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Abstract

We model the process of exceedances of Ebullition of methane from wetlands in the sedge-grass marsh, South Bohemia, Czech Republic by a Mixed Poisson process with mixing variable that is Pareto distributed. We investigate the properties of this process and describe it as a particular case of a counting process. We define Mixed Poisson Pareto random variable, Exponential-Pareto and Erlang-Pareto distribution and investigate their properties.

Keywords: Ebullition of methane, Mixed Poisson processes, Renewal process, Pareto distribution.

1 Introduction

In general, ebullition is bubble transport of gasses from places with a high gas production or concentration to neighboring environment mainly in the soil-water-air interfaces. Ebullition is typical process for direct gas transport in wetland or aquatic ecosystems where accumulated gas in deeper sediments transferred directly to the atmosphere via gas bubbles. There are many factors that can affect bubble formation and their releasing. One of them is a low solubility of methane in water. Solubility of CO_2 in water is 500-600 times greater than that of methane (see Yamamoto et al. [1]). Next main physical factors which affect ebullition are temperature, hydrostatic pressure, atmospheric pressure of air above soil or water and wind. Pressure affect we can record during formation of gas bubbles and their following releasing. Methane bubbles are trapped in soil pores and at once releasing at the time where pores filled with a high pressure of gas. Bubbles suffer little dissolution during releasing from the sediment through the water column. Martens and Klump [2] reported only a 15% change in bubble volume trough 3 m of water column.

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Modeling the dependence of the ebullition of methane on time is useful from ecological point of view. In Jordanova et al. [3] we have modeled this dependence by a time series model. The trend component is estimated by Ordinary Least Squared technique. The noise component is presented by sum of an infinite moving average model with Pareto-like positive and negative parts of the innovations and independent identically distributed (i.i.d.) innovations with similar tail behavior. Pareto tails have been justified also by robust tests for normality against Pareto tails (see Stehlík et al. [4]). Such a moving average time series could be considered as born by a point process. The sums, extremes, exceedances and first passage times as characteristics of this point processes, as well as the behavior of the sample covariance function have been investigated in Davis and Resnick [5] and Resnick [6]. They consider a sequence of non-negative i.i.d. innovations $Z_1, Z_2, ...$ with regularly varying d.f. G, i.e. such that

$$(1) \qquad 1 - G(x) + G(-x) = P(|Z_k| > x) = x^{-\alpha}.L(x),$$

where $\alpha > 0$, $L(x)$ is a slowly varying function at ∞ and

$$(2) \qquad \frac{P(Z_k > x)}{P(|Z_k| > x)} \to p \in (0,1) \ and \ \frac{P(Z_k \leq -x)}{P(|Z_k| > x)} \to q = 1 - p, \ as \ x \to \infty.$$

Denote by

$$(3) \qquad \mathbf{X}_n = \sum_{j=0}^{\infty} c_j \mathbf{Z}_{n-j}, \quad -\infty < n < \infty,$$

a stationary sequence of moving averages, where at least one of the real numbers $c_j, j = 0, 1, ...$ is positive and there exists $\delta \in (0,1)$, $\delta < \alpha$ such that

$$(4) \qquad \sum_{j=0}^{\infty} |c_j|^{\delta} < \infty.$$

For a sequence $a_1, a_2, ...$ such that

$$(5) \qquad nP(|Z_1| > a_n.x) \to x^{-\alpha}, \ for \ all \ x > 0,$$

i.e. $a_n = inf\{x : P(|Z_1| > x) \leq \frac{1}{n}\}$, Davis and Resnick [5] prove that although the sequence $\{X_n, \ -\infty < n < \infty\}$ does not obligatory satisfy the condition D' of Leadbetter [7] the point process

$$(6) \qquad \sum_{k=1}^{\infty} \varepsilon_{(k/n, X_k/a_n)} \Longrightarrow \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} \varepsilon_{(t_k, j_k.c_i)} \quad t \to \infty,$$

where $\{(t_k, j_k) : \ k = 1, 2, ...\}$ is a homogeneous in time Poisson point process on $(0, \infty) \times R \backslash \{0\}$, with mean measure μ such that

$$\mu(dt, dx) = dt \times \lambda(dx), \ \lambda(dx) = \alpha.p.x^{-\alpha-1}.I_{(0,\infty)}(x)(dx) + \alpha.q.(-x)^{-\alpha-1}.I_{(-\infty,0)}(x)(dx).$$

They also note that "any stationary $ARMA$ process driven by a noise sequence of regularly varying tail probabilities will satisfy the hypotheses" of their theorems.

The main properties of the sequence of i.i.d. innovations follow from the investigations of Resnick [8] and Weissman [9]. They prove that

$$\sum_{k=1}^{\infty} \varepsilon_{(k/n, Z_k/a_n)} \Longrightarrow \sum_{k=1}^{\infty} \varepsilon_{(t_k, j_k)} \quad t \to \infty. \tag{7}$$

In this paper we show that the intensity of the observed Poisson processes of exceedances is not a constant. This intensity is usually considered as a constant because the parameters of the process are estimated using only one sample path. Our conclusions come from the fact that if we resample the data, in such a way that to preserve the dependence structure of the process and its main properties we obtain the process of the innovations that is an uncountable mixture of a moving average processes and Lomax(shifted Pareto) mixing random variable (r.v.) plus i.i.d. innovations. We investigate the properties of the limiting point processes of exceedances. Particularly the counting process of the exceedances over a high threshold turns out to be a Mixed Poisson process (MPP) with the inflated Lomax (Shifted Pareto) mixing variable. The time-grid that we use is very helpful. From ecological point of view, reaching a certain grid threshold we are touching a micro-hierarchical level in the context of Addiscott and Mirza [10].

As a byproduct of our observations we define a MPP with Pareto mixing variable and obtain some of its properties. We consider it as a particular case of a counting process with dependent additive increments. We define Mixed Poisson Pareto random variable. It describes the distribution of the number of the "events" up to time t in the case of the Pareto mixing variable. Exponential-Pareto distribution appears as a distribution of the length of the interval between consecutive events. Erlang-Pareto distribution is the distribution of the moment of the n-th "event". When we consider only exceedances of the mixed moving average process the corresponding distributions appeared to be inflated. Some properties of the defined variables are investigated. We obtain the relation between the probability mass function(p.m.f.) of a MPP and p.m.f. of MPP with shifted mixing variable.

The Mixed Poisson processes and their properties are considered e.g. in Karlis and Xekalaki [11] and in the monograph by Grandell [12]. The MPP with Gamma mixing variable is a particular case of a counting process, related with a random time changed renewal process. The times between renewals of this process are uncountable mixture of exponentials with gamma mixing variable. It is well known that this distribution coincides with Pareto distribution. However due to the common mixing variable they are dependent. The random change of time is just the random scale change. The distribution of the time intersections of such a MPP with Gamma mixing variable is a Negative binomial. Our theoretical results with respect to the probability theory are analogous to these investigations.

2 Microergodicity effects on ebullition

To interpret the above mentioned phenomena in statistical terminology, let us consider the data and model (8) described in Jordanova et al. [3]. Therein we denoted the time series of the sample at the moment $t > 0$ by $Em(t)$, where Em stands for the abbreviation of "emissions". In this section we have considered a quadratic model with time as the entire regressor,

$$(8) \quad Em_{\lambda}(t) = a + bt + c.t^2 + X_{\lambda}(t) + Z_{\lambda}(t) - Z_{-,\lambda}(t) + \epsilon_{\lambda}(t).I_{[-c_1,c_2]}(\epsilon_{\lambda}(t)), \quad t \geq 0,$$

where $a \in \mathbb{R}$, $b \in \mathbb{R}$, $c \in \mathbb{R}$, $X_{\lambda}(t)$ is a moving average process, $\epsilon_{\lambda}(t) \sim N(0, \sigma^2)$ relates to standard diffusion, $Z_{\lambda}(t) = Pareto(\alpha_1)I_{Pareto(\alpha_1)>c_2}(t)$ and $Z_{-,\lambda}(t) = Pareto(\alpha_2)I_{Pareto(\alpha_2)>c_1}(t)$, where $I_A(t)$ is an indicator function of the set A.

Here λ is the intensity of the point process of exceedances of the process

$$Em_{S,\lambda}(t) = Em_{\lambda}(t) - a - bt - c.t^2, \quad t \geq 0$$

over a high threshold. As convolution of two homogeneous in time Poisson processes(HPP) this point process is again HPP. (The index "S" comes from "Stationary part of the process $Em_{\lambda}(t)$"). Assuming that λ is a constant, the properties of the corresponding point process of exceedences over a high threshold are well described in Davis and Resnick [5] and Resnick [6] and references therein. The intensity of the process of exceedences over a high threshold of $Em_{S,\lambda}(\frac{t}{\lambda})$ is 1. The point estimators of a, b, c, α_1 and α_2, as well as the estimators of the coefficients of the moving average process are given in Jordanova et al. [3].

Consider the process $Em_{S,\lambda}(t)$. If we observe only one sample path of the process, then λ is a constant. Due to the approximation (6) λ have to be the same for any sample path. By homogeneity in time of this process we could make different time subinterval and to check if λ is the same. In view of stationarity of $Em_{S,\lambda}(t)$ if λ is replaced with a random variable Λ that is independent on the noise components included in the process, the new process $Em_{S,\Lambda}(t)$ is again stationary. Therefore we include mixing variable Λ and check if it is a constant or not.

