LINEAR ALGEBRA—A GEOMETRIC INTRODUCTION I

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Preface In these notes we have tried to present the theory of matrices, vector spaces and linear operators in a form which preserves a balance between an approach which is too abstract and one which is merely computational. The typical linear algebra course in the early sixties tended to be very computational in nature, with emphasis on the calculation of various canonical forms, culminating in a treatment of the Jordan form without mentioning the geometrical background of the constructions. More recently, such courses have tended to be more conceptual in form beginning with abstract vector spaces (or even modules over rings!) and developing the theory in the Bourbaki style. Of course, the advantage of such an approach is that it provides easy access and a very efficient approach to a great body of classical mathematics. Unfortunately, this was not exploited in many such courses, leaving the abstract theory a torso, deprived of any useful sense for many students. The present course is an attempt to combine the advantages of both approaches. It begins with the theory of linear equations. The method of Gaußian elimination is explained and, with this material as motivation, matrices and their arithmetic are introduced. The method of solution and its consequences are re-interpreted in terms of this arithmetic. This is followed by a chapter on analytic geometry in two and three dimensions. Two by two matrices are interpreted as geometric transformations and it is shown how the arithmetic of matrices can be used to obtain significant geometrical results. The material of these first two chapters and this dual interpretation of matrices provides the basis for a meaningful transition to the axiomatic theory of vector spaces and linear mappings.

After this introductory material, I have felt free to use increasingly higher levels of abstraction in the remainder of the book which deals with determinants, the eigenvalue problem, diagonalisation and the Jordan form, spectral theory and multilinear algebra. I have also included a brief introduction to the complex numbers. The book contains several topics which are not usually covered in introductory text books—of which we mentioned generalised inverses (including Moore-Penrose inverses), singular values and the classification of the isometries in $\mathbf{R}^2$ and $\mathbf{R}^3$. It is also accompanied by three further volumes, one on elementary geometry (based on affine transformations) one containing a further set of exercises and one an introduction to abstract algebra and number theory.

I have taken pains to include a large number of worked examples and exercises in the text. The latter are of two types. Each set begins with routine computational exercises to allow the reader to familiarise himself with the concepts and proofs just covered. These are followed by more theoretical exercises which are intended to serve the dual purpose of providing the student with more challenging problems and of introducing him to the rich array of mathematics which has been made accessible by the theory developed.

1 LINEAR EQUATIONS AND MATRIX THEORY

1.1 Systems of linear equations, Gaußian elimination

The subject of our first chapter is the classical theory of linear equations and matrices. We begin with the elementary treatment of systems of linear equations. Before attempting to derive a general theory, we consider a concrete example:

EXAMPLES. Solve the system

$$\begin{array}{rcrcrcrl} 3x & - & y & & & = & 6 & \quad (1) \\ x & + & 3y & - & 2z & = & 1 & \quad (2) \\ 2x & + & 2y & + & 2z & = & 2. & \quad (3) \end{array}$$

We use the familiar method of successively eliminating the variables: replacing (2) by $3 \cdot (2) - (1)$ and (3) by $3 \cdot (3) - 2(1)$ we get the system:

$$\begin{array}{rcrcrcrl} 3x & - & y & & & = & 6 & \quad (4) \\ & & 10y & - & 6z & = & -3 & \quad (5) \\ & & 8y & + & 6z & = & -6. & \quad (6) \end{array}$$

We now proceed to eliminate y in the same manner and so reduce to the system:

$$\begin{array}{rcrcrcrl} 3x & - & y & & & = & 6 & \quad (7) \\ & & 10y & - & 6z & = & -3 & \quad (8) \\ & & & & 54z & = & -18 & \quad (9) \end{array}$$

and this system can be solved "backwards" i.e. by first calculating z from the final equation, then y from (8) and finally x. This gives the solution

$$z = -\frac{1}{3} \quad y = -\frac{1}{2} \quad x = \frac{11}{6}.$$

(Of course, it would have been more sensible to use (2) in the original system to eliminate x from (1) and (3). We have deliberately chosen this more mechanical course since the general method we shall proceed to develop cannot take such special features of a concrete system into account).

In order to treat general equations we introduce a more efficient notation:

$$\begin{array}{ccccccc} a_{11}x_1 & + & \dots & + & a_{1n}x_n & = & y_1 \\ & & \vdots & & & & \vdots \\ a_{m1}x_1 & + & \dots & + & a_{mn}x_n & = & y_m \end{array}$$

is the general system of m equations in n unknowns. The a_{ij} are the **coefficients** and the problem is to determine the values of the x for given $y_1, \dots, y_m$.

In this context we can describe the method of solution used above as follows. (For simplicity, we assume that $m = n$). We begin by subtracting suitable multiples of the first equation from the later ones in such a way that the coefficients of x_1 in these equations vanish. We thus obtain a system of the form

$$\begin{array}{ccccccccc} b_{11}x_1 & + & b_{12}x_2 & + & \dots & + & b_{1n}x_n & = & z_1 \\ & & b_{22}x_2 & + & \dots & + & b_{2n}x_n & = & z_2 \\ & & & & \vdots & & & & \vdots \\ & & b_{n1}x_2 & + & \dots & + & b_{nn}x_n & = & z_n. \end{array}$$

(with new coefficients and a new right hand side). Ignoring the first equation, we now have a reduced system of $(n-1)$ equations in $(n-1)$ unknowns and, proceeding in the same way with this system, then with the resulting system of $(n-2)$ equations and so on, we finally arrive at one of the form:

$$\begin{array}{ccccccccc} d_{11}x_1 & + & d_{12}x_2 & + & \dots & + & d_{1n}x_n & = & w_1 \\ & & d_{22}x_2 & + & \dots & + & d_{2n}x_n & = & w_2 \\ & & & & & & \vdots & & \vdots \\ & & & & & & d_{nn}x_n & = & w_n \end{array}$$

which we can solve simply by calculating the value of x_n from the last equation, substituting this in the second last equation and so working backwards to x_1.

This all sounds very simple but, unfortunately, reality is rather more complicated. If we call the coefficient of x_i in the i-th equation after $(i-1)$ steps in the above procedure the i-th **pivot element**, then the applicability of our method depends on the fact that the i-th pivot is non-zero. If it *is* zero, then two things can happen.

Case 1: the coefficient of x_i in a later equation (say the j-th one) is non-zero. Then we simply exchange the i-th and j-th equations and proceed as before.

Case 2: the coefficients of x_i in the i-th and all following equations vanish. In this case, it may happen that the system has *no* solution.

EXAMPLES. Consider the systems

$$\begin{array}{ccccccccc} x & + & y & + & 2z & = & -2 \\ 3x & + & 3y & - & 4z & = & 6 \\ 2x & - & y & + & 6z & = & -1 \end{array} \qquad \begin{array}{ccccccccc} 2x & + & 3y & - & z & = & 1 \\ 6x & + & y & + & z & = & 3 \\ 4x & + & 2y & & & = & -1. \end{array}$$

If we apply the above method to the first system we get successively

$$\begin{array}{rcrcrcr} x & + & y & + & 2z & = & -2 \\ & & & & -10z & = & 12 \\ & & -3y & + & 2z & = & 3 \end{array}$$

i.e.

$$\begin{array}{rcrcrcr} x & + & y & + & 2z & = & -2 \\ & & -3y & + & 2z & = & 3 \\ & & & & -10z & = & 12 \end{array}$$

and this is solvable.

In the second case, we get:

$$\begin{array}{rcrcrcr} 2x & + & 3y & - & z & = & 1 \\ & & -8y & + & 4z & = & 0 \\ & & -4y & + & 2z & = & -3. \end{array}$$

At this stage we see at a glance that the system is not solvable (the last two equations are incompatible) and indeed the next step leads to the system

$$\begin{array}{rcrcrcr} 2x & + & 3y & - & z & = & 1 \\ & & -8y & + & 4z & = & 0 \\ & & & & 0 & = & 6 \end{array}$$

and the third pivot vanishes. Note that the vanishing of a pivot element does not *automatically* imply the non-solvability of the equation. For example, if the right hand side of our original equation had been

$$\begin{array}{c} 4 \\ 8 \\ 6 \end{array}$$

then it would indeed have been solvable as the reader can verify for himself.

Hence we see that if a pivot element is zero in the non-trivial manner of case 2 above, then we require a more subtle analysis to decide the solvability or non-solvability of the system. This will be the main concern of this Chapter whereby the method and result will be repeatedly employed in later ones. A useful tool will be the so-called **matrix formalism** which we introduce in the next section.

EXAMPLES. Solve the following systems (if possible):

$$\begin{array}{rcrcrcr} 2x & + & 4y & + & z & = & 1 \\ 3x & + & 5y & & & = & 1 \\ 5x & + & 13y & + & 7z & = & 5 \end{array} \qquad \begin{array}{rcrcrcr} x & + & 5y & + & 2z & = & 9 \\ x & + & y & + & 7z & = & 6 \\ & & -3y & + & 4z & = & -2. \end{array}$$

Solution: In the first case we subtract 7 times the first equation from the third one and get

$$-9x - 15y = -2$$

which is incompatible with the second equation.

