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Exact testing of the scale parameter with the missing time-to-failure information

Milan Stehlík

in Cooperation with
Comenius University Bratislava, Department of Probability and Statistics,

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Abstract

We provide the exact likelihood ratio testing procedure of the scale parameter of the Erlang and gamma distribution when there is a missing time-to-failure information. This is an important result because the asymptotical χ^2-test is oversized an thus inappropriate especially for small merged samples. The small merged samples can arise also for a large sample sizes when individual times-to-failure are not available. Data sets with missing time-to-failure data can arise from field data collection systems. Real data and simulated examples are provided to illustrate the methods discussed.

AMS Subject Classification: Primary 62E15; Secondary 62P30, 62H15, 93C41

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1 Introduction

Reliability prediction plays a major role in many reliability programs across government and industry. The reliability prediction is the process of forecasting, from available failure-rate information, the realistically achievable reliability of a part, component, subsystem or system (for details see [1]). Standards based reliability (see e.g. www.weibull.com) predictions relies on defining failure rates for the components of a system based on predefined standards, depending on the types of components, the use environment, the way the components are connected and the reliability prediction standard. These component failure rates are then used to obtain an overall system failure rate. Several standards have been introduced by various governments and industry organizations to assist in conducting this type of analysis.

Complete data indicates that all of the units under the test failed and the time-to-failure for each unit is known. Therefore, complete information is known regarding the entire sample. However, data collection is generally performed passively by the system owner and such type of data collection is often uncontrolled and important details are not always recorded or they can be lost. The actual times-to-failure are often not recorded even though the failure itself has been carefully noted. E.g. for a variety of reasons over 90% of the data in the Reliability Analysis Center (RAC) does not have the individual failure times recorded (see [2]). Many large organizations such as the national airlines or train systems and utility companies develop reliability databases to track the field reliability on the systems they operate and maintain. The magnitude of such efforts often leads to compromises in the level of details tracked on the system and component failures. For the assessment of component reliability, field data has many distinct advantages (see [2]). For all of the advantages of the field data, there are also disadvantages, including incomplete or inaccurate data reporting and others. Several of these disadvantages are described in more detail by [3]. The disadvantage to be addressed in [2] is the fact that the individual times-to-failure are often missing. There has been other research concerned with the use of data with missing attributes. In [7] was developed a simulation model to observe the behavior of grouped data and test an exponential distribution assumption. Coit and Dey (see [4]) have developed and demonstrated a hypothesis test to evaluate an exponential distribution assumption when there is missing time-to-failure data. In [2] the grouped exponential data was modeled using a k-Erlang distribution.

In this paper we provide the exact likelihood ratio (LR) test of the scale hypothesis

$$H_0 : \lambda = \lambda_0 \text{ versus } H_1 : \lambda \neq \lambda_0 \tag{1}$$

to support the reliability prediction in the model with missing time-to-failure information when times-to-failure are exponentially, Erlang or generalized gamma distributed. The data is often available only in the form of r_j collective failures observed T_j cumulative hours with no further delineation or detail available (see [2]). Quantities r_j and T_j are known but the individual failure times are not. Analysts may have many of these merged data records available for the same component.

In particular a reliability practitioner could be interested in conducting the hypothesis (1) test, e.g. to see whether the field reliability has significantly changed from its current level, and λ_0 could be the previously observed exponential parameter. Then, a significant shift in the exponential parameter could trigger an exploratory reliability investigation into failure causes and mechanisms, if it got worse.

Such a test could be useful also for a mean time to failure (MTTF) analysis. A component or a system with exponential lifetime and rate parameter λ has MTTF $1/\lambda$. Moreover, such a test could be employed also by analysis of a system described by a Markov diagram with only one route through the diagram and constant transition rates. Such a system has MTTF consisting of the sum of mean times spent in each state, i.e.

$$MTTF = \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \dots$$

(for more about MTTF see [17]).

Table 1 presents a data set of this type, the data on aircraft indicator lights from the database RAC (see [2]). Individual time-to-failure is not available, however, the total number of failures and the cumulative operating hours are recorded. Here T_j is the jth cumulative operating time with r_j failures, i.e.

$$T_j = X_1 + ... + X_{r_j}$$

where X_i is the ith time-to-failure.