Lomax Fitting

Further on we consider the fitting of the mixing variable. For estimating the mixing variable we use both, the strong law of large numbers for Homogeneous Poisson processes with finite mean

$$\frac{N_u(t)}{t} \to \lambda_u, \quad t \to \infty \ a.s.,$$

and the strong law of large numbers for the MPP with finite mean of the time intersections, i.e.

$$\frac{N_{\alpha,\delta,u}(t)}{t} \to \Lambda_{\alpha,\delta,u}, \quad t \to \infty \ a.s.,$$

where $\{N_u(t) : t \geq 0\}$ is the number of exceedencess over the threshold u of the process $Em_{S,\lambda}(t)$, i.e. λ is the mean number of exceedences of a unit time interval. $\{N_{\alpha,\delta,u}(t) : t \geq 0\}$ is the number of exceedencess over the threshold u of the Mixed process $Em_{S,\Lambda_{\alpha,\delta,u}}(t)$ and $\Lambda_{\alpha,\delta,u}$ is the mean number of exceedences during unit time interval of this process. Due to the fact that in the model, described in Jordanova et al.[3], we consider only one sample path, λ is a constant. Practically if we consider many sample paths it would be a random variable, and we denote this r.v. by $\Lambda_{\alpha,\delta,u}$. In order to obtain many sample paths we form subsamples of this sample. To save the dependence structure of the process the observations in a subsample are consecutive. The first sample contains all the data set. The second sample contains the first half of the data. The third sample contains the second half of the data. The fourth, fifth and sixth sample contain correspondingly the first, second and third third of the data and so on. Finally we divide the interval to 10 equal consecutive parts and obtain totaly 55 samples. In any of the intervals we calculate the mean number of exceedances, so we obtain 55 realizations of the r.v. $\Lambda_{\alpha,\delta,u}$.

Case 1. $u = 0.005$. The following table contains the mean number of exceedencess over the threshold $u = 0.005$ in the corresponding subintervals described above.

0.04327									
0.07290	0.18278								
0.09764	0.54650	0.27404							
0.16603	0.36450	0.00000	0.36556						
0.26412	0.26048	0.00000	0.63007	1.37152					
0.31710	0.31289	1.09456	0.00000	0.54887	0.00000				
0.14883	0.30435	1.27759	0.00000	0.00000	0.64065	0.00000			
0.16985	0.49809	0.72901	0.00000	0.00000	0.00000	0.73113	0.00000		
0.19099	0.91416	0.58583	1.63948	0.00000	0.00000	1.13305	2.46638	0.00000	
0.21292	1.58851	0.52249	1.82775	0.00000	0.00000	0.00000	1.26316	2.74960	0.00000

In Jordanova et al. [3] we obtain that $\alpha > 2$. Therefore we can use the moment estimators of the parameters of the inflated Lomax distribution. Denote by p the probability of the event "There is no exceedences of u in the unit time interval", then

$$F_{\Lambda_{\alpha,\delta,u}}(x) = p + (1-p).(1 - (1 + \frac{x}{\delta})^{-\alpha}), \quad x \geq 0.$$

$$E\,\Lambda_{\alpha,\delta,u} = \frac{\delta.(1-p)}{\alpha - 1},$$

$$E\,\Lambda^2_{\alpha,\delta,u} = \frac{2.\delta^2.(1-p)}{(\alpha-2).(\alpha-1)^2},$$

The estimator of p is a result of the Strong Law of Large numbers.

$$\widehat{p} = \frac{The\ number\ of\ zeros}{The\ number\ of\ observations} = \frac{18}{55} = 0.3272727$$

$$\widehat{\alpha} = \frac{2.m^2_{n,1}}{(1-\widehat{p}).m^2_{n,2}} + 2 = 3.146497,$$

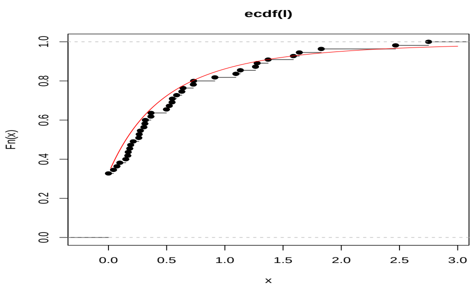

Figure 1: The plot of the empirical c.d.f. on the data above the threshold $u = 0.005$ and the corresponding theoretical Lomax c.d.f.

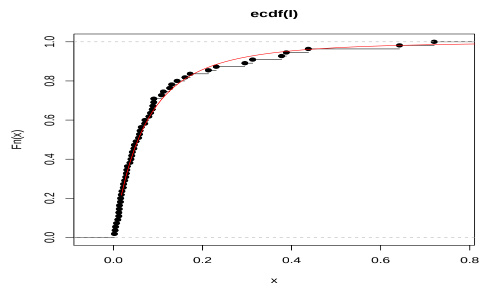

Figure 2: The plot of the empirical cdf on the data above the threshold $u = 0.0005$ and the corresponding theoretical Lomax c.d.f.

$$\widehat{\delta} = \frac{(\alpha - 1).m_{n,1}}{1 - \widehat{p}} = 1.785954,$$

where $m_{n,1}$ is the average of the observations and $m_{n,2}$ is the empirical second moment. The plot of the empirical c.d.f. and corresponding estimated theoretical c.d.f. are given on Figure 1.

Practically the random variable $\Lambda_{\alpha,\delta,u}$ describes the overdispersion of the process.

Case 2. For $u = 0.0005$ the mean numbers of exceedences over a unit time interval, calculated by the corresponding subsample are given in the following table.

0.00212									
0.00370	0.00985								
0.00475	0.01755	0.01911							
0.00687	0.01605	0.04225	0.03008						
0.01299	0.01202	0.04110	0.10784	0.03937					
0.02266	0.01211	0.04597	0.08693	0.12634	0.05085				
0.02768	0.01347	0.04682	0.07012	0.17230	0.29507	0.05983			
0.05837	0.01559	0.03704	0.09064	0.11203	0.23077	0.38802	0.07080		
0.05634	0.02198	0.02888	0.07965	0.31250	0.13078	0.21313	0.72010	0.08333	
0.06236	0.03125	0.02517	0.09054	0.16051	0.14286	0.43750	0.37743	0.64216	0.08850

The moment estimators of the corresponding parameters are

$$\widehat{p} = \frac{The\ number\ of\ zeros}{The\ number\ of\ observations} = 0$$

$$\widehat{\alpha} = \frac{2.m_{n,1}^2}{(1 - \widehat{p}).m_{n,2}^2} + 2 = 2.705181,$$

$$\widehat{\delta} = \frac{(\alpha - 1).m_{n,1}}{1 - \widehat{p}} = 0.1892441,$$

The plots of the empirical c.d.f. and the corresponding estimated theoretical c.d.f. are given on Figure 2.

3 Mixed Poisson process

In this section we consider a Mixed Poisson Process with Pareto mixing variable. We investigate some of the properties of this process and describe it as a particular case of a counting process with dependent additive increments. We define Mixed Poisson Pareto random variable. It describes the distribution of the number of the "events" up to time t. Exponential-Pareto distribution appears as a distribution of the length of the interval between consecutive "events". Erlang-Pareto distribution is the distribution of the moment of the n-th "event". Some properties of the defined variables are investigated.

We denote by $\stackrel{d}{=}$ coincidence in distribution.

Denote by $\Lambda_{\alpha,\delta}$, $\alpha > 0$, $\delta > 0$, a Pareto distributed r.v. with cumulative distribution function

$$F_{\Lambda_{\alpha,\delta}}(x) = \left\{ \begin{array}{ccc} 0 & , & x \leq \delta \\ 1 - \frac{\delta^\alpha}{x^\alpha} & , & x > \delta \end{array} \right. \tag{9}$$

Briefly $\Lambda_{\alpha,\delta} \sim Pareto(\alpha, \delta)$. It is well known that,

$$E\Lambda_{\alpha,\delta}^k = \frac{\alpha.\delta^k}{\alpha - k}, \quad \alpha > k, \; k \in \mathbb{R}, \tag{10}$$

$$\phi_\alpha(s) := Ee^{-s.\Lambda_{\alpha,\delta}} = \alpha.(\delta.s)^\alpha.\Gamma(-\alpha, \delta.s), \quad s > 0, \tag{11}$$

where $\Gamma(x,t) = \int_t^\infty y^{x-1}.e^{-y}dy$, x is the upper incomplete gamma function and

$$\lim_{s \downarrow 0} s^\alpha.\Gamma(-\alpha, s) = \alpha^{-1}.$$

Comments with respect to $\Gamma(x,t)$, could be found in Olver et al [13] or in in Nadarajah and Kotz [14] where (11) is proved. We will use also Generalized exponential integral

$$E_p(z) = z^{p-1}.\Gamma(1-p, z),$$

for $z \in R$ and $p \in R$. Some of its properties and numerical tables could be found in Milgram [15] or Olver [13].

Let $\mathcal{A}$ be a sigma algebra with right - continuous filtration. Consider a probability space $\mathbf{\Omega} = (\Omega, \mathcal{A}, \mathbb{P})$.

Definition 1. Let N be a standard homogeneous Poisson process in $\mathbf{\Omega}$, ($EN(1) = 1$) and $c(t)$ be a non-negative, non-decreasing and continuous function, not obligatory starting from the coordinate beginning and $c(t) \to \infty$, $t \to \infty$. Denote by $\Lambda_{\alpha,\delta}$ a r.v. that have c.d.f. (9). Assume $\Lambda_{\alpha,\delta}$ and N are independent. We call the random process

$$\{N_{\alpha,\delta}(t); t \geq 0\} = \{N(\Lambda_{\alpha,\delta}.c(t)); t \geq 0\}$$

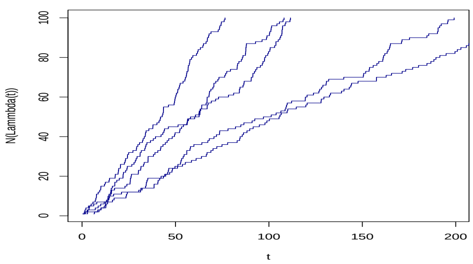

Figure 3: Five sample paths of a homogeneous MPPP-process, (i.e. $c(t) = t$) with parameters $\alpha = 2$ and $\delta = 0.5$.