In the second case, we subtract the 2nd one from the first one and get

$$4y - 5z = 3.$$

This, together with the third equation, gives $y = 2, z = 1$. Hence $x = -3$ and this solution is unique.

Exercises: 1) Solve the following systems (if possible):

$$\begin{array}{ccccccccc} y_1 & + & y_2 & & & & & = & a \\ & & y_2 & + & y_3 & & & = & b \\ & & & & y_3 & + & y_4 & = & c \\ y_1 & & & & & + & y_4 & = & d. \end{array}$$

$$\begin{array}{ccccccccc} 2x & + & y & + & 2z & - & w & = & 0 \\ 6x & + & 8y & + & 12z & - & 13w & = & -21 \\ 10x & + & 2y & + & 2z & + & 3w & = & 21 \\ -4x & & & + & z & - & 3w & = & -13. \end{array}$$

2) Which of the following two systems of equations has a solution?

$$\begin{array}{ccccccc} 2x & + & 4y & + & z & = & 1 \\ 3x & + & 5y & & & = & 1 \\ 5x & + & 13y & + & 7z & = & 5 \end{array} \qquad \begin{array}{ccccccc} x & + & 5y & + & 2z & = & 9 \\ x & + & y & + & 7z & = & 6 \\ & - & 3y & + & 4z & = & -2. \end{array}$$

3) For which values of a, b, c does the system

$$\begin{array}{ccccccc} x & + & y & + & z & = & 1 \\ ax & + & by & + & cz & = & 1 \\ a^2x & + & b^2y & + & c^2z & = & 1. \end{array}$$

have a solution? In case it does, calculate it explicitly.

4) For which values of a does the system

$$\begin{array}{ccccccc} x & + & y & - & z & = & 1 \\ 2x & + & 3y & + & az & = & 3 \\ x & + & ay & + & 3z & = & 2 \end{array}$$

have a unique solution? 5) For which values of a, b, c and d is the equation

$$\begin{array}{rcrcrcl} 2x & + & y & + & 2z & = & a \\ x & + & 2y & + & z & = & b \\ -x & - & y & & & = & c \\ & & y & & & = & d \end{array}$$

solvable?
6) Show that the system

$$\begin{aligned} ax + by &= e \\ cx + dy &= f \end{aligned}$$

is solvable for every value of e and f if and only if $ad - bc \neq 0$. The solution is then unique and is given by the formula

$$x = \frac{ed - bf}{ad - bc} \qquad y = \frac{af - ce}{ad - bc}.$$

7) Find suitable constants $a, b, c, \ldots$ so that

- $1 + 2 + \cdots + n = an^2 + bn + c$;
- $1 + 3 + \cdots + (2n - 1) = an^2 + bn + c$;
- $1^2 + 2^2 + \cdots + n^2 = an^3 + bn^2 + cn + d$.

(Of course, the constants may be different in each case).

1.2 Matrices and their arithmetic

Consider again our original system

$$\begin{array}{ccccccc} 3x & - & y & + & 0.z & = & 6 \\ x & + & 3y & - & 2z & = & 1 \\ 2x & + & 2y & + & 2z & = & 2. \end{array}$$

The information contained in these equations can be reduced to two schemes of numbers—the coefficients of the unknowns—which can be written thus: the coefficients of the left hand side

$$\begin{bmatrix} 3 & -1 & 0 \\ 1 & 3 & -2 \\ 2 & 2 & 2 \end{bmatrix}$$

and the right hand side

$$\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}.$$

For the general system

$$\begin{array}{ccccccc} a_{11}x_1 & + & \dots & + & a_{1n}x_n & = & y_1 \\ \vdots & & & & \vdots & & \\ a_{m1}x_1 & + & \dots & + & a_{mn}x_n & = & y_m \end{array}$$

the corresponding schemes are:

$$\begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix}$$

resp.

$$\begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}.$$

We call such an array

$$\begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix}$$

an $m \times n$ **matrix**. m is the number of **rows**, n the number of **columns** of the matrix which can be written in shortened form as $[a_{ij}]_{i=1,j=1}^{m,n}$ or simply $[a_{ij}]$

when it is not necessary to specify m and n. We use capital letters A,B,C,... to denote matrices: thus $A = [a_{ij}]$ means that the (i, j)-th element (i.e. the element in the i-th row and j-th column) of A is a_{ij}.

Similarly,

$$\begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}$$

is an $m \times 1$ matrix. Such matrices are called **column matrices** for obvious reasons. Thus the j-th column of $A = [a_{ij}]$ is the $m \times 1$ matrix

$$\begin{bmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{bmatrix}.$$

Similarly, the i-th row is the $1 \times n$ **row matrix**

$$\begin{bmatrix} a_{i1} \ldots a_{in} \end{bmatrix}.$$

If A_i (resp. B_j) is the i-th row (resp. j-th column) of A it is sometimes convenient to write A in the forms:

$$\begin{bmatrix} A_1 \\ \vdots \\ A_m \end{bmatrix}$$

or

$$\begin{bmatrix} B_1 \ldots B_n \end{bmatrix}.$$

EXAMPLES.

$$\begin{bmatrix} 6 & -3 & 7 \\ 1 & 1 & -1 \end{bmatrix}$$

is a 2×3 matrix. Its second row is

$$\begin{bmatrix} 1 & 1 & -1 \end{bmatrix}$$

and its third column is

$$\begin{bmatrix} 7 \\ -1 \end{bmatrix}.$$

The $(2, 3)$-rd element is -1.

We list some important special matrices resp. types of matrices. Note the systems of equations to which they correspond.

1) the $m \times n$ **zero matrix**

$$\begin{bmatrix} 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{bmatrix}$$

(i.e. the matrix with all entries zero). This corresponds to the system

$$\begin{bmatrix} 0 \cdot x_1 & + & \dots & + & 0 \cdot x_n & = & y_1 \\ & \vdots & & & \vdots & & \\ 0 \cdot x_1 & + & \dots & + & 0 \cdot x_n & = & y_m \end{bmatrix}$$

which is solvable only when the right hand side vanishes. Then *any* n-tuple $(x_1, \dots, x_n)$ is a solution. We denote this matrix by $0_{m,n}$ or, less pedantically, by 0.
2) the $n \times n$ matrix

$$\begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & & & \vdots \\ 0 & & \dots & 1 \end{bmatrix}$$

which has one's in the main diagonal and zeros elsewhere. This matrix is called the **unit matrix**, denoted by I_n or simply by I. It corresponds to the system

$$\begin{bmatrix} x_1 & + & 0\cdot & x_2 & + & \dots & + & 0 \cdot x_n & = & y_1 \\ & & & & \vdots & & & \vdots & & \\ & & & & & & & x_n & = & y_n \end{bmatrix}$$

which always has a unique solution (in fact, it *is* its own solution!).

It is customary to introduce the so-called **Kronecker δ-symbol** δ_{ij} where $\delta_{ij} = 1$ if $i = j$, $\delta_{ij} = 0$ otherwise. Then we can describe I succinctly by the formula $I = [\delta_{ij}]$.
3) diagonal matrices: these are square matrices (i.e. $n \times n$ matrices for some n) of the form

$$\begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}$$

i.e. matrices whose only non-zero elements are in the main diagonal. We use the shorthand $\operatorname{diag}(\lambda_1, \dots, \lambda_n)$ to denote this matrix. It corresponds to the system:

$$\begin{bmatrix} \lambda_1 & \dots & & = & y_1 \\ & \vdots & & \vdots & \\ & & \lambda_n x_n & = & y_n. \end{bmatrix}$$

If all of the λ_i are non-zero, then this always has a unique solution. If any of them vanish, then it is solvable only for those right hand sides which vanish at each equation where the λ_i vanishes and the solution is not unique since the corresponding x_i can be chosen arbitrarily.
4) (upper) triangular matrices. These are square matrices $A = [a_{ij}]$ for which $a_{ij} = 0$ if $i > j$ i.e. they have the form

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \vdots & & & \vdots \\ 0 & 0 & \dots & a_{nn} \end{bmatrix}.$$

Lower triangular matrices are defined similarly, the condition being that $a_{ij} = 0$ if $i < j$.

Note that in the case of n equations in n unknowns, the elimination process described above is intended to replace a general system by one with a triangular matrix, for the very good reason that the solution of such an equation (if it exists) can be read off directly, starting with the last equation. Once again it is precisely the condition that the diagonal elements $a_{11}, \dots, a_{nn}$ do not vanish which is required to ensure uniqueness and existence of solutions for systems with triangular matrices.

