21. Consider a degenerate ultra-relativistic electron gas at vanishing temperature $T = 0$. The dispersion relation reads $\epsilon = cp$, where ϵ is the single-particle energy, c the speed of light, and $|\mathbf{p}| = p$ the absolute value of the single-particle momentum.

(a) Show that the radius p_F of the Fermi sphere in momentum space and the associated Fermi energy ϵ_F are given by

$$p_F = \hbar \left(3\pi^2 \rho\right)^{\frac{1}{3}} \qquad \text{and} \qquad \epsilon_F = \hbar c \left(3\pi^2 \rho\right)^{\frac{1}{3}} ,$$

respectively. Here $\rho = N/V$ is the particle-number density.

(b) Prove that the total energy E/N per particle may be cast in the form

$$\frac{E}{N} = \frac{3}{4} \hbar c \left(3\pi^2 \rho\right)^{\frac{1}{3}} = \frac{3}{4} \epsilon_F$$

and that

$$p \rho^{-1} = \frac{1}{3} \frac{E}{N} .$$

22. Consider a degenerate ultra-relativistic electron gas at a temperature $T > 0$. The dispersion relation reads $\epsilon = cp$, where ϵ is the single-particle energy, c the speed of light, and $|\mathbf{p}| = p$ the absolute value of the single-particle momentum. The grand-canonical thermodynamic potential Ω is given by

$$\Omega = -2 k_B T \sum_{\mathbf{p}} \ln \left\{ 1 + \mathrm{e}^{\beta(\mu - cp)} \right\} ,$$

where μ is the chemical potential. By transforming the summation $\sum_{\mathbf{p}}$ to an integration show that

$$\Omega = -\frac{V}{\pi^2 (\hbar c)^3} k_B T \int_0^\infty \epsilon^2 \ln \left\{ 1 + \mathrm{e}^{\beta(\mu - \epsilon)} \right\} d\epsilon .$$

Use this expression and $\Omega = -pV$ to prove that

$$PV = \frac{E}{3} ,$$

where E is the total energy. This is the same equation as in 21.(b) but for $T > 0$.

23. The trace (Spur) of an operator $\hat{A}$ is defined as

$$\mathrm{Tr}\{\hat{A}\} = \sum_m \langle \Psi_m | \hat{A} | \Psi_m \rangle ,$$

where the $|\Psi_m\rangle$ are a complete set of orthonormalized basis functions of the Hilbert space (Tr=Sp).

(a) Show that the trace is independent of the choice of the complete set of orthonormalized basis functions, i.e.

$$\mathrm{Tr}\{\hat{A}\} = \sum_l \langle \Phi_l | \hat{A} | \Phi_l \rangle$$

for any other complete set of orthonormalized basis functions $|\Phi_l\rangle$.

(b) Prove that

$$\mathrm{Tr}\{\hat{A}\hat{B}\} = \mathrm{Tr}\{\hat{B}\hat{A}\}\,,$$

irrespective of whether the two operators $\hat{A}$ and $\hat{B}$ commute or not.

(c) Consider a gas of diatomic molecules with rotational energy levels

$$E_l = \frac{\hbar^2}{2I}\, l(l+1)\,,$$

which are $(2l+1)$-degenerate. Show that at low temperatures the canonical partition function is given by

$$Z \approx 1 + 3\exp\left(-\frac{\hbar^2}{I k_B T}\right)$$

and at high temperatures by

$$Z = \frac{2 I k_B T}{\hbar^2}\,.$$

At high temperatures the sum $\sum_l$ may be replaced by an integral $\int_0^\infty dl$ (why?).

24. Consider photons at a temperature T in a volume V. The dispersion relation reads $\epsilon = cp = \hbar c k = \hbar\omega$, where ϵ is the single-particle energy, c the speed of light, $|\mathbf{p}| = p$ the magnitude of the single-particle momentum, k the wave-number, and $\omega = ck$. The degeneracy factor g is 2, since photons are transversal with two independent polarization directions. Since the total number of photons is not conserved, the chemical potential μ vanishes (see Landau/Lifschitz, Statistische Physik, Chap. 63). Therefore $F = \Omega$ with Ω the grand-canonical thermodynamic potential. Derive the expression

$$F = -VT^4 \frac{k_B^4}{3\pi^2(\hbar c)^3} \int_0^\infty \frac{x^3}{\mathrm{e}^x - 1}\, dx$$

for the Helmholtz free energy. Employing

$$\int_0^\infty \frac{x^3}{\mathrm{e}^x - 1}\, dx = \frac{\pi^4}{15}$$

show that

$$F = -\frac{4\sigma}{3c} V T^4$$

with the Stedan-Boltzmann constant

$$\sigma = \frac{\pi^2 k_B^4}{60 \hbar^3 c^2}\,.$$