on E, we define a seminorm $\tilde{p}$ on E/F by putting

$$\tilde{p}(x) = \inf\{p(x') : x' \in E \quad \pi(x') = x\}$$

where π denotes the natural mapping form E onto E/F.

We shall frequently use the simple fact that if (E, S) resp. (E_1, S_1) are locally convex spaces with F resp. F_1 a closed subspace of E resp. E_1 and T is a continuous linear mapping from E into E_1 which maps F into F_1, then T lifts to a continuous linear mapping $\tilde{T}$ from E/F into E_1/F_1.

Finite products: If (E_i, S_i) $(i = 1, \dots, n)$ is a finite family of locally convex spaces, then the **product structure**on $E = \prod_{i=1}^{n} E_i$ is that one which is defined by the seminorms of the form

$$(x_1, \dots, x_n) \mapsto p_1(x_1) + \dots + p_n(x_n)$$

where $p_1, \dots, p_n$ is any choice of seminorms, whereby $p_i \in S_i$. The corresponding topology on E is simply the Cartesian product topology.

Projective structures: Suppose that E is a vector space and that (E_α, S_α) is an indexed family of locally convex spaces. Suppose that for each $\alpha \in A$ we are given a linear mapping T_α from E into E_α and that these mappings separate E i.e. if x in E is non-zero, then there is an α so that $T_\alpha(x) \neq 0$. Then the family of all seminorms of the form

$$x \mapsto p_\alpha(T_\alpha(x))$$

where $\alpha \in A$ and p_α is in S_α, define a locally convex structure on E called the **projective structure** induced by the T_α. The corresponding topology on E is none other than the initial topology induced by the T_α. Hence a linear mapping from a locally convex space F into E is continuous if and only if for each α, the composition $T_\alpha \circ F$ from F into E_α is continuous.

For example, if S is a locally compact space, then the locally convex structure on $C(S)$ is precisely the projective structure induced by the restriction mappings from $C(S)$ into the Banach spaces $C(K)$ as K runs through the family $\mathcal{K}(\mathcal{S})$.

Another important example is that of infinite products. If $(E_\alpha, S_\alpha)_{\alpha \in A}$ is a family of locally convex spaces and E is the Cartesian product of the underlying sets, then for each $J \in \mathcal{F}(\mathcal{A})$, the family of finite subsets of A, there is a natural projection π_J from E onto $\prod_{\alpha \in J} E_\alpha$. These can be used to give E a projective locally convex structure, under which it is called the **locally convex product** of the E_α.

We now consider in some details two more abstract methods of constructing locally convex spaces–projective and inductive limits. This will be particularly useful later since the spaces which we shall discuss are constructed in one of those ways from classical Banach spaces and this simplifies the task of presenting them and their properties.

Projective limits: Let (E_α, S_α) be a family of locally convex spaces indexed by a directed set A so that for each $\alpha \in A$ there is a continuous linear mapping $\pi_{\beta\alpha} : E_\beta \to E_\alpha$ and the following compatibility conditions are fulfilled:

- for each $\alpha \in A$ $\pi_{\alpha\alpha}$ is the identity mapping on E_α;
- if $\alpha \leq \beta \leq \gamma$, then $\pi_{\beta\alpha} \circ \pi_{\gamma\beta} = \pi_{\gamma\alpha}$.

Then

$$E_0 = \{x = (x_\alpha) \in \prod E_\alpha : \pi_{\beta\alpha}(x_\beta) = x_\alpha \text{ for } \alpha \leq \beta\}$$

is a subspace of the product space, in fact a closed subspace. We call E_0, with the locally convex structure induced from the product, the **projective limit** of the E_α. We also write π_α for the restriction of the projection from the product onto E_α to E_0.

The space which we have just constructed is characterised by the following *universal* property: if F is a locally convex space and $(T_\alpha)_{\alpha \in A}$ is a family of continuous linear mappings, whereby T_α maps F into E_α and we have $T_\alpha = \pi_{\beta\alpha} \circ T_\beta$ whenever $\alpha \leq \beta$, then there is a unique continuous linear mapping T from F into E so that $T_\alpha = \pi_a \circ T$ for each a. On the other hand, every such mapping from F into E arises in this way. The proof is easy. If x is an element of F, we simply define Tx to be the thread $(T_\alpha(x))_{\alpha \in A}$.

The following are examples of projective limits:

I. The product $\prod_{\alpha \in A} E_\alpha$ can be regarded as the projective limit of the family of finite products $\{\prod_{\alpha \in H} E_\alpha : J \in \mathcal{F}(\mathcal{S})\}$. here $\mathcal{F}(\mathcal{A})$ is directed by inclusion and the linking mappings $\pi_{J',J} (J \subset J')$ are the natural projections.

II. Intersections: Let X be a vector space and $(E_\alpha)_{\alpha\in A}$ an indexed family of vector subspaces, provided with locally convex structures so that if $\alpha \leq \beta$, then $E_a \subset E_b$ and the natural injection from E_α into E_β is continuous. Then the projective limit of the family (E_α) can be identified (as a vector space) with the intersection of the subspaces and so the latter has a natural locally convex structure. It is called the (locally convex) intersection of the E_α. We shall be especially interested in the following examples of this construction: III. The space of continuous functions. Let S be a locally compact space and direct $\mathcal{K}(\mathcal{S})$, the family of compacta of S, by inclusion. if $K \subset K'$, $\rho_{K',K}$ denotes the restriction operator from $C(K')$ into $C(K)$. Then $\{C(K), \rho_{K',K}\}$ is a projective system and its projective limit can be identified with $(C(S), S_{\mathcal{K}})$.

We remark that a projective limit of a system of complete spaces is itself complete, as a closed subspace of a product.

We shall repeatedly use the fact that each complete space has a natural representation as a projective limit of a spectrum of Banach spaces. This is done as follows. Suppose that (E, S) is such a space. We construct, for each p in S, a Banach space E_p in the following manner. Firstly, we factor our N_p, the null-space of p, and use p to define a norm on E/N_p in the natural way (i.e. we define??????). E_p is then defined to be the completion of E/N_p with this norm. Now S is directed by the pointwise ordering i.e. $p \leq q$ if $p(x) \leq q(x)$ for each x in E and it $p \leq q$, then $N_q \subset N_p$ and so there is a natural mapping from E/N_q into E/N_p and this lifts to a linear contraction ω_{qp} from E_q into E_p. (E_p, ω_{qp}) is a projective spectrum of Banach spaces and its projective limit is isomorphic to E. This can be seen as follows. There is a mapping ω_p from E into E_p for each p (namely, the projection from E onto $E?N_p$) and the operator $\emptyset : x \mapsto (\omega_p(x))_p)_{p\in S}$ defines a mapping from E into the space of threads in the product $\prod_{p\in S} E_p$ i.e. into E. In fact, this is an isomorphism from E onto a dense subspace of E and so is surjective since E is complete.

The representation $E = \varprojlim\{E_p : p \in S\}$ of E as a projective limit of Banach spaces is called its **canonical representation**. Note that if S_1 is a subfamily of S which generates it and is directed on the right, then $(E_p : p \in S_1)$ is also a projective spectrum and exactly the same argument shows that its limit is also E.

E satisfies the following property: every continuous linear mapping T from E into a complete space F has a unique extension to a mapping $\hat{T}$ from $\hat{E}$ into F. As usual, this property determines $\hat{E}$ up to isomorphism and so we are justified in calling it *the* completion of E.

We now discuss a method of constructing spaces which is dual to that of the formation of projective limits.

Inductive structures: Let E be a vector space, $(E_\alpha)_{\alpha \in A}$ a family of locally convex spaces and, for each $\alpha \in A$, T_α a linear mapping from E_α into E. We suppose further that the union of the ranges $T_\alpha(E_\alpha)$ of these mappings is E. For each set $\gamma = (p_\alpha)$ of seminorms indexed by A, whereby $p_\alpha \in S_\alpha$, we define a seminorm p_γ on E by putting

$$p_\gamma(x) = \inf\{\sum_{\alpha \in J} p_\alpha(x_\alpha)\}$$

the infimum being taken over the representations of x as a finite sum $\sum_{\alpha \in J} T_\alpha(x_\alpha)$. The set S of all such seminorms does not in general separate E. Hence we may have to have recourse to the trick mentioned above and use these seminorms to define a locally convex structure on the quotient space E/N_S. (We remark that there are pathological examples where it *is* necessary to take this quotient space–however, this will not be the case for the spaces which we shall construct in this manner).

The structure on E has the property that a linear mapping from E/N_S into a locally convex space F is continuous if and only if $T_\alpha \circ \pi_S \circ T$ is continuous from E_α into F fro each $\alpha \in A$. Further, an absolutely convex subset U of $E?N_S$ is a τ_S-neighbourhood of zero if and only if $(\pi_S \circ T_\alpha)^{-1}(U)$ is a neighbourhood of zero in E_α for each α.

The cases in which shall be interested are the following:

Direct sums: As usual, $(E_\alpha)_{\alpha \in A}$ is an indexed family of locally convex spaces. $\bigoplus_{\alpha \in A} E_\alpha$ denotes the vector space direct sum i.e. the subspace of $\prod_{\alpha \in A} E_\alpha$ consisting of those vectors (x_α) for which at most finitely many of the x_α are non-zero. Now if $J \in \mathcal{J}(\mathcal{A})$, there is a natural injection from $\prod_{\alpha \in J} E_\alpha$ into $\bigoplus_{\alpha \in A} E_\alpha$. hence we can provide the latter with the corresponding inductive locally convex structure. It is then called the **(locally convex) direct sum** of the E_α. (Note that this is finer than the structure induced from the Cartesian product–in particular, it is not necessary to take a quotient space in the construction of the inductive structure).

Inductive limits: There is a construction dual to that of projective limits which we now describe. (E_α) is a family of locally convex spaces indexed by a directed set A and for each $\alpha \leq \beta$ there is a continuous linear mapping $i_{\alpha\beta} : E_\alpha \to E_\beta$ such that the following conditions are satisfied:

- for each α, $i_{\alpha\alpha}$ is the identity;
- if $\alpha \leq \beta \leq \gamma$, then $i_{\beta\gamma} \circ i_{\alpha\beta} = i_{\alpha\gamma}$.

Let N be the closed subspace of the direct sum $\bigoplus_{\alpha \in A} E_\alpha$ which is generated by elements of the form

$$\{i_\alpha(x) - i_\beta \circ i_{\alpha\beta}(x) : \alpha \leq \beta, x \in E_\alpha\}$$

where i_α denotes the injection from E_α into the direct sum. Then $\bigoplus_{\alpha \in A} E_\alpha / N$, with the quotient structure inherited from the direct sum, is called the **inductive limit** of the spectrum $(E_\alpha, i_{\alpha\beta}$. It is characterised by the following universal property. For each space F and each family (T_α) of continuous linear mappings (where T_α maps E_α into F) which satisfies the conditions $T_\alpha = T_\beta \circ i_{\alpha\beta}$ $(\alpha \leq \beta)$, there is a unique continuous linear mapping T from E into F so that $T_\alpha = T \circ \pi_N \circ i$ for each α.

Examples: I. Unions: X is a vector space, $(E_\alpha)_{\alpha \in A}$ an indexed set of subspaces, each with a locally convex structure so that if $\alpha \leq \beta$, then $E_\alpha \subset E_\beta$ and the inclusion is continuous. Suppose that there is a Hausdorff topology τ on X whose restriction to each E_α is coarser than τ_{S_α}. Then the inductive limit can be identified with the union and the existence ot τ ensures that the corresponding family of seminorms on E separates points. E, with the inductive structure, is the **(locally convex) union** of the E_α. We shall be interested in the following particular examples.

We now consider the Hahn-Banach theorem for locally convex spaces. As we have seen, a suitable form of this result is true for all such spaces. This ensures that they can be equipped with a satisfactory duality theory, a fact which is of particular consequence in the Schwartzian distribution theory. We begin with an alternative proof of the Hahn-Banach theorem which illustrates the type of argument which can often be used to reduce results on locally convex spaces to the corresponding ones for normed spaces. We recall that the **dual** of a locally convex space, which we denote by E', is the space of all continuous linear forms on E (i.e. the continuous linear mappings from E into the canonical one-dimensional space $\mathbf{R}$ or $\mathbf{C}$).

Proposition 1 *Let f be a linear form on the locally convex space (E, S). Then the following are equivalent:*

- *f is continuous;*
- *there exists a p in S so that $|f(x)| \leq 1$ if $x \in U_p$'*
- *there is a $p \in S$ so that $|f| \leq p$;*
- *$|f|$ is a continuous seminorm on E;*

- *there is a p in S and an $\tilde{f} \in E_p'$ so that f factorises over $\tilde{f}$ i.e. $f = \tilde{f} \circ \omega_p$;*

- *$\mathrm{Ker} f$ is closed.*

In the same way we can show that if M is a set of linear forms on E, then the following are equivalent:

- M is equicontinuous at zero;

- M is equicontinuous on E;

- there is a p in S so that $|f| \leq p$ for each $f \in M$;

- there is a p in S and a norm bounded subset of E_p' $\tilde{M}$ so that $M = \tilde{M} \circ \omega_p$.

