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Statistical Methods

Tutorial in WS 2017/18 for Monday, 13.11.17
from 12:00-13:30 in S2 053

Tutorial 4

Moment and maximum likelihood estimators, Baysian inference

Exercise 19 (Moment and maximum likelihood estimators)
Solve the exercise on slide 14 of week 6.

Exercise 20 (1-parameter exponential family)
Consider the 1-parameter natural exponential family for $\mathbf{X}$ given by

$$f(\mathbf{x};\eta) = \mathbb{1}_{x\in A^n} e^{\eta T(\mathbf{x}) + d_0(\eta) + S(\mathbf{x})}$$

with normalizing constant

$$d_0(\eta) = -\ln\left(\int_A e^{\eta T(\mathbf{x}) + S(\mathbf{x})} d\mathbf{x}\right).$$

Prove that

$$M_{T(\mathbf{X})}(s) = \mathbb{E}[e^{s\,T(\mathbf{X})}] = e^{d_0(\eta) - d_0(\eta+s)} < \infty,$$
$$\mathbb{E}[T(\mathbf{X})] = -d_0'(\eta),$$
$$\mathbb{V}[T(\mathbf{X})] = -d_0''(\eta),$$

for all natural parameter $\eta(=\xi(\theta)), \eta + s \in H$, with $H := \{\eta \in \Theta' : d_0(\eta) < \infty\}$.

Exercise 21 (Bayesian inference)
Consider the Bayesian inference framework. Let $x_i \in \mathbb{N}_0$ for $i = 1, ..., n$ be independent observations from a Geometric distribution with parameter $p \in (0, 1]$. Assume the prior distribution of p with probability density function

$$\pi(p) = p^{\alpha-1}(1-p)^{\beta-1}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \tag{1}$$

with hyperparameters $\alpha, \beta > 0$ and Γ denoting the gamma function. Show that the posterior distribution of p is of the same family as (1) and find the parameters of this posterior distribution. What happens to the posterior mean if $n \to \infty$?

Exercise 22 (Biased vs. unbiased estimators)
Let X_i for $i = 1, ..., n$ be iid random variables from $U[0, \theta]$ with $\theta \in \mathbb{R}^+$. From Exercise 18 we know that $\hat{\theta}_{MLE} = X_{(n)}$. Show that $\hat{\theta}_{MLE}$ is a biased estimator for θ. Consider the unbiased estimator $\hat{\theta}$. Show that $MSE(\hat{\theta}, \theta) < MSE(\hat{\theta}_{MLE}, \theta)$.

Exercise 23 (Moment estimators)
Let $X_1, \ldots, X_n$ be iid random variables from $Bin(k, p)$ with parameters $k \in \mathbb{N}$ and $p \in [0, 1]$. Determine the moment estimators of k and p. Are they good estimators?

Exercise 24 (Maximum likelihood estimators)
Let $X_1, \ldots, X_n$ be iid random variables from a continuous $U(0, \theta)$ with $\theta \in \mathbb{R}^+$. Derive the maximum likelihood estimator of θ.