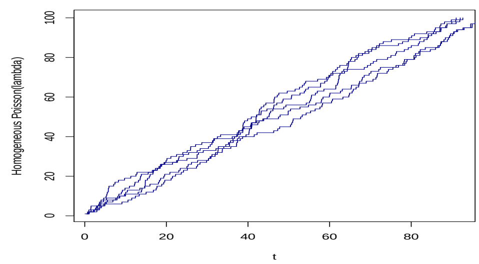

Figure 4: Five sample paths of a homogeneous Poisson process, with parameter $\lambda = 1$.

a **Mixed Poisson Process with Pareto mixing variable** (MPPP-processes). Briefly

$$\{N_{\alpha,\delta}(t); t \geq 0\} \sim MPPP(\alpha, \delta; c(t)), \quad \alpha > 0,\ \delta > 0.$$

Figure 3 displays 5 sample paths of such a process. Due to the over-dispersion we observe the bigger difference between the trajectories of MPPP-process than between the trajectories of the homogeneous Poisson process with constant intensity given in Figure 4. The mean of both processes coincides, when $t = 1$.

As a particular case of Mixed Poisson processes, the MPPP-processes have the following properties (See e.g. Mikosch [16])

1.) it only has a finite number of jumps on any finite time interval;

2.) it has dependent additive increments;

3.) if $c(t) = t$ it is homogeneous in time;

4.) it is over-dispersed;

5.) it has the order statistics property;

6.) it has Markov property.

The next theorem describes the distribution of the time intersections and finite dimensional distributions (f.d.ds) of the Mixed Poisson Process with Pareto mixing variable.

Theorem 1. If $\{N_{\delta,\alpha}(t); t \geq 0\} \sim MPPP(\alpha, \delta; c(t))$, $\alpha > 0$, $\delta > 0$, then

a.) for all $t > 0$

$$P(N_{\alpha,\delta}(t) = k) = \alpha.\frac{(\delta.c(t))^{\alpha}}{k!}\Gamma(k-\alpha, \delta.c(t)) = \alpha.\frac{(\delta.c(t))^{k}}{k!}E_{1-k+\alpha}(\delta.c(t)), \quad k = 0, 1, ...$$

b.) For $n \in \mathbb{N}$, $0 \le t_1 < t_2 < ... < t_n$, $k_i = 0, 1, ...$, $i = 1, 2, ..., n$

$$(12) \qquad P(N_{\alpha,\delta}(t_1) = k_1, N_{\alpha,\delta}(t_2) = k_1 + k_2, ..., N_{\alpha,\delta}(t_n) = k_1 + ... = k_n) =$$

$$= \frac{\alpha.\delta^{\alpha}.c(t_1)^{k_1}.(c(t_2) - c(t_1))^{k_2}...(c(t_n) - c(t_{n-1}))^{k_n}}{k_1!...k_n!.c(t_n)^{k_1+...+k_n-\alpha}}.\Gamma(k_1+...+k_n-\alpha, \delta.c(t_n)) =$$

$$= \frac{\alpha.\delta^{k_1+...+k_n}.c(t_1)^{k_1}.(c(t_2) - c(t_1))^{k_2}...(c(t_n) - c(t_{n-1}))^{k_n}}{k_1!...k_n!}.E_{1-(k_1+...+k_n)-\alpha}(\delta.c(t_n)).$$

c.) If $c(t) = t$, then for all $t > 0$, $N_{\alpha,\delta}(t) \stackrel{d}{=} N_{\alpha,\delta.t}(1)$.

Proof: We use the Total probability formula.
a.) $P(N_{\alpha,\delta}(t) = k) = P(N(\Lambda_{\alpha,\delta}.c(t)) = k) =$

$$= \int_{\delta}^{\infty} P(N(y.c(t)) = k)P_{\Lambda_{\alpha,\delta}}(y)dy = \int_{\delta}^{\infty} \frac{(y.c(t))^k}{k!}e^{-y.c(t)}\alpha.y^{-\alpha-1}\delta^{\alpha}dy =$$

$$= \frac{\alpha.\delta^{\alpha}}{k!}\int_{\delta.c(t)}^{\infty} z^k.e^{-z}.\frac{z^{-\alpha-1}}{(c(t))^{-\alpha-1}}\frac{dz}{c(t)} = \alpha.\frac{(\delta.c(t))^{\alpha}}{k!}\Gamma(k-\alpha, \delta.c(t)),$$

and $y = z/c(t)$.

b.) Consider $n \in \mathbb{N}$, $0 \le t_1 < t_2 < ... < t_n$, $k_i = 0, 1, ...$, $i = 1, 2, ..., n$. The independence and homogeneity of the Poisson process N and the Pareto distribution of $\Lambda_{\alpha,\delta}$ entail

$P(N_{\alpha,\delta}(t_1) = k_1, N_{\alpha,\delta}(t_2) = k_1 + k_2, ..., N_{\alpha,\delta}(t_n) = k_1 + ... + k_n) =$

$$= \int_{\delta}^{\infty} P(N(y.c(t_1)) = k_1, N(y.c(t_2)) = k_1+k_2, ..., N(y.c(t_n)) = k_1+... = k_n)P_{\Lambda_{\alpha,\delta}}(y)dy =$$

$$= \int_{\delta}^{\infty} P(N(y.c(t_1)) = k_1).P(N(y.(c(t_2)-c(t_1))) = k_2)...P(N(y.(c(t_n)-c(t_{n-1}))) = k_n)P_{\Lambda_{\alpha,\delta}}(y)dy =$$

$$= \int_{\delta}^{\infty} \frac{(y.c(t_1))^{k_1}}{k_1!}.e^{-y.c(t_1)}.\frac{(y.(c(t_2) - c(t_1)))^{k_2}}{k_2!}.e^{-y.(c(t_2)-c(t_1))}...\frac{(y.(c(t_n) - c(t_{n-1})))^{k_n}}{k_n!}.$$

$$.e^{-y.(c(t_n)-c(t_{n-1}))}.\alpha.\delta^{\alpha}.y^{-\alpha-1}dy =$$

$$= \frac{\alpha.\delta^{\alpha}.c(t_1)^{k_1}.(c(t_2) - c(t_1))^{k_2}...(c(t_n) - c(t_{n-1}))^{k_n}}{k_1!...k_n!}\int_{\delta}^{\infty} y^{k_1+...+k_n-\alpha-1}.e^{-y.c(t_n)}dy =$$

$$= \frac{\alpha.\delta^{\alpha}.c(t_1)^{k_1}.(c(t_2) - c(t_1))^{k_2}...(c(t_n) - c(t_{n-1}))^{k_n}}{k_1!...k_n!.c(t_n)^{k_1+...+k_n-\alpha}}.\Gamma(k_1 + ... + k_n - \alpha, \delta.c(t_n)).$$

c.) Follows immediately by a.).

3.1 MPPP process and M_{PP} distribution

Definition 2. We call a random variable ξ with probability mass function (p.m.f.)

$$P(\xi = k) = \frac{\alpha.\delta^\alpha}{k!}\Gamma(k-\alpha,\delta), \quad k = 0, 1, ... \tag{13}$$

Mixed Poisson Pareto r.v. with parameters $\alpha > 0$**,** $\delta > 0$. Briefly $\xi \sim M_{PP}(\alpha, \delta)$.

Remark: 1. Due to the following considerations, the $M_{PP}(\alpha, \delta)$ distribution is proper.

$$\sum_{k=0}^{\infty} \frac{\alpha.\delta^\alpha}{k!}\Gamma(k-\alpha,\delta) = \alpha.\delta^\alpha \sum_{k=0}^{\infty} \frac{1}{k!}\int_\delta^\infty e^{-y}.y^{k-\alpha-1}dy =$$

$$= \alpha.\delta^\alpha. \int_\delta^\infty y^{-\alpha-1}.e^{-y}\sum_{k=0}^{\infty}\frac{y^k}{k!}dy = \alpha.\delta^\alpha. \int_\delta^\infty y^{-\alpha-1}dy = 1.$$

The limit interchange is valid here, as we can see from Lebesgue Theorem on dominating convergence, having a sequence $f_n(y) = \sum_{k=0}^{n} \frac{y^k}{k!}$.

Any M_{PP} r.v. could be presented as a Mixed Poisson distributed r.v. with Pareto mixing variable.

Theorem 2. Let $\alpha > 0$ and $\delta > 0$. If $\xi_{\alpha,\delta} \sim M_{PP}(\alpha, \delta)$, then there exists a probability space and two random variables $\Lambda_{\alpha,\delta}$ and $N_{\Lambda_{\alpha,\delta}}$ defined on it such that $\Lambda_{\alpha,\delta} \sim Pareto(\alpha, \delta)$,

$$P(N_{\Lambda_{\alpha,\delta}} = k|\Lambda_{\alpha,\delta} = y) = \frac{y^k}{k!}e^{-y},\ y > \delta,\ k = 0, 1, ...$$

and $\xi_{\alpha,\delta} \stackrel{d}{=} N_{\Lambda_{\alpha,\delta}}$.

