Dr. Dmitry Efrosinin
JKU Linz, Institute for Stochastics

Statistical Methods

Tutorial in WS 2017/18 for Monday, 06.11.17
from 12:00-13:30 in S2 053

Tutorial 2

Continuous distribution families, Chebyshev's inequality and sufficient statistics

Exercise 7 (Memoryless property of exponential distribution)
The distribution function of an exponentially distributed random variable X with rate $\lambda > 0$ is

$$F_X(x) := \mathbb{P}(X \leq x) = 1 - e^{-\lambda x}$$

for $x > 0$. Derive its probability density function and compute the mean and the variance of X. Show the memoryless property of X, i.e.

$$\mathbb{P}(X > s | X > t) = \mathbb{P}(X > s - t)$$

for $s > t$.

Exercise 8 (Gamma distribution and Chebyshev's inequality)

1. Let X_1, X_2 be independent Gamma distributed random variables where $X_i \sim \Gamma(\alpha, \beta_i)$ with scale $\alpha > 0$ and shapes $\beta_i > 0$ for $i = 1, 2$. Prove that $X_1 + X_2$ is Gamma distributed and compute the new shape and scale parameters.
2. Let X be a random variable with finite mean and variance. Show that

$$\mathbb{P}(|X - \mathbb{E}[X]| > \epsilon) \leq \frac{\text{Var}(X)}{\epsilon^2}$$

for all $\epsilon > 0$.

Exercise 9 (Limiting result)
Let X_i be iid random variables with $X_i \sim U(0,1)$ for $i = 1, ..., n$ and let $X_{(n)} := \max_{1 \leq i \leq n} X_i$. Show that $n(1 - X_{(n)})$ converges in distribution to $Exp(1)$.

Exercise 10 (Sufficient statistic 1)
Let X_i be independent normally distributed random variables where $X_i \sim N(\mu_i, \sigma_i^2)$ with $\mu_i \in \mathbb{R}$ and $\sigma_i > 0$ for $i = 1, ..., n$.

1. Prove that

$$\bar{X} := \frac{1}{n}\sum_{i=1}^{n} X_i$$

is normally distributed and derive the mean and the variance of $\bar{X}$.

2. Let $\mu_i = \mu$ and $\sigma_i = \sigma$ with $\mu \in \mathbb{R}$ and $\sigma > 0$ for all $i = 1, ..., n$. Show that $T(\mathbf{X}) = \bar{X}$ is a sufficient statistic for μ when σ is known by using the 1st Theorem for sufficient statistics.

Exercise 11 (Exponential family)
Do the distributions $Pois(\lambda)$ with $\lambda > 0$ and $Geom(p)$ with $p \in [0, 1]$ belong to an exponential family? If yes, define all the involved quantities, i.e. $c(\theta), h(x), \xi_j(\theta)$ and $T_j(x)$ for $j = 1, ..., k$ and $k \in \mathbb{N}$.

Exercise 12 (Characteristic function for normal distribution)
Prove the following property: The random variable X is normally distributed with parameters μ and σ^2 if and only if the corresponding characteristic function $\phi_X(t)$ is of the following form:

$$\phi_X(t) = \mathbb{E}[e^{itX}] = e^{it\mu - \frac{t^2\sigma^2}{2}}.$$