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Risk measurement models in finance management

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Abstract

In many practical investment situations the amount of available memory stock data is extremely huge. Thus many investors are attracted to base their decisions on the information "currently available in their minds" (see Nocetti (2005,2006)). The main aim of this letter is to provide some dimensionality or complexity reduction analytic tool which is based on the information theory concepts. Particularly, we discuss a various risk measurement models possibly of application in the risk management. First we recall the model of Markowitz who gave the concept of mean variance efficient frontier to find all efficient portfolios that maximize the expected returns and minimize the risk. We also recall Markovian risk measures. Some measures of portfolio analysis based on entropy mean-variance frontier and maximum entropy model in risk sharing are proposed and studied. Risk aversion index and Pareto-optimal sharing of risk sharing are explained. In view of these facts it is very interesting to study how the investor should make investments so that his total expected return is maximized and risk of loosing his capital is minimized. We also link these measures with its probabilistic rationale.

1 Introduction

Every investor wants to maximize his profits by selecting proper strategy for investment. There are investments like government and bank securities, real estate, mutual funds and blue chips stocks which have low return but are relatively safe because of a proven record of non-volatility in price fluctuations. On the other hand, there are investments which bring high returns, but may be prone to a great deal of risk and the investor makes loss in case the investment goes sour. To overcome the above mentioned problem the investor should invest his funds in a spread of low and high risk securities in such a way that the total expected return for all his investments is maximized and at the same time the risk of losing his capital is minimized. Since the various outcomes as well as the probabilities of these outcomes and the return on a unit amount invested in each security are known, therefore, there is not much difficulty in maximizing the expected return. However, the main problem is to overcome risk factor. The earliest measure proposed regarding risk factor was variance of the returns on all investments in the portfolio and was based on the argument that risk increases with variance. (8) gave the concept of mean-variance efficient frontier and this enabled him to find all the efficient portfolios which maximize the expected returns and minimize the variance.

Also other standard models of portfolio selection under parameter uncertainty are typically based on the assumption that investors learn about the true data generating process of asset returns using all available information. This assumption requires investors to have up to date databases of extremely large size. (10) argued, however, that many investors do not use databases as econometricians, but make decisions based on the information currently available in their minds. In line with this argument, Nocetti (2005) presents a model where individuals exert mental effort to estimate the parameters of an economic model, by retrieving observations from a stock of memories.

In our paper we would like to provide approaches how to reduce the "dimensionality of the problem" of finding the optimal investment, which is typically not uniquely defined. The main principle we use is the information theory approach. (6) made a brief account of application of entropy optimization principles in minimizing risk in portfolio analysis. (4) have applied these principles in characterizing crop area distributions for optimal yield. In this paper we make a study of various risk measure models and discuss their utilities in finance management. The paper is organized as follows: In section 2 we discuss the Markowitz Mean-Variance-Efficient Frontier and interpret it in the context of maxEnt optimization. In section 3 we discuss the Maximum Entropy Mean-Variance Frontier. In section 4 we introduce the concept of Markovian risk measure. In section 5 we discuss the Risk Aversion Index. In section 6 we discuss the Pareto-Optimal Sharing of Risks. Finally in section 7 the Maximum Entropy principle in Risk Sharing is discussed.

2 Markowitz Mean-Variance-Efficient Frontier

Let p_j be the probability of j-th outcome for $j = 1, 2, ..., m$ and r_{ij} be the return on i-th security for $i = 1, 2, n$, when j-th outcome occurs. Then the expected return on the $i - th$ security is $r_i = \sum_{j=1}^{m} p_j r_{ij}$, $i = 1, 2, , n$. Variance and covariance of returns are given by $\sigma_i^2 = \sum_{j=1}^{m} p_j (r_{ij} - r_i)^2, i = 1, 2, , n$ and $\rho_{ik}\sigma_i\sigma_k = \sum_{j=1}^{m} p_j (r_{ij} - r_i)(r_{kj} - r_k), i, k = 1, 2, , n; i \neq k$. A person decides to invest proportions $x_1, x_2, , x_n$ of his capitals in n securities. If $x_i \geq 0$ for all i and $\sum_{i=1} x_i = 1$, then the mean E and variance V of the expected returns are driving indicators for investor.