Table 1: Airplane indicator light reliability data

Failures	T_j	Cumulative operating time (hours)
2	T_1	51 000
9	T_2	194 900
8	T_3	45 300
8	T_4	112 400
6	T_5	104 000
5	T_6	44 800

We made the following assumptions

- times-to-failure are independent, identically distributed
- times-to-failure are exponentially (section 2.1), Erlang and gamma (section 2.2) distributed with unknown scale parameter λ. The exponential distribution often occurs in the modelling of the time-to-failure, see e.g. the software-reliability model of Moranda developed further by Gaudoin and Soler (see [8]). Erlang and gamma distribution are also often used time-to-failure distributions (see [2]).
- repair times are insignificant compared to operating time
- system repair does not degrade or otherwise affect the reliability of the unfailed components.

The paper is organized as follows. In Section 2 we discuss the exact LR test of the scale hypothesis (1) and the exact power function of such test. There we consider the all cases of times-to-failure distribution, exponential, Erlang and general gamma. In Section 3 we provide the case study of the time processing dataset plotted in Table 1. Two simulated data examples are also provided to illustrate the methods discussed. To maintain the continuity of the explanation the properties of the Lambert W function are included into the Appendix.

2 Exact test of the scale with the missing times-to-failure

2.1 Times-to-failure are exponentially distributed

When time-to-failure X_i is distributed according to the exponential distribution with unknown scale parameter λ, then $T_j = \sum_{i=1}^{r_j} X_i$ is distributed in accordance with an Erlang distribution with the same scale parameter and shape parameter equal to r_j. Let us shortly remember mentioned distributions. The family of density functions

$$f(y|\lambda) = \frac{\lambda(\lambda y)^{k-1}}{(k-1)!} e^{-\lambda y}, \ y > 0, \ k = 1, 2, 3, ...$$

is referred to as the family of Erlang (k, λ)-distributions. The length of time interval required in order to collect k arrivals from a Poisson process is distributed according to the Erlang distribution with the shape parameter k. Erlang distribution is frequently used in queueing systems (see [9]). The special case of Erlang distribution, Erlang $(1, \lambda)$, is exponential distribution with scale parameter λ.

Let us consider the LR test of the hypothesis (1). Let T_j, $j = 1, .., J$, is the jth cumulative operating time with r_j failures, i.e.

$$T_j = X_{j,1} + ... + X_{j,r_j}$$

where $X_{j,i}$ is the ith time-to-failure corresponding to the jth cumulative operating time. Then the LR Λ of the hypothesis (1) test is defined by

$$\Lambda = \frac{\max_{\lambda \in H_0} \prod_{j=1}^{J} f(t_j, r_j, \lambda)}{\max_{\lambda \in H_1} \prod_{j=1}^{J} f(t_j, r_j, \lambda)} \tag{2}$$

where $H_0 = \{\lambda_0\}$, $H_1 = (0, \lambda_0) \cup (\lambda_0, +\infty)$ and $f(t_j, r_j, \lambda)$ denotes the density of the Erlang (r_j, λ)-distribution.

The cumulative operating time with r_j failures, t_j, $j = 1, .., J$, is Erlang (r_j, λ)-distributed. The application of the Theorem 4 in [13] provides the exact cumulative distribution function of the test statistics. The following theorem provides the LR test statistics and its exact distribution to enable the exact testing procedure of the scale hypothesis (1).