We can now state the hahn-banach theorem for locally convex spaces. Note that the quantitative aspect is replaced by the fact that equicontinuous sets of functionals can be lifted simultaneously to equicontinuous families.

Proposition 2 *Proposition (Hahn-Banach theorem) Let M be an equicontinuous family of linear forms on a subspace F of the locally convex space E. Then there is an equicontinuous family M_1 of E' so that M is the set of restrictions of the members of M_1 to F.*

PROOF. We choose a seminorm p as in 4) above and apply the Hahn-Banach theorem for normed spaces to find a bounded family $\tilde{M}_1$ which extends the subset $\tilde{M}$ of the dual of F_p. Then $M_1 = \tilde{M}_1 \circ \omega_p$ has the required property.

Exactly as in the case of normed spaces, this result has a number of corollaries which we list without proofs:

- a linear form f in the dual of F can be lifted to one in the dual of E;

- if x_0 is an element of E and G is a closed subspace of E which does not contain x_0, then there exists an f in E' so that $f - 0$ on G and $f(x_0) = 1$;

- if $x_0 \in E$ and p is a continuous seminorm on E so that $p(x_0) \neq 0$, then there is a continuous linear form f on E so that $f(x_0) = 1$ and $p(x_0)f(x) \leq p(x)$ for each $x \in E$;

- let $x_1, \dots, x_n$ be linearly independent elements of a locally convex space E. Then there exists elements $f_1, \dots, f_n$ in E' so that $f_i(x_i) = 1$ for each i‘ and $f_i(x_j) = 0$ for each distinct pair i, j;

- let A be a closed, absolutely convex subset of a locally convex space which does not contain the point x_0. Then there is a continuous linear form f on E so that $f(x_0) > 1$ and $|f(x)| \leq 1$ for each x in A. In other words, a point x lies in the closed, absolutely convex hull of a set B if and only if for each continuous linear form f on E which is less than one in absolute value on B, we have $|f(x)| \leq 1$.

We now consider some topological and uniform concepts (such as completeness, compactness etc.) in the context of locally convex spaces and show how they may be characterised using the canonical representation of the space E as a projective limit of Banach spaces:

Definition: A subset M of a locally convex space (E, S) is said to be S-**complete** (or simply complete) if it is complete for the uniform structure induced by S. It is S-**bounded** if each seminorm p in S is bounded on M and S-**compact** (resp. relatively compact) if it is compact (relatively) for the topology τ_S. Finally it is S-**precompact** if it is relatively compact in the completion $\hat{E}$ of E. of course, we shall omit the prefix S in the above notation unless it is not clear from the context which locally convex structure we are dealing with.

The following comments on these definitions are obvious:

- in the definition of boundedness, it is sufficient to check that each p in a generating family of seminorms is bounded on M;
- we have the implications

  $$\text{compact} \Rightarrow \text{relatively compact} \Rightarrow \text{precompact} \Rightarrow \text{bounded}$$

  and

  $$\text{compact} \Rightarrow \text{complete}.$$

- if M is complete, then it is τ_S closed. On the other hand, every closed subset of a complete set is complete;
- if we denote by $\mathcal{B}_\mathcal{S}$ the collection of bounded subsets of E, then this family has the following stability properties: if $B, C \in \mathcal{B}_\mathcal{S}$ and $\lambda > 0$, then $\lambda B \in \mathcal{B}_\mathcal{S}, \mathcal{B} + \mathcal{C} \in \mathcal{B}_\mathcal{S}$ and $\Gamma(B) \in \mathcal{B}_\mathcal{S}$ (where $\Gamma(B)$ denotes the closed, convex hull of B).

Less obvious is the following stability property of completeness with respect to changes in the topology:

Proposition 3 *Let S and S_1 be locally convex structures on a vector space E which are such that the corresponding duals coincide. Then they define the same family of bounded sets on E.*

PROOF. It suffices to show that S and $S_w(E)$, the weak structure on E, define the same bounded sets. Clearly, if a set is S-bounded, it is weakly bounded. On the other hand, if B is weakly bounded, then $\omega_p(B)$ is weakly bounded in E_p for each p in S. By the uniform boundedness theorem, the latter set is also norm-bounded. Since this holds for each p in S, B is S-bounded.

■

If B is a bounded, absolutely convex subset of a locally convex space E, then E_B, the linear space spanned by B, is the union $\bigcup_{n\in\mathbf{N}} nB$ of multiples of B and so we can define the **Minkowski functional**

$$\| \ \|_B : x \mapsto \inf\{\rho > l0 : x \in \rho B\}$$

on B. This is a norm and the canonical injection from $(E_B, \| \ \|_B)$ into (E, S) is continuous. Thus in a certain sense, a bounded subset of a locally convex space is just an injection of a normed space into the latter.

The following is a simple criterium for E_B to be a Banach space.

Proposition 4 *If B is S-complete, then E_B is a Banach space.*

We now turn to some elementary properties of compact subsets of a locally convex space:

Proposition 5 *A complete subset A of a locally convex space E is compact if and only if for each p in S (or even in some generating subfamily of S), its image $\omega_p(A)$ is compact (or even relatively compact) in the normed space E_p.*

PROOF. The necessity is clear. For the sufficiency, consider the mapping $x \mapsto (\omega_p(x))_{p\in S}$ from E into the product $\prod_{p\in S} E_p$. This is a homeomorphism and A is mapped onto a closed subset of the product (since it is complete) and so onto a closed subset of $\prod_{p\in S} \omega_p(A)$, which is compact by Tychonov's theorem.

■

Proposition 6 *Corollary A subset A of E is precompact if and only if for each p in S, its image $\omega_p(A)$ in E_p is relatively compact (i.e. totally bounded.*

If we apply the definition of total boundedness in a uniform space to a locally convex space, we obtain the following condition; A is totally bounded in E if for leach p in S there is a finite subset $x_1, \dots, x_n$ of A so that $A \subset \bigcup_{i=1}^n (x_i + U_p)$. Now this is equivalent to the fact that the $\omega_p(A)$ are totally bounded in the E_p ($p \in S$). Since a subset of a metric space is totally bounded if and only if it is precompact, the same holds for subsets of locally convex spaces and so compactness is equivalent to being complete and totally bounded. This fact immediately implies that if B is a compact subset of a locally convex space, then its closed convex hull $\Gamma(B)$ is compact if and only if it is complete (which is always the case if E itself is complete). For the convex (or even absolutely convex) hull of a totally bounded set is easily seen to be totally bounded. Since a bounded subset of a locally convex space E is always $S_w(E')$-precompact (bounded subsets of $\mathbf{R}$ being precompact), it follows that such a bounded set is weakly compact if and only if it is weakly complete. The next result is a generalisation of the theorem of Alaoglu on the compactness of the unit ball of a dual banach space with respect to the weak star topology.

Proposition 7 *Let E be a vector space with algebraic dual E. Then E is $S_w(E)$ complete. hence if E is a locally convex space, then for each $p \in S$*

$$B_p = \{f \in E' : |f'| \leq p\}$$

is a $S_w(E)$-compact subset of E'.

PROOF. The first statement follows immediately from the trivial fact that the pointwise limit of linear functionals on E is itself linear. For the second, we note firstly that each B_p is an equicontinuous subset of the dual (and, in fact, a set is equicontinuous if and only if it is contained in such a B_p). We now regard B_p as a subset of $C(E), S_{\mathcal{K}})$, the locally convex space of continuous functions on E (the latter regarded as a topological space with the discrete topology). Since E is discrete, the topology τ_c of compact convergence coincides with that of pointwise convergence. Now B_p is clearly relatively compact in the complete space $C(E)$. On the other hand it is closed there since the pointwise limit of a net in B_p is also in this set since the latter is defined by pointwise estimates. hence B_p is compact for the topology of pointwise convergence in $C(E)$ and this coincides with the topology induced by $S_w(E)$.

■

One of the consequences of the Hahn-Banach theorem is that if E is a locally convex space, then (E, E') forms a dual pair and so E can be provided with a series of locally convex topologies as described above. In particular, in addition to the weak topology $\sigma(E, E')$, we can consider the strong topology $\underline{(E, E')}$ and the Mackey topology $\tau(E, E')$. The weak topology and the Mackey topology enjoy privileged positions as two extreme topologies in the following sense:

Proposition 8 *If (E, F) is a dual pair, then a locally convex topology τ on E has F as dual space if and only if τ is finer than $\sigma(E, F)$ and coarser than $\tau(E, F)$. In particular, $\sigma(E, E')$ and $\tau(E, E')$ are the weakest resp. the strongest locally convex topologies on E which have E' as dual.*

3 Convex bornological spaces

For many aspects of the theory of Banach spaces or their generalisations, the concept f boundedness plays a more fundamental role than topological ones. For this and other reasons, it is convenient to use a second generalisation of banach spaces which is based on boundedness and is, in a certain sense, dual to that of local convexity.

Definition: A subset B of a vector space E is a **pseudo-disc** if it is absolutely convex. It is a **disc** if it is absolutely convex and contains no non-trivial subspace of E. If B is a pseudo-disc, then the subspace $E_B = \bigcup_{n \in \mathbf{N}} nB$ is a semi-normed space provided with the Minkowski functional

$$\| \ \|_B : x \mapsto \inf\{\rho > 0 : x \in \rho B\}.$$

If the semi-normed space $(E_B, \| \ \|_B)$ is complete, we call B a **Banach pseudo-discc** and if it is, in addition, a Banach space, then B is a **Banach disc**.

A convex bornological structure on E is a family $\mathcal{B}$ of discs in E with the properties that

- $E = \bigcup \mathcal{B}$;
- $\mathcal{B}$ is directed on the right by inclusion i.e. if $B, C \in \mathcal{B}$, there is a $D \in \mathcal{B}$ so that $B \cup C \subset D$;
- if $B \in \mathcal{B}, \lambda > \prime$, then $\lambda B \in \mathcal{B}$;
- if $C \in \mathcal{B}$ and B is a disc contained in C, then $B \in \mathcal{B}$.

A **convex bornological space** is a pair $(E, \mathcal{B})$ where E is a vector space and $\mathcal{B}$ is a convex bornological structure thereon. E is complete if each $B \in \mathcal{B}$ is contained in a Banach disc in $\mathcal{B}$ (i.e. the Banach discs in $\mathcal{B}$ form a **basis** for $\mathcal{B}$ in the obvious sense). A subset B of E is then said to be **bounded** if it is contained in some set C of $\mathcal{B}$. By the first axiom above singletons are bounded. Hence by the second, finite subsets are also bounded.

It follows easily from the definition that the sum of two bounded sets in E is bounded and that if B is a disc in E, then it is in $\mathcal{B}$ if and only if it is bounded.

If $(E,)$ is a convex bornological space and we define

$$\mathcal{B}^{\text{comp}} = \{\mathcal{B} \in \mathcal{B} : \mathcal{B} \text{ is contained in a Banach disc of } \mathcal{B}\},$$

then $(E, \mathcal{B}^{\text{comp}})$ is a complete convex bornological space. We call it the **complete space associated with** $(E, \mathcal{B})$ **({it not** the completion which is very different and will be treated below).

A linear mapping T between spaces $(E, \mathcal{B})$ and $(F, \mathcal{B}_\infty)$ is said to be **bounded** if it carries bounded sets into bounded sets. Note that if B is a Banach disc in E and T is a linear mapping from E into F, then $T(B)$ is a Banach disc (provided that it is a disc). For the latter is the case if and only if $T_{-1}(0) \cap E_B$ is $\| \|_B$-closed in E_B. Hence if T is bounded from $(E, \mathcal{B})$ into $(F, \mathcal{B}_\infty)$, then it is also bounded as a mapping on the structures $\mathcal{B}^{\text{comp}}$ and $\mathcal{B}_\infty^{\text{comp}}$.

Using the bornological structure, we can introduce a notion of convergence as follows: a sequence (x_n) in E is said to be **Mackey-convergent** to a point x if there is a B in $\mathcal{B}$ so that $\{x_n\} \cup \{x\}$ lies in E_B and $x_n \to x$ in the normed space $(E_B, \| \|_B)$. We then write $x_n \to x$.

A subset M of E is said to be **Mackey closed** if $M \cap E_B$ is $\| \|_B$-closed for each $B \in \mathcal{B}$. Clearly this is equivalent to the fact that if (x_n) is a sequence in M which is Mackey convergent to a point in E, then the latter is also in M. It is easy to check that $x_n \to x$ is equivalent to the fact that there is a sequence $\lambda_n)$ of positive scalars which converge to zero so that the sequence

$$\frac{x_n - x}{\lambda_n}$$

is bounded. From this one can deduce that the boundedness of a linear operator is equivalent to the fact that it preserves mackey-convergence i.e. if $x_n \to x$ in E, then $Tx_n \to Tx$ in F.

