Turning points in Nearring Theory – and a short poem by G.P.

7th February, 2014

19th Century - Spectacular achievements

1843 ff. HAMILTON's Quaternions **H**
1877 FROBENIUS: Finite dimensional division algebras over **R:** only **R, C, H, O**
1873 HERMITE: e is a transcendental number
1882 LINDEMANN : π is a transcendental number; squaring the circle is impossible.
[In 1885 LINDEMANN was HILBERT's supervisor]

1875 – 1884 TH München: Modellierkabinett KLEIN, BRILL

1899 (2nd ed. 1903) HILBERT, Grundlagen der Geometrie

1890 University of Chicago

founded by American Baptist Education Society
and oil magnate John D. ROCKEFELLER
Motto: Crescat scientia, Vita excolatur –
Let science grow, and [thus human] life be enriched

Land donated by owner of legendary
Chicago Department Store Marshall Field
Opening day: 1 October 1892

Private elite University, emphasis on graduate courses and research.
First President (1891 - 1906): William Rainey HARPER (1856 - 1906), a scholar from Yale (biblical languages).

First Professors of Mathematics:
Oskar BOLZA, Hermann MASCHKE, E. H. MOORE

Eliakim Hastings MOORE
(*1862 Marietta/Ohio, + 1932 Chicago/Ill.)

1879 Student at Yale (Mathematics, Astronomy)
1885 PhD (Yale)
1885/86 Studies at Göttingen and Berlin (KRONECKER, WEIERSTRASS)
Tutor Northwestern University and Yale
1892 Prof. of Mathematics and Acting Head of Dept. at newly founded University of Chicago
1896 – 1931 (retirement) Head of Department
Transformed this Dept. into a center of mathematical research in USA

Students of E.H.MOORE were among many others:
Oswald VEBLEN (1880 - 1960), Leonard E. DICKSON (1874 - 1954), Garrett BIKHOFF (1884 - 1944)

1901/03 President AMS

Research of E.H.MOORE:

- 1902 HILBERT's system of axioms is redundant.
- With Heinrich WEBER (1842 – 1913) extended research on fields to <u>finite</u> structures.
- Introduces 'Pseudo-Inverse' of a matrix (1920 EHM., 1955 Roger PENROSE)

1893 **Columbian Exposition Chicago**
In connection with exposition:
Scientific Congresses and University exhibitions (presentations).
A „German University exhibition room“ was presented, with many mathematical models and instruments.
Catalogue by von W. Von DYCK

Mathematical Congress (August 22 – August 26, 1893)
Committee: BOLZA, MASCHKE, MOORE, WHITE.
Attended by Felix KLEIN (1849 - 1925) as „Beauftragter“ (official representative of German Government)

Evanston Colloquium Lectures by Felix KLEIN at Northwestern University (August 28 – Sept. 9, 1893) [Evanston/Illinois just north of Chicago, home of Northwestern U]

Visiting tour (one month) of Felix KLEIN to the Universities in the East of the USA

Strategic decision by E.H.MOORE:
Area/line of research for Chicago Dept. of Math – for the time being:
Axiomatics of algebra and geometry
(joining the „postulationalists“). MOORE himself contributed.

1901 DICKSON, Linear groups with an exposition of the Galois Field Theory. Published with Teubner, Leipzig.- Important contribution to the theory of **finite simple groups.** Cf. K.V.H.Parshall, Intelligenvcer **13**, no. 10 (1991)

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1905: WEDDERBURN (on visit from Scotland):
Finite division algebras have commutative multiplication, (hence they are fields)

DICKSON: Finite proper near-fields do exist.
Consequence: A distributive law cannot be proved from the rest of axioms of a division algebra, even in the finite case.
TAMS **6** (1905), Nachr. Göttingen 1905; Nachr. Göttingen 1905

Let $K := GF(p^2)$, p *an odd prime.* Change multiplication on K to

$$x \bullet y := \begin{cases} xy, & \text{if } x \text{ is a non-square in } K \\ x\,y^p, & \text{if } x \text{ is a square} \end{cases}$$