Markowitz suggested that $x_1, x_2, , x_n$ be chosen to maximize E and to minimize V or alternatively, to minimize V keeping E at a fixed value.

Corresponding to each vector $(x_1, x_2, , x_n)$, there are certain values of E and V, so that corresponding to each portfolio, there is unique point in the E-V plane.

3 Maximum Entropy Mean-Variance Frontier

One of the investor's objective is to diversify his portfolio so that out of all points on the mean-variance efficient frontier, he chooses that portfolio for which his investments in different stocks as equal as possible i.e. to make $R_1, R_2, , R_m$ as equal as possible among themselves. Any departure of $R_1, R_2, .., R_m$ from equality is considered a measure of risk which can be minimized if we choose $x_1, x_2, , x_n$ so as to maximize the entropy measure $-\sum_{j=1}^{m} \frac{R_j}{\sum_{j=1}^{m} R_j} \ln[\frac{R_j}{\sum_{j=1}^{m} R_j}]$. Since this does not include p_j's, therefore, we can modify the principle to say that $p_j R_j$'s should be as equal as possible i.e. the entropy of the probability distribution $p_j R_j / \bar{R}$ should be as large as possible, where $\bar{R}$ is the mean return on investment. For this we maximize $-\sum_{j=1}^{m} \frac{p_j R_j}{\bar{R}} \ln[\frac{p_j R_j}{\bar{R}}]$ subject to $\sum_{j=1}^{m} p_j R_j = \bar{R}$. Applying Lagrange's method of multipliers, we get $p_j R_j = \bar{R}/m$. Thus according to our first principle $R_j = \bar{R}$, while according to second principle $R_j = (1/p_j)\bar{R}/m$. If $p_j = 1/m$ i.e. if the outcomes are equally likely, the two principles give the same results. Again since we want R_j's to be as equal as possible we want the probability distribution $P_j = \frac{p_j R_j}{\bar{R}}$ to be as close to the probability distribution p_j as possible. So we chose $x_1, x_2, , x_n$ to minimize either $D(P_j, p_j)$ or $D(p_j, P_j)$. If we use Kullback and Leibler 's measure (see (7)), then we have $D(P_j, p_j) = \sum_{j=1}^{m} p_j R_j \ln R_j - \ln \bar{R}$. Since $\ln \bar{R}$ is constant, therefore, it implies that $\sum_{j=1}^{m} p_j R_j \ln R_j$ should be as small as possible. This is the third principle. Next to minimize $D(p_j, P_j)$ we again apply Kullback-Leibler's measure and get $\sum_{j=1}^{m} p_j \ln(p_j/P_j)$ or $\sum_{j=1}^{m} p_j \ln(p_j R_j)$ should be as small as possible, which is fourth principle. We can also use Harvda and Charvat's measure of directed divergence or cross-entropy (see (3)). In that case we have to minimize $\frac{1}{\alpha-1}(\sum_{j=1}^{m} P_j^{\alpha} p_j^{1-\alpha} - 1)$ or $\frac{1}{\alpha-1}(\sum_{j=1}^{m} p_j^{\alpha} P_j^{1-\alpha} - 1)$. Thus according to 5th and 6th principle, we choose $x_1, x_2, , x_n$ to minimize respectively $\frac{1}{\alpha-1} E(R^{1-\alpha} - 1)$ or $\frac{1}{\alpha-1} E(R^{\alpha} - 1)$ where $R = p_j/P_j$.