Returning to the topic of linear equations, we can now streamline our notation by writing the general equation in the form

$$AX = Y$$

where A is the $m \times n$ matrix $[a_{ij}]$ and X is

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

and Y

$$\begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}$$

. Up to this point, our introduction of the matrix formalism has brought nothing more than a simplification of the notation. We shall now introduce an "arithmetic" for matrices which is closely connected with properties of the corresponding systems of equations and which will provide the machinery that we require for a theory of such systems. More precisely, we shall equip certain sets of matrices with an addition and a multiplication as follows:

Addition: We add two $m \times n$ matrices A and B simply by adding their respective components. The result is denoted by $A + B$. In symbols:

$$[a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}].$$

For example:

$$\begin{bmatrix} 6 & 3 \\ 2 & -1 \\ 1 & -1 \end{bmatrix} + \begin{bmatrix} 7 & 9 \\ -2 & 2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 13 & 12 \\ 0 & 1 \\ 1 & 2 \end{bmatrix}.$$

Note that we only add matrices of the same type (i.e. with the same numbers of rows and columns). Thus the sum of a 3×1 and a 1×2 matrix is not defined.

Addition of matrices possesses many properties which are reminiscent of those of addition of real numbers e.g.

- $A + B = B + A$ (commutativity);
- $A + (B + C) = (A + B) + C$ (associativity);
- $A = 0 = 0 + A = A$;
- $A + (-A) = 0$ where $-A$ is the matrix $[-a_{ij}]$.

In future we shall write $A - B$ for the matrix $A + (-B)$.

These properties are verified by noting that the corresponding equations hold element for element. For example, $A + B = B + A$ since

$$A + B = [a_{ij} + b_{ij}] = [b_{ij} + a_{ij}] = B + A.$$

Another operation which will be useful is that of **scalar multiplication** i.e. the multiplication of each element of A by a given λ in $\mathbf{R}$. The result is denoted by λA (i.e. $\lambda A = [\lambda a_{ij}]$).

For example:

$$3 \begin{bmatrix} 1 & 6 & 1 \\ -1 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 18 & 3 \\ -3 & 6 & -3 \end{bmatrix}.$$

This multiplication has the following properties:

- $\lambda(A+B) = \lambda A + \lambda B$;
- $(\lambda\mu)A = \lambda(\mu A)$;
- $(\lambda + \mu)A = \lambda A + \mu A$;
- $1 \cdot A = A$, $(-1) \cdot A = -A$.

We now consider **multiplication** of matrices. Here it is not quite so clear *a priori* how the product of two matrices should be defined and we begin with some remarks which may help to motivate the formal definition. If we write our general system of equations in the form $AX = Y$ as above, then it is natural to define the product of an $m \times n$ matrix $A = [a_{ij}]$ and an $n \times 1$ matrix X as above to be the $m \times 1$ matrix

$$\begin{bmatrix} a_{11}x_1 + & \cdots + & a_{1n}x_n \\ \vdots & & \vdots \\ a_{m1}x_1 + & \cdots + & a_{mn}x_n \end{bmatrix}.$$

Then if B is the $n \times p$ matrix of the form

$$\begin{bmatrix} 0 & 0 & \dots & 0 & b_{1k} & 0 & \dots & 0 \\ \vdots & & & & & \vdots & & \vdots \\ 0 & 0 & \dots & 0 & b_{mk} & 0 & \dots & 0 \end{bmatrix}$$

(where the column of b's is in the k-th column), it is natural to define AB to be the $m \times p$ matrix

$$\begin{bmatrix} 0 & 0 & \dots & 0 & a_{11}b_{1k} + \cdots + a_{1n}b_{nk} & 0 & \dots & 0 \\ \vdots & & & & & \vdots & & \vdots \\ 0 & 0 & \dots & 0 & a_{m1}b_{1k} + \cdots + a_{mn}b_{nk} & 0 & \dots & 0 \end{bmatrix}.$$

Now if we assume that multiplication obeys the usual rules of arithmetic we can calculate the general product AB where $B = [b_{jk}]$ as follows: We write B as a sum of p matrices of the above form i.e. each with one non-vanishing column. We then multiply out as if the usual distributive law holds.

We illustrate this with the example:

$$\begin{bmatrix} -1 & 6 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \\ 1 & 1 & 3 \end{bmatrix}.$$

We express the product as

$$\begin{bmatrix} -1 & 6 \\ -2 & 1 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 3 \end{bmatrix} \right).$$

If we multiply out each term using the above rule and simplify, we obtain the matrix

$$\begin{bmatrix} 5 & 7 & 16 \\ -1 & 3 & -1 \end{bmatrix}.$$

This leads naturally to the following definition: if $A = [a_{ij}]$ is an $m \times n$ matrix and $B = [b_{jk}]$ an $n \times p$ matrix, then their **product** AB is defined to be the $m \times p$ matrix $C = [c_{ik}]$ where

$$c_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk}.$$

That is, the (i, k)-th element of C is obtained by running along the i-th row of A and the k-th column of B, multiplying successively the corresponding elements and then taking the sum. From this description it is clear that the product AB is only defined when the number of columns of A equals the number of rows of B. In particular, the expression $A \cdot A$ (which we shorten to A^2) is defined only when $m = n$ i.e. when A is a square matrix. Then we can define higher powers $A^3, A^4, \dots, A^k, \dots$ of A in the obvious way.

Before discussing further properties of matrices, we illustrate the definition of multiplication with a simple example:

Examples. Calculate the products

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$\begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$$

resp.

$$\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$$

$$\begin{bmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{bmatrix}.$$

The product of the first two matrices is

$$\begin{bmatrix} \cos\theta\cos\phi - \sin\theta\sin\phi & -\cos\theta\sin\phi - \sin\theta\cos\phi \\ \sin\theta\cos\phi + \cos\theta\sin\phi & -\sin\theta\sin\phi + \cos\theta\cos\phi \end{bmatrix}$$

which is equal to

$$\begin{bmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) \\ \sin(\theta+\phi) & \cos(\theta+\phi) \end{bmatrix}.$$

Similarly,

$$\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} \begin{bmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{bmatrix} = \begin{bmatrix} \cos(\theta-\phi) & -\sin(\theta-\phi) \\ \sin(\theta-\phi) & \cos(\theta-\phi) \end{bmatrix}.$$

(We shall give a geometric interpretation of these formulae in the next chapter).

This definition of multiplication can also be motivated by the following considerations. Suppose that we have two systems:

$$BX = Y \quad \text{and} \quad AY = Z.$$

where the unknowns in the second equation are the right hand side of the first. Then eliminating Y from these equations leads to the equation $CX = Z$ where $C = AB$ is the product defined above as a little arithmetic shows.

Example Consider the systems

$$\begin{bmatrix} 3x & + & 6y & + & 4z & + & 2w & = a \\ x & + & 7y & + & 8z & + & 9w & = b \\ -4x & - & y & + & 2z & + & 3w & = c \end{bmatrix}$$

resp.

$$\begin{bmatrix} u & & & = x \\ 2u & + & 4v & = y \\ 6u & - & v & = z \\ 2u & + & v & = w. \end{bmatrix}$$

Substituting we get the system

$$\begin{bmatrix} 3u & + & 6(2u+4v) & + & 4(6u-v) & + & 2(2u+v) & = a \\ u & + & 7(2u+4v) & + & 8(6u-v) & + & 9(2u+v) & = b \\ -4u & - & (2u+4v) & + & 2(6u-v) & + & 3(2u+v) & = c. \end{bmatrix}$$

with matrix

$$\begin{bmatrix} 43 & 22 \\ 81 & 29 \\ 12 & -3 \end{bmatrix} = \begin{bmatrix} 3 & 6 & 4 & 2 \\ 1 & 7 & 8 & 9 \\ -4 & -1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 4 \\ 6 & -1 \\ 2 & 1 \end{bmatrix}.$$

We now collect some simple properties of matrix multiplication:

1) matrix multiplication is associative i.e. $(AB)C = A(BC)$. (Here, as elsewhere, we tacitly assume, when we write such a formula, that A, B and C satisfy the conditions necessary to ensure that all products which appear are defined. In this case, this means that A is of type $m \times n$, B of type $n \times p$ and C of type $p \times r$ for suitable m,n, p and r).

Since the above equation is not quite so obvious as those encountered so far, we bring the proof.