Proof: Consider the probability space born by $N_0(1)$, where

$$\{N_0(t); t \geq 0\} \sim MPPP(\alpha, \delta; c(t) = t).$$

Then by Theorem 1 a.), $N_0(1) \stackrel{d}{=} \xi_{\alpha,\delta}$. By the Formula of total probability

$$P(N_{\Lambda_{\alpha,\delta}} = k) = \int_\delta^\infty P(N_{\Lambda_{\alpha,\delta}} = k|\Lambda_{\alpha,\delta} = y)P_{\Lambda_{\alpha,\delta}}(y)dy = \int_\delta^\infty \frac{y^k}{k!}e^{-y}\alpha.y^{-\alpha-1}\delta^\alpha dy =$$

$$= \alpha.\frac{\delta^\alpha}{k!}\Gamma(k-\alpha,\delta),\ k = 0, 1, ...$$

Theorem 3. If $\xi_{\alpha,\delta} \sim M_{PP}(\alpha, \delta)$, $\alpha > 0$, $\delta > 0$ then,

a.) $E\xi_{\alpha,\delta} = \frac{\alpha.\delta}{\alpha-1}, \quad \alpha > 1.$

b.) $E\xi^2_{\alpha,\delta} = \frac{\alpha.\delta}{\alpha-1} + \frac{\alpha.\delta^2}{\alpha-2}, \quad \alpha > 2.$

c.) $Var\ \xi_{\alpha,\delta} = \frac{\alpha.\delta}{\alpha-1} + \frac{\alpha.\delta^2}{\alpha-2} - \left(\frac{\alpha.\delta}{\alpha-1}\right)^2, \quad \alpha > 2.$

d.) $Ee^{-s.\xi_{\alpha,\delta}} = \alpha.\delta^\alpha \sum_{k=0}^\infty \frac{e^{-sk}}{k!}\Gamma(k-\alpha,\delta) = \alpha.(\delta(1-e^{-s}))^\alpha.\Gamma(-\alpha,\delta.(1-e^{-s})) =$
$= \alpha.E_{\alpha+1}(\delta.(1-e^{-s})),\ \alpha > 0.$

e.) $Ez^{\xi_{\alpha,\delta}} = \alpha.\delta^\alpha \sum_{k=0}^\infty \frac{z^k}{k!}\Gamma(k-\alpha,\delta) = \alpha.(\delta(1-z))^\alpha.\Gamma(-\alpha,\delta.(1-z)) = \alpha.E_{\alpha+1}(\delta.(1-z)),\ \alpha > 0.$

Proof: We will use that $\xi_{\alpha,\delta} \stackrel{d}{=} N_{\Lambda_{\alpha,\delta}}$, where $N_{\Lambda_{\alpha,\delta}}$ is described in Theorem 2.

a.) By Double expectations theorem or by properties of the MPP, and by (10) we obtain

$$E\xi_{\alpha,\delta} = EN_{\Lambda_{\alpha,\delta}} = \int_\delta^\infty E(N_{\Lambda_{\alpha,\delta}}|\Lambda_{\alpha,\delta} = y).P_{\Lambda_{\alpha,\delta}}(y)dy = E\Lambda_{\alpha,\delta} = \frac{\alpha.\delta}{\alpha-1},\ \alpha > 1.$$

b.) $E\xi_{\alpha,\delta}^2 = EN_{\Lambda_{\alpha,\delta}}^2 =$

$$= \int_\delta^\infty E(N_{\Lambda_{\alpha,\delta}}^2|\Lambda_{\alpha,\delta} = y).P_{\Lambda_{\alpha,\delta}}(y)dy = \int_\delta^\infty (y+y^2).P_{\Lambda_{\alpha,\delta}}(y)dy =$$

$$= E\Lambda_{\alpha,\delta} + E\Lambda_{\alpha,\delta}^2 = \frac{\alpha.\delta}{\alpha-1} + \frac{\alpha.\delta^2}{\alpha-2}, \quad \alpha > 2.$$

c.) $Var\ \xi_{\alpha,\delta} = E\xi_{\alpha,\delta}^2 - (E\xi_{\alpha,\delta})^2$.

d.) By definition of Laplace Transform and as in the proof of Theorem 2, we have

$$Ee^{-s.\xi_{\alpha,\delta}} = Ee^{-s.N_{\Lambda_{\alpha,\delta}}} = \sum_{k=0}^\infty e^{-sk}P(N_{\Lambda_{\alpha,\delta}} = k) = \alpha.\delta^\alpha.\sum_{k=0}^\infty \frac{e^{-sk}}{k!}\Gamma(k-\alpha,\delta).$$

From the other side, by the properties of the Mixed Poisson processes we obtain

$$Ee^{-s.\xi_{\alpha,\delta}} = Ee^{-\Lambda_{\alpha,\delta}(1-e^{-s})} = \alpha.(\delta(1-e^{-s}))^\alpha.\Gamma(-\alpha,\delta(1-e^{-s})) = \alpha.E_{\alpha+1}(\delta.(1-e^{-s})).$$

e.) Analogously to d.).

Remark: 1. In Section 4.2. of the paper [17], Willmot considers the moment generating function which coincides with ours.

3.2 MPPP process as a counting process.

Let $\Lambda_{\alpha,\delta} \sim Pareto(\alpha,\delta)$ and $\tau_{\Lambda_{\alpha,\delta}}$ be a random variable defined on the same probability space and with the following conditional cumulative distribution function (c.d.f.):

$$P(\tau_{\Lambda_{\alpha,\delta}} < x|\Lambda_{\alpha,\delta} = y) = 1 - e^{-xy}, \quad x > 0,\ y > \delta.$$

The last means that $\tau_{\Lambda_{\alpha,\delta}}$ is an uncountable mixture of exponentials with Pareto Mixing variable.

$$P(\tau_{\Lambda_{\alpha,\delta}} < x) = 1 - P(\tau_{\Lambda_{\alpha,\delta}} \geq x) = 1 - \int_\delta^\infty P(\tau_{\Lambda_{\alpha,\delta}} \geq x | \Lambda_{\alpha,\delta} = y) P_{\Lambda_{\alpha,\delta}}(y) dy =$$

$$= 1 - \alpha.\delta^\alpha. \int_\delta^\infty e^{-xy}.y^{-\alpha-1} dy = 1 - \alpha.(x.\delta)^\alpha. \int_{x.\delta}^\infty e^{-z}.z^{-\alpha-1} dz =$$

$$= 1 - \alpha.(x.\delta)^\alpha.\Gamma(-\alpha, x.\delta), \; x > 0.$$

Definition 3. We call a random variable $\tau_{\alpha,\delta}$ with c.d.f.

$$(14) \qquad P(\tau_{\alpha,\delta} < x) = 1 - \alpha.(x.\delta)^\alpha.\Gamma(-\alpha, x.\delta) = 1 - \alpha.E_{\alpha+1}(x.\delta), \quad x > 0,$$

Exponentially-Pareto distributed with parameters $\alpha > 0$ and $\delta > 0$. Briefly $\tau_{\alpha,\delta} \sim EP(\alpha, \delta)$.

Theorem 4. If $\tau_{\alpha,\delta} \sim EP(\alpha, \delta)$, $\alpha > 0$, $\delta > 0$, then

a.) density has the form

$$P_{\tau_{\alpha,\delta}}(x) = \alpha.\delta^\alpha. \left(-\alpha.x^{\alpha-1}.\Gamma(-\alpha, x.\delta) + exp\{-\delta.x\}\delta^{-\alpha}.x^{-1}\right) = \alpha.\delta.E_\alpha(\delta.x), \; x > 0,$$

b.) $P_{\tau_{\alpha,\delta}}(x) = \frac{1}{x}.P_{\tau_{\alpha,\delta.x}}(1)$, $x > 0$.

c.) $E\tau_{\alpha,\delta} = \frac{\alpha}{\delta.(1+\alpha)}$.

d.) $E\tau^k_{\alpha,\delta} = \frac{\alpha.k!}{\delta^k(k+\alpha)}, \quad k = 1, 2, ...$

e.) $Var\tau_{\alpha,\delta} = \frac{\alpha.(2+2\alpha+\alpha^2)}{\delta^2(\alpha+1)^2(\alpha+2)}$.

f.) $Ee^{-t.\tau_{\alpha,\delta}} = \alpha.\delta^\alpha.t^{-\alpha} \int_{\delta/t}^\infty z^{-\alpha}(1+z)^{-1} dz = \alpha.\delta. \int_0^\infty e^{-t.y} E_\alpha(\delta.y) dy$.

g.) Scale property: $\tau_{\alpha,\delta} \stackrel{d}{=} \delta.\tau_{\alpha,1}$.

h.) Let E_1 be an exponential random variable with parameter 1 and $\Lambda_{\alpha,\delta} \sim Pareto(\alpha, \delta)$. Assume E_1 and $\Lambda_{\alpha,\delta}$ are independent, then

$$\tau_{\alpha,\delta} \stackrel{d}{=} \frac{E_1}{\Lambda_{\alpha,\delta}}.$$

Proof: a.) Using the derivative of the c.d.f. and the results of Section 8.19 in Olver [13] we obtain we obtain

$$P_{\tau_{\alpha,\delta}}(x) = \alpha.\delta^\alpha. \left(-\alpha.x^{\alpha-1}.\Gamma(-\alpha, x.\delta) + exp\{-\delta.x\}\delta^{-\alpha}.x^{-1}\right) =$$

$$= \alpha.\delta(-\frac{\alpha.(\delta.x)^\alpha}{\delta.x}\Gamma(-\alpha, x.\delta) + \frac{e^{-\delta.x}}{\delta.x}) = \alpha.\delta(-\frac{\alpha}{\delta.x}E_{\alpha+1}(\delta.x) + E_0(\delta.x)) = \alpha.\delta.E_\alpha(\delta.x).$$

Proof of b.) and c.) is straightforward.

d.) We use the construction of this r.v. given in the beginning of the Section.

$$E\tau^k_{\alpha,\delta} = \int_\delta^\infty E(\tau^k_{\Lambda_{\alpha,\delta}}|\Lambda_{\alpha,\delta} = y)P_{\Lambda_{\alpha,\delta}}(y)dy = \int_\delta^\infty \frac{k!}{y^k}\alpha.\delta^\alpha.y^{-\alpha-1}dy = \frac{\alpha.k!}{\delta^k(\alpha+k)}.$$

e.)$Var\tau_{\alpha,\delta} = E\tau^2_{\alpha,\delta} - (E\tau_{\alpha,\delta})^2$.