Theorem 1 *Let t_j, $j = 1, .., J$ is the jth cumulative operating time with r_j failures and times-to-failure are independent, exponentially distributed with the unknown scale parameter λ. Then the Wilks statistics of the hypothesis (1) test has the form*

$$-2 \ln \Lambda = 2G_\omega(\lambda_0 \sum_{j=1}^{J} t_j) - 2G_\omega(\omega), \tag{3}$$

where $\omega = \sum_{j=1}^{J} r_j$ and for $u > 0$ let us introduce a function

$$G_u(x) = \begin{cases} x - u \ln x, & \text{for } x > 0, \\ 0, & \text{for } x \leq 0. \end{cases}$$

The exact cumulative distribution function (cdf) of the Wilks statistics $-2 \ln \Lambda$ has under the H_0 the form

$$F_N(\rho) = \begin{cases} \mathcal{F}_\omega(-\omega W_{-1}(-e^{-1-\frac{\rho}{2\omega}})) - \mathcal{F}_\omega(-\omega W_0(-e^{-1-\frac{\rho}{2\omega}})), & \rho > 0, \\ 0, & \rho \leq 0. \end{cases} \tag{4}$$

Here W_k, $k = -1, 0$, is the k-th branch of the Lambert W function (see Appendix) and $\mathcal{F}_\omega$ is the cdf of the gamma distribution with the shape parameter ω and scale parameter 1.

The LR test of the hypothesis (1) is unbiased, uniformly the most powerful (UUMP), see [10]. Moreover, if all $r_j = 1$ then the exact likelihood ratio test is asymptotically optimal in the Bahadur sense (see [11, 12] and [13]).

The Wilks statistics $-2 \ln \Lambda$ of the LR test of the hypothesis (1) is under H_0 asymptotically χ_1^2-distributed (see [16]) and the test based on this asymptotics is oversized. The oversizing of the asymptotical test can be defined as the difference between $\alpha_{e,N}$ and α, where $\alpha_{e,N} = 1 - F_N(\chi^2_{\alpha,1})$ and $\chi^2_{\alpha,1}$ denotes $(1-\alpha)$-quantile of the asymptotical χ_1^2-distribution, F_N is the exact cdf of the Wilks statistics $-2 \ln \lambda$ of the LR test of the hypothesis (1) under the H_0. Here α is the size of the test given from the Wilks asymptotics while $\alpha_{e,N}$ is the exact size of the same test. The table giving the oversizing of the asymptotical test for small samples when the observations are distributed exponentially can be found in [14].

In other words, the oversizing of the asymptotical test means, that for critical constants the inequality holds

$$c_{a,\alpha} < c_{N,\alpha},$$

where $c_{a,\alpha}$ is the critical constant of the χ_1^2 α-sized asymptotical test based on the Wilks statistics and $c_{N,\alpha}$ is the critical constant of the α-sized exact test based on the the Wilks statistics when the sample size is N.

It means that for all λ_0 such that

$$c_{a,\alpha} < -2\ln\Lambda \leq c_{N,\alpha}$$

we reject the null hypothesis $H_0 : \lambda = \lambda_0$ statistically incorrectly. In other words, there is no α-level significant change in reliability driven by changes of λ_0 but we oversize the level of significance of the test by inappropriate use of χ^2_1-asymptotics.

The determination of the hypothesis (1) LR test power function based on the χ^2-asymptotics is also inaccurate for small samples. Therefore we recommend to compute the exact power of the LR test of the hypothesis (1) by the use of the following theorem:

Theorem 2 *The exact power $p(\gamma, \alpha)$ of the LR test based on the Wilks statistics of the hypothesis (1) in the situation considered in Theorem 1 on the level α at the point λ of the alternative has the form*

$$p(\lambda,\alpha) = 1 - \mathcal{F}_\omega(-\omega\frac{\lambda}{\lambda_0}W_{-1}(-e^{-1-\frac{c_\alpha}{2\omega}})) + \mathcal{F}_\omega(-\omega\frac{\lambda}{\lambda_0}W_0(-e^{-1-\frac{c_\alpha}{2\omega}})), \quad (5)$$

where c_α denotes the critical value of the exact test of the hypothesis (1) on the level α.