The following are simple examples of convex bornological spaces:

I. If (E, S) is a locally convex space, then we can associate to it several natural bornologies–for example

$\mathcal{B}_\mathcal{S}$ – the family of bounded discs of E;

$\mathcal{B}_{\sqrt{\rfloor}}$–the family of precompact discs in E;

$\mathcal{B}_{\dashv\rfloor}$–the family of relatively discs in E;

$\mathcal{B}_{\{\rangle\backslash}$–the finite bornology which consists of those discs B for which E_B is finite dimensional.

$\mathcal{B}_{\mathcal{S}}$ is called the **von Neumann bornology** of E. Obviously, $\mathcal{B}_{\text{fin}} \subset \mathcal{B}_{\text{ac}} \subset \mathcal{B}_{\text{pc}} \subset \mathcal{B}_{\mathcal{S}}$. $\mathcal{B}_{\text{fin}}$ and $\mathcal{B}_{\text{ac}}$ are always complete, while $\mathcal{B}_{\mathcal{S}}$ is complete provided that (E, S) is.
II. On the other hand, the dual E' also has several natural bornologies, in particular, the structure $\mathcal{B}_{\text{equ}}$ which consists of the equicontinuous discs. This space is complete and the sets of the form B_p $(p \in S)$ form a basis.
III. many of the natural vector spaces of functions have a bornology which is defined by the order structure as follows. We suppose that the vector space E consists of a subspace of the space of real functions on a given set S and is such that if x and y are in E, then so is $\max(|x|, |y|)$. if x is an element of E, then

$$B_x = \{y \in E : |y| \leq \xi \}$$

is a disc in E. We define the **normal bornology** $\mathcal{B}_\backslash$ on E to be the one consisting of those discs in E which are contained in some B_x $(x \in E)$.

The usual constructions can be carried out on bornological spaces. Thus a subspace F of such a space $(E, \mathcal{B})$ also has a natural bornology which consists of those discs in $\mathcal{B}$ which are contained in F. Similarly, the quotient space E/F has a structure which is defined as follows: Denote by π_F the natural projection from E onto E/F. If B is a disc in E, then $\pi_F(B)$ is a pseudo-disc in E/F and is a disc if and only if $E_B \cap F$ $(= E_B \cap \pi_B^{-1}(0))$ is $\| \|_B$-closed. hence in particular, if $B \in \mathcal{B}$ and F is Mackey-closed, then it is always a disc. Thus if F satisfies this property, then the family $\{\pi_F(B) : B \in \mathcal{B}\}$ is a bornology, called the **quotient bornology**. It is complete if E is.

Projective structures: Let E be a vector space $(E_\alpha, \mathcal{B}_\alpha)$ a family of bornological spaces and, for each $\alpha \in A$, T_α a linear mapping from E into E_α. If the T_α separate E, then we can define a bornology $\mathcal{B}$ on E by calling a disc B in E bounded if each of its images $T_\alpha(B)$ are bounded in E_α. $\mathcal{B}$ is called the **projective structure** induced by the mappings $\{T_\alpha\}$. It is characterised by the fact that a linear mapping T from a space F into E is bounded if and only if the compositions $T_\alpha \circ T$ are. As in the case of locally convex spaces, one can construct the following special types by first carrying out the st-theoretical construction and then using the above to provide the result with a bornology: the product of a family of spaces, the projective limit of a spectrum and the intersection. For future reference, we note that the projective limit of a spectrum is a Mackey-closed subspace of their product nd so is complete fi the component spaces are.

Inductive structures: We have a vector space E and a family of linear mappings (T_α) where T_α maps the convex bornological space $(E_\alpha)\mathcal{B}_\alpha)$ into

E. We assume that E is the union of the spaces $\{T_\alpha(E_\alpha)\}$ and we define a pseudo-bornology on E as follows: a pseudo-disc B is in $\mathcal{B}$ if and only if there is a finite subset $J = \{\alpha_1, \dots, \alpha_n\}$ of A and, for each α_i in J a $B_{\alpha_i} \in \mathcal{B}\rangle$ so that

$$B \subset T_{\alpha_1} + \cdots + T_{\alpha_n}(B_{\alpha_n}).$$

We call the associated bornological space $(E/N_{\mathcal{B}}, \tilde{\mathcal{B}})$ the **inductive structure** defined by the T_α.

Using this one can define the direct sum of a family of convex bornological spaces, the inductive limit of a spectrum and unions. In contrast to the case of locally convex spaces, it is true and elementary that the inductive limit of a spectrum of complete spaces is complete.

4 The canonical representation:

Recall that we have shown that every complete locally convex space has a natural representation as a projective limit of banach spaces and that we used this fact to construct completions of general spaces. The following construction is dual to this. Suppose that $(E, \mathcal{B})$ is a convex bornological space. Then the normed spaces E_B ($B \in \mathcal{B}$) form an inductive spectrum of normed spaces under the natural inclusion mappings $i_{BC} : E_B to E_C$ for $B \subset C$. it follows immediately from the definition of inductive limits that E is the inductive limit of this system. This is called the **canonical representation** of E.

In order to construct the completion of E, we consider the inductive system

$$\{i_{BC} : \hat{E}_B \to \hat{E}_C, B \subset C\}$$

which is obtained by completing the components of the original spectrum As we shall see below, the linking mappings (which we continue to denote by i_{BC}) need no longer be injections. nevertheless, the limit $\hat{E}$ of the above spectrum is a complete bornological space which is called the **completion** of E.

We have a natural mapping $\emptyset : E \to \hat{E}$ which satisfies the usual universal property, namely that every bounded linear mapping T from E into a complete space F factors over $\emptyset$. However, the fact that the i_{BC} are no longer necessarily injective is reflected in the fact that the mapping $\emptyset : E \to \hat{E}$ need no longer be injective, as is shown by the following example.

Example: We denote by E_n the subspace of $C([\frac{-1}{n}, \frac{1}{n}])$ which consists of those functions which are the restrictions of polynomials which vanish at zero. The completion $\hat{E}_n$ of E_n (under the supremum norm) is the space of continuous functions on $[\frac{-1}{n}, \frac{1}{n}]$ which vanish at zero. (E_n) can be regarded as an increasing sequence of normed spaces and its union is thus a convex bornological space. The underlying vector space consists of the polynomials which vanish at zero. We shall now show that the completion of this space is trivial i.e. consists only of the zero vector. For this completion, $\hat{E}$, is the set theoretical inductive limit of the E_n, provided with a pseudo-bornology, with the largest bounded subspace factored out. Now the set theoretical inductive limit is the set of germs of continuous functions at zero which vanish at zero. More precisely,

Now we claim that the whole space is bounded and so the factor space is trivial as claimed. This follows from the fact that any germ which vanishes at zero is the restriction to some neighbourhood of zero of a continuous function, say on $[-1, 1]$, which is bounded by one.

We remark that there is a large class of bornological spaces in which this type of pathology cannot occur.

Definition: A space $(E, \mathcal{B})$ is said to be **proper** if there is a basis $\mathcal{B}_\infty$ of the bornology which is such that each B in $\mathcal{B}_\infty$ is Mackey closed. in E. Then if E is a proper space, the natural mapping from E into its completion $\hat{E}$ is an injection. This follows from the elementary fact that if $i : E \to F$ is a continuous injection from a normed space E into a normed space F such that $i(B)$ is closed in F (where B is the unit ball of E), thant the extension of i to a mapping form $\hat{E}$ into $\hat{F}$ is also injective.

We complete our brief discussion of bornological spaces with a simple closed graph theorem. A disc B in $(E, \mathcal{B})$ is said to be a **associated with** $\mathcal{B}$ if $B \sqcup C$ is a ball for each C in $\mathcal{B}$. This is equivalent to the existence of a bornology $\mathcal{B}_\infty$ on E which contains $\mathcal{B}$ and also C. E is defined to be **co-barrelled** if it contains each Banach disc which is associated with it.

Proposition 9 *A complete space with a countable basis is co-barrelled.*

PROOF. Let C be a Banach disc associated with $\mathcal{B}$ and let (B_n) be an increasing basis for $\mathcal{B}$. Then $E_C = \bigcup E_{B_n \cap C}$ and so there is an m in $\mathbf{N}$ for which $E_{B_m \cap C}$ is of second category in E_C. Since this is a Banach space, it follows from a version of the closed graph theorem that $E_C = E_{B_m \cap C}$ and so C is absorbed by B_m.

■

In order to apply this result to obtain a closed graph theorem we note that if $T : E \to F$ is a linear mapping between convex bornological spaces,

then its graph is Mackey-closed if and only if for each B in the bornology of E, $T(B)$ is associated with the bornology of F.
PROOF. Suppose that B is bounded in E and C in F. The mapping

$$(x, y) \mapsto (-Tx) + y$$

is a surjection from $E_B \times F_C$ into $E_{T(B)\sqcup C}$ and its kernel is $(E_B \times F_C) \cap \Gamma(T)$ where $\Gamma(T)$ is the graph of T. This mapping induces a linear isomorphism between $E_B \times F_C/(E_B \times F_C) \cap \Gamma(T)$ and $F_{T(B)\sqcup C}$ which is an isometry for the natural semi-norms (when $E_B \times F_C$ is given the sum semi-norm). Now T has a Mackey-closed graph if and only if $(E_B \times F_C) \cap \Gamma(T)$ is closed n $E_B \times F_C$ for each B and C and this is equivalent to the fact that $T(B) \sqcup C$ is a disc for each B and C.

■

The following closed graph theorem follows now immediately from the above facts;

Proposition 10 *A linear mapping T from a complete space into a co-barrelled space is bounded if and only if its graph is mackey-closed in the product.*

As a Corollary, we see that the result holds in particular when the image space is complete and countably generated.

One can deduce the usual variants of the closed graph theorem (isomorphism and epimorphism theorems) form this result.

5 Duality

We now consider duality for locally convex and convex bornological spaces. We regard the dual of a locally convex space as a convex bornological space and *vice versa*:

Definition: Suppose that (E, S) is a locally convex space and $(F, \mathcal{B})$ is a convex bornological space. We define structures on their duals as follows. On E', the dual of E, we have already defined a bornology, the equicontinuous bornology, which we denoted by $\mathcal{B}_{\text{equ}}$. F^b denotes the set of bounded linear mappings from F into the one-dimensional space $\mathbf{R}$ (or $\mathbf{C}$ if we are working with complex spaces). The latter has a natural vector space structure and we define a locally convex structure on it by putting, for $B \in \mathcal{B}$,

$$p_B : f \mapsto \sup\{|f(x)| : x \in B\}.$$

This is clearly a seminorm on F^b and the set of all such seminorms $\{p_B : B \in \mathcal{B}\}$ generates a locally convex structure which we denote by $S_{\mathcal{B}}$. Thus if

$$E = \varprojlim\{E_p : p \in S\} \quad \text{and} \quad F = \varinjlim\{F_B : B \in \mathcal{B}\}$$

are the canonical representations for these spaces (resp. in the case of E its completion), then the duals E' and F^b are the limits of the dual systems i.e.

$$E' = \varinjlim\{E'_p : p \in S\} \quad \text{and} \quad F^b = \varprojlim\{F'_B : B \in \mathcal{B}\}$$

these limits being taken in the sense of bornological spaces, resp. locally convex spaces. This immediately implies the fact that E' and F^b are complete.

The duality theory for general convex bornological spaces suffers from the lack of a suitable form of the Hahn-Banach theorem for bounded forms. In fact, the dual of a convex bornological space can be trivial as the next example shows:

Example: consider the space $F = S([0,1])$ of equivalence classes of Lebesgue measurable functions on the unit interval. Then F, supplied with the normal bornology (cf. ???) has a trivial dual space as the reader can verify.

Hence, in order to obtain a satisfactory duality theory, it is necessary to impose some additional condition on our spaces.

Definition: — convex bornological space F is **regular** if its dual separates F i.e. for each non-zero x in F there is an f in F^b so that $f(x) \neq 0$. Another way of saying this is that (F, F^b) forms a dual pair. This in turn implies that there is a locally convex structure on F (namely, the weak one $S_w(F^b)$) for which each B in $\mathcal{B}$ is bounded. Conversely, if such a structure exists, then its dual separates F and hence so does the larger space F^b so that F is regular.

it follows immediately from the definition that a regular space is proper. Also the characterisation in terms of the existence of a suitable locally convex structure shows that products, direct sums and subspaces of regular spaces are regular. This is not true of quotient spaces.

if a space F is not regular, then we can factor away the non-regular part in the usual way by putting

$$N_F = \{x \in F : f(x) = 0 \quad \text{for each} \quad f \in F^b\}.$$

Then N_F is mackey-closed and we call the quotient F/N_F (which is clearly regular), the regular space associated to F. it has the usual universal property (namely, that every bounded linear mapping from F into a regular space F_1 lifts to a mapping on F/N_F).