Proof. Put $A = [a_{ij}]$, $B = [b_{jk}]$ and $C = [c_{kl}]$. Then

$$AB = \left[\sum_{j=1}^{n} a_{ij} b_{jk} \right]_{ik}$$

and

$$(AB)C = \left[\sum_{k=1}^{p} \left(\sum_{j=1}^{n} a_{ij} b_{jk} \right) c_{kl} \right]_{il}.$$

Similarly

$$A(BC) = \left[\sum_{j=1}^{n} a_{ij} \left(\sum_{k=1}^{p} b_{jk} c_{kl} \right) \right]_{il}.$$

Hence we must show that

$$\sum_{k=1}^{p} \left(\sum_{j=1}^{n} a_{ij} b_{jk} c_{kl} \right) = \sum_{j=1}^{n} \left(\sum_{k=1}^{p} a_{ij} b_{jk} c_{kl} \right).$$

This follows from the general fact that if $[d_{jk}]$ is an $n \times p$ matrix, then

$$\sum_{k=1}^{p} \left(\sum_{j=1}^{n} d_{jk} \right) = \sum_{j=1}^{n} \left(\sum_{k=1}^{p} d_{jk} \right).$$

The above relation is clear. We are simply summing all elements of the matrix in two ways—as the sum of the row sums respectively as the sum of the column sums. In our example, we put $d_{jk} = a_{ij} b_{jk} c_{kl}$ with i and l fixed.
2) multiplication is distributive over addition i.e.

$$A(B + C) = AB + AC \quad (A + B)C = AC + BC.$$

One negative property which has important repercussions is:
3) multiplication is *not* commutative i.e. we do not have the equation $AB = BA$ in general. For suppose that A is an $m \times n$ matrix and B an $n \times p$ matrix. Then if $p \neq m$, BA is not even defined. If $p = m$ but $m \neq n$, then AB is an $m \times m$ matrix while BA is $n \times n$ so they are certainly not equal. This leaves the only interesting case as the one where $m = n = p$ i.e. where A and B are square. Even in this case it can (and usually *does*) happen that AB and BA are distinct. Thus

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$$

while

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

4) I_n is a unit for multiplication: more precisely, if A is an $m \times n$ matrix, then

$$A \cdot I_n = A = I_m \cdot A.$$

One question which plays a central role in the theory of matrices is that of the invertibility of a matrix. We say that an $n \times n$ matrix A is **invertible** if there is an $n \times n$ matrix B so that $AB = BA = I$. Note that if we can find such a B we can reduce the question of solving the equation $AX = Y$ to one of matrix multiplication since $X = BY$ is then a solution. (For $AX = A(BY) = (AB)Y = I \cdot Y = Y$). We shall examine the relationship between invertibility of A and solvability of the equation $AX = Y$ in more detail later. In the meantime we continue our remarks on inverses. We note that there is at most *one* B with the property that $AB = BA = I$. For if $AC = CA = I$, then

$$B = B \cdot I = B(AC) = (BA)C = I \cdot C = C.$$

Hence if A is invertible, we can refer to *the* inverse of A and denote it by A^{-1}.

Of course, not every $n \times n$ matrix is invertible (otherwise every system of equations would be solvable). The following are simple examples of non-invertible 2×2 matrices:

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

as can easily be checked by direct calculation since the product of these matrices by a typical 2×2 matrix

$$\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

on the right produces the results:

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} b_{21} & b_{22} \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} b_{11} + b_{21} & b_{12} + b_{22} \\ b_{11} + b_{21} & b_{21} + b_{22} \end{bmatrix}$$

and no choice of the b's will give the unit matrix.

In general, it doesn't make sense to talk of an inverse for a non-square matrix A but the concept of a **left inverse** B i.e. an $n \times m$ matrix so that $BA = I_n$ *is* meaningful—as is that of a right inverse which is defined analogously. Note that the argument used above to show that a matrix has a unique inverse shows that if an $n \times n$ matrix A has a right inverse *and* a left one, then they are automatically equal and so provide an *inverse* for A. (Later we shall see that if a square matrix has a left *or* right inverse, then this is automatically an inverse). The same argument also shows that a non-square matrix cannot have both a left and a right inverse.

We now investigate in more detail the connection between the existence of inverses and the solvability of systems of equations.

Proposition 1 *Let A be an $m \times n$ matrix and suppose that the equations $AX = E_k$ are solvable for $k = 1, \dots, m$ where E_k is the k-th column vector of I_m (i.e. the vector with 0's everywhere except for a 1 in the k-th row). Then A has a right inverse.*

PROOF. We let the $n \times 1$ column vector X_k be a solution of $AX = E_k$. Then $B = [X_1 \dots X_m]$ is a right inverse for A since

$$A[X_1 \dots X_m] = [AX_1 \dots AX_m] = [E_1 \dots E_m].$$

Of course if A has a right inverse B then the equation $AX = Y$ always has a solution, namely the vector $X = BY$. We can summarise the situation as follows:

Proposition 2 *A has a right inverse if and only if the system $AX = Y$ is solvable for any choice of the right hand side Y.*

We illustrate this result by considering some special matrices whose right inverses (and hence inverses by the above remark since the matrices of our examples are square) can be calculated with relative ease:
1) the diagonal matrix $A = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$. Clearly this matrix has a right inverse if and only if each λ_i is non-zero and then its right inverse is the diagonal matrix $\operatorname{diag}(\lambda_1^{-1}, \dots, \lambda_n^{-1})$.
2) $n \times n$ matrices of the form

$$\begin{bmatrix} \lambda & 1 & 0 & \dots & 0 \\ 0 & \lambda & 1 & \dots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \dots & \lambda \end{bmatrix}.$$

As we shall see later, these play an important role in matrix theory and so we shall denote the above matrix by $J_n(\lambda)$. The k-th column of its right inverse is the solution of the equation

$$\begin{bmatrix} \lambda x_1 + x_2 + & & \dots & = 0 \\ & & \vdots & \vdots \\ & \lambda x_k + x_{k+1} + & \dots & = 1 \\ & & \vdots & \vdots \\ & & \lambda x_n & = 0 \end{bmatrix}$$

and this is easily seen to be the vector

$$\begin{bmatrix} (-1)^{k-1}\lambda^{-k} \\ \\ \vdots \\ \frac{1}{\lambda} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

(where the term $\dfrac{1}{\lambda}$ is in the k-th row) provided that λ is non-zero.

Hence the (right) inverse of $J_n(\lambda)$ has the form

$$\begin{bmatrix} \frac{1}{\lambda} & -\frac{1}{\lambda^2} & & \dots & \frac{(-1)^{n-1}}{\lambda^n} \\ 0 & \frac{1}{\lambda} & & \dots & \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \dots & \frac{1}{\lambda} \end{bmatrix}.$$

3) the above two examples are triangular matrices. By considering the corresponding system of equations for the general triangular matrix

$$\begin{bmatrix} a_{11} & a_{12} & & \dots & a_{1n} \\ 0 & a_{22} & & \dots & a_{2n} \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \dots & a_{nn} \end{bmatrix}$$

one sees that it has a (right) inverse if and only if the diagonal elements are non-zero.

We now continue with our discussion of operations on matrices by considering polynomial functions thereof. We have already mentioned the fact that for a square matrix A we can form its powers A^k in the obvious way. It is then easy to see how to define polynomial functions of A. For example, if

$$p(t) = t^2 + 3t + 1$$

and

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

then

$$\begin{aligned} p(A) &= A^2 + 3A + I && (1)\\ &= \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 3 \\ 0 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} && (2)\\ &= \begin{bmatrix} 5 & 5 \\ 0 & 5 \end{bmatrix}. && (3) \end{aligned}$$

The general definition is as follows: if

$$p(t) = a_0 + a_1 t + \cdots + a_k t^k$$

then $p(A)$ is the $n \times n$ matrix

$$a_0 I + a_1 A + \cdots + a_k A^k.$$

A simple but important fact is that if p_1 and p_2 are polynomials, then

$$p_1(A)p_2(A) = p_1p_2(A)$$

where p_1p_2 is the product of the polynomials p_1 and p_2. From this it follows that $p_1(A)$ and $p_2(A)$ commute i.e. matrices which are both polynomial functions of the same matrix commute.

For suppose that

$$p_1(t) = a_0 + a_1 t + \cdots + a_k t^k$$

$$p_2(t) = b_0 + b_1 t + \cdots + b_l t^l$$

Then the product polynomial p_1p_2 has the form:

$$\begin{aligned} p_1p_2(t) &= a_0b_0 + (a_1b_0 + a_0b_1)t + \cdots + a_kb_lt^{k+l} && (4)\\ &= \sum_{r=0}^{k+l} c_r t^r \quad \text{where} \quad c_r = \sum_{i+j=r} a_ib_j. && (5) \end{aligned}$$

On the other hand, if we multiply out the expressions for $p_1(A)$ and $p_2(A)$ we see that the result is $\sum_{r=0}^{k+l} c_r A^r$ where the c_r are as above and this is just $p_1p_2(A)$.

EXAMPLES.