f.)$Ee^{-t.\tau_{\alpha,\delta}} =$

$$= \int_\delta^\infty E(e^{-t.\tau_{\Lambda_{\alpha,\delta}}}|\Lambda_{\alpha,\delta} = y)P_{\Lambda_{\alpha,\delta}}(y)dy = \int_\delta^\infty \frac{y}{y+t}\alpha.\delta^\alpha.y^{-\alpha-1}dy =$$

For $y = zt$,

$$= \alpha.\delta^\alpha.\int_\delta^\infty (y+t)^{-1}y^{-\alpha}dy = \alpha.\delta^\alpha.t^{-\alpha}\int_{\delta/t}^\infty z^{-\alpha}(1+z)^{-1}dz.$$

From the other side by definition of the expectation and a.) we obtain the last part of this statement.

g.) It follows by Definition 3.

h.) We use the Total probability formula and (11). Let $x > 0$, then

$$P(\frac{E_1}{\Lambda_{\alpha,\delta}} \geq x) = \int_\delta^\infty P(\frac{E_1}{\Lambda_{\alpha,\delta}} \geq x|\Lambda_{\alpha,\delta} = y)P_{\Lambda_{\alpha,\delta}}(y)dy =$$

$$= \int_\delta^\infty P(E_1 \geq x.y)P_{\Lambda_{\alpha,\delta}}(y)dy = \int_\delta^\infty e^{-x.y}.P_{\Lambda_{\alpha,\delta}}(y)dy = Ee^{-x.\Lambda_{\alpha,\delta}} =$$

$$= \alpha.(\delta.x)^\alpha.\Gamma(-\alpha,\delta.x), \quad x > 0.$$

Remark: 1. In his paper Felgueiras [18] uses the notion "Pareto scale mixture of the X variable" for description of the variable that is fraction of independent r.v. X and $Beta(\alpha, 1)$ distributed r.v. As is discussed in that paper, this distribution coincides with the one of the product of X and independent of it Pareto r.v. Therefore due to the property described above in Th. 4., h.) we do not use just the name "Pareto mixture of the exponential r.v." in Definition 3 or if we use it, we have to notice that the word "scale" in the description of Felgueiras [18] is substantial. "Pareto scale mixture of the X variable" and "Pareto mixture of the X variable" are two different probability laws.

2. The Property h.) in the above Th. 4. describes clearly the relation between the Exponentially-Pareto distribution and other continuous distributions, e.g. due to the fact that the reciprocal value of a Pareto distributed r.v. coincides with the power distribution the property h.) in the above Th. 4. could be explained also in the terms of the power distribution.

3. When $\alpha = n \in N$ and $\delta = 1$ this distribution coincides with the Exponential integral distribution, for $\nu = 1$ and $m = 1$, see Meijer and Baken [19].

4. In view of the definition of the Hypergeometric functions we could express Property f.) as

$$Ee^{-t.\tau_{\alpha,\delta}} = \frac{t}{t+\delta}.F(1,1;\alpha+1;\frac{t}{t+\delta}).$$

Definition 4. Let $(\tau_{\alpha,\delta,1}, \tau_{\alpha,\delta,2}, ..., \tau_{\alpha,\delta,n})$, be a random vector with
(15)

$$P(\tau_{\alpha,\delta,1} \geq x_1, \tau_{\alpha,\delta,2} \geq x_2, ..., \tau_{\alpha,\delta,n} \geq x_n) = \alpha.[\delta(x_1+...+x_n)]^{\alpha}.\Gamma(-\alpha, \delta(x_1+...+x_n)),$$

$$x_i \geq 0,\ i = 1, 2, ..., n,$$

we call $(\tau_{\alpha,\delta,1}, \tau_{\alpha,\delta,2}, ..., \tau_{\alpha,\delta,n})$ **Exponentially-Pareto distributed random vector**. Briefly $(\tau_{\alpha,\delta,1}, \tau_{\alpha,\delta,2}, ..., \tau_{\alpha,\delta,n}) \sim \mathbf{EP}(\alpha, \delta)$.

Remark: 1. In terms of the Generalized exponential integral (15) coincides with the expression

$$P(\tau_{\alpha,\delta,1} \geq x_1, \tau_{\alpha,\delta,2} \geq x_2, ..., \tau_{\alpha,\delta,n} \geq x_n) = \alpha.E_{\alpha+1}((x_1 + ... + x_n).\delta).$$

Theorem 5. If $(\tau_{\alpha,\delta,1}, \tau_{\alpha,\delta,2}, ..., \tau_{\alpha,\delta,n}) \sim \mathbf{EP}(\alpha, \delta)$ then

a.) for all $i = 1, 2, ..., n$, $\tau_{\alpha,\delta,i} \sim EP(\alpha, \delta)$.

b.) Any subvector of this vector is again $\mathbf{EP}(\alpha, \delta)$ distributed random vector.

c.) Let $E_1, E_2, ..., E_n$ be i.i.d. exponentially distributed random variables with parameter 1 and $\Lambda_{\alpha,\delta} \sim Pareto(\alpha, \delta)$. Assume $E_1, E_2, ..., E_n$ and $\Lambda_{\alpha,\delta}$ are independent, then

$$\{\tau_{\alpha,\delta,1}, \tau_{\alpha,\delta,2}, ..., \tau_{\alpha,\delta,n})\} \stackrel{d}{=} \{\frac{E_1}{\Lambda_{\alpha,\delta}}, \frac{E_2}{\Lambda_{\alpha,\delta}}, ..., \frac{E_n}{\Lambda_{\alpha,\delta}}\}.$$

d.) There exist a probability space, generated by $\Lambda_{\alpha,\delta} \sim Pareto(\alpha, \delta)$ and r.v. $(\tau_1, \tau_2, ..., \tau_n)$ defined by

$$P(\tau_1 \geq x_1, \tau_2 \geq x_2, ..., \tau_n \geq x_n) = \int_{\delta}^{\infty} e^{-y.(x_1+...+x_n)} P_{\Lambda_{\alpha,\delta}}(y)dy,\ x_i \geq 0,\ i = 1, 2, ..., n.$$

such that

$$(\tau_{\alpha,\delta,1}, \tau_{\alpha,\delta,2}, ..., \tau_{\alpha,\delta,n}) \stackrel{d}{=} (\tau_1, \tau_2, ..., \tau_n).$$

Proof: a.) Replace redundant variables in (15) by zero and obtain the survival function corresponding to (14).

b.) Analogously to a.) we obtain (15) for the corresponding random vector and corresponding variables.

c.) We use the Total probability formula and (11). Let $x_1 \geq 0, ..., x_n \geq 0$, then

$$P(\frac{E_1}{\Lambda_{\alpha,\delta}} \geq x_1, ..., \frac{E_n}{\Lambda_{\alpha,\delta}} \geq x_n) = \int_{\delta}^{\infty} P(\frac{E_1}{\Lambda_{\alpha,\delta}} \geq x_1, ..., \frac{E_n}{\Lambda_{\alpha,\delta}} \geq x_n | \Lambda_{\alpha,\delta} = y) P_{\Lambda_{\alpha,\delta}}(y)dy =$$

$$= \int_{\delta}^{\infty} P(E_1 \geq x_1.y)...P(E_n \geq x_n.y).P_{\Lambda_{\alpha,\delta}}(y)dy = \int_{\delta}^{\infty} e^{-(x_1+...+x_n).y}.P_{\Lambda_{\alpha,\delta}}(y)dy =$$

$$= Ee^{-(x_1+...+x_n).\Lambda_{\alpha,\delta}} = \alpha.(\delta.(x_1+...+x_n))^{\alpha}.\Gamma(-\alpha, \delta.(x_1+...+x_n)).$$

d.) Use the Total probability formula.

□

Define

$$T_{\alpha,\delta,n} = \tau_{\alpha,\delta,1} + \tau_{\alpha,\delta,2} + ... + \tau_{\alpha,\delta,n} \tag{16}$$

and denote the corresponding counting process by

$$\{N_{\alpha,\delta}(t);\, t \geq 0\} = \{sup\{i \geq 0 : T_{\alpha,\delta,i} \leq t\},\, t \geq 0\}. \tag{17}$$

The next theorem describes the distribution of the sum $T_{\alpha,\delta,n}$ in a particular case.

Theorem 6. Let for all $n = 1, 2, ...,$ $(\tau_{\alpha,\delta,1}, \tau_{\alpha,\delta,2}, ..., \tau_{\alpha,\delta,n}) \sim \mathbf{EP}(\alpha, \delta)$ and $T_{\alpha,\delta,n}$ be the random variable, described in (16), then

a.)

$$P(T_{\alpha,\delta,n} \geq x) = \alpha.(\delta.x)^{\alpha}.\sum_{i=0}^{n-1} \frac{\Gamma(i-\alpha, \delta.x)}{i!}, \quad x \geq 0 \tag{18}$$

b.)

$$P_{T_{\alpha,\delta,n}}(x) = \alpha.\delta.\frac{(\delta.x)^{\alpha-1}}{(n-1)!}\Gamma(n-\alpha, \delta.x) = \frac{\alpha.\delta^n.x^{n-1}}{(n-1)!}.E_{\alpha-n+1}(\delta.x), \quad x \geq 0.$$

c.) $P_{T_{\alpha,\delta,n}}(x) = \frac{1}{x} P_{T_{\alpha,\delta.x,n}}(1)$, $x > 0$.