Having defined duals, we can proceed to define second duals or biduals. Thus E'^b is the dual of the bornological space E'. As such it has a natural locally convex structure under which it is complete. Similarly, the bidual $F^{b\prime}$ of a bornological space is the bounded dual of F' and is a complete bornological space. As in the case of normed spaces, there are natural mappings

$$J_E : E \to E'^b \quad \text{and} \quad J_F : F \to F^{b\prime}$$

where if $x \in E, y \in F$,

$$J_F(x) : f \mapsto f(x) \quad (f \in E');$$

and

$$J_F(y) : g \mapsto g(y) \quad (g \in F^b).$$

J_E is injective but this is the case for J_F if and only if F is regular. It is a simple exercise in duality to show that J_E is an isomorphism from the locally convex space E into its bidual. Similarly, if F is a regular bornological space, J_F is an isomorphism from F onto a subspace of its bidual.

We remark here that the usual definition of the bidual of a locally convex space does not coincide with the one given here. We have chosen the latter because of the last property which does not always hold for the usual embedding of E into its locally convex bidual E''.

It is often useful to bear the following description of these biduals in mind. If E is a locally convex space, then the dual of E' is a subspace of E'^*, its algebraic dual. In fact, it is the space

$$\bigcup \{V^{oo} : V \quad \text{is a neighbourhood of zero in} \quad E\}$$

where the first polar is taken in E', the second in E'^*.

Similarly, if F is a regular convex bornological space, we can identify its bidual with the subspace $\bigcup\{B^{oo} : B \in \mathcal{B}\}$ of F^{b*}. Using this fact, and standard manipulations with polar sets, we obtain the following results on reflexivity (where a locally convex space is **reflexive** if J_E is surjective and hence an isomorphism from E onto its bidual. Similarly, a convex bornological space is defined to be reflexive if it is regular and J_F is surjective).

Proposition 11 *A locally convex space E is reflexive if and only if it has a basis of absolutely convex neighbourhoods of zero consisting of $\sigma(E, E')$ complete sets. A regular bornological space F is reflexive if and only if $\mathcal{B}$ is generated by a family of $\sigma(F, F^b)$-compact sets.*

6 Metrisable and Fréchet spaces:

These are spaces which have representations $E = \varprojlim E_n$ of a spectrum of Banach spaces which is indexed by $\mathbf{N}$. More precisely, a space with this property is called a **Fréchet space**. A general (i.e. non-complete) space is **metrisable** if its completion is a Fréchet space. less pedantically, they are those locally convex spaces whose structures are generated by countably many semi-norms. Of course, the name comes from the fact that this definition is equivalent to the fact that τ_S is metrisable. For suppose that this condition holds. Then 0 has a countable basis of neighbourhoods (which we can suppose to be absolutely convex) and their Minkowski functionals generate the locally convex structure. On the other hand, if E is metrisable in the above sense, then $\hat{E}$ is representable as the limit $\varprojlim E_n$ of a countable spectrum of Banach spaces and hence is homeomorphic to a subspace of the product $\prod E_N$ (as is E itself). hence it suffices to show that the latter is metrisable as a topological space. But the metric

$$d(x, y) = \sum \frac{1}{2^n} \frac{\|x_n - y_n\|}{1 + \|x_n - y_n\|}$$

where $x = (x_n)$ and $y = (y_n)$.

The closed graph theorem and its usual variants are also valid for Fréchet spaces. The reason is that the structure of a Fréchet space can be defined by a so-called **paranorm** and these are sufficiently similar to norms to allow us to carry over the proofs of these results from the case of banach space with only small changes. In fact, the results hold for an even wider class of classes which we now introduce:

Definition: A **metric linear space** is a vector space E, provided with a metric d which is translation invariant (that is, satisfies the condition

$$d(x + x_0, y + x_0) = d(x, y) \quad (x, y, x_0 \in E)$$

and is such that the mapping $(\lambda, x) \mapsto \lambda x$ from $\mathbf{R} \times E$ into E is continuous for the topology induced by d. A **paranorm** on a linear space is a mapping p from E into $\mathbf{R}^+$ so that

- $p(x) = 0$ if and only if $x = 0$;
- $p(x + y) \leq p(x) + p(y)$;
- $p(\lambda_n x) \to 0$ for every x in E and every null sequence of scalars.

if p is a paranorm on a space, then the mapping $d_p : (x, y) \mapsto p(x - y)$ is a translation invariant metric on E and (E, d_p) is a metric linear space. On the other hand, if (E, d) is such a space, then $pp : x \mapsto d(x, 0)$ is a paranorm. Thus the notions of a metric linear space and a space with paranorm are equivalent.

If the linear space E with paranorm is such that the metric space (E, d_p) is complete, it is called an F-space. Of course, every Banach space and indeed every Fréchet space is an F-space. An example of an F-space which is not a Fréchet space is $S(\mu)$, the set of equivalence classes of measurable functions on a measure space (Ω, μ). The mapping

$$x \mapsto \int \frac{|x|}{1 + |x|} \, d\mu$$

is a paranorm on $S(\mu)$. The reader can check that $S(\mu)$ is complete under this paranorm and that the corresponding notion of convergence is convergence in measure. The canonical example is provided by the case of lebesgue measure. In this case, if U is a neighbourhood of zero, then the absolutely convex hull of U is the whole space. This easily implies that the only continuous linear form on $S(\mu)$ is the zero form (since the set $\{|f| \leq 1\}$ is a neighbourhood of zero). Thus $S(\mu)$ cannot be locally convex (and so is not a Fréchet space).

The usual constructions shows that if (E, p) is a paranormed space, then so is each subspace and each quotient by a closed subspace. Also a countable product of paranormed spaces is paranormed (but not a non-trivial direct sum or an uncountable product). The same remark holds for F-spaces (where we only consider closed subspaces of course).

Our claim is that suitable versions of the classical theorems of Banach hold for paranormed spaces. We shall simply state these results without proof–those for Banach spaces can be carried over with only slight changes involving the substitution of norms by paranorms. We begin with the Banach-Steinhaus theorem. Here we use the term **bounded** to indicate a subset B of a paranormed space (F, p) for which $\sup\{p(x) : x \in B\} < \infty$.

Proposition 12 *Let E be an F-space and F an F-space or a locally convex space. Then if M is a family of continuous linear mappings from E into F which is bounded on the points of a set A of second category in E, M is equicontinuous. Hence if a sequence (T_n) of continuous linear mappings from E into F is such that the pointwise limit exists, then the latter is continuous.*

The open mapping theorem holds in the following form:

Proposition 13 *Let E and F b4 F-spaces, T a continuous linear mapping from E into F whose range $T(E)$ is of second category in F. Then T is open and surjective.*

As usual, version of the closed graph theorem and the isomorphism theorem can immediately be deduced from this result.

The following result about bounded sets resp. convergence sequences in metrisable, locally convex spaces is often useful:

Proposition 14 *Let (x_n) be a null-sequence resp. (B_n) a sequence of bounded sets in a metrisable locally convex space E. Then there exists*

- *a sequence (λ_n) of positive scalars which tends to infinity and is such that $\lambda_n x_n \to 0$;*
- *a sequence (λ_n) of positive scalars so that $\bigcup_n \lambda_n B_n$ is bounded.*

PROOF. We prove (1). The proof of (2) is similar. We choose an increasing sequence (p_n) of seminorms which generate the structure of E. For each k in $\mathbf{N}$ there is an n_k so that $p_k(x_n) \leq \frac{1}{k}$ if $n \geq n_k$. We can also suppose that $n_{k+1} \geq n_k$ for each k. Define the sequence (λ_n) as follows: $\lambda_n = \sqrt{k}$ where k is that positive integer for which $n_k \leq n < n_{k+1}$. Clearly this sequence increases to infinity and $\lambda_n x_n \to o$ since $p_k(\lambda_n x_n) \leq \frac{1}{\sqrt{k}}$ if $n \geq n_k$.

Barrelled and bornological spaces We have just seen that some classical results on banach spaces are also valid for Fréchet spaces. Not surprisingly, they fail to hold for all locally convex spaces and we shall now isolate those classes of space for which they *do* hold. We say that a family M of seminorms on a locally convex space (E, S) is **pointwise bounded** (resp. **bounded**) if $\sup\{p(x) : p \in M\} < \infty$ for each x in E (res. $\sup\{p(x) : p \in M, x \in B\} < \infty$ for each bounded set B of E). In particular, a single seminorm is bounded if it is bounded on bounded subsets of E (of course, every continuous seminorm has this property). if M is a pointwise bounded family of seminorms, then $\sup M$ (i.e. the mapping $x \mapsto \sup\{p(x) : p \in M\}$ is also a seminorm on E. The next result shows that the difference between these two boundedness conditions is rather fine. We precede its statement with the introduction of the some notation. A **barrel** in a locally convex spaces is a closed, absolutely convex, absorbing subset. In the proof of the following result, we use the fact that a barrel U in a banach space E is a neighbourhood of zero and so absorbs the unit ball B_E i.e. there is a $\rho > 0$ so that $B_E \subset \rho U$. For $E = \bigcup_{n \in \mathbf{N}} nU$ and so, by Baire's theorem, some nU (and hence U itself) has interior. It follows easily from this fact that U is a neighbourhood of zero.

Proposition 15 *Lemma Let E be a locally convex space in which every bounded closed subset is complete. Then any pointwise bounded family M of continuous seminorms on E is bounded.*

PROOF. Let B be a bounded subset of E which we can choose without loss of generality to be a complete disc (and so a banach disc). Then

$$U = \bigcap_{p \in M} U_p \cap E_B$$

is a barrel. For it is clearly closed and absolutely convex while the fact that it is absorbing is just the pointwise boundedness of M. Hence by the above it absorbs the unit ball of E_B which is just B.

■

It is traditional to call the class of spaces which appears in the above formulation (i.e. those in which closed, bounded subsets are complete) **quasi-complete**. Of course this is a weaker condition than completeness. Thus a reflexive Banach space with the weak topology is quasi-complete but not complete. We now introduce various classes of locally convex spaces by introducing suitable closure properties of families of seminorms. A set M of such seminorms on E is said to be **sup-closed** (res. **σ-sup-closed**) if the supremum of every pointwise bounded subset (res. every countable, pointwise bounded subset) lies in M. We define weakly sup-closed and weakly σ-sup-closed by demanding that the same properties hold for the corresponding *bounded* subsets of M. A locally convex space (E, S) is **barrelled** if S is sup-closed; **infra-barrelled** if S is weakly sup-closed; **σ-barrelled** if S is σ-sup-closed and finally **σ-infra-barrelled** if S is weakly σ-sup-closed.

It follows from the above Lemma that in the context of quasi-complete spaces, there is no difference between the concepts of barrelledness and infra-barrelledness (res. between σ-barrelledness and σ-infra-barrelledness).

Using duality theory and the correspondence between semi-norms and neighbourhoods of zero, the above definitions can be reformulated as follows:
I. E is barrelled if and only if every barrel in E is a neighbourhood of zero resp. every $S_\beta(E', E)$-bounded subset of E' is equicontinuous;
II. E is infra-barrelled if and only if every barrel in E which absorbs bounded sets is a neighbourhood resp. every $S_\beta(E', E)$-bounded subset of E' is equicontinuous;
III. E is σ-barrelled if and only if for every sequence (U_n) of closed, absolutely convex neighbourhoods of zero in E, $U = \bigcup_{n \in \mathbf{N}} U_n$ is a neighbourhood of zero whenever it is absorbing resp. for every sequence (B_n) of equicontinuous subsets of E', the union $B = \bigcup B_n$ is equicontinuous whenever it is bounded. There are corresponding reformulations of the definition of σ-infra-barrelledness.

We shall now show that suitable forms of the Banach-Steinhaus theorems are valid for these types of spaces.

Proposition 16 *If E is a σ-barrelled (resp. a σ-infra-barrelled) and (H_n) is a sequence of equicontinuous sets of linear mappings from E into a locally convex space F, then its union $\bigcup_{n \in \mathbf{N}} H_n$ is equicontinuous if it is pointwise bounded (resp. bounded). Hence if E is σ-barrelled and (T_n) is a pointwise convergent sequence of continuous linear mappings from E into F, the limit mapping is also continuous.*

Here the subset M of $L(E, F)$ is **pointwise bounded** if for each x in E, the set $Mx = \{Tx : T \in M\}$ is a bounded subset of F and it is **bounded** if for each bounded set B in E, $M(B) = \{Tx : T \in M, x \in B\}$ is bounded in F. The Lemma follows immediately from the simple observation that such a set M is pointwise bounded (resp. bounded, resp. equicontinuous) if and only if for each equicontinuous subset A of F', $M'(A)$ is weakly bounded (resp. strongly bounded, resp. equicontinuous) in E' (where M' denotes the set of mappings $\{T' : T \in M\}$).