1) If $A = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$, then

$$p(A) = \operatorname{diag}(p(\lambda_1), \dots, p(\lambda_n)).$$

2) If p can be split into linear factors

$$(t - \lambda_1) \cdots (t - \lambda_k)$$

then

$$p(A) = (A - \lambda_1 I) \cdots (A - \lambda_k I).$$

3) If A is the matrix $J_n(\lambda)$ introduced above, then

$$p(A) = \begin{bmatrix} p(\lambda) & p'(\lambda) & \dots & \frac{p^{(n-1)}(\lambda)}{(n-1)!} \\ 0 & p(\lambda) & \dots & \frac{p^{(n-2)}(\lambda}{(n-2)!} \\ \vdots & & & \vdots \\ 0 & 0 & \dots & p(\lambda) \end{bmatrix}.$$

Perhaps the easiest way to see this is to note first that it suffices to verify it for the monomials t^r $(r = 0, 1, 2, \dots)$ and this can be proved quite easily using induction on r. One interesting way to do it is as follows: $J_n(\lambda)$ can be expressed as the sum

$$\begin{bmatrix} \lambda & 0 & \dots & 0 \\ 0 & \lambda & \dots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \dots & \lambda \end{bmatrix} + \begin{bmatrix} 0 & 1 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}$$

i.e. as $\lambda I_n + J_n(0)$ where both matrices commute. This allows us to use the following version of the binomial theorem to calculate the powers of $J_n(\lambda)$. Let A and B be *commuting* $n \times n$ matrices. Then for $r \in \mathbf{N}$,

$$(A + B)^r = \sum_{k=0}^{r} \binom{r}{k} A^k B^{r-k}.$$

This is proved by induction exactly as in the scalar case. We apply this to the above representation of $J_n(\lambda)$ and note that for $r \leq n$, the r-th power of $J_n(0)$ is the matrix with zeros everywhere except for the diagonal which begins at the r-th element of the first row and which contains 1's. If $r \geq n$, then $J_n(0)^r = 0$.

A final operation on matrices whose significance will become clear later is that of **transposition**. If $A = [a_{ij}]$ is an $m \times n$ matrix, then the transposed matrix A^t is the $n \times m$ matrix $[b_{ij}]$ where $b_{ij} = a_{ji}$ i.e. the rows of A^t are just the columns of A. For example

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}^t = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 4 & 1 \end{bmatrix}.$$

The following simple properties can be easily checked:

- $(A + B)^t = A^t + B^t$;
- $(AB)^t = B^t A^t$.
- if A is invertible, then so is A^t and $(A^t)^{-1} = (A^{-1})^t$ (for $(A^t)(A^{-1})^t = (A^{-1} \cdot A)^t = I^t = I$);
- $(A^t)^t = A$.

Examples. Calculate the inverse of the $n \times n$ matrix

$$\begin{bmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & 1 & \dots & 1 \\ \vdots & & & & \vdots \\ 1 & 1 & 1 & \dots & 0 \end{bmatrix}.$$

This corresponds to the system

$$\begin{bmatrix} & x_2 & + & & \dots & + & x_n & = & y_1 \\ x_1 & + & x_3 & & \dots & + & x_n & = & y_2 \\ & \vdots & & & & & \vdots & & \\ x_1 & + & x_2 & & \dots & + & x_{n-1} & = & y_n. \end{bmatrix}$$

Adding, we get the additional equation

$$x_1 + \dots + x_n = \frac{1}{n-1}(y_1 + \dots + y_n)$$

and so

$$\begin{aligned} x_1 &= \frac{1}{n-1}(y_1 + \dots + y_n) - y_1 && (6) \\ &= \frac{1}{n-1}(-(n-2)y_1 + y_2 + \dots y_n) && (7) \end{aligned}$$

etc. Hence the required inverse is the matrix

$$\frac{1}{n-1}\begin{bmatrix} -(n-2) & 1 & \dots & 1 \\ 1 & -(n-2) & \dots & 1 \\ \vdots & & & \vdots \\ 1 & 1 & \dots & -(n-2) \end{bmatrix}.$$

Exercises :
1) What are the matrices of the following systems

$$\begin{bmatrix} x+y & =0 & a_1x_1 & + & \dots & + & a_nx_n & =0 \\ y+z & =0 & a_1x_2 & + & \dots & + & a_nx_1 & =0 \\ z+u & =0 & & & & \vdots & & \\ u+x & =0 & a_1x_n & + & \dots & + & a_nx_{n-1} & =0? \end{bmatrix}$$

2) Calculate AB and BA where

$$A = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix}.$$

3) Give examples of matrices A, B, C, D, E, F, G, J and K so that

- $A^2 = -I$;
- $B^2 = 0$ but $B \neq 0$;
- $CD = -DC$ but neither is zero;
- $EF = 0$ but each element of E resp. F is non-zero;
- $G^3 = 0$ but $G^2 \neq 0$;
- $J = J^t$, $K = K^t$ but $(JK)^t \neq JK$.

4) Let

$$C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

be a matrix with $c_{11} + c_{22} = 0$. Show that there are 2×2 matrices A and B with $C = AB - BA$. (Why have we insisted on the condition $c_{11} + c_{22} = 0$?)
5) Determine all 2×2 matrices

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

for which $A^2 = 0$. Do the same for those A which are such that $A^2 = A$ resp. $A^2 = \mathrm{Id}$.
6) Calculate $A^2 - 4A - 5I$ where

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 3 & 1 & 3 \\ 2 & 3 & 1 \end{bmatrix}.$$

Verify directly that $p(A)$ and $q(A)$ commute where A is as above and $p(t) = 6t^2 + 7t - 2$ and $q(t) = t^2 - 2t + 1$.
7) Show that the matrix

$$A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

is invertible if $a^2 + b^2 > 0$. Calculate its inverse.
8) Show that if

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

then

$$A^2 - (a_{11} + a_{22})A + (a_{11}a_{22} - a_{21}a_{12})I = 0.$$

9) Determine all 2×2 matrices A

- which commute with
$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$
;
- which commute with *all* 2×2 matrices.

10) If

$$A = \begin{bmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{bmatrix} \qquad X = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

show that

- $AX = 0$;
- $A^2 = XX^t - (a^2 + b^2 + c^2)I$;
- $A^3 = -(a^2 + b^2 + c^2)A$.

Calculate A^{921}.
11) Calculate $p(C)$ where

$$p(t) = a_0 + a_1 t + \cdots + a_{n-1} t^{n-1}$$

and C is the $n \times n$ matrix

$$\begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & & \vdots \\ 1 & 0 & 0 & \dots & 0 \end{bmatrix}.$$

12) If A and B are $n \times n$ matrices and we put $[A, B] = AB - BA$, show that

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0.$$

13) Show that if A is an $n \times n$ matrix with $a_{ij} = 0$ for $i \leq j$, then $A^n = 0$.
14) Show that if an $m \times n$ matrix A is such that $A^t A = 0$, then $A = 0$.
15) Let C denote the matrix of example 11) and let A be the typical $n \times n$ matrix. Calculate CAC, CAC^t, CAC^{-1}.

If $A_p = C^p + C^{-p}$, show that

$$\begin{aligned} A_p &= A_{n-p} && (8) \\ A_p A_q &= A_{p+q} + A_{p-q} && (9) \\ A_{p+1} &= A_1 A_p - A_{p-1}. && (10) \end{aligned}$$

16) If A is a $n \times n$ matrix, we define A_s ($s \in \mathbf{R}$) to be $(sI - A)^{-1}$ whenever this inverse exists. Show that

$$(t - s) A_t A_s = A_s - A_t.$$

17) Sei T the $n \times n$ matrix

$$\begin{bmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \vdots & & & & 0 \\ 0 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{bmatrix}.$$

If A is the general $n \times n$ matrix $[a_{ij}]$, calculate AT, TA, TAT, $A^t T$, $TA^t T$. Describe those matrices B which commute with T.
18) Show that the product of two triangular matrices is triangular. Show that if the diagonal elements of a triangular matrix are all positive, then

the matrix is invertible and the same holds for its inverse (i.e. it is upper triangular with positive diagonal elements).

19) If A is an $n \times n$ matrix, the **trace** of A is defined to be the sum $\sum_{i=1}^n a_{ii}$ of the diagonal elements (written $\operatorname{tr} A$). Show that

- $\operatorname{tr}(A+B) = \operatorname{tr} A + \operatorname{tr} B$;
- $\operatorname{tr}(lA) = l \operatorname{tr} A$;
- $\operatorname{tr}(AB) = \operatorname{tr}(BA)$;
- $\operatorname{tr}(P^{-1}AP) = \operatorname{tr} A \quad (P \text{ invertible})$.

Show that there do not exist $n \times n$ matrices A and B with $AB - BA = I$ (cf. Exercise 4)).

20) Let A and C be 2×2 matrices so that $C^2 = A$. Show that

$$(\operatorname{tr} C)C = A + \Delta I \text{ and } (\operatorname{tr} C)^2 = \operatorname{tr} A + 2\Delta$$

where $\Delta = \sqrt{a_{11}a_{22} - a_{12}a_{21}}$. Use this to calculate such a C if

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}.$$

Show that in the general case, C must have the form

$$\pm(\operatorname{tr}(A) \pm 2\Delta)^{-\frac{1}{2}} \begin{bmatrix} a_{11} \pm \Delta & a_{12} \\ a_{21} & a_{22} \pm \Delta \end{bmatrix}$$

provided that the expression in the bracket is positive (use exercise 8)). (In this formula, the sign of Δ must be the same at each occurrence. There are four possible square roots in the general case).