Corollary 1 *By employing of the maximum entropy principle the investor minimizes the risk of any departure of $R_1, R_2, .., R_m$ from equality. Then Thus according to our first principle $R_j = \bar{R}$, while according to second principle $R_j = (1/p_j)\bar{R}/m$. If $p_j = 1/m$ i.e. if the outcomes are equally likely, the two principles give the same results.*

4 Markovian risk measure

Particularly, one can be interested in the risk measure based on the Markov inequality, so called Markovian risk measure (see (2)) The Markov inequality can be generalized as follows. Let X be a random variable, $a \in R$. Let $\Phi(x, y)$ be any Lebesgue measurable bivariate function and $\nu(x)$ any non-negative and non-decreasing function such that $E[\nu(X)] < \infty$ and $E[\Phi(X, y)\nu(X)] < \infty$ for all relevant y. Then

$$P\{X \geq a\} \leq \frac{E[\Phi(X, a)\nu(X)]}{E[\nu(X)]} \tag{1}$$

Supposing $X \geq 0$ with probability 1, $a \geq 0$, $\Phi(x, y) = x^r/y^r$ and $\nu = 1$ in Markov inequality, one gets classical Markov inequality as a special case of (1).

By a risk measure π we understand a mapping from the set of risk (random) variables to the set of real numbers.

In what follows we put $X = S$, where S is a risk variable. Using (1) it is not difficult to prove that, under certain conditions, for some $\alpha, 0 \leq \alpha \leq 1$ there exists a minimal value π_M such that

$$P[S > \pi_M] \leq E[\Phi(S, \pi_M)\nu(S)]/E[\nu(S)] \leq \alpha \leq 1.$$

This value is the solution of the equation

$$E[\Phi(S, \pi_M)\nu(S)]/E[\nu(S)] = \alpha$$

and is called a *Markovian risk measure of the risk variable S at level α*. When $\alpha = 1$, the equation

$$E[\Phi(S, \pi_M)\nu(S)]/E[\nu(S)] = 1 \tag{2}$$

is called the *unifying equation.* Many well-known insurance premium principles and corresponding risk measures follow from (2) as special cases, i.e. the mean value principle, the zero-utility premium principle and the Swiss premium calculation principle (see also (2)). This approach can also be used when seeking the "best" (in a sense suitably defined) strategy of insurance companies how to avoid ruin. For details see e.g.(11).

5 Risk Aversion Index

Let us consider a lottery in which the returns are $x_1, x_2, ..., x_n$ with probabilities $p_1, p_2, ..., p_n$ so that mean monetary return is

$$\bar{x} = \sum_{i=1}^{n} p_i x_i. \tag{3}$$

It may be noted that the utility of an amount x is not always proportional to x. If the monetary value is doubled, for some persons the utility increases, but it is less than double of the previous one. Such persons are also called risk-averse and those for which the utility is more than doubled will be called risk-prone. Thus the attitude to risk of every person is characterized by $u(x)$. For risk-averse persons, $u(x)$ increases at a decreasing rate i.e. $u``(x) < 0$ or $u(x)$ is concave function, while for risk-prone persons $u``(x) > 0$ and $u(x)$ is a convex function and for risk-neutral persons $u``(x) = 0$.

Pratt [8] and Arrow [1] defined a risk-aversion index (RAI) as

$$RAI = -(u``(x)/u`(x)). \tag{4}$$

It can be easily verified that if $u(x) = log(x)$, then $RAI = 1/x > 0$ and if $u(x) = e^x$, then $RAI = -1 < 0$ and $RAI = 0$ in case $u(x) = x$.