Most of the spaces that we will have course to use will be barrelled. This follows from the fact that Fréchet spaces are barrelled and that the class of barrelled spaces have strong stability properties. The first assertion follows from the Baire category theorem (we essentially proved it for the case of banach spaces above). With regard to the second, it follows immediately from the definition that the inductive structure defined by a family of barrelled spaces is barrelled. Hence quotients and direct sums are barrelled. It is also true that products of barrelled spaces are barrelled as we shall now prove:

Proposition 17 *Lemma Let $(E_\alpha)_{\alpha \in A}$ be a family of locally convex spaces and let p be a lower semi-continuous seminorm on $\prod_{\alpha \in A} E_\alpha$. Then p has finite support i.e. there is a finite subset J of A so that $p(x) = 0$ for each $x = (x_\alpha)$ with $x_\alpha = 0$ for $\alpha \notin J$.*

PROOF. The statement is clearly equivalent to the fact that U_p contains $\prod_{\alpha \notin J} E_\alpha$. Since this set is closed and $\oplus_{\alpha \notin J} E_\alpha$ is dense in this product, this in turn is equivalent to the fact that U_p contains $\oplus_{\alpha \notin J} E_\alpha$. If this were not the case for any finite subset J of A, then we could construct inductively a sequence (x_n) in $\oplus_{\alpha \in A} E_\alpha$ so that 1) $x_n \notin nU_p (n \in \mathbf{N})$;

2) if $J_n = \{\alpha \in A : (x_n)_\alpha \neq 0\}$, then the J_n's are disjoint. now let $K_n - \{\lambda_\alpha (x_n)_\alpha : \alpha \in J_n, |\lambda_\alpha| \leq 1\} \subset \prod_{\alpha \in J_n} E_\alpha$. Each K_n is absolutely convex and compact and hence so is the product $K = \prod_n K_n$ as a subset of the Cartesian product). But p is not bounded on K and this is a contradiction.

■

Using this Lemma, we can show that products of families of barrelled spaces are barrelled by reducing to the case of finite products which coincide with direct sums.

Similar remarks apply to other types of barrelledness condition. Of course, it is not true that closed subspaces of barrelled spaces are barrelled (otherwise every complete locally convex space, as a closed subspace of a product of Banach spaces, would be barrelled).

The next class of spaces to be discussed are the so-called bornological locally convex spaces (we are forced to use this rather turgid notation to distinguish the spaces which we now introduce form the convex bornological spaces discussed above).

Definition: A locally convex space E is **bornological** if every absolutely convex set which absorbs bounded sets is a neighbourhood of zero. Thus normed spaces are bornological. The definition is equivalent to the following variations:

Every metrisable locally convex space is bornological. To prove this we us criterium 4) above. Suppose that T is a linear mapping from a metrisable space E into a Banach space which maps zero sequences into bounded ones. Then if (x_n) converges to zero in E, there is a sequence (λ_n) of positive scalars which tends to infinity and is such that $\lambda_n x_n \to 0$. Then $T(\lambda_n x_n)$ is bounded and so $T(x_n) = \frac{1}{\lambda_n} T(\lambda_n x_n)$ tends to zero i.e. T is continuous.

Inductive limits, in particular direct sums and quotients of bornological spaces are clearly bornological. Closed subspaces of such spaces need not have the same property. The question of when products of such spaces are bornological is rather more delicate. In fact, the problem of where a product $\prod_{\alpha \in A} E_\alpha$ of non-trivial bornological spaces is bornological reduces to the same one for $\mathbf{R}^A$. The latter is of set theoretical nature and the answer depends on the axioms of set theory that one uses.

Proposition 18 *Lemma let A be a set so that $\mathbf{R}^A$ is a bornological locally convex space with the product structure. Then if $\{E_\alpha\}_{\alpha \in A}$ is a family of locally convex spaces and T is a bounded linear mapping from the product of the E_α into a locally convex space F so that T vanishes on each E_α, then T is the zero mapping.*

PROOF. Let $x = (x_\alpha)$ be a typical element of the product. Then $E_x = \prod_{\alpha \in A} \mathbf{R} x_\alpha$ where $\mathbf{R} x_\alpha$ is the subspace of E_α spanned by x_α is a locally convex subspace of the product which is isomorphic to $\mathbf{R}^{A_1}$ where A_1 is the set of those α for which $x_\alpha \neq 0$. Of course, the latter space is a quotient of $\mathbf{R}^A$ and so is also bornological. Now the restriction of T to E_x is bounded and so continuous. Hence T vanishes on E_x (since it is continuous and vanishes on a dense subspace) and so $T(x) = 0$.

■

Proposition 19 *If A is such that $\mathbf{R}^A$ is bornological, then the same is true for each Cartesian product $\prod_{\alpha \in A} E - \alpha$ of bornological locally convex spaces which is indexed by A. In particular, this holds for any countable product $\prod_{n \in \mathbf{N}} E_n$.*

PROOF. consider a bounded linear mapping form the product into a banach space F. Then we claim that there is a finite subset J of A so that T vanishes on each E_α with $\alpha \notin J$ (in other words, T has finite support). For if this were not the case, we could find a sequence (α_n) of distinct elements of A and for leach n in $\mathbf{N}$ in E_{α_n} so that $\|Tx_n\| geqn$. Then if we regard each x_n as an element in the product space in the obvious way, the sequence (x_n) is bounded but its image in F is unbounded.

We can now write the product of the E_α as

$$\prod_{\alpha \in A \setminus J} E_\alpha \times \prod_{\alpha \in J} E_\alpha.$$

By the above lemma, T vanishes on the first term. its restriction to the second term is bounded and so continuous since this is a finite product and so is bornological. If we put the two parts together again, we see that T is continuous.

■

7 (DF)-spaces

This is a class of spaces which is in a certain sense dual to the class of Fréchet spaces (the initials stand for "dual Fréchet"). The definition is based on the following property of the strung duals of Fréchet spaces:

Proposition 20 *Let E be a metrisable locally convex space. Then*

- *E' has a countable basis of equicontinuous sets;*
- *the strong dual $(E', S_\beta(E))$ is σ-infra-barrelled.*

PROOF. 1) is clear since if the sequence (p_n) generates the structure of E, then (B_{p_N}) is a basis for the equicontinuous subsets of E'.
2) Suppose that (U_n) is a sequence of neighbourhoods of zero in E' where $U = \bigcap_n U_n$ absorbs bounded sets. We shall show that U is a $S_\beta(E)$-neighbourhood of zero by constructing a $\sigma(E', E)$-closed, absolutely convex set W which absorbs bounded sets and is contained in U. For by the bipolar theorem, this set is the polar of its polar in E and the latter is a bounded set there.

This is done as follows. For each n in $\mathbf{N}$ we choose a positive number ϵ_n so that $2\epsilon_n B_{p_n} \subset U$ and let C_n denote the absolutely convex hull of $\epsilon_1 B_{p_1} \cup \cdots \cup \epsilon_k B_{p_n}$. By the definition of $S_\beta(E)$, we can find, for each n, a $\sigma(E', E)$-closed, absolutely convex $S_\beta(E)$-neighbourhood W_n of zero which is contained in $\frac{1}{2}U_n$. Then $W_n + C_N$ is contained in U_n and hence $W = \bigcap_n (W_n + C_n)$ is contained in U. This set has the required property. The crucial point is that $W_n + C_n$ is $\sigma(E', E)$-closed as the sum of a closed and a compact set in a locally convex space.
This leads to the following definition:

Definition: A **(DF)-space** is a locally convex space (E, S) so that

- the von Neumann bornology $\mathcal{B}_S$ is of countable type;
- E is σ-infra-barrelled.

A **weak (DF)-space** is a space which satisfies 1) and

- "2')" every bounded sequence in E' is equicontinuous.

In fact, most of our examples will be (DF)-spaces in virtue of possessing property 1) and being barrelled. The more general definition above is used in order to accomodate *all* duals of Fréchet spaces (since these need not be barrelled).

The situation between Fréchet spaces and (DF)-spaces is completely symmetrical– the dual of a (DF)-space (even a weak (DF)-space) being a Fréchet space. For the latter is certainly metrisable and if (f_n) is a strongly Cauchy sequence, then it certainly has a weak limit which is a linear functional on E but not *a priori* necessarily continuous. however, $(fn)4$, being bounded, is equicontinuous and so its pointwise limit is continuous.

We remark that Baire's theorem can be used to show that a Fréchet space which satisfies the countability condition 1) in the definition above is necessarily a banach space. Thus the only spaces which are simultaneously Fréchet spaces and (DF)-spaces are the Banach spaces.

We shall now show that whereas mappings between Fréchet space resp. (DF)-spaces can have a rather complex structure, those between mixed spaces (i.e. from a Fréchet space into a (DF)-space or *vice versa*) are more tractable. More precisely, they facto over a banach space and so are in the spaces of type L_{sb} discussed above.

Proposition 21 *Lemma Let E be a (DF)-space and (U_n) a sequence of neighbourhoods of zero in E. Then there exists a neighbourhood U of zero in E which is absorbed by each U_n.*

PROOF. We can suppose that U_n is closed and absolutely convex. The polar sets U_n^o are bounded in its strong dual which is a Fréchet space and so there exist positive scalars (λ_n) with the property that $\bigcup \lambda_n U_n^o$ is bounded. Hence it is equicontinuous by the second defining property of (DF)-spaces. The polar of this set is the required neighbourhood.

■

Proposition 22 *If T is a continuous linear mapping from a (DF)-space E into a Fréchet space F, then T is in $L_{sb}(E, F)$. In other words, T factors as follows*

$$E \to E_p \to F$$

over some E_p.

PROOF. Let (p_n) be a family of seminorms which generates the structure of F and let V_n be a neighbourhood of zero in E so that $T(V_n) \subset U_{p_n}$. Choose a neighbourhood V as in the Lemma i.e. which is absorbed by each V_+n. Then $T(V)$ is absorbed by each U_{p_n} and so is bounded. The result now follows.

■

Proposition 23 *Let T be a continuous linear mapping from a Fréchet space E into a complete weak (DF)-space F. Then T is in $L_{sb}(E, F)$.*

PROOF. T induces a bilinear form o $E \times F'$ by a standard construction, namely the form

$$u_T : (x, f) \mapsto f(Tx) = T'(f(x)).$$

It is clear that this form is separately continuous and so, by the corollary to the closed graph theorem given above it is jointly continuous (since E and F' are Fréchet space–the latter provided with the strong topology). Thus there exist a seminorm p on E and a bounded disc B in F so that

$$|U_T(x, f)| \leq 1 \quad \text{if} \quad p(x) \leq 1 \quad \text{and} \quad f \in B^o.$$

This translates into the condition $T(U_p) \subset B^{oo}$ for T and this implies the required result.

■

We complete this section with some remarks on stability properties of (DF)-spaces. Finite products of such spaces are clearly also (DF)-spaces but the same does not hold for (non-trivial) infinite products. Direct sums of sequence of (DF)-spaces are (DF) as are quotient spaces. hence the inductive limits of countable spectra of (DF)-spaces have the same property. it is not true in general that subspaces (even closed ones) of (DF)-spaces are (DF).

8 Special classes of locally convex spaces described as inductive or projective limits

In this section we shall consider a number of special types of locally convex space which will occur frequently in the later chapters and are conveniently described by the possibility of representing them as special types of inductive or projective limits:
I. **Nachbin spaces:** We use this as a generic term for various classes of spaces which are defined by growth conditions with respect to suitable weights.

Let S be a locally compact space. A **Nachbin family** on S is a family (u_α) of strictly positive, continuous functions on S which is monotone decreasing. This means that the indexing family A is a directed set and $u_\alpha \geq u_\beta$ whenever $\alpha \leq \beta$. (At this point we remark that the following discussion can be carried out in more generality. Thus it suffices to demand semi-continuity for the functions and the condition that each u_α be strictly positive can be replaced by the demand that for each $s \in S$ there is an α so that $u_\alpha(s) > 0$ and the u_α be non-negative). For each α, we consider the normed space $u_\alpha C^b(S)$ i.e. the set of functions in $C(S)$ for which there is a $K > 0$ so that $|u| \leq K u_\alpha$. This is a normed space with norm defined as the smallest K for which such an inequality holds. In fact, it is a Banach space since the mapping $x \mapsto xu$ is an isometry from $C^b(S)$ onto $u_\alpha C^b(S)$.
This leads to the following definition:

Definition: A **(DF)-space** is a locally convex space (E, S) so that

- the von Neumann bornology $\mathcal{B}_S$ is of countable type;
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Proposition 24 *Lemma Let E be a (DF)-space and (U_n) a sequence of neighbourhoods of zero in E. Then there exists a neighbourhood U of zero in E which is absorbed by each U_n.*

PROOF. We can suppose that U_n is closed and absolutely convex. The polar sets U_n^o are bounded in its strong dual which is a Fréchet space and so there exist positive scalars (λ_n) with the property that $\bigcup \lambda_n U_n^o$ is bounded. Hence it is equicontinuous by the second defining property of (DF)-spaces. The polar of this set is the required neighbourhood.

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Proposition 26 *Let T be a continuous linear mapping from a Fréchet space E into a complete weak (DF)-space F. Then T is in $L_{sb}(E,F)$.*

PROOF. T induces a bilinear form o $E \times F'$ by a standard construction, namely the form

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If (E, F) is a dual pair, and B is a subset of E, we define B^o, the **polar** of B in F, by the equation

$$B^o = \{y \in F : |\langle x, y\rangle| \leq 1 \text{ for } x \text{ in } B\}.$$

Similarly, we can define the polar in E of a subset of A of F (also written A^o). Note that if $B \subset E$ is bounded for the duality, then the polar of B is the unit ball of the seminorm p_B.