Proof

The critical region based on the Wilks statistics of the LR test of the hypothesis (1) on the level of significance α has the form

$$W_c = \{t \in T : -2\ln\Lambda(t) > c\} \text{ such that } P\{W_c|\lambda = \lambda_0\} = \alpha,$$

where T denotes the sample space. The power $p(\lambda_1, \alpha)$ of the test of the hypothesis (1) at the point λ_1 of the alternative is equal to $P\{W_c|\lambda = \lambda_1\}$. Applying Theorem 4 in [13] we obtain the equality

$$1 - P\{W_c|\lambda = \lambda_1\} = \mathcal{F}_\omega(-\omega\frac{\lambda_1}{\lambda_0}W_{-1}(-e^{-1-\frac{c_\alpha}{2\omega}})) - \mathcal{F}_\omega(-\omega\frac{\lambda_1}{\lambda_0}W_0(-e^{-1-\frac{c_\alpha}{2\omega}})).$$

The other possibility is employing of the Theorem 4 in [15]. Thus we obtain (5). This completes the proof. □

2.2 k-Erlang and gamma distributed times-to-failure

In this subsection we discuss the case when component time-to-failure is distributed in accordance with a gamma distribution. The Erlang (k, λ)-distribution is a special case of gamma distribution with shape parameter $k = 2, 3, ..$ and scale parameter λ.

The limitations of the field data and the simplicity of the exponential maximum likelihood estimator (MLE) have been used to rationalize the exponential distribution in applications where it would seemingly be a poor choice (for more see [2]). The constant hazard function associated with the exponential distribution is not intuitively appropriate for some failure mechanisms which can be attributed to the accumulation of stress, such as fracture, fatigue, corrosion and wear mechanisms.

The gamma distribution is a flexible distribution that can model many particular component failure mechanisms.

Actually we have proved more by proofs of Theorem 1 and 2. Thus we can constitute the following statement about the LR exact test of the scale hypothesis (1) for the general case when the operating times are gamma distributed.

Theorem 3 *Let t_j, $j = 1, .., J$ is the sequence of independent cumulative operating times and t_j is gamma distributed with known shape parameter r_j and unknown scale parameter λ. Then the Wilks statistics of the hypothesis (1) test has the form (3) where $\omega = \sum_{j=1}^{J} r_j$. The exact cumulative distribution function of the Wilks statistics $-2\ln\Lambda$ has under the H_0 the form (4). The exact power $p(\gamma, \alpha)$ of the LR test based on the Wilks statistics of the hypothesis (1) on the level α at the point λ of the alternative has the form (5) where c_α denotes the critical value of the exact test of the hypothesis (1) on the level α.*

3 Illustrative examples

The first example is the airplane indicator light example presented in Table 1. This data is from RAC database (see [2]). The other examples use simulated data.

3.1 Real data example

In [2] we can found the MLE of shape parameter ($\hat{v} = 0.7$) and scale parameter ($\hat{\lambda} = 0.0000484$) of the gamma distributed times-to-failure of the airplane indicator light data presented in Table 1. We have $\omega = \sum_{j=1}^{6} r_j = 38 \times 0.7 = 26.6$ and $\sum_{j=1}^{6} T_j = 552\ 400$. Let us consider the testing problem

$$H_0 : \lambda = 0.00003207 \text{ versus } H_1 : \lambda \neq 0.00003207 \tag{6}$$

at the level of significance $\alpha = 0.05$. The critical value $c_{0.05}$ of the exact LR test of the hypothesis (6) is $c_{0.05} = 3.86550298$. The value of the Wilks statistics of the LR test of the hypothesis (6) is $-2\ln\Lambda = 3.855303 < c_{0.05}$. Therefore the null hypothesis is accepted at the level 0.05.

The power function $p(\lambda, 0.05)$ of the LR test of the hypothesis (6) has the form

$$1 - \mathcal{F}_{26.6}\Big(-\frac{26.6\lambda}{0.00003207} W_{-1}(-e^{-\frac{57.06550298}{53.2}})\Big) + \mathcal{F}_{26.6}\Big(-\frac{26.6\lambda}{0.00003207} W_0(-e^{-\frac{57.06550298}{53.2}})\Big)$$

and is for $\lambda \in (0.00001, 0.00008)$ displayed in Figure 1.