Next, we explain how the expression (4) can be obtained by two different methods. We define $\bar{x} = \sum_{i=1}^{n} p_i x_i$ as certain monetary equivalent (CME) and also define $\star X$ by $u(X + \star X) = \sum_{i=1}^{n} p_i U(X + x_i)$, where X is the positive initial capital. This can be written as

$$u(X + \ X + \bar{x} - \bar{x}) = \sum_{i=1}^{n} p_i U(X + \bar{x} + x_i - \bar{x}) \tag{5}$$

or

$$u(X+\bar{x}) + (\star X - \bar{x})u`(X+\bar{x}) + ((\star X - \bar{x})^2/2!)u`(X+\bar{x}) + ... = u(X+\bar{x}) +$$

$$+ \sum_{i=1}^{n} p_i(x_i - \bar{x})u`(X+\bar{x}) + \sum_{i=1}^{n} p_i((x_i - \bar{x})^2/2!)u``(X+\bar{x}) + ...$$

Neclecting $(\star X - \bar{x})^2$ and higher orders, we have

$$(\bar{x} - \star X) = -(1/2)(u``(X+\bar{x})\sigma_x^2/u`(X+\bar{x}) = (1/2)RAI\sigma_x^2. \tag{6}$$

Thus CME exceeds $\star X$ by an amount proportional to RAI and this arises due to the attitude to risk of investor. The concept of RAI can be generalized for $u(x, y)$ and we get

$$RAI = r_{11}\sigma_x^2 + 2r_{12}\sigma_x\sigma_y + r_{22}\sigma_y^2, \tag{7}$$

where risk averse functions are

$$r_{11} = -(1/2)(u_{xx}/(u_x^2+u_y^2)^{1/2}, r_{12} = -u_{xy}/(u_x^2+u_y^2)^{1/2}, r_{22} = -(1/2)(u_{yy}/(u_x^2+u_y^2)^{1/2} \quad (8)$$

This can be further generalized for $u(x_1, x_2, ..., x_n)$ to get

$$RAI = -(1/2)(\sum_{i=1}^{n} r_{ii}\sigma_i^2 + 2\sum_{i=1}^{n}\sum_{j=1}^{m} r_{ij}\sigma_i\sigma_j), \quad (9)$$

where

$$r_{ij} = (\partial^2 u/\partial x_i \partial x_j)(\sum_{i=1}^{n}(\partial u/\partial x_i)^2)^{1/2}. \quad (10)$$

If risk aversion index for two variables is 0, then

$$u_{xx}\sigma_x^2 + 2u_{xy}\sigma_x\sigma_y + u_{yy}\sigma_y^2 = 0, \quad (11)$$

which is an elliptic partial equation of second order.

6 Pareto-Optimal Sharing of Risks

A number m of persons agree to share risks in a business on basis of optimal sharing of risks and profits in such a manner that no individual can increase his expected utility without decreasing the expected utilities of others.

Let a risk have n possible states $s_1, s_2, ..., s_n$ with payments $x_1, x_2, ..., x_n$ and with probabilities $p_1, p_2, ..., p_n$. Let payments be partitioned among m individuals whose utility functions are $u_1, u_2, ..., u_m$. Let x_{ij} be the payment of jth individual in case of ith outcome, then the expected utility of this partitioned risk is given by

$$\bar{u}_j = \sum_{i=1}^{n} p_i u_i x_{ij}, j = 1, 2, ..., m, \quad (12)$$

where $\sum_{i=1}^{m} x_{ij} = x_i$.

We can plot the m expected utilities in m dimensional space. If the m expected utilities are negative, then no partition is acceptable because $(0, 0, ..., 0)$ will be preferred by all. In case all u_i's are positive, we maximize

$$\lambda_1\bar{u}_1 + \lambda_2\bar{u}_2 + ... + \lambda_m\bar{u}_m \quad (13)$$

subject to $\sum_{i=1}^{n} \lambda_j = 1$, $\lambda_j > 0$. Thus we get a linear hyperplane

$$\lambda_1\bar{u}_1 + \lambda_2\bar{u}_2 + ... + \lambda_m\bar{u}_m = k(\lambda_1, \lambda_2, ..., \lambda_m). \quad (14)$$

The envelope of this hyperplane gives the equation of the Pareto optimal hyperplane. All points of this hyper-surface are accepted but which point is chosen depends on the relative bargaining power of the partner or they can choosethe point of intersection with the line

$$\bar{u}_1 = \bar{u}_2 = ... = \bar{u}_m. \quad (15)$$

Thus this equitable Pareto optimal sharing can be obtained instead of individual. We can have groups fighting for increasing their social, political or economic utilities and arriving at Pareto Optimal Equilibria. When these equilibria are disturbed, new Pareto optimal positions have to be obtained.