The following rules for the manipulation of polars follow immediately from the definition:

- $(\lambda B)^o = \left(\frac{1}{\lambda}\right) B^o$;
- if $(B_\alpha)_{\alpha \in A}$ is a family of subsets of E, then

$$\left(\bigcup_\alpha B_\alpha\right)^o = \bigcap B_\alpha^o$$

  ;
- if $A \subset B$, then $B^o \subset A^o$;
- if M is a *subspace* of E, then M^o coincides with the annihilator of M i.e. the space

$$\{y \in F : \langle x, y\rangle = 0 \text{ for } x \in M\}$$

  and so, in particular, is also a subspace.

We remark that the polar B^o of a set, being the intersection of sets of the form $\{y \in F : |\langle x, y\rangle| \leq 1\}$ which are weakly closed, is itself weakly closed. In fact, this property characterises those absolutely convex sets which are polars as the following result states:

Proposition 27 *The bipolar B^{oo} of a subset of E (i.e. the polar in E of the subset B^o of F) is the $\sigma(E, F)$-closed, absolutely convex hull of B.*

PROOF. By the above remark, $\Gamma(B)$, the weakly closed absolutely convex hull of B, is contained in B^{oo}. Now suppose that $x_0 \notin \Gamma(B)$. Then by the Hahn-Banach theorem applied to E with the weak structure $S_w(F)$ there is a $y \in F$ so that $\langle x, y\rangle \leq 1$ if $x \in B$ and $\langle x_0, y\rangle > 1$. Hence $y \in B^o$ and so $x_0 \not inB^{oo}$.

■

The same proof demonstrates the following:

Proposition 28 *Let B be a subset of a locally convex space E. Then the τ_S-closed, absolutely convex hull of B is its bipolar $B^o o$ in the dual pair (E, E').*

Proposition 29 *Corollaries 1. If S and S_1 are two locally convex structures on a vector space E so that the duals of E under the corresponding topologies coincide, then an absolutely convex subset of E is τ_S-closed if and only if it is τ_{S_1}-closed. 2. A subspace M of a locally convex space E is dense if and only if its polar M^o in E' is the trivial space $\{0\}$. 3. If $\{B_\alpha : \alpha \in A\}$ is a family of weakly closed, absolutely convex subsets of a locally convex space E, then $(\bigcap_\alpha B_\alpha)^o$ is the weakly closed, absolutely convex hull of $\bigcup_\alpha B_\alpha^o$.*

At a later point, we shall require the following simple constructions. Suppose that B and C are pseudo-discs in the vector space E. We define $B \vee C$ to be the convex hull of the union of B and C, resp. $B \wedge C$ to be their intersection. Then both of these sets are pseudo-discs and the corresponding spans satisfy the conditions:

$$E_{B \vee C} = E_B + E_C, E_{B \wedge C} = E_B \cap E_C.$$

The corresponding norms can be expressed in terms of the Minkowski functions of B and C as follows:

$$\| \ \|_{B \vee C} : z \mapsto \inf\{\|x\|_B + \|y\|_C : z = x + y, x \in E_B, y \in E_C\} \quad (1)$$

$$\| \ \|_{B \wedge C} : z \mapsto \max\{\|z\|_B, \|z\|_C\}. \quad (2)$$

This implies that the normed space $E_{B \vee C}$ is isometrically isomorphic to the subspace

$$N_1 = \{(x, x) : x \in E_{B \wedge C}\}$$

of the product $E_B \times E_C$ (provided with the norm

$$(x, y) \mapsto \max(\|x\|_B, \|y\|_C)$$

and that $E_{B \vee C}$ is isometrically isomorphic to the quotient space $E_B \times E_C / N_2$ where

$$N_2 = \{(x, -x) : x \in E_{B \wedge C}\}$$

where $E_B \times E_C$ is now provided with the norm

$$(x, y) \mapsto \|x\|_B + \|y\|_C.$$

This implies that if B and C are discs, then $B \vee C$ is a disc if and only if N_1 (or, equivalently, N_2) is closed in $E_B \times E_C$. Hence if B and C are Banach discs and this condition holds, then $B \vee C$ (and $B \wedge C$) are Banach discs.

Adjoints of linear mappings are defined exactly as in the case of normed spaces. Thus if $T : E \to E_1$ is a continuous linear mapping between locally convex spaces, T' is the bounded mapping from E_1' into E' defined by the formula

$$T' : f \mapsto T \circ f.$$

Similarly we can define the adjoint U^b of a bounded linear mapping between convex bornological spaces. It is continuous for the corresponding locally convex structures.

In order to discuss some simple general properties of such adjoints, it is convenient to return to the context of dual pairs. If (E, F) and (E_1, F_1) are such pairs and T is a linear mapping form E into E_1, an **adjoint** for T is a mapping $U : F_1 \to F$ so that $\langle Tx, y\rangle = \langle x, Uy\rangle$ for each $xinE$ and $y \in F_1$. T can have at most one adjoint and it is necessarily linear. This concept includes those of the adjoint for continuous linear mappings between locally convex spaces resp. bounded linear mappings between regular bornological spaces. In fact, the existence of an adjoint can be characterised by the following continuity condition. T has an adjoint if and only if it is $\sigma(E, F) - \sigma(E_1, F_1)$-continuous. For if it is continuous in this sense, then its locally convex adjoint is an adjoint in the above sense. On the other hand, if T has an adjoint U, then the linear form $x \mapsto \langle Tx, y\rangle$ is weakly continuous (since it is equal to the form $x \mapsto \langle x, Uy\rangle$). Since this is true for each $y \in F_1$, T is weakly continuous.

It follows easily from this that if $T : E \to F$ is a continuous linear mapping between two locally convex spaces, then T is also continuous for the weak topologies.

We remark that the above definition is symmetric in T and U i.e. if U is an adjoint for T, then T is an adjoint for U.

The following simple formulae are useful in the development of duality theory for linear mappings: Here T and U are as above and A is a subset of E while B is a subset of F:

- $T(A)^o = U^{-1}(A^o)$;
- $T(A) \subset B$ implies $U(b^o) \subset A^o$;
- if A is weakly closed and absolutely convex, and B is likewise, then $U(B^o) \subset A^o$ implies that $T(A) \subset B$.

(1) and (2) are proved by simple manipulations with the definitions. (3) follows from an application of the bipolar theorem. In particular, we have the corollary that U is injective if and only if $T(E)$ is weakly dense in E_1.

One of the advantages of the duality theory presented here is the fact that it behaves well with respect to various methods for constructing new spaces. Thus we have:

- if E_1 is a closed subspace of a locally convex space E, then the adjoint of the natural projection

$$\pi : E \to E/E_1$$

  is an isomorphism between the bornological spaces E_1' and the subspace E_1^o of E';

- if E and E_1 are as above (without the proviso that E_1 be closed), then the adjoint of the injection from E_1 into E is the restriction operator from E' into E_1'. Its kernel is E_1^o and so it generates an injection from E'/E_1^o into E_1' which is, in fact an isomorphism between the corresponding convex bornological spaces (this is just the Hahn-Banach theorem in disguise);

- suppose that $\{(E_\alpha, S_\alpha)\}_{\alpha \in A}$ is a family of locally convex spaces. Then each element $f = (f_\alpha)$ of the cartesian product $\prod_{\alpha \in A} E'_\alpha$ defines a linear form

$$f : (x_\alpha) \mapsto \sum_\alpha f_\alpha(x_\alpha)$$

  on the locally convex direct sum E_0. Once again this is an isomorphism between the product (with the product bornology) and the dual of the direct sum.

- in a similar way, one can show that the dual of a cartesian product $\prod_{\alpha \in A} E_\alpha$ of a family of locally convex spaces is naturally isomorphic as a convex bornological space to the direct sum $\oplus_{\alpha \in A} E'_\alpha$ of the duals.

We leave to the reader the task of examining the case of the duals of constructed bornological spaces.

Suppose that (E, S) is a locally convex space. We shall now give a description of its completion by means of duality. We define $\tilde{E}$ to be the set of those linear forms on E' whose restrictions to each B_p ($p \in A$) are $\sigma(E', E)$-continuous. It is easy to see that $\tilde{E}$ is a complete locally convex space with the topology of uniform convergence on the equicontinuous subsets of E' (i.e. the topology induced on E as a subspace of E'^b). Of course, E is embedded in $\tilde{E}$ as a locally convex subspace. The result of Grothendieck is as follows:

Proposition 30 *E is dense in $\tilde{E}$ and so the latter space is its completion.*

Proof. Consider the dual pair (E, E'). By the very definition of $\tilde{E}$, the weak topologies $\sigma(E', \tilde{E})$ and $\sigma(E', E)$ coincide on the equicontinuous sets B_p $(p \in S)$. Hence each B_p is $\sigma(E', \tilde{E})$-compact and so, by Mackey's theorem, the dual of $\tilde{E}$ is E'. This implies the denseness of E in $\tilde{E}$ since it is clearly $\sigma(E, E')$-dense there.

∎

If we apply this result to a locally convex space

We bring some brief remarks on structures on spaces of operators. At first sight, the natural setting consists of spaces of type $L(E, E_1)$ of continuous linear mappings between two locally convex spaces resp. $L_b(F, F_1)$ of bounded linear mappings between two convex bornological spaces. The sad truth is that there is no really satisfying structure on such spaces once one gets away from the setting of normed spaces. We shall find that it will suffice to consider mixed spaces of the following types (where E is a locally convex space and F is a convex bornological space):

- $L_{\text{sb}}(E, F)$ is the space of linear mappings form E into F for which there is a p in S and a B in $\mathcal{B}$ with $T(U_p) \subset B$;
- $L_{\text{wb}}(F, E)$ is the space of linear mappings from F into E which are such that for each B in $\mathcal{B}$ $T(B)$ is bounded in E.

The letters w and s stand for weak and strong. These spaces have convenient limit representations. If E and F are complete and so have representations

$$E = \lim\{E_p : p \in S\} \text{ and } F = \lim\{F_B : B \in \mathcal{B}\},$$

then we have

$$L_{\text{sb}}(E, F) = \lim_{p \in S} \lim_{B \in CalB} L(E_p, F_B)$$

resp.

$$L_{\text{wb}}(E, F) = \lim_{p \in S} \lim_{B \in \mathcal{B}} L(F_B, E_p).$$

At this stage these limits hold in the sense of vector spaces but it is natural to impose on $L_{\text{sb}}(E, F)$ the corresponding bornology as an inductive limit of the Banach spaces $L(E_p, F_B))$ so that they also hold in the category of convex bornological spaces. Thus a subset M of $L_{\text{sb}}(E, F)$ is bounded, by definition if there is p in S and a $B \in \mathcal{B}$ so that $T(U_p) \subset B$ for each T in M. Similarly, the structure on $L_{\text{wb}}(E, F)$ is that of uniform convergence on the sets of $\mathcal{B}$. It follows immediately form the above representations that both of these spaces are complete.

Despite the apparent artificiality of these definitions, in many of the cases which we shall investigate we shall be dealing with pairs (E, E_1) of locally convex spaces where the space $L(E, E_1)$ of continuous linear operators from E into E_1 coincides with either $L_{\text{wb}}(E, E_1)$ or $L_{\text{sb}}(E, E_1)$ where, in the first case we regard E as a convex bornological spaces and in the second E_1 (both with the von Neumann bornology). Thus the former is true if E is a (locally convex) inductive limit of normed spaces (such spaces will be considered in more detail below) and the latter is true if E is a Fréchet space and E_1 is a (DF)-space (or *vice versa*). Once again we refer to later sections for the terminology.

We now define the Nachbin space $N(S; u_\alpha)$ to be the intersection of these spaces, regarded as a locally convex space with the projective limit structure. Thus this is a complete, locally convex space. It is even a Fréchet space in the case that the net (u_α) has a countable, cofinal subfamily.

The typical example is where $S = \mathbf{R}$ and (u_α) consists of the sequence of functions

$$t \mapsto (1 + |t|^2)^{-n}.$$

Then $N(S; u_n)$ is the space of rapidly decreasing continuous functions on $\mathbf{R}$ (i.e. those functions which go to zero at infinity faster than any polynomial).

The following construction is dual to the one above. In this case we have a co-Nachbin family i.e. a net (u_α) as above, with the difference that it is increasing. Then we define the space $CN(A; u_\alpha)$ to be those continuous functions which satisfy an inequality of the type

$$|x| \leq K u_\alpha$$

for some $K > 0$ and some $\alpha \in A$. Thus

$$CN(S; u_\alpha) = \bigcup_\alpha u_\alpha C^b(S)$$

and we regard it as a locally convex space, with the inductive limit structure. Note that all topologies in sight are finer than that of pointwise convergence on S and so there are no problems of non-Hausdorffness for the inductive limit topology.