Proof: a.) Let $x \geq 0$. By the definition of $T_{\alpha,\delta,n}$, Theorem 5, the form of the survival function of the Erlang distribution with parameters (n, y) and Pareto (α, δ) distribution density function,

$$P(T_{\alpha,\delta,n} \geq x) = \alpha.\delta^{\alpha}.\int_{\delta}^{\infty} \sum_{i=0}^{n-1} \frac{e^{-yx}.(yx)^i}{i!} y^{-\alpha-1}dy =$$

For $t = xy$

$$= \alpha.\delta^{\alpha}.\sum_{i=0}^{n-1} \frac{1}{i!} \int_{\delta}^{\infty} e^{-yx}.(yx)^i y^{-\alpha-1}dy = \alpha.(\delta.x)^{\alpha}.\sum_{i=0}^{n-1} \frac{1}{i!} \int_{x.\delta}^{\infty} e^{-t}.t^{i-\alpha-1}dy.$$

b.) Analogously to a.)

$$P_{T_{\alpha,\delta,n}}(x) = \alpha.\delta^{\alpha}.\int_{\delta}^{\infty} y^n \frac{x^{n-1}}{(n-1)!} e^{-x.y} y^{-\alpha-1}dy = \alpha.\delta.\frac{(\delta.x)^{\alpha-1}}{(n-1)!}\Gamma(n-\alpha, \delta.x).$$

Definition 5. Let $T_{\alpha,\delta,n}$ be a r.v. with survival function (18). We call such a r.v. **Erlang-Pareto** distributed with parameters $\alpha > 0$, $\delta > 0$ and $n \in \mathbb{N}$. Briefly $T_{\alpha,\delta,n} \sim ErlP(\alpha, \delta, n)$.

Remarks 1. $ErlP(\alpha, \delta, 1)$ distribution coincides with $EP(\alpha, \delta)$ distribution.

2. When $\delta = 1$ and $\alpha \in \{n, n+1, ...\}$ this distribution coincides with the Exponential integral distribution, for $\nu = n$, $m = 1$ and $n_0 = \alpha - n + 1$ see Meijer and Baken [19].

Theorem 7. Let $T_{\alpha,\delta,n} \sim ErlP(\alpha, \delta, n)$, then

a.) $ET_{\alpha,\delta,n} = \frac{n.\alpha}{\delta.(1+\alpha)}$.

b.) $ET^k_{\alpha,\delta,n} = \frac{\alpha.(n+k-1)!}{(n-1)!.\delta^k(\alpha+k)}, \quad k = 1, 2, ...$

c.) $VarT_{\alpha,\delta,n} = \frac{\alpha.n.(n+(\alpha+1)^2)}{\delta^2.(\alpha+2)(\alpha+1)^2}$.

d.)

$$Ee^{-t.T_{\alpha,\delta,n}} = \frac{\alpha.\delta^\alpha.}{t^\alpha} \int_{\delta/t}^{\infty} \frac{z^{n-\alpha-1}}{(1+z)^n} dz.$$

e.) Let $T_{1,n}$ be an Erlang distributed random variable with parameters 1 and n and $\Lambda_{\alpha,\delta} \sim Pareto(\alpha, \delta)$. Assume $T_{1,n}$ and $\Lambda_{\alpha,\delta}$ are independent, then

$$T_{\alpha,\delta,n} \stackrel{d}{=} \frac{T_{1,n}}{\Lambda_{\alpha,\delta}}.$$

f.) Let $\{N_{\alpha,\delta}(t), t \geq 0\} \sim MPPP(\alpha, \delta, c(t) = t)$, then

$$P_{T_{\alpha,\delta,n}}(x) = \delta.P(N_{\alpha-1,\delta}(x) = n-1), \quad x > 0.$$

Proof: a.) We use that $ET_{\alpha,\delta,n} = E(\tau_{\alpha,\delta,1} + \tau_{\alpha,\delta,2} + ... + \tau_{\alpha,\delta,n}) = n.E\,\tau_{\alpha,\delta,1}$ and apply Theorem 4, c.).

b.) By Theorem 6, a) and Theorem 5, c.),
$ET^k_{\alpha,\delta,n} = E(\tau_{\alpha,\delta,1} + \tau_{\alpha,\delta,2} + ... + \tau_{\alpha,\delta,n})^k =$

$$= \int_\delta^\infty \frac{(n+k-1)!}{(n-1)!y^k} \alpha.\delta^\alpha.y^{-\alpha-1} dy = \frac{\alpha.\delta^\alpha.(n+k-1)!}{(n-1)!} \int_\delta^\infty y^{-\alpha-k-1} dy =$$

$$= \frac{\alpha.(n+k-1)!}{(n-1)!.\delta^k(\alpha+k)}.$$

c.) $VarT_{\alpha,\delta,n} = E\,T^2_{\alpha,\delta,n} - (E\,T_{\alpha,\delta,n})^2$.

d.) By Theorem 6, a) and Theorem 5, c.), for $y = t.z$,

$$Ee^{-t.T_{\alpha,\delta,n}} = \int_\delta^\infty E(e^{-t.T_{\alpha,\delta,n}} | \Lambda_{\alpha,\delta} = y).P_{\Lambda_{\alpha,\delta}}(y) dy = \int_\delta^\infty \frac{y^n}{(y+t)^n} \alpha.\delta^\alpha.y^{-\alpha-1} dy =$$

$$= \frac{\alpha.\delta^\alpha.}{t^\alpha}\int_{\delta/t}^{\infty}\frac{z^{n-\alpha-1}}{(1+z)^n}dz.$$

e.) By the Total probability formula and (11) for $x \geq 0$,

$$P_{\frac{T_{1,n}}{\Lambda_{\alpha,\delta}}}(x) = \int_\delta^\infty P_{\frac{T_{1,n}}{\Lambda_{\alpha,\delta}}}(x|\Lambda_{\alpha,\delta} = y)P_{\Lambda_{\alpha,\delta}}(y)dy =$$

$$= \int_\delta^\infty y.P_{T_{1,n}}(x.y).P_{\Lambda_{\alpha,\delta}}(y)dy = \alpha.\delta^\alpha.\int_\delta^\infty y\frac{(x.y)^{n-1}}{(n-1)!}e^{-x.y}y^{-\alpha-1}dy = \alpha.\delta.\frac{(\delta.x)^{\alpha-1}}{(n-1)!}\Gamma(n-\alpha,\delta.x).$$

Theorem 8. Let $\tau_{\alpha,\delta,1}, \tau_{\alpha,\delta,2}, ...,$ be a sequence of random variables with

$$(\tau_{\alpha,\delta,1}, \tau_{\alpha,\delta,2}, ..., \tau_{\alpha,\delta,n}) \sim \mathbf{EP}(\alpha,\delta),\ \ for\ all\ n \in \mathbb{N}.$$

Define $\{N_{\alpha,\delta}(t),\ t \geq 0\}$ as in (17), then

$$\{N_{\alpha,\delta}(t),\ t \geq 0\} \sim MPPP(\alpha,\delta;t).$$

Proof: Let $t > 0$ and $k \in 0, 1, ...$ By (17) we have,

$$P(N_{\alpha,\delta}(t) = k) = P(T_{\alpha,\delta,n+1} \leq t < T_{\alpha,\delta,n}) = P(T_{\alpha,\delta,n} \leq t) - P(T_{\alpha,\delta,n+1} \leq t).$$

By Theorem 6 and Theorem 1. a.). we obtain equality of one dimensional marginals.

Analogously for the f.d.ds of the process $\{N_{\alpha,\delta}(t),\ t \geq 0\}$.

$\square$

Consider $T_{\alpha,\delta,n}$, defined in (16). Denote by

$\eta_{b,\alpha,\delta}(t) = t - T_{\alpha,\delta,N_{\alpha,\delta}(t)}$ - the length of the period $(T_{N_{\alpha,\delta}(t)}, t]$ since the last "event" occur, and by

$\eta_{f,\alpha,\delta}(t) = T_{\alpha,\delta,N_{\alpha,\delta}(t)+1} - t$ - the length of the period $(t, T_{N_{\alpha,\delta}(t)+1}]$ until the next "event" occur.

The next theorem describes the distributions of these two random variables.

Theorem 9. Suppose $\tau_{\alpha,\delta,1}, \tau_{\alpha,\delta,2}, ...,$ is a sequence of random variables with

$$(\tau_{\alpha,\delta,1}, \tau_{\alpha,\delta,2}, ..., \tau_{\alpha,\delta,n}) \sim \mathbf{EP}(\alpha,\delta),\ \ for\ all\ n \in \mathbb{N}$$

and $T_{\alpha,\delta,n}$ is the sequence, defined in (16). Then for all $t > 0$,

a.) for $x_1 \in [0, t]$

$$P(\eta_{b,\alpha,\delta}(t) \geq x_1) = \alpha(x_1.\delta)^\alpha.\Gamma(-\alpha, \delta.x_1);$$

b.) $\eta_{f,\alpha,\delta}(t) \sim EP(\alpha,\delta)$;

c.) for $x_1 \in [0, t]$ and $x_2 > 0$

$$P(\eta_{b,\alpha,\delta}(t) \geq x_1, \eta_{f,\alpha,\delta}(t) \geq x_2) = \alpha((x_1 + x_2).\delta)^\alpha.\Gamma(-\alpha, \delta.(x_1 + x_2)).$$

Proof: c.) Consider a homogeneous Poisson process $N_\lambda := \{N_\lambda(t), t \geq 0\}$ with intensity $\lambda > 0$ and $\Lambda_{\alpha,\delta} \sim Pareto(\alpha, \delta)$ that are independent, defined on one and the same probability space and

$$\{N_{\alpha,\delta}(t),\, t \geq 0\} \stackrel{d}{=} \{N_{\Lambda_{\alpha,\delta}}(t),\, t \geq 0\}.$$

Then $P(\eta_{b,\alpha,\delta}(t) < x_1, \eta_{f,\alpha,\delta}(t) < x_2) = P(t-x \leq T_{\alpha,\delta,N_{\alpha,\delta}(t)} \leq t, t < T_{\alpha,\delta,N_{\alpha,\delta}(t)+1} \leq t + x_2) =$

$$= P(N_{\alpha,\delta}(t-x_1, t] \geq 1, N_{\alpha,\delta}(t, t+x_2] \geq 1) = \int_\delta^\infty P(N_y(t-x_1, t] \geq 1, N_y(t, t+x_2] \geq 1) P_{\Lambda_{\alpha,\delta}}(y)dy =$$

$$= \int_\delta^\infty (1 - e^{-y.x_1}).(1 - e^{-y.x_2}).\alpha.\delta^\alpha.y^{-\alpha-1}dy =$$

$$= 1 - \int_\delta^\infty e^{-y.x_1}\alpha.\delta^\alpha.y^{-\alpha-1}dy - \int_\delta^\infty e^{-y.x_2}\alpha.\delta^\alpha.y^{-\alpha-1}dy + \int_\delta^\infty e^{-y.(x_1+x_2)}\alpha.\delta^\alpha.y^{-\alpha-1}dy =$$

$$= 1 - \alpha(x_1.\delta)^\alpha.\Gamma(-\alpha, \delta.x_1) - \alpha(x_2.\delta)^\alpha.\Gamma(-\alpha, \delta.x_2) + \alpha((x_1+x_2).\delta)^\alpha.\Gamma(-\alpha, \delta.(x_1+x_2)).$$

a.) is immediate consequences from c.) when replace x_2 with zero.

b.) is immediate consequences from c.) when replace x_1 with zero.