21) A square matrix A is said to be **stochastic** (resp. **doubly stochastic**) if and only if $a_{ij} \geq 0$ for each i and j and the sums of its rows are 1 (resp. the sums of its rows and columns are 1). Show that if the positivity condition holds, then this is equivalent to the fact that $Ae = e$ (resp. $Ae = e = A^t e$) where e is the column matrix all of whose entries are 1. Deduce that the product of two stochastic (resp. doubly stochastic) matrices is stochastic (resp. doubly stochastic).

22) Verify the identity

$$A^r - B^r = \sum_{j=0}^{r-1} A^j (A - B) B^{r-1-j}$$

valid for $n \times n$ matrices A and B and $r \in \mathbf{N}$.
23) Let A be an invertible $n \times n$ matrix such that the row sums of A are constant. Show that this constant is non-zero and that the inverse of A also has constant row sums. What is the common value of these sums (in dependence on the corresponding quantity for A)?
24) Suppose that f is a smooth function and define

$$L(f) = \begin{bmatrix} f & 0 & 0 & \dots & 0 \\ f' & f & 0 & \dots & 0 \\ \vdots & & & & \vdots \\ \frac{f^{(n-1)}}{(n-1)!} & & & \dots & f \end{bmatrix}.$$

Show that $L(fg) = L(f)L(g)$.

1.3 Matrix multiplication, Gaußian elimination and hermitian forms

We now turn to a more detailed study of the method used to solve the general system $AX = Y$ in the light of the arithmetical operations. This consisted in the following procedure. At the i-th step, we examined the last $(m - i + 1)$ equations and located that one which had the lowest indexed non-zero coefficient (in general there will be several of them). We exchanged one of these equations with the i-th one and arranged (by division) for it to have leading coefficient 1. By subtracting suitable multiples of the i-th equation from the later ones we ensured that their leading coefficients all lay to the right of that of the i-th equation. The procedure stopped after that step r for which the last $(m-r)$ equations were trivial (i.e. were such that all coefficients were zero). The matrix of the new system then had the following form

$$\begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_r \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

where the i-th row B_i is

$$\begin{bmatrix} 0 & 0 & \dots & 0 & \mathbf{1} & \tilde{a}_{i,j_i+1} & \dots & \tilde{a}_{in} \end{bmatrix}$$

for some strictly increasing sequence $1 \leq j_1 < j_2 < \cdots < j_r \leq n$ where the pivotal 1's (which are in bold type) are in the columns $j_1, j_2, \dots, j_r$. A matrix with this structure is said to be in **Hermitian form** (or to be an **echelon matrix**).

Note that in carrying out the algorithm we apply at each step one of the following three so-called **elementary operations** on the rows of A:
I. Addition of λ times row i to row j for some scalar λ. In symbols:

$$\begin{bmatrix} A_1 \\ \vdots \\ A_i \\ \vdots \\ A_j \\ \vdots \\ A_m \end{bmatrix} \mapsto \begin{bmatrix} A_1 \\ \vdots \\ A_i \\ \vdots \\ A_j + \lambda A_i \\ \vdots \\ A_m \end{bmatrix}.$$

(This was used to eliminate unwanted coefficients).
II. Exchanging row i with row j:

$$\begin{bmatrix} A_1 \\ \vdots \\ A_i \\ \vdots \\ A_j \\ \vdots \\ A_m \end{bmatrix} \mapsto \begin{bmatrix} A_1 \\ \vdots \\ A_j \\ \vdots \\ A_i \\ \vdots \\ A_m \end{bmatrix}.$$

(This was used to obtain a non-zero pivot by exchanging equations).
III. Multiplying the i-th row by a non-zero scalar λ.

$$\begin{bmatrix} A_1 \\ \vdots \\ A_i \\ \vdots \\ A_m \end{bmatrix} \mapsto \begin{bmatrix} A_1 \\ \vdots \\ \lambda A_i \\ \vdots \\ A_m \end{bmatrix}$$

(in order to obtain 1 as the leading coefficient).

We now make the simple observation that each of these operations can be realised by left multiplication by a suitable $m \times m$ matrix:
I. Left multiplication by the matrix P_{ij}^{λ} where the latter is the matrix $[a_{kl}]$ with $a_{kl} = 1$ if $k = l$, $a_{kl} = \lambda$ if $k = i$ and $l = j$ and $a_{kl} = 0$ otherwise.
II. Left multiplication by U_{ij} where the latter is the matrix $[a_{kl}]$ with $a_{kl} = 1$ if $k = l$ and k is neither i or j, $a_{kl} = 1$ if $k = i$, $l = j$ or $k = j$, $k = l$ and $a_{kl} = 0$ otherwise.
III. Left multiplication by M_i^{λ} where the latter is the matrix $[a_{kl}]$ with $a_{kl} = 1$ if $k = l$ and both are distinct from i and $a_{kl} = \lambda$ if $k = l = i$. Otherwise $a_{kl} = 0$.

Note that the above matrices are those which one obtains from the unit matrix I_m by applying the appropriate row operations. Matrices of the above form are called **elementary matrices.** Each of them is invertible and we have the relationships:

$$(P_{ij}^{\lambda})^{-1} = P_{ij}^{-\lambda} \qquad U_{ij}^{-1} = U_{ij} \qquad (M_i^{\lambda})^{-1} = M_i^{\frac{1}{\lambda}}.$$

Since each step in the reduction to Hermitian form is accomplished by left multiplication by an invertible matrix, we have the following result:

Proposition 3 *Let A be an $m \times n$ matrix. Then there exists an invertible $m \times m$ matrix B so that $\tilde{A} = BA$ has Hermitian form.*

PROOF. For suppose that the reduction employs k steps whereby the i-th step involves left multiplication by the matrix P_i. Then we have

$$P_k \dots P_2 P_1 A = \tilde{A}$$

and so $B = P_k \dots P_1$ is the required matrix. That B is invertible follows from the following simple result:

Proposition 4 *Let P_1, ...,, P_k be invertible $m \times m$ matrices. Then their product*

$$P_k \dots P_1$$

is invertible and its inverse is $P_1^{-1} \dots P_k^{-1}$.

PROOF. We simply multiply the product on the left and on the right by the proposed inverse and cancel (notice the order of the factors in the inverse).

We remark that neither B or $\tilde{A}$ are uniquely determined by A.

We now examine the transformed equation $\tilde{A}X = Z$. Note that we now have a new right hand side $Z = BY$ i.e. Z is obtained from Y by applying the same row operations. The equation is then solvable if and only if $z_{r+1} = \dots = z_m = 0$ and we can then write down the solution explicitly as follows

$$
\begin{aligned}
& x_{j_r+1}, \dots, x_n \quad \text{are arbitrary;} && (11)\\
& x_{j_r} = z_r - \tilde{a}_{r,j_r+1} x_{j_r+1} - \dots - \tilde{a}_{rn} x_n; && (12)\\
& x_{j_{r-1}+1}, \dots, x_{j_r-1} \quad \text{are arbitrary;} && (13)\\
& x_{j_{r-1}} = z_{r-1} - \tilde{a}_{r-1,j_{r-1}+1} x_{j_r-1} + 1 - \dots - \tilde{a}_{r-1,n} x_n; && (14)\\
& \vdots && (15)\\
& x_{j_1} = z_1 - \tilde{a}_{1,j_1+1} x_{j_1+1} - \dots - \tilde{a}_{1n} x_n; && (16)\\
& x_1, \dots, x_{j_1-1} \quad \text{are arbitrary.} && (17)
\end{aligned}
$$

(Note that the solutions then contain $(n-r)$ free parameters—the x_i which correspond to those columns which do not contain a pivotal 1).

Since the general system

$$AX = Y$$

is equivalent to

$$\tilde{A}X = Z \quad \text{where} \quad Z = BY$$

we see that it is solvable if and only if the transformed column matrix Z is such that $z_{r+1} = \dots = z_m = 0$ and the solution can be written down as above.

A useful consequence of this analysis is the following: The equation $AX = Y$ has a solution for *any* right hand side Y if and only if no row of the hermitian form $\tilde{A}$ vanishes. The equation $AX = 0$ has only the trivial solution $X = 0$ if and only if each column of A has a pivotal 1. Then $m \geq n$ and the last $n - m$ rows vanish.

Once again, the solution contains $n - r$ free parameters. In particular, if $n > r$ the homogeneous equation $\tilde{A}X = 0$ (and hence $AX = 0$) has a non trivial solution (i.e. a solution other than $X = 0$). Since this is automatically the case if $m < n$, we get the following result.

Proposition 5 *Every homogeneous system $AX = 0$ of m equations in n unknowns has a non-trivial solution provided that $m < n$.*

In some applications it is often useful to be able to calculate the matrix B which reduces A to Hermitian form. An economical way to do this is to keep track of the matrices inducing the row operations by extending A to the $m \times (m + n)$ matrix $[A \quad I_m]$. If we apply to the whole matrix the row operations employed to bring A into Hermitian form, then the required matrix B will be the right hand $m \times m$ block at the end of the process.