7 Maximum Entropy principle in Risk Sharing

The Pareto optimal boundary gives infinity of solutions and we need one more criterion to get a unique solution. This is possible by considering that payments are divided as Kapur [4] uniformly as possible subject to other constraints. For this we maximize the following measure of entropy as suggested by Kapur [4]:

$$H^\star = -\sum_{i=1}^{n} p_i \sum_{j=1}^{m} x_{ij}/x_i \ln(x_{ij}/x_i) = -\sum_{i=1}^{n} p_i/x_i \sum_{j=1}^{m} x_{ij} \ln x_{ij} + constant \tag{16}$$

Thus out of all Pareto Optimal solutions we choose that one which maximizes H^*. Raiffa [9] has shown that the Pareto Optimal solution is obtained by maximizing $\sum_{j=1}^{m} \lambda_j \bar{u}_j = \sum_{j=1}^{m} \lambda_j \sum_{i=1}^{n} p_i u_j(x_{ij}) = \sum_{i=1}^{n} p_i \sum_{j=1}^{m} \lambda_j u_j(x_{ij})$ subject to $\sum_{j=1}^{m} x_{ij} = x_i, \sum_{j=1}^{m} \lambda_j = 1$. This will determine x_{ij} in term of $\lambda_1, .., \lambda_m$. Since normalization $\sum_{j=1}^{m} \lambda_j = 1$, therefore $H^\star$ is function of $\lambda_1, .., \lambda_{m-1}$. We choose $\lambda_1, .., \lambda_{m-1}$ satisfying $0 \le \lambda_j \le 1$ for $j = 1, 2, .., m-1$ and $0 \le \sum_{j=1}^{m-1} \lambda_j \le 1$ to maximize H^*.

Example 7.0.1 ***Special Case of Exponential Utility Function***
Let us consider $u_j(x) = 1 - \exp(-x/c_j), j = 1, 2, ..., m$. *We maximize* $\sum_{i=1}^{n} p_i \lambda_j (1 - \exp(-x_{ij}/c_j))$ *subject to*

$$\sum_{j=1}^{m} x_{ij} = x_i, \sum_{j=1}^{m} \lambda_j = 1. \tag{17}$$

Following Lagrange's method of multiplier, we get

$$x_{ij}/c_j = x_i/c - \sum_{j=1}^{m} c_j/c \ln(\lambda_j/c_j) + \ln(\lambda_j/c_j), \tag{18}$$

where $c = \sum_{j=1}^{m} c_j$. *Substituting in (16) and differentiating w.r.t.* λ_k

$$\frac{\partial H^*}{\partial \lambda_k} = \sum_{i=1}^{n} p_i \frac{c_k}{\lambda_k} [\sum_{j=1}^{m} (1 + \ln x_{ij} \frac{c_j}{c} - \ln(1 + x_{ij})]$$

Since $\sum_{k=1}^{m} \lambda_k = 1$, *this gives*

$$\frac{c_1(A - B_1)}{\lambda_1} = \frac{c_2(A - B_2)}{\lambda_2} = ... = \frac{c_n(A - B_n)}{\lambda_n} = CA - \sum_{j=1}^{m} B_j C_j \tag{19}$$

where

$$A = \sum_{i=1}^{m} p_i \sum_{j=1}^{m} \ln(1 + x_{ij}) \frac{c_j}{c}, B_k = \sum_{i=1}^{n} p_i \ln(1 + x_{ij}) \tag{20}$$

Using (17, 18, 19) and (20) we can solve for $x'_{ij}s$ and $\lambda'_j s$.

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