Let us illustrate the oversizing of the asymptotical χ_1^2-test. The critical value of the χ_1^2-test is $c_{a,0.05} = 3.841459$ and

$$c_{a,0.05} < c_{0.05}.$$

It means that for all λ_0 such that

$$c_{a,0.05} < -2\ln\Lambda \leq c_{0.05}$$

we reject the null hypothesis $H_0 : \lambda = \lambda_0$ statistically incorrectly. In other words, there is no change in reliability driven by λ_0, but we oversize the level of significance of the test by inappropriate use of χ_1^2-asymptotics. In our example such a situation appears for instance if $\lambda_0 = 0.00003207$.

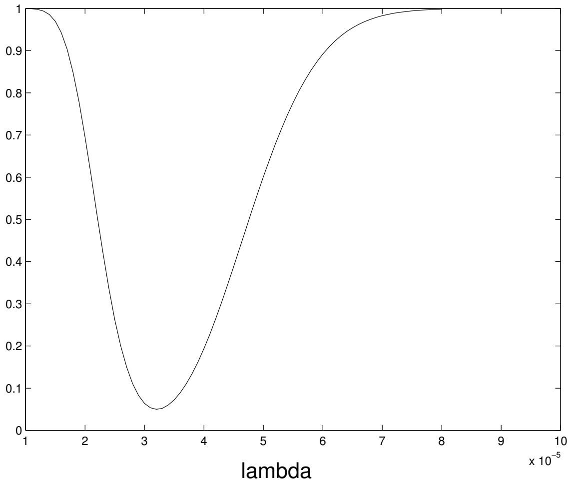

Figure 1: The power function.

3.2 Simulated examples

Here we use the simulated data from examples 2 and 3 in [2].

Table 2: Simulated data

Failures	T_j	Cumulative time in exponential case	Cumulative time in Erlang case
2	T_1	28 131	46 170
5	T_2	61 363	83 170
6	T_3	64 995	88 950
8	T_4	98 859	110 530
8	T_5	145 683	103 210
9	T_6	37 607	93 010

The exponentially distributed times-to-failure are simulated with the scale parameter $\lambda = 0.000\ 068\ 79$. We have $\omega = \sum_{j=1}^{6} r_j = 38$ and $\sum_{j=1}^{6} T_j = 436\ 638$. The critical value of the exact LR test of the hypothesis

$$H_0 : \lambda = 0.00006217965 \text{ versus } H_1 : \lambda \neq 0.00006217965 \tag{7}$$

is $c_{0.05} = 3.858319$. We have $-2\ln\Lambda = 3.851893 < c_{0.05}$, but $-2\ln\Lambda > c_{a,0.05}$. We accept the H_0 at the level of significance $\alpha = 0.05$. The χ_1^2-asymptotics is misleading and rejecting H_0, because of oversizing.

The power function $p(\lambda, 0.05)$ of the LR test of the hypothesis (7) has the form

$$1 - \mathcal{F}_{38}\left(-\frac{38\lambda}{0.00006217965}W_{-1}\left(-e^{-\frac{79.8583}{76}}\right)\right) + \mathcal{F}_{38}\left(-\frac{38\lambda}{0.00006217965}W_0\left(-e^{-\frac{79.8583}{76}}\right)\right)$$

and is for $\lambda \in (0.00001, 0.00013)$ displayed in Figure 2.

The 3-Erlang times-to-failure are simulated with the scale parameter $\lambda = 0.000\ 2064$. Then we have $\omega = \sum_{j=1}^{6} r_j = 114$ and $\sum_{j=1}^{6} T_j = 535\ 240$. The critical value of the exact LR test of the hypothesis

$$H_0 : \lambda = 0.00017624 \text{ versus } H_1 : \lambda \neq 0.00017624 \tag{8}$$

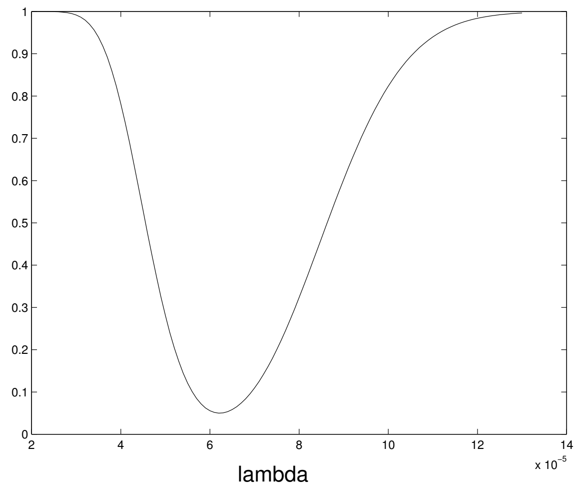

Figure 2: The power function.