EXAMPLES. We calculate a Hermitian form for the matrix

$$\begin{bmatrix} 2 & 3 & -1 & 1 \\ 3 & 2 & -2 & 2 \\ 5 & 0 & -4 & 4 \end{bmatrix}.$$

We get successively:

$$\begin{bmatrix} 2 & 3 & -1 & 1 \\ 3 & 2 & -2 & 2 \\ 5 & 0 & -4 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{3}{2} & -\frac{1}{2} & \frac{1}{2} \\ 3 & 2 & -2 & 2 \\ 5 & 0 & -4 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{3}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{5}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{15}{2} & -\frac{3}{2} & \frac{3}{2} \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{3}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & \frac{1}{5} & -\frac{1}{5} \\ 0 & 5 & 1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{3}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & \frac{1}{5} & -\frac{1}{5} \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

(In the second last step we multiplied the last row by $\dfrac{2}{3}$ to simplify the arithmetic).

EXAMPLES. For

$$A = \begin{bmatrix} 5 & 3 & 8 & 9 \\ 2 & -1 & 2 & 3 \\ 1 & 0 & 1 & 0 \\ 3 & 4 & 5 & 6 \end{bmatrix}$$

we find an invertible 4×4 matrix B so that BA has Hermitian form. We extend A as above and obtain successively the following matrices:

$$\begin{bmatrix} 5 & 3 & 8 & 9 & 1 & 0 & 0 & 0 \\ 2 & -1 & 2 & 3 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 3 & 4 & 5 & 6 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 5 & 3 & 8 & 9 & 1 & 0 & 0 & 0 \\ 2 & -1 & 2 & 3 & 0 & 1 & 0 & 0 \\ 3 & 4 & 5 & 6 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 3 & 3 & 9 & 1 & 0 & -5 & 0 \\ 0 & -1 & 0 & 3 & 0 & 1 & -2 & 0 \\ 0 & 4 & 2 & 6 & 0 & 0 & -3 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 3 & 0 & 1 & -2 & 0 \\ 0 & 3 & 3 & 9 & 1 & 0 & -5 & 0 \\ 0 & 4 & 2 & 6 & 0 & 0 & -3 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 3 & 0 & 1 & -2 & 0 \\ 0 & 0 & 3 & 18 & 1 & 3 & -11 & 0 \\ 0 & 0 & 2 & 18 & 0 & 4 & -11 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 3 & 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 6 & \frac{1}{3} & 1 & -\frac{11}{3} & 0 \\ 0 & 0 & 1 & 9 & 0 & 2 & -\frac{11}{2} & \frac{1}{2} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 3 & 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 6 & \frac{1}{3} & 1 & -\frac{11}{3} & 0 \\ 0 & 0 & 0 & 3 & -\frac{1}{3} & 1 & -\frac{11}{6} & \frac{1}{2} \end{bmatrix}.$$

Hence

$$BA = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where

$$B = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 \\ \frac{1}{3} & 1 & \frac{11}{3} & 0 \\ -\frac{1}{9} & \frac{1}{3} & -\frac{11}{18} & \frac{1}{6} \end{bmatrix}.$$

We saw earlier that existence of a right inverse can be characterised in terms of the existence of solutions of the system $AX = Y$. We shall now relate the existence of a left inverse to uniqueness of solutions for the associated homogeneous system.

Proposition 6 *Let A by an $m \times n$ matrix. Then A possesses a left inverse if and only if the equation $AX = 0$ has only the trivial solution $X = 0$. In this case the solution of the inhomogeneous equation $AX = Y$ is unique (if it exists).*

PROOF. Suppose that B is a left inverse for A i.e. $BA = I_n$. Then if $AX = 0$, $BAX = 0$ i.e. $X = 0$.

Now suppose that the uniqueness condition holds. Then we know that the Hermitian form $\tilde{A}$ must have a pivotal 1 in *each* column i.e. it has the form

$$\begin{bmatrix} \tilde{A}_1 \\ 0 \end{bmatrix}$$

where $\tilde{A}_1$ is an upper triangular square matrix with 1's in the diagonal. Now $\tilde{A}_1$ has an inverse, say P and then the $n \times m$ matrix $\tilde{P} = [P \quad 0]$ is a left inverse for $\tilde{A}$. From this it easily follows that A itself has a left inverse. For if $\tilde{A} = BA$ where B is invertible, then $\tilde{P}B$ is a left inverse for A since $\tilde{P}BA = \tilde{P}\tilde{A} = I$.

If we relate this to the discussion above we see that a square matrix has a left inverse if and only if it has a right inverse, both being equivalent to the fact that the diagonal elements of a Hermitian form are all 1's. These remarks suggest the following algorithm for calculating the inverse of a square matrix A. We reduce the matrix to Hermitian form $\tilde{A}$. A is invertible if and only if $\tilde{A}$ is an upper triangular matrix with 1's in the diagonal i.e. of the

form
$$\begin{bmatrix} 1 & b_{12} & b_{13} & \dots & b_{1n} \\ 0 & 1 & b_{23} & \dots & b_{2n} \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}.$$
Now we can reduce this matrix to the unit matrix by further row operations. Thus by subtracting b_{12} times the first row from the second one we eliminate the element above the diagonal one. Carrying on in the obvious way, we can eliminate all the non-diagonal elements. If we once more keep track of the row operations with the aid of an added right hand block, we end up with a matrix B which is invertible and such that $BA = I$. B is then the inverse of A.

Examples. We calculate the inverse of
$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & 1 & 0 \\ -2 & 2 & 1 \end{bmatrix}.$$
We expand the matrix as suggested above:
$$\begin{bmatrix} 2 & 1 & 1 & 1 & 0 & 0 \\ 4 & 1 & 0 & 0 & 1 & 0 \\ -2 & 2 & 1 & 0 & 0 & 1 \end{bmatrix}.$$
Successive row operations lead to the following sequence of matrices:
$$\begin{bmatrix} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -2 & -2 & 1 & 0 \\ 0 & 3 & 2 & 1 & 0 & 1 \end{bmatrix}$$
$$\begin{bmatrix} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -2 & -2 & 1 & 0 \\ 0 & 0 & -4 & -5 & 3 & 1 \end{bmatrix}$$
$$\begin{bmatrix} 2 & 0 & -1 & -1 & 1 & 0 \\ 0 & -1 & -2 & -2 & 1 & 0 \\ 0 & 0 & -4 & -5 & 3 & 1 \end{bmatrix}$$
$$\begin{bmatrix} 2 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & -1 & 0 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & -4 & -5 & 3 & 1 \end{bmatrix}$$

$$\left[\begin{array}{cccccc} 1 & 0 & 0 & \frac{1}{8} & \frac{1}{8} & -\frac{1}{8} \\ 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & \frac{5}{4} & -\frac{3}{4} & -\frac{1}{4} \end{array}\right].$$

Hence the inverse of A is

$$\left[\begin{array}{ccc} \frac{1}{8} & \frac{1}{8} & -\frac{1}{8} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{5}{4} & -\frac{3}{4} & -\frac{1}{4} \end{array}\right].$$

We conclude this section with some general remarks on the method of Gaußian elimination.

I. In carrying out the elimination, it was necessary to arrange for the pivot elements to be non-zero. In numerical calculations, it is advantageous to arrange for it to be as large as possible. This can be achieved by using a so-called column pivotal search which means that while choosing the pivotal element in the k-th row, we replace the latter by that row which follows it and which has the largest initial element (in absolute value).

An even more efficient method is that of a complete pivot search. Here one arranges for the largest element in the bottom right hand block to take on the position of the pivotal element at the top left hand corner (of course, this involves exchanging rows *and* columns).

II. If the pivot element at each stage is non-zero, in which case no row exchanges are necessary, then the matrix B which transforms A into hermitian form $\tilde{A}$ is lower triangular. In particular, if A is $n \times n$ (and so invertible), then A can be factorised as $A = LU$ where L is lower triangular (it is the inverse of B) and U is upper triangular (it is the Hermitian form of A). This is call a **lower-upper factorisation** of A. Notice that then the equation $AX = Y$ is equivalent to the two auxiliary ones

$$LZ = Y$$

and

$$UX = Z$$

which can be solved immediately since their matrices are triangular.

Not every invertible matrix has a lower-upper factorisation—for example the 2×2 matrix

$$\left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right]$$

has no such factorisation. However, as we know from the method of elimination, if A is invertible, then we can arrange, by permuting the rows of A, for the pivotal elements to be non-vanishing. This is achieved by left multiplication by a so-called **permutation matrix** P i.e. the matrix which is obtained by permuting the rows of I_n in the same manner. Hence a general invertible matrix A has a factorisation $A = QLU$ where Q is a permutation matrix (the inverse of the P above), L is lower triangular and U is upper triangular.

