25. Show the following properties of the trace:
(i) for matrices A, B, C, the trace is cyclic: $\text{Tr}[ABC] = \text{Tr}[BCA] = \ldots$;
(ii) the trace of A can be computed as $\text{Tr}[A] = \sum_i \lambda_i$ where λ_i are the eigenvalues of A;
(iii) $\det(\exp(A)) = \exp(\text{Tr}[A])$.

26. We consider a system of 2 non-interacting atoms (temperature T) are in a large volume V (periodic boundary conditions). The 2 atoms obey either Bose or Fermi statistics.

(a) Show that the canonical partition function is given by

$$Q = \frac{1}{2}\frac{V^2}{\lambda^6}\left(1 \pm \frac{1}{2^{3/2}}\frac{\lambda^3}{V}\right)$$

(b) Show that the density distribution, i.e. the "diagonal" part of the 2-body density matrix, is given by

$$\rho(\mathbf{r}_1, \mathbf{r}_2) \equiv \rho(\mathbf{r}_1, \mathbf{r}_2; \mathbf{r}'_1, \mathbf{r}'_2)_{\mathbf{r}_1=\mathbf{r}'_1, \mathbf{r}_2=\mathbf{r}'_2} = \frac{1}{Q\lambda^6}\left(1 \pm e^{-\frac{2\pi}{\lambda^2}(\mathbf{r}_1-\mathbf{r}_2)^2}\right)$$

[Hint: first calculate the 2-body density operator $e^{-\beta H}$ in the plane wave basis; then use Fourier transformation.]

where "+" and "−" hold for bosons and fermions, and where λ is the thermal wave length: $\lambda = (\frac{2\pi\hbar^2}{mk_BT})^{1/2}$. Compare the results for the Bose and Fermi case with the case of Boltzmann statistics, also as function of T. Which system will be more likely to "condense" at low T, the Bose or the Fermi system?

Experimental realization of this condensation was achieved in 1995 (nobel prize 2001) for 2000 Rb atoms at 20 nK.

28. The 1-dimensional Ising model:
N spins $\vec{\sigma}_i$ are aligned in a row. We assume periodic boundaries, *i.e.* the row is closed to a circle such that $\sigma_{N+1} = \sigma_1$. We regard only the z-component of each spin $\sigma_i \equiv \vec{\sigma}_i|_z$ which can have 2 values, ± 1. The Ising Hamiltonian is given by

$$H = -\epsilon\sum_{i=1}^{N}\hat{\sigma}_i\hat{\sigma}_{i+1} - B\sum_{i=1}^{N}\hat{\sigma}_i = -\epsilon\sum_{i=1}^{N}\hat{\sigma}_i\hat{\sigma}_{i+1} - \frac{B}{2}\sum_{i=1}^{N}(\hat{\sigma}_i + \hat{\sigma}_{i+1})$$

ϵ is the interaction strength between spins.

(a) Discuss the Hamiltonian H. What is the meaning of ϵ and B?

(b) We define an operator O in spin space by

$$\langle\sigma_i|O|\sigma_{i+1}\rangle = e^{\beta[\epsilon\sigma_i\sigma_{i+1} + \frac{B}{2}(\sigma_i+\sigma_{i+1})]}$$

which, in the basis of σ-eigenstates $|\sigma_i\rangle = |\pm\rangle$, is a 2×2-matrix. Using O and the fact that $|\sigma_i\rangle$ is a full basis of the space of spin i, calculate the partition function Q and the free energy A in the canonical ensemble.

(c) Calculate the free energy per particle A/N in the limit of large N. From that, calculate the magnetization per particle M/N. Is there sponaneous magnetization for $B = 0$?

(d) Which limit does $\epsilon = 0$ correspond to? Compare M/N for finite ϵ and for $\epsilon = 0$ (which one is larger and why?). What happens to M/N in the limit of very large ϵ?

FIN