Then $(K, +, \bullet)$ is a proper nearfield (left nf.), satisfying

$$a \bullet (b + c) = a \bullet b + a \bullet c$$

Note: the map $x \to x^p$ is an automorphism of $GF(p^2)$

Some structure theorems on finite nearfields:
Order is a prime power, addition is commutative

Later: Generalization of this approach: „Dicksonian nfs“

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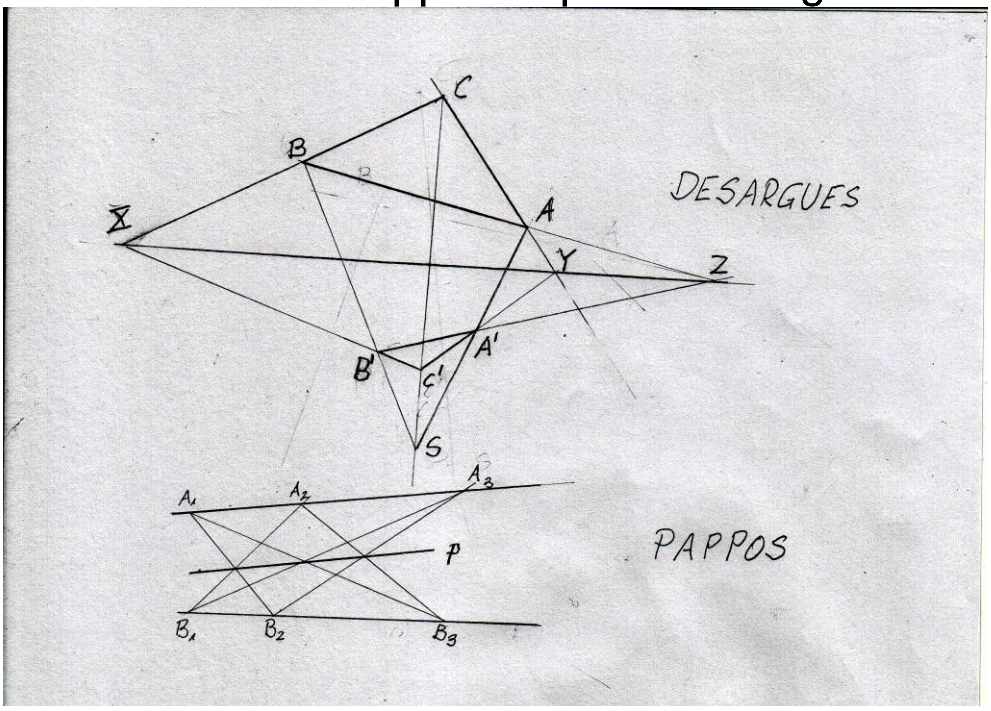
VEBLEN and WEDDERBURN (Publ. 1907) : Construction of a finite projective plane, which does not satisfy the Desargues axiom nor the Pappos-Pascal axiom.

1905 HESSENBERG: Pappos implies Desargues



1931 CARMICHAEL: Finite sharply 2-transitive permutation groups are precisely
the groups of affine selfmaps

$$x \rightarrow ax + b \quad (a \neq 0)$$

of a nearfield (Recognition Thm.)

1930-ies ZASSENHAUS: Classisfication
- of finite sharply 3-tra groups,
- of finite nearfields (sharply 2-tra Gruppen).
- Simple 2-tra groups.

[important for the classification of finite simple groups, completed in 1982 (ASCHBACHER, GORENSTEIN) and quite a few others].

1938 WIELANDT: Nearrings for the first time. Structure of simple 0-symmetric Nearrings with Minimum Condition.

1981 ZASSENHAUS (1912 – 1991) [Honorarprofessor JKU Linz]: There exist infinitely many infinite „non-Dicksonian“ nearfields.

After World War II

- Nearfields and Incidence Groups: KARZEL and his students.
- Distributively generated NR: Hanna NEUMANN, Alan(Albrecht) FRÖHLICH, R.R.LAXTON and others
- Structure theory: (WIELANDT), D.W. BLACKETT and others

What is a structure theory ?

- Determine standard examples with the property, that other “structures” (Groups, NR, Rings ...) may be reduced to these standard examples (the bricks of a wall);
- Determine “small” structures, which may be used to build up more complex structures;
- Decomposition theorems;
- Determine possible homomorphic images;
- How to dispose of unpleasant substructures/epimorphic images? (“Radikals”)
- Classification Theorems and “Theorems of recognition” Example: Every group with 65 elements is the - essentially unique – cyclic group of order 65.
- Special problems.