is $c_{0.05} = 3.84707364949$. We have $-2\ln\Lambda = 3.842721 < c_{0.05}$, but $-2\ln\Lambda > c_{a,0.05}$. We accept the H_0 at the level of significance $\alpha = 0.05$. The χ_1^2-asymptotics is misleading and rejecting H_0, because of oversizing.

The power function $p(\lambda, 0.05)$ of the LR test of the hypothesis (8) has the form

$$1 - \mathcal{F}_{114}(-\frac{114\lambda}{0.00017624}W_{-1}(-e^{-\frac{231.8471}{228}})) + \mathcal{F}_{114}(-\frac{114\lambda}{0.00017624}W_0(-e^{-\frac{231.8471}{228}}))$$

and is for $\lambda \in (0.0001, 0.0003)$ displayed in Figure 2.

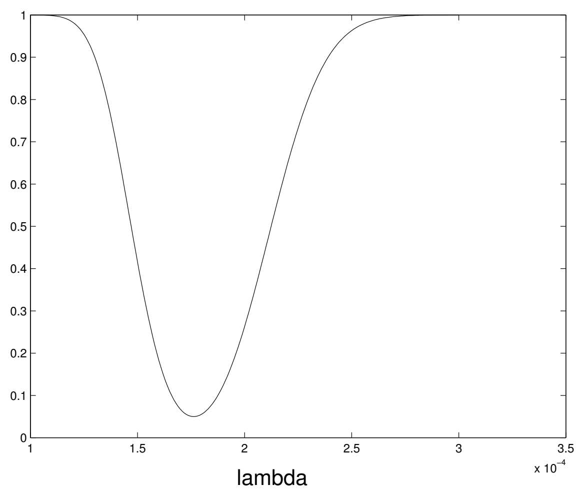

Figure 3: The power function.

4 Conclusion

An exact testing procedure of the hypothesis (1) has been developed when the times-to-failure are independent, exponentially, Erlang or gamma distributed and time-to-failure information is missing. Data sets with missing time-to-failure data can arise from field data collection systems. The advantages of our approach are:

- we provide the procedure for the exact LR testing of the hypothesis (1)
- we provide the power-function in the explicit analytical form
- the exact LR test of the hypothesis is UUMP
- the provided procedure could be easily implemented to the computational softwares in terms of Lambert W function and gamma cumulative distribution function.

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Appendix

The Lambert W function (LW function) is defined to be the multivalued inverse of the complex function $f(y) = ye^y$. As the equation $ye^y = z$ has an infinite number of solutions for each (non-zero) value of $z \in \mathbf{C}$, the LW has an infinite number of branches. Exactly one of these branches is analytic at 0. Usually this branch is referred to as the principal branch of the LW and is denoted by W or W_0. The other branches all have a branch point at 0. These branches are denoted by W_k where $k \in \mathbf{Z} \setminus \{0\}$. The principal branch and the pair of branches W_{-1} and W_1 share an order 2 branch point at $z = -e^{-1}$. The principal branch W is real-valued for $z \in (-e^{-1}, \infty)$ and the branch W_{-1} is real-valued on the interval $(-e^{-1}, 0)$. For all the branches other than the principal branch the branch cut dividing them is the negative real axis. The branches are numbered up and down from the real axis. A detailed discussion of the branches of the LW can be found in [5]. Since the LW function has many applications in pure and applied mathematics, the branches of the LW are implemented to many mathematical softwares, e.g. the Maple, Matlab, Mathematica and Mathcad. For instance, the Figures 1,2 and 3 are made in Matlab. For more information about the implementation and some computational aspects see [6].