Note: Due to the fact that the support of $\eta_{b,\alpha,\delta}(t)$ is bounded (it is $[0, t]$), the distribution of $\eta_{b,\alpha,\delta}(t)$ is truncated Exponentially Pareto distributed with parameters $\alpha > 0$ and $\delta > 0$.

4 The Mixed Poisson Lomax Process

The relation between the discussed Lomax distribution and the discussed Pareto distribution is just shifting. Particularly for $\alpha > 0$ and $\delta > 0$,

$$\Lambda_{\alpha,\delta} \sim Pareto(\alpha, \delta) \Longleftrightarrow \Lambda_{\alpha,\delta} - \delta \sim Lomax(\alpha, \delta).$$

Therefore we start our discussion in this section with the relation between the p.m.f. of a MPP with mixing variable Λ and the p.m.f. of the time intersections of the MPP process with shifted mixing variable $\Lambda - \delta$, where $P(\Lambda - \delta \geq 0) = 1$.

Theorem 10. Let Λ be a random variable that is a.s. greater than $\delta > 0$ and let N be a HPP with intensity 1, independent on Λ and defined on the same probability space. Then

$$P(N(\Lambda - \delta) = k) = e^\delta \sum_{j=0}^{k} \frac{(-\delta)^{k-j}}{(k-j)!} P(N(\Lambda) = j), \quad k = 0, 1, ... \tag{19}$$

Proof: By the total probability formula

$$P(N(\Lambda - \delta) = k) = \int_\delta^\infty P(N(y-\delta) = k) P_\Lambda(y)dy = \int_\delta^\infty \frac{(y-\delta)^k}{k!} e^{-(y-\delta)}.P_\Lambda(y)dy =$$

$$= e^{\delta}\int_{\delta}^{\infty}\frac{e^{-y}}{k!}\sum_{j=0}^{k}\frac{k!}{j!(k-j)!}.y^{j}.(-\delta)^{k-j}.P_{\Lambda}(y)dy = e^{\delta}\sum_{j=0}^{k}\frac{(-\delta)^{k-j}}{j!(k-j)!}.\int_{\delta}^{\infty}e^{-y}.y^{j}.P_{\Lambda}(y)dy =$$

$$= e^{\delta}\sum_{j=0}^{k}\frac{(-\delta)^{k-j}}{(k-j)!}P(N(\Lambda)=j).$$

Now we will express the p.m.f. of the time intersections of a Mixed Poisson Lomax Process in two different ways.

Corollary: Let $\Lambda_{\alpha,\delta}\sim Pareto(\alpha,\delta)$, i.e. $\Lambda_{\alpha,\delta}-\delta\sim Lomax(\alpha,\delta)$. Then

a.)

$$P(N(\Lambda_{\alpha,\delta}-\delta)=k)=\alpha.\delta^{k}.e^{\delta}.\sum_{j=0}^{k}\frac{(-1)^{k-j}}{(k-j)!.j!}.E_{\alpha-j+1}(\delta),\quad k=0,1,...$$

b.)

$$P(N(\Lambda_{\alpha,\delta}-\delta)=k)=\alpha.\frac{\delta^{k}}{k!}.\int_{0}^{\infty}z^{k}.e^{-\delta.z}.(1+z)^{-\alpha-1}dz,\quad k=0,1,...$$

Proof: a.) By Theorem 1, a.) and the Poisson distribution of $N(y)$, when $y\geq 0$,

$$P(N(\Lambda_{\alpha,\delta}-\delta)=k)=e^{\delta}\sum_{j=0}^{k}\frac{(-\delta)^{k-j}}{j!(k-j)!}.\alpha.\delta^{\alpha}.\int_{\delta}^{\infty}e^{-y}.y^{j-\alpha-1}dy=$$

$$= e^{\delta}\sum_{j=0}^{k}\frac{(-\delta)^{k-j}}{(k-j)!}.\frac{\alpha.\delta^{\alpha}}{j!}\Gamma(j-\alpha,\delta)=\alpha.\delta^{k}.e^{\delta}.\sum_{j=0}^{k}\frac{(-1)^{k-j}}{(k-j)!.j!}.E_{\alpha-j+1}(\delta). \tag{20}$$

b.) By the formula of total probability and $\Lambda_{\alpha,\delta}-\delta\sim Lomax(\alpha,\delta)$,

$$P(N(\Lambda_{\alpha,\delta}-\delta)=k)=\int_{0}^{\infty}P(N(y)=k).\alpha.\delta^{\alpha}.(\delta+y)^{-\alpha-1}dy=$$

$$=\int_{0}^{\infty}\frac{y^{k}}{k!}.e^{-y}.\alpha.\delta^{\alpha}.(\delta+y)^{-\alpha-1}dy=\alpha.\int_{0}^{\infty}\frac{(\delta.z)^{k}}{k!}.e^{-\delta.z}.(1+z)^{-\alpha-1}dz=$$

$$=\alpha.\frac{\delta^{k}}{k!}.\int_{0}^{\infty}z^{k}.e^{-\delta.z}.(1+z)^{-\alpha-1}dz, \tag{21}$$

where $y=\delta.z$. □

Particularly

$$P(N(\Lambda_{\alpha,\delta}-\delta)=0)=e^{\delta}.P(N(\Lambda_{\alpha,\delta})=0),$$

$$P(N(\Lambda_{\alpha,\delta}-\delta)=1)=e^{\delta}.(P(N(\Lambda_{\alpha,\delta})=1)-\delta.P(N(\Lambda_{\alpha,\delta})=0)).$$

Remark: 1. In view of (20) and (21)

$$e^{\delta}.\sum_{j=0}^{k}\frac{(-1)^{k-j}}{(k-j)!.j!}.E_{\alpha-j+1}(\delta)=\int_0^\infty\frac{z^k}{k!}.e^{-\delta.z}.(1+z)^{-\alpha-1}dy,\quad k=0,1,...$$

The distribution of the times between exceedances $\tau_{\Lambda_{\alpha,\delta}-\delta}$ and the probability density function of the time of the n-th exceedance $\widetilde{T}_{\alpha,\delta,n}$ in the case of Lomax mixing variable, i.e. if the counting process of the exceedances is $N(\Lambda_{\alpha,\delta}-\delta)$ is given in the next theorem.

Theorem 11. Let $\Lambda_{\alpha,\delta}\sim Pareto(\alpha,\delta)$, i.e. $\Lambda_{\alpha,\delta}-\delta\sim Lomax(\alpha,\delta)$. Then

a.)

$$P(\tau_{\Lambda_{\alpha,\delta}-\delta}\geq x)=\alpha.e^{x.\delta}.(x.\delta)^{\alpha}.\Gamma(-\alpha,\delta.x)=\alpha.e^{x.\delta}.E_{\alpha+1}(\delta.x),\quad x>0.$$

b.)

$$P_{\widetilde{T}_{\alpha,\delta,n}}(x)=\frac{\alpha.\delta^n.x^{n-1}}{(n-1)!}.\int_0^\infty t^n.(t+1)^{-\alpha-1}.e^{-x.t.\delta}dt.$$

Proof. a.) For $y=\delta.z$,

$$P(\tau_{\Lambda_{\alpha,\delta}-\delta}\geq x)=\alpha.\delta^{\alpha}.\int_0^\infty e^{-x.y}(y+\delta)^{-\alpha-1}dy=\alpha.\int_0^\infty e^{-x.\delta.z}(1+z)^{-\alpha-1}dz=$$

$$=\alpha.e^{x.\delta}.\int_1^\infty e^{-x.u.\delta}u^{-\alpha-1}du=$$

where $z+1=u$. For $t=x.u.\delta$

$$=\alpha.e^{x.\delta}.(x.\delta)^{\alpha}.\int_{x.\delta}^\infty e^{-t}.t^{-\alpha-1}dt=\alpha.e^{x.\delta}.(x.\delta)^{\alpha}.\Gamma(-\alpha,\delta.x)=\alpha.e^{x.\delta}.E_{\alpha+1}(\delta.x),\quad x>0.$$

b.) For $y=z.\delta$, and identically distributed $\tau_{\Lambda_{\alpha,\delta}-\delta,1}$, $\tau_{\Lambda_{\alpha,\delta}-\delta,2}$, ..., such that

$$\widetilde{T}_{\alpha,\delta,n}=\tau_{\Lambda_{\alpha,\delta}-\delta,1}+...+\tau_{\Lambda_{\alpha,\delta}-\delta,n},$$

by formula of total probability, the Erlang p.d.f. and Pareto p.d.f.,

$$P_{\widetilde{T}_{\alpha,\delta,n}}(x)=P_{\tau_{\Lambda_{\alpha,\delta}-\delta,1}+...+\tau_{\Lambda_{\alpha,\delta}-\delta,n}}(x)=\alpha.\delta^{\alpha}.\int_\delta^\infty(y-\delta)^n.\frac{x^{n-1}}{(n-1)!}.e^{-x.(y-\delta)}.y^{-\alpha-1}dy=$$

$$=\frac{\alpha.\delta^{\alpha}.e^{x.\delta}.x^{n-1}}{(n-1)!}.\int_1^\infty(z.\delta-\delta)^n e^{-x.z.\delta}.(z.\delta)^{-\alpha-1}dz.\delta=\frac{\alpha.\delta^n.x^{n-1}.e^{x.\delta}}{(n-1)!}.\int_1^\infty(z-1)^n e^{-x.z.\delta}.z^{-\alpha-1}dz=$$

$$=\frac{\alpha.\delta^n.x^{n-1}}{(n-1)!}.\int_0^\infty t^n.(t+1)^{-\alpha-1}.e^{-x.t.\delta}dt,$$