The obvious modifications of this method can be used to define spaces of measurable functions or sequence which are characterised by growth conditions. Thus if (Ω, μ) is a measure space and (u_α) is a net as before in this context (that is, we assume that the functions are measurable, rather than continuous), then we can define, for any $p \in [1, \infty[$, the spaces

$$u_\alpha L^p(\mu) \text{ and } NL^p(u_\alpha),$$

the first space being the image of $L^p(\mu)$ under the mapping $x \mapsto x u_\alpha$ (with the appropriate norm), the second being the intersection of these spaces. The latter is a complete, locally convex space and is Fréchet if the net (u_α) has a countable, cofinal subset.

Similarly if (u_α) is increasing, we define $CNL^p(u_\alpha)$ to be the union of the $u_\alpha L^p(\mu)$. In this connection, we mention that $CN(S; u_\alpha)$ and $CNL^p(S; u_\alpha)$ can be regarded as convex bornological spaces in the natural way—the above spaces being then the corresponding locally convex spaces. Also we have that $u_\alpha L^p(\mu)$ is just the $L^p(\nu)$-space defined by the measure ??? (with weight ??? with respect to ????). Also we have the following duality results:

If we specialise to the case where the measure space is $\mathbf{N}$ with counting measure, we obtain the famous Köthe spaces (also variously called *Stufenräume* and *gestufte Räume* resp. echelon spaces) which are defined as follows:
(LF)**-spaces:** These are, by definition, locally convex spaces which are obtainable as the inductive limit of a sequence

$$(i_n : E_n \to E_{n+1})$$

of Fréchet spaces, whereby the linking mappings i_n are isomorphisms from the E_n onto a (necessarily closed) subspace of E_{n+1}. It is customary to assume that no i_n is onto (in order the case where the sequence is stationary after finitely many steps, in which case the inductive limit is a Fréchet space). It follows from Baire's theorem that in this case, the inductive limit is never a Fréchet space (see below). Before proceeding further, we remark that such an inductive limit E has many properties of Fréchet spaces. Thus since a linear mapping from E into a space F is continuous if and only it its restriction to each E_n is continuous there, it follows easily that T is continuous if it is sequentially continuous resp. if and only if it is bounded. (Note that we can and will assume that each E_n is a subspace of E_{n+1} so that the set theoretical inductive limit is just the union. Hence it makes sense to talk of the restriction of F to E_n).

The facts about (LF)-spaces which we shall require are the following:

Proposition 31

- *The topology on $E = \bigcup E_n$ is Hausdorff i.e. in the formation of the inductive limit, we do not require to go over to a quotient of the union;*
- *Each E_n is a locally convex subspace of E i.e. the inductive limit topology on the latter induces the original topology on each E_n;*
- *Each E_n is closed in E;*
- *A subset of E is bounded if and only if it is a subset of some E_n (and is bounded there);*

- *A sequence (x_n) in E converges to zero if and only if there is an n so that the sequence lies in E_n and converges to zero there;*

- *a subset of E is compact if and only if it is a subset of some E_n and is compact there;*

- *E is complete.*

Examples of (LF)-spaces are ϕ (regarded as the union of the spaces E_n where the latter is the set of sequences which vanish at those coordinates m which are greater than n) and the space $C_{00}(\mathbf{R})$ of continuous functions with compact support on $\mathbf{R}$. We regard the latter space as the union of the E_n where E_n is the Banach space of those continuous functions on $\mathbf{R}$ with support in $[-n, n]$.

If (E_n) is a sequence of Fréchet spaces, then we can regard their direct sum as an (LF)-space with an obvious generalisation of the method used for ϕ (which is the special case where each E_n is one-dimensional).

The spaces which will most occupy our attention in the theory of distributions are the test functions of Schwartz which are defined as follows:

The epimorphism theorem holds for (LF)-spaces:

Proposition 32 *Let $E = \bigcup E_n$ and $F = \bigcup F_n$ be (LF)-spaces. Then if $T : E \to F$ is a surjective continuous linear mapping, it is open.*

PROOF. We use a typical Baire theorem argument to reduce to the case of a mapping between Fréchet spaces. Let G_{mn} be the subspace $E_m \cap T^{-1}(F_n)$ of E_m. Of course, it is closed. We have $T(G_{mn}) = T(E_m) \cap F_n$ and so, since T is surjective, F_n is the union (over m) of the $T(G_{mn})$. Hence there is an m_0 so that $T(G_{m_0 n})$ is of second category in F_n.

Then it follows form the classical epimorphism theorem between Fréchet spaces that T maps $G_{m_0 n}$ onto F_n as an open mapping. This easily implies the result.

∎

Of course, one can deduce from this result in the usual way a closed graph theorem and isomorphism theorems for mappings between (LF)-spaces.

Further result are about spaces $L(E, F)$, completeness, duality, reflexivity, (DLF)-spaces.

10 Komatsu and Silva spaces:

We now propose to study briefly classes of spaces which in a certain sense are at the opposite extreme from the (LF)-spaces. The pleasant properties of the latter were due to the fact that the linking mappings were isomorphisms. We shall now consider spaces which are defined by spectra of weakly compact and compact mappings.

Definition A projective or inductive spectrum of a sequence of Banach spaces is called a **Komatsu spectrum** if the linking mappings are weakly compact. If they are even norm compact, they are called **Silva spectra**. **An** (FK)**-space** is a locally convex space which is a projective limit of a Komatsu sequence. Dually, a locally convex space which is an inductive limit of an inductive Komatsu sequence is called a (DFK)**-space**. The concepts of (FS)- and (DFS)-spaces are defined accordingly, using Silva sequences.

11 Partitions of unity:

We have seen that most locally convex spaces have natural representations as projective or inductive limits of simpler ones. This means that they are closed subspaces of products resp. quotients of direct sums of these simpler spaces and so inherit those properties which are stable under one of these two sets of conditions. In fact, many of the limits which we shall meet in practice have a richer structure which ensures that they are not only subspaces (resp. quotient spaces) but direct summands i.e. simultaneously subspaces and quotient spaces. This has important consequences for their structure. We begin by recalling some simple facts in connection with complemented subspaces. Recall that a subspace E_1 of a locally convex space is **complemented** if there is a continuous linear projection from E onto E_1 (i.e. P maps E into E_1 and $P(x) = x$ for $x \in E_1$). Note the following:

- if $P \in L(E)$ is a projection i.e. $P^2 = P$, then $E_1 = E$ and $E_2 = \ker P$ are complemented subspaces of E (E_2 is the range of the mapping $(\mathrm{Id} - P)$ which is also a projection;
  - E_1 and E_2 are automatically closed (E_1 is the kernel of $(\mathrm{Id} - P)$, E_2 that of P);
  - the mapping
    $$x \mapsto (Px, x - Px)$$
    is an isomorphism from E onto $E_1 \times E_2$ (its inverse is the mapping $(y, z) \mapsto y + z$);
  - E_1 is isomorphic to the quotient space E/E_1.

The dual situation is as follows: We have a pair E and E_1 of locally convex spaces and a continuous linear mapping $T : E \to E_1$. If T has a right inverse i.e. a continuous linear S from E_1 into E so that $TS = \text{Id}$, (so that T, in particular, must be surjective), then T is a quotient mapping and E_1 is isomorphic to a complemented subspace of E. Further every bounded subset of E_1 is the image under T of a bounded subset of E.

It follows easily form the above considerations that, for example, a complemented subspace of a barrelled space is barrelled. The same remark holds for (DF)-spaces and the various variants of barrelledness.

We shall begin by discussing partitions of unity for inductive limits. Before doing so, we examine some less elementary properties of direct sums. First we note that a direct sum of complete spaces is itself complete. This follows form the following description of the direct sum as a subspace of the (complete) cartesian product.

Two further facts which we shall require are the following:

- a subset B of a direct sum $\oplus_{\alpha \in A} E_\alpha$ of locally convex spaces is bounded if and only if there is a finite subset J of A so that B is contained in the finite product $\prod_{\alpha \in J} E_\alpha$ (regarded as a subset of the direct sum) and bounded there. A similar description of the compact subsets of the direct sum is valid and follows immediately from the first sentence;
- a sequence (x_n) in the direct sum converges to zero if and only if it is contained in a finite product $\prod_{\alpha \in J} E_\alpha$ as in (1) and is convergent there.

We are now ready for the definition of a **partition of unity.** We have an inductive system

$$\{i_{\alpha\beta} : E_\alpha \to E_\beta, \alpha \leq \beta\}$$

of locally convex spaces. i_α denotes the natural map from E_α into E, the limit of this spectrum. A partition of unity for this system is a set (U_α) of continuous linear mappings whereby U_α maps E into E_α and we have

- for each $\beta \in A$ the set of those α in A so that $U_\alpha \circ i_\beta \neq 0$ is finite;
- if $x \in E$, then $x = \sum_{\alpha \in A} i_\alpha \circ U_\alpha(x)$.

The important point about such partitions is that they ensure that the resulting inductive limit is a complemented subspace of the direct sum.

Proposition 33 *With the above notation, the space E is a complemented subspace of the direct sum $\oplus_{\alpha\in A}E_\alpha$.*

PROOF. Consider the mapping

$$U : x \mapsto (U_\alpha(x))$$

from E into the product of the E_α's. In fact U maps E into the direct sum by the first condition and it is clearly linear and continuous. By the second condition it is a right inverse of the natural projection from $\oplus E_\alpha$ onto E.

■

As a Corollary, we obtain the following result on inductive limits with partitions of unity:

Proposition 34 *Corollary With the above notation, E is complete if each E_α is and a subset B is bounded if and only if there is a finite subset J of A and, for each $\alpha \in F$, a bounded subset B of E_α so that*

$$B \subset \sum_{\alpha\in A} i_\alpha(B_\alpha).$$

Corresponding results hold for compact subsets resp. convergent sequences in E.

We now bring some examples of partitions of unity. The standard one is the following: let S be a paracompact locally compact space. The fact that S is paracompact means that we can find a covering (U_α) of S by relatively compact open set, so that for α fixed $U_\alpha \cap U_\beta = \emptyset$ except for finitely many β (i.e. the covering is **locally finite**). Then we can find a so-called partition (ϕ_α) of unity on S which is subordinate to (U_α) i.e. a family of continuous mappings form S into $[0,1]$ so that the support of ϕ_α is contained in U_α for each α and the functions sum to one (note that this sum is finite at each point of S).

We regard the space $C_{00}(S)$ as the limit of the $C_{K_J}(S)$ where the index set is $\mathcal{J}(\mathcal{A})$, the family of finite subsets of A, whereby $K_J = \overline{\bigcup_{\alpha\in J} U_\alpha}$. We define mappings U_J from $C_{00}(S)$ into $C_{K_J}(S)$ by $x \mapsto x(\sum_{\alpha\in J}\phi_\alpha)^{-1}$. This is a partition of unity in the above sense.

We now turn to projective limits. Let

$$\{\pi_{\beta\alpha} : E_\beta \to E_\alpha, \alpha, \beta \in A, \alpha \leq \beta\}$$

be a projective system of locally convex spaces, E its projective limit, π_α the natural projection from E into E_α. A **partition of unity** for this system is a family (T_α) of continuous linear mappings (whereby T_α maps E_α into E) which satisfies the conditions:

- for each $\alpha \in A$, the set of those $\beta \in A$ for which $\pi_\alpha \circ T_\beta \neq 0$ is finite;
- $\sum_\alpha T_\alpha(x) = x$ for each x in E.

Proposition 35 *With the above notation, E is isomorphic to a complemented subspace of the product $\prod_{\alpha\in A} E_\alpha$.*

PROOF. If $x = (x_\alpha)$ is an element of the cartesian product, then $\sum_{\alpha\in A} T_\alpha x_\alpha$ converges in E. For it suffices to show that $\sum_\alpha \pi_\beta T_\alpha(x_\alpha)$ converges in E_β for each $\beta \in A$ and this follows immediately form (1). Thus we can define a linear mapping

$$T : (\xi_a) \mapsto \sum_\alpha T_\alpha(x_\alpha)$$

from the product into E which is clearly continuous (by the same argument). It is also a left inverse for the injection from E into the product i.e. a projection from the cartesian product onto E.

∎

Proposition 36 *Corollary In this situation, E is barrelled if each E_α is. If A is countable, then it is bornological (resp. a (DF)-space) if each E_α is.*

12 Webs:

We now turn to the topic of webs in locally convex spaces. We have seen that the closed graph theorem can be generalised to Fréchet spaces without essential changes. Unfortunately, many of the spaces which occur in applications are not metrisable (in particular, duals and inductive limits of Fréchet spaces are usually no longer metrisable). This led Grothendieck to pose the problem of specifying a large class of locally convex spaces for which these theorems are valid and which is closed under the formation of duals, countable inductive limits and countable projective limits. This problem was elegantly solved by de Wilde who introduced the concept of a web (*reseau*) for locally convex spaces.

Definition A **web** on a vector space E is a family $\{B(n_1, \dots, n_k)\}$ of absolutely convex subsets, indexed by $\mathbf{N}^{(\mathbf{N})} = \bigcup_{k \in \mathbf{N}} \mathbf{N}^k$, the family of finite sequences of natural numbers, so that

$$E = \bigcup_{n_1 \in \mathbf{N}} B(n_1)$$

and

$$B(n_1, \dots, n_{k-1}) = \bigcup_{n_k \in \mathbf{N}} B(n_1, \dots, n_k)$$

for all suitable multi-indices.