Central tool: Density Theorem – may be formulated in

various versions, according to the given situation.
(Cf. Theorem of Riemann-Roch in the Theory of Riemann Surfaces, algebraic curves, …)
Papers by Wielandt, Laxton, Ramakotaiah, GBt, **Fuchs-Pilz, Aichinger, Wendt** and others

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<u>Groups of fixed point free (fpf) automorphims</u>
Let *G* be a group of automorphisms of the group *(Γ, +).*
G is called *fixed point free* iff *gγ = γ* implies
g = 1 or γ = o .

<u>One version of the Density Theorem</u>:
Let *N* be a NR (Non-Ring) with 1 consisting from mappings of a group *(Γ, +)* into itself, which leave o fixed. Assume that *Γ* has no "*N*-sugroup" except {o } und *Γ* itself. Then the N-endomorphisms of *Γ* are the zero mapping and a group *G* of fixed point free automorphisms. Let *α, β, γ ...* be finitely many elements of Γ \ {o } from pairwise different orbits under *G*, and let *α′, β′, γ′,* … be (finitely many) arbitrary elements of *Γ.* <u>Then:</u>
N contains an element *n* with the property
αn = α′ , βn = β′ , γn = γ′, ... *(a property of interpolation !)*

This theorem has numerous consequences.

Planar NR and Block Designs

A NF *F* is called *planar* iff the equartion $ax = bx + c$ $(a \neq b)$ has exactly one solution in *F*. (Think of the unique intersection point of two non-parallel lines !). Anshel and Clay discovered a meaningful .
generalization to nrs.

Ferrero and Clay could show: Let (Φ, N) be a pair consisting of a finite group $(N, +)$ and the fixed point free group of automorphisms Φ of N (acting from the left). Such a pair is frequently called a *Ferrero-Pair*.)
Then one may, using Φ , define a multiplication on N in such a way that $(N, +, \cdot)$ turns into a *planar* *NR*, and all planar Nrs may be obtained in this way.

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Furthermore, the orbits B of Φ on $N \setminus \{0\}$ together with their "translated sets" $B + n$ form a block design.
These block designs may be applied, to increase the crops of farmers in Upper Austria.

The incidence matrices of these block designs yield interesting codes - a wide area of investigation.
Papers by Ferrero, Clay, ***Fuchs-Hofer-Pilz und Pilz***

Radicals

Let **C** be a class of structures of the same type (associative rings, nrs, groups, etc.). A subclass ***R*** of ***C*** is called a (KUROSH-AMITSUR) *radical* of ***C*** iff

I) ***R*** is homomorphically closed;

II) every A ε **C** contains a unique ***R***-ideal ***R***(A) which contains any other ***R***-ideal of A;

III) ***R***(A/***R***(A)) = 0

Theorem of ANDERSON-DIVINSKY-SULINSKI (Canad.J.Math. **17**(1965):

Let ***R*** be any radical of associative rings.

If *I* is an ideal of the associative ring *A*, then ***R****(I)* is an ideal of *A*.

This theorem is fundamental in radical theory of rings.

It implies, that ***S***:= ***SR*** := {*A ε* **C** | ***R****(A) = 0* },

the *"semi-simple" class of* ***R*** , is *hereditary* in the following sense: If *I* is an ideal of *A ε* ***S***, then *I ε* ***S.***

This result is no longer true, if we consider nearrings and their radicals. Hence the Theorem of Anderson-Divinsky-Sulinski is no longer true for nearrings.

1982 Wiegandt and GBt, Kaarli and GBt

There exists a vast literature on radical theory in general, and on radicals of nearrings in particular.

And Günter Pilz and his group ?
a) Algebra Dept. of JKU center of nr. theory, actively participating in many lines of research.
(Cf. Chicago some decades earlier).
b) Feeling for “Geneses and applications” (Jim CLAY);
c) Propagating the theory in papers and monographs, duely stressing the applications.

Poem by G.P.
If you try to non-linearize
You will find the nearrings nice