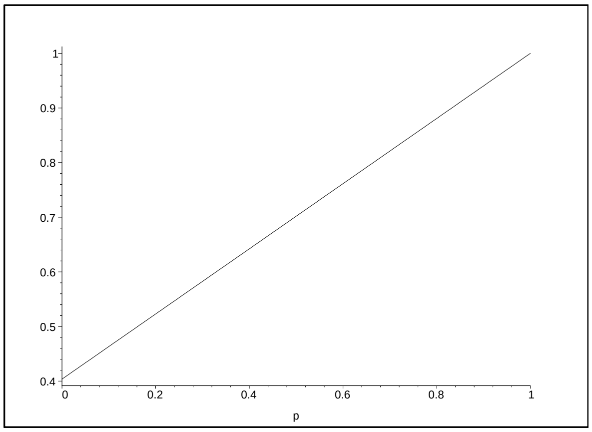

Figure 5: $P(N((\Lambda_{1,1} - 1).I\{\eta = 1\}) = 0)$ for various p.

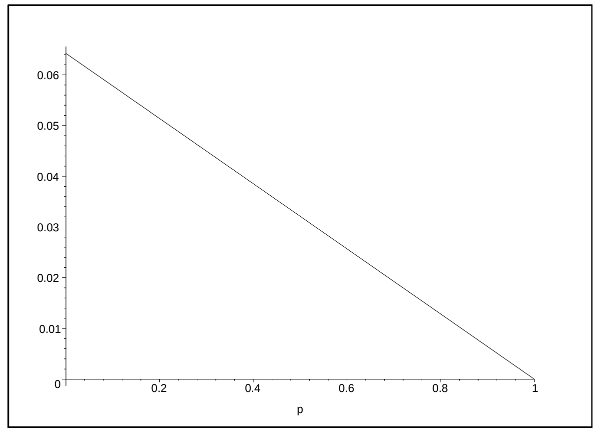

Figure 6: $P(N((\Lambda_{1,1} - 1).I\{\eta = 1\}) = 3)$ for various p.

where $t = z - 1$. $\square$

Finally in this section we discuss the case of the inflated Lomax mixing variable.

Suppose that the random variable η with

$$P(\eta = i) = \begin{cases} p & , \quad i = 0 \\ 1-p & , \quad i = 1 \end{cases}$$

is independent on $\Lambda_{\alpha,\delta}$ and defined on the same probability space. Here $p \in [0, 1]$. Due to $N(0) = 0$ a.s. and the formula of total probability we have

$$P(N((\Lambda_{\alpha,\delta} - \delta).I\{\eta = 1\}) = 0) = p + (1-p).P(N(\Lambda_{\alpha,\delta} - \delta) = 0) = p + (1-p).\alpha.\int_0^\infty e^{-\delta.z}.(1+z)^{-\alpha-1}dz =$$

$$= p + (1-p).\alpha.\delta^\alpha.e^\delta \int_\delta^\infty e^{-t}.t^{-\alpha-1}dt = p + (1-p).\alpha.\delta^\alpha.e^\delta.\Gamma(-\alpha, \delta) = p + (1-p).\alpha.e^\delta.E_{\alpha+1}(\delta).$$

For $k = 1, 2, ...$

$$P(N((\Lambda_{\alpha,\delta} - \delta).I\{\eta = 1\}) = k) = (1-p).P(N(\Lambda_{\alpha,\delta} - \delta) = k) = (1-p).\alpha.\frac{\delta^k}{k!}.\int_0^\infty z^k.e^{-\delta.z}.(1+z)^{-\alpha-1}dz.$$

These probabilities as functions of p are illustrated on Figures 5 and 6.

Let us now find the tail functions of the inflated Exponential Lomax and inflated Erlang Lomax distributions.

If $\tau_0 = 0$ a.s., then $P(\tau_{(\Lambda_{\alpha,\delta} - \delta).I\{\eta=1\}} \geq x) = 1$ for $x \leq 0$ and

$$P(\tau_{(\Lambda_{\alpha,\delta} - \delta).I\{\eta=1\}} \geq x) = (1-p).\alpha.e^{x.\delta}.(x.\delta)^\alpha.\Gamma(-\alpha, \delta.x) = (1-p).\alpha.e^{x.\delta}.E_{\alpha+1}(\delta.x),$$

for $x > 0$.

Analogously suppose $\tau_{(\Lambda_{\alpha,\delta}-\delta).I\{\eta=1\}}$, $\tau_{(\Lambda_{\alpha,\delta}-\delta).I\{\eta=1\},1}$, $\tau_{(\Lambda_{\alpha,\delta}-\delta).I\{\eta=1\},2}$, ..., are identically distributed random variables, such that given $\Lambda_{\alpha,\delta}$ and η are independent. Denote by

$$\widetilde{T}_{\alpha,\delta,n,p} := \tau_{(\Lambda_{\alpha,\delta}-\delta).I\{\eta=1\},1} + ... + \tau_{(\Lambda_{\alpha,\delta}-\delta).I\{\eta=1\},n}.$$

Then by the formula of total probability

$$P_{\widetilde{T}_{\alpha,\delta,n,p}}(x) = P_{\tau_{(\Lambda_{\alpha,\delta}-\delta).I\{\eta=1\},1}+...+\tau_{(\Lambda_{\alpha,\delta}-\delta).I\{\eta=1\},n}}(x) = (1-p).\frac{\alpha.\delta^n.x^{n-1}}{(n-1)!}.\int_0^\infty t^n.(t+1)^{-\alpha-1}.e^{-x.t.\delta}dt.$$

5 Discussion

Present paper introduces point process model for ebullition of methane emissions from sedge-grass marsh station located in South Bohemia, Czech Republic. From the biochemical point of view addressing of probability structure of ebullition is a very important issue, since it allows experimenter to better understand tradeoff between classical diffusion and of methane and superdiffusion of bubbles released from water. We provide simple Mixed Poisson Process with Pareto mixing for building an accurate model since the data are possessing a clear heavy- tails pattern in residuals. To our best knowledge this is the first probabilistically completely described model for ebullition.

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References

[1] Yamamoto S, Alcauskas JB, Crozier TE. Solubility of methane in distilled water and seawater. J Chem Eng Data, 21 (1976) 78-80.

[2] Martens C.S., Klump J.V. Biogeochemical cycling in an organic-rich coastal basin 1.Methane sediment-water exchange processes. Geochimica et Cosmochimica Acta, 44 (1980) 471-490.

[3] Jordanova, P., Dušek, J., Stehlík, M., Modeling methane emission by the infinite moving average process, Chemometrics and Intelligent Laboratory Systems, 122 (2013) 40-49.

[4] Stehlík, M., Střelec, L., Thulin, M. On robust testing for normality in chemometrics, Chemometrics and Intelligent Laboratory Systems, under review.

[5] R. Davis, S. Resnick, Limit theory for moing averages of random variable with regularly varying tail probabilities, The Anals of Probability, 13(1) (1985) 179-195.

[6] S.I. Resnick, Heavy-Tail Phenomena, Springer, New York (2007).

[7] M.R.Leadbetter, On extreme values in stationary sequences, Z. Wahrsch. verw. Gebiete, 28 (1974) 289-303.

[8] S.I. Resnick, Weak convergence to extremal processes, Ann. Probab. 3 (1975) 951-960.

[9] I. Weissman, Multivariate extremal processes generated by independent non-identically distributed random variable, J. Applied Probab. 12 (1975) 477-487.

[10] T.M. Addiscott and N.A. Mirza, New paradigms for modelling mass transfers in soils, Soil & Tillage Research 47 (1998) 105-109.

[11] Karlis, D., Xekalaki, E., Mixed Poisson Dostributions, International Statistical Review, 73(1) (2005) 35-58.

[12] Grandell, J. Mixed Poisson Processes. Chapman and Hall, London, (1997).

[13] Olver, Fr. W.J., D.W. Lozier, R.F. Boisvert, Ch. W. Clark, NIST Handbook on Mathematical Functions, Cambridge University Press, (2010).

[14] Nadarajah, S., Kotz S., On the Laplace transform of the Pareto distribution Queueing Systems, 54(4) (2006) 243-244.

[15] Milgram,M. S., The generalized integro-exponential function, Math. Comp. 44,(1985) 443-458.

[16] Mikosch, T., Non-Lide Insurance Mathematics: An Introduction with Stochastic Processes, Springer-Verlag, Berlin Heidelberg (2004).

[17] Willmot, G., E., On recursive evaluation of mixed Poisson probabilities and related quantities, Scandinavian Actuarial Journal, 2 (1993) 114-133.

[18] Felgueiras, M., M., Explaining the seismic moment of large earthquakes by heavy and extremely heavy tailed models, International Journal on Geomathemtics, 3 (2012) 209-222.

[19] Meijer, J.W. and Baken, N.H.G., The exponential integral distribution, Statistics & Probability Letters 5 (1987) 209-211.

[20] R Development Core Team, R: A Language and Environment for Statistical Computing, R Foundation for Statistical Computing, ISBN 3-900051-07-0, http://www.R-project.org