If E has a locally convex structure S, then a web is an S**-web** if for each sequence (n_k) of integers, there is a sequence (ρ_k) of positive numbers so that whenever (x_k) is a sequence in E with $x_k \in B(n_1, \dots, n_k)$ for each k, ,then $\sum_k \rho_k x_k$ converges absolutely in E.

Before continuing with the consequences of the existence of a web, we bring some examples. We begin with the remark that every Fréchet space has a web. For if (p_n) is an increasing basis for S, then the sets

$$B(n_1, \dots, n_k) = \bigcap_{r=1}^{k} n_r U_{p_r}$$

form a web as the reader can check (we can take $\rho_k = 2^{-k} n_k$).

We remark further that closed subspaces of spaces with webs have webs (we simply intersect the components of the web with the subspace). Also if $T : E \to F$ is a continuous linear surjection and E has a web, then so does F (we take the images of the components of the web in E). In particular, a quotient of a space with a web has a web. Also if we weaken the topology of a space with a web, then we doe not lose the web.

One can also show that the countable product and countable direct sum of spaces with a web themselves have webs. This and the above implies that the existence of a web is stable under the formation of countable projective or inductive limits.

The concept of a web is designed to allow a proof of the following closed graph theorem:

Proposition 37 *Let E be a Banach space, F a locally convex space with an S-web. Then a linear mapping T from E into F is continuous if and only if it has a closed graph.*

PROOF. Let T be such a mapping. From the first condition on a web, we have

$$E = \bigcup_{n \in \mathbf{N}} T^{-1}(B(n)).$$

Hence there is an n_1 with $T^{-1}(B(n_1))$ of second category. Continuing in the obvious way, we construct a sequence (n_k) so that $T^{-1}B(n_1, \dots, n_k)$ is of second category for each k. Let (ρ_k) be a sequence of positive numbers as in the definition of an S-web. Choose a $p \in S$. By the usual argument in the proof of the closed graph theorem, $\overline{T^{-1}(U_p)}$ is a neighbourhood. Hence the proof will be complete if we can show that

$$\overline{T^{-1}(U_p)} \subset (1+\epsilon)T^{-1}(U_p)$$

for some suitable positive number ϵ. In fact, we can prove that this is the case for *any* positive ϵ. To do this, note that for each k in $\mathbf{N}$ there is an integer m_k so that

$$T^{-1}(B(n_1, \dots, n_k) \cap m_k U_p)$$

is of second category. We make the ρ_k smaller, if necessary, so that we have the inequality $\sum_k \rho_k m_k < \epsilon$. Now put

$$D_k = T^{-1}(\rho_k B(n_1, \dots, n_k) \cap \rho_k m_k U_p).$$

since $\overline{D_k}$ is an absolutely convex, closed set which is not of first category, its interior is non-empty and so, by the usual argument, contains $\epsilon_k B_k$ for some $\epsilon_k > 0$. We can assume that ϵ_k decreases to zero. We now proceed to show that

$$\overline{T^{-1}(U_p)} \subset (1+\epsilon)T^{-1}(U_p).$$

Choose x_0 from the left hand side. Then there is an $x_1 \in T^{-1}(U_p)$ with $\|x_1 - x_0\| \leq \epsilon_1$ and so $x_1 - x_0 \in \bar{D}_1$. Hence there exists an $x_2 \in D_1$ with $\|x_0 - x_1 - x_2\| \leq \epsilon_2$ and so $x_0 - x_1 - x_2 \in \bar{D}_2$. Continuing inductively, we get a sequence (x_i) with

$$x_{k+1} in D_i \text{ and } x_0 - \sum_{r=1}^{k} x_r \in \epsilon_k B_E \subset \bar{D}_k.$$

Then $x_0 = \sum_k x_k$. On the other hand,

$$T(x_1) \in U_p \text{ and } T(x_{k+1} \in \rho_k B(n_1, \dots, n_k) \cap \sum \rho_k m_k U_p$$

and so $\sum T(x_k)$ converges in F and $\sum_k T(x_k) \in (1+\epsilon)U_p$. Then $T(x_0) = \sum T(x_k)$ since T has a closed graph and so $T(x_0) \in (1+\epsilon)U_p$. ∎

It is perhaps worth remarking that in the above proof we have only used the fact that the graph of T is sequentially closed.

As usual, this result has a number of corollaries and variants of which we quote two without proofs.

Proposition 38 *Corollary The same result holds if we assume that E is an ultrabornological locally convex space (i.e. an inductive limit of a spectrum of Banach spaces).*

Proposition 39 *Corollary Let T be a continuous, surjective linear mapping from a locally convex space E with an S-web onto an ultrabornological locally convex space. Then T is open. In particular, if T is injective, then it is an isomorphism.*

We remark briefly on webs in convex bornological spaces.

We close this section with a factorisation theorem for weakly compact mappings which will be useful later. We shall use the following criterium for weak compactness:

Proposition 40 *Lemma Let K be a weakly closed subset of a Banach space E so that for each positive ϵ there is a weakly compact set K_ϵ so that $K \subset K_\epsilon + B_E$. Then K is itself weakly compact.*

PROOF. Since K is then obviously bounded, its $\sigma(E'', E')$-closure in E'' is $\sigma(E'', E')$-compact. Hence it is sufficient to show that this closure actually lies in E and coincides with K). But for each $\epsilon > 0$, the weak closure lies in the set $K_\epsilon + \epsilon B_{E''}$ (since the latter set is closed as the sum of two compact sets). Hence each point in this closure has distance at most ϵ from E. Since this holds for each positive ϵ, it follows that each point in the closure is in E.

■

From this we can deduce the following version which is valid for locally convex spaces:

Proposition 41 *A weakly closed subset of the complete locally convex space E is weakly compact if the following condition is satisfied: for each $p \in S$, there is a weakly compact subset K_p so that K is contained in $K_p + S_p$.*

The following version which is valid for compact subsets of a Banach space is more elementary.

Proposition 42 *A closed subset K of a Banach space is norm compact if for each positive ϵ there is a norm-compact set K_ϵ so that $K \subset K_\epsilon + \epsilon B_E$.*

Proposition 43 *Let K be an absolutely convex, weakly compact subset of the unit ball of a Banach space $(E, \| \ \|)$. Then there is a closed, absolutely convex subset C of B_E which contains K and is such that $(E_C, \| \ \|_C)$ is reflexive. If K is compact, we can construct C so that K is compact in E_C and C is compact in $(E, \| \ \|)$. In this case, $(E_C, \| \ \|_C)$ is separable.*

PROOF. Denote by W_n the closed, absolutely convex subset $2^n K + \dfrac{1}{n} B_E$ of E and by $\| \|_n$ its minkowski functional. The latter is equivalent to the original norm and so E_n, the space E with W_n as unit ball, is a Banach space. Now put

$$C = \{x \in E : \sum \|x\|_n^2 \leq 1\}.$$

This is a closed, absolutely convex subset of E and a simple calculation show that K is contained in C. Also we have the inclusion

$$C \subset 2^n K + \frac{1}{n} B_E$$

for each n. Hence C is weakly compact by the above lemma. We shall show that the weak topologies $\sigma(E, E')$ and $\sigma(E_C, {E_C}')$ coincide on C and this will conclude the first part.

We begin by noting that the diagonal mapping

$$x \mapsto (x, x, x, \dots)$$

is an isometric embedding from E_C onto a closed subspace of the ℓ^2-sum $F = \ell^2 \sum (E_n)$ of the E_n. Hence, since the dual of the latter is $\ell^2 \sum {E_n}'$, it suffices to show that $\sigma(F, F')$ coincides with $\sigma(E, E')$ on C. Now the direct sum $\oplus {E_n}'$ is a norm dense subset of F' and so the weak topology $\sigma(F, \oplus {E_n}')$ agrees with $\sigma(F, F')$ on the bounded set C. But the latter topology coincides with $\sigma(E, E')$ on C since the restriction of a form on $\oplus {E_n}'$ to C is defined by an element of E'.

We now turn to the second part. If K is norm-compact, then it follows as above that C is also norm-compact. We show that K is compact in E_C. It will suffice to show that it is precompact. Choose a positive ϵ. We shall construct an ϵ-net for K (with respect to $\| \|_C$). First note that if $x \in K$, then $\|x\|_n \leq 2^{-n}$. Hence there is an $N > 0$ so that $\sum_{n=N+1} \|x_n\|^2 \leq \frac{\epsilon^2}{3}$ for each $x \in K$. Now the norm $\left(\sum_{n=1}^N \| \|_n^2\right)^{1/2}$ is equivalent to $\| \|$ and so there is a finite set $\{x_1, \dots, x_k\}$ in K so that for each element x of K there is an i with

$$\left(\sum_{n=1}^{N} \|x - x_i\|^2\right) < \frac{\epsilon^2}{3}.$$

Then we have $\|x - x_i\|_C \leq \epsilon$.

To show that $(E_C, \| \|_C)$ is separable, note that since C is norm compact, the norm topology agrees with the weak topology $\sigma(E, E')$ on C and so the latter is metrisable. However, as we know, $\sigma(E, E')$ agrees with $\sigma(E_C, E_C{}')$ on C and so the latter is also metrisable. From this it follows that $(E_C, \| \|_C)$ is separable.

■

With these results we can now prove the following factorisation theorem:

Proposition 44 *Let E and F be Banach spaces, T a continuous linear operator from E into F. Then*

- *if T is weakly compact, it factors over a reflexive Banach space i.e. there is a reflexive Banach space G and continuous (and hence weakly compact) operators*

  $$R : E \to G, \quad S : G \to F$$

  so that $T = S \circ R$;
- *if T is compact, we can find G, R and S as above where G is now separable and reflexive, while R and S are compact.*

PROOF. We prove (1)—the proof of (2) is similar. Since T is weakly compact, the closure K of $T(B_E)$ is an absolutely convex, weakly compact subset of F. It is no loss of generality to assume that it is contained in B_F. Let C now be the set constructed in the above proof. Then we can take G to be F_C, S to be the natural injection form F_C into F and R to be T, regarded as a mapping from E into F_C.

■

13 Distributions on $\mathbf{R}^n$

Of course the methods used above can be carried over to the case of distributions on $\mathbf{R}^n$ and we conclude this section with some brief remarks on this subject. We use the following notation: a **compact interval** in $\mathbf{R}^n$ is a set of the form $I_1 \times \cdots \times I_n$ where each I_k is a compact (non-degenerate) interval of the line. If $r = (r_1, \ldots, r_n)$ is a multi-index i.e. an element of $\mathbf{N}^n$, then $C^r(I)$ denotes the set of those x in $C(I)$ for which the partial derivative $D^r(= D_1^{r_1} \ldots D_n^{r_n})$ exists and is continuous. $C^r(I)$ has a natural Banach space structure and D^r is a continuous linear mapping from it into $C(I)$. D^r has a continuous right inverse I^r, which is implemented by repeated integration. We write $P^r(I)$ for the set of functions of the form $p_1 + \cdots + p_n$ where p_j is a polynomial in s_j of degree at most $r_j - 1$ whose coefficients are continuous functions in the remaining variables

$$s_1, \ldots, s_{j-1}, s_{j+1}, \ldots, s_n.$$

A **space of distributions on** I is a vector space $C^{-\infty}(I)$ which contains $C(I)$ as a vector subspace and is provided with a sequence $(D_1, \ldots, D_n)$ of partial differentiation operators which coincide with the classical ones on the smooth functions. Further we suppose that each x in $C^{-\infty}(I)$ has a representation of the form $x = \tilde{D}^r X$ for some continuous function X and some multi-index r. If x is a distribution which is simultaneously a continuous function, then $\tilde{D}^r x = 0$ implies that x is in $P^r(I)$.

The reader will not be surprised to learn that such a space exists and is essentially uniquely determined by these properties. This is proved by means of a construction which is analogous to the one used for distributions on the line. We remark only that the following properties of $P^r(I)$ are useful in its construction:

- if x is in $P^r(I)$, then $I^p(x)$ is in $P^{r+p}(I)$;
- if x is in $P^r(I)$ and in $C^p(I)$, then $D^p(x) \in P^{r-p}(I)$;
- $P^r(I)$ is closed in $C(I)$.

Of course, only (3) is non-trivial and we leave its proof to the reader.

We can regard integrable functions as distributions, in particular, the Heaviside function

$$H_0^n : (s_1, \ldots, s_n) \mapsto H(s_1) \ldots H(s_n).$$

This allows us to define the delta-function

$$\delta_0 = D_1 \dots D_n H_0^n.$$

An obvious variation on the construction above then allows us to consider Radon measure as distributions. Translations, tensor products, restrictions are then defined as follows;

If U is an open subset of $\mathbf{R}^n$, we define $C^{-\infty}(U)$ to be the set of compatible families (x_I) where x_I is a distribution on I and I runs through the family of all compact intervals in U.