Exercises: 1) Calculate a Hermitian form for the matrices:

$$\begin{bmatrix} 2 & 3 & -1 & 1 \\ 3 & 2 & -2 & 2 \\ 5 & 0 & -4 & 4 \end{bmatrix} \qquad \begin{bmatrix} 1 & -2 & 3 & -1 \\ 2 & -1 & 2 & 2 \\ 3 & 1 & 2 & 3 \end{bmatrix}.$$

2) Find an invertible matrix B so that BA is in Hermitian form where

$$A = \begin{bmatrix} 5 & 3 & 8 & 9 \\ 2 & -1 & 2 & 3 \\ 1 & 0 & 1 & 0 \\ 3 & 4 & 5 & 6 \end{bmatrix}.$$

3) Calculate the general solutions of the systems:

$$\begin{bmatrix} 2x & + & 3y & - & z & + & w & = & 3 \\ 3x & + & 2y & - & 2z & + & 2w & = & 0 \\ 5x & & & - & 4z & + & 4w & = & 6 \end{bmatrix}$$

resp.

$$\begin{bmatrix} x & + & 2y & + & 3z & + & 4w & = & 10 \\ 2x & + & 3y & + & z & + & w & = & 3. \end{bmatrix}$$

4) Calculate the inverses of the matrices

$$\begin{bmatrix} 1 & 0 & 2 \\ 2 & 2 & 2 \\ -1 & 3 & 4 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ a & 1 & 0 & 0 & 0 \\ 0 & b & 1 & 0 & 0 \\ 0 & 0 & c & 1 & 0 \\ 0 & 0 & 0 & d & 1 \end{bmatrix}.$$

5) Show that every invertible $n \times n$ matrix is a product of matrices of the

following forms:

$$\begin{bmatrix} 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & & & \vdots & \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & k & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

$$\begin{bmatrix} \lambda & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

(where $k \in \mathbf{R}$ and $\lambda \in \mathbf{R} \setminus \{0\}$).

6) We have seen that if the pivot elements of the matrix do not vanish, then we have a representation $A = LU$ of A as a product of a lower and an upper triangular matrix. It is convenient to vary this representation a little in order to obtain a unique one. Show that if A is as above, then it has a unique representation as a product LDU where D is a diagonal matrix and L and U are as above, except that we require them to have 1's in their main diagonals.

1.4 The rank of a matrix

In preparation for later developments we review what we have achieved so far in a more abstract language. We are concerned with the mapping

$$f_A : X \mapsto AX$$

from the set of $n \times 1$ matrices into the set of $m \times 1$ matrices. We note some of its elementary properties:

- it is **additive** i.e.

$$f_A(X_1 + X_2) = f_A(X_1) + f(X_2).$$

  Hence we can obtain a solution of the equation $AX = Y_1 + Y_2$ by taking the sum of solutions of $AX = Y_1$ resp. $AX = Y_2$;

- the mapping is **homogeneous** i.e. $f_A(\lambda x) = \lambda f_A(X)$.

We can combine these two properties into the following single one:

$$f_A(\lambda_1 X_1 + \lambda_2 X_2) = \lambda_1 f_A(X_1) + \lambda_2 f_A(X_2).$$

Mappings with this property are called **linear**. It then follows that for scalars $\lambda_1, \dots, \lambda_r$ and $n \times 1$ matrices $X_1, \dots, X_r$ we have:

$$f_A(\lambda_1 X_1 + \dots + \lambda_r X_r) = \lambda_1 f_A(X_1) + \dots + \lambda_r f_A(X_r).$$

In the light of these facts it is convenient to introduce the following notation: an $n \times 1$ matrix X is called a (column) vector. A vector of the form $\lambda_1 X_1 + \dots + \lambda_r X_r$ is called a **linear combination** of the vectors X_i. The X_i are **linearly independent** if there are no scalars $\lambda_1, \dots, \lambda_r$ (not all zero) so that $\lambda_1 X_1 + \dots \lambda_r X_r = 0$. Otherwise they are **linearly dependent**.

These terms are motivated by a geometrical interpretation which will be treated in detail in the following chapters. We note now that
1) the equation $AX = Y$ has a solution if and only if Y is a linear combination of the columns of A. For the fact that the equation has a solution $x_1, \dots, x_n$ can be expressed in the equation

$$x_1 A_1 + \dots + x_n A_n = Y$$

where the A_i are the columns of A.
2) any linear combination of solutions of the homogeneous equation $AX = 0$ is itself a solution. This equation has the unique solution $X = 0$ if and only if the columns of A are linearly independent.

3) if X is a solution of the equation $AX = Y$ and X_0 is *any* solution of the corresponding homogeneous equation $AX = 0$, then $X_0 + X$ is also a solution of $AX = Y$. Conversely, every solution of the latter is of the form $X_0 + X$ where X_0 is a solution of $AX = 0$. Hence in order to find *all* solutions of $AX = Y$ it suffices to find *one* particular solution and to find *all* solutions of the homogeneous equation.
4) the equation $AX = Y$ has at most one solution if and only if the homogeneous equation $AX = 0$ has only the trivial solution $X = 0$ i.e. if and only if the columns of A are linearly independent.

The reader will notice that all these facts follow from the linearity of f_A.

In the light of these remarks it is clear that the following concept will play a important role in the theory of systems of equations:

Definition: Let A be an $m \times n$ matrix. The **column rank** of A is the maximal number of linearly independent columns in A (we shall see shortly that this is just the number r of non-vanishing rows in the Hermitian form of A).

In principle one could calculate this rank as follows: we investigate successively the columns A_1, $A_2, \dots$ of A and discard those ones which are linear combinations of the preceding ones. Eventually we obtain a matrix $\tilde{A}$ whose columns are linear independent. Then the column rank of $\tilde{A}$ (and also of A) is r, the number of columns of $\tilde{A}$.

We can also define the concept of row rank in an analogous manner. Of course it is just the column rank of A^t. We shall now show that the row rank and the column rank coincide so that we can talk of the **rank** of A, written $r(A)$. In order to do this we require the following Lemma:

Lemma 1 *Let $A = [A_1 \dots A_n]$ be an $m \times n$ matrix and suppose that $\tilde{A} = [A_1 \dots A_{j-1} A_{j+1} \dots A_n]$ is obtained by omitting the column A_j which is a linear combination of $A_1, \dots, A_{j-1}$. Then the row ranks of A and $\tilde{A}$ coincide.*

PROOF. The assumption on A means that it can be written in the form

$$A = \begin{bmatrix} A_1 & \dots & A_{j-1} & \lambda_1 A_1 + \dots + \lambda_{j-1} A_{j-1} & A_{j+1} & \dots & A_n \end{bmatrix}$$

for suitable scalars $\lambda_1, \dots, \lambda_{j-1}$. Now suppose that some linear combination of the rows of $\tilde{A}$ is zero. Then since the elements $\{a_{ij}\}_{i=1}^m$ of the extra column in A are obtained as linear combinations of the components of the corresponding rows, we see that the same linear combination of the rows of A vanish and this means that a set of rows of $\tilde{A}$ is linearly independent if and only if the corresponding rows of A are.

Proposition 7 *If A is an $m \times n$ matrix, then the row rank and the column rank of A coincide.*

PROOF. We apply the method described above to reject suitable columns of A and obtain an $m \times s$ matrix B with linearly independent columns, where s is the columns rank of A. We now proceed to apply the same method to the rows of B to obtain an $r \times s$ matrix with independent rows where r is the row rank of B and so, by the Lemma, also of A. now we know that there exist at most s linearly independent s-vectors and so $r \leq s$ (this follows from the result on the existence of non-trivial soutions of homogeneuos systems). Similarly $s \leq r$ and the proof is done.

Remark: Usually the most effective method of calculating the rank of a matrix A is as follows: one calculates a Hermitian form $\tilde{A}$ for A. Then the rank of A is the number of non-vanishing rows of $\tilde{A}$. For each elementary row operation clearly leaves the row rank of A unchanged and so the row ranks of A and $\tilde{A}$ coincide. But the non-zero rows of a matrix in Hermitian form are obviously linearly independent and so the rank of $\tilde{A}$ is just the number of its non-vanishing rows.

EXAMPLES. We illustrate this by calculating the rank of

$$\begin{bmatrix} 3 & -1 & 4 & 6 \\ 2 & 0 & -2 & 6 \\ 0 & 3 & 1 & 4 \\ 0 & 1 & 1 & -3 \\ 2 & 2 & 2 & 0 \end{bmatrix}.$$

We reduce to hermitian form with row operations to obtain successively the matrices:

$$\begin{bmatrix} 3 & -1 & 4 & 6 \\ 2 & 0 & -2 & 6 \\ 0 & 3 & 1 & 4 \\ 0 & 1 & 1 & -3 \\ 2 & 2 & 2 & 0 \end{bmatrix}$$

resp.

$$\begin{bmatrix} 1 & 0 & -1 & 3 \\ 0 & 1 & 1 & -3 \\ 0 & 3 & 1 & 4 \\ 2 & 2 & 2 & 0 \\ 3 & -1 & 4 & 6 \end{bmatrix}$$

resp.

$$\begin{bmatrix} 1 & 0 & -1 & 3 \\ 0 & 1 & 1 & -3 \\ 0 & 3 & 1 & 4 \\ 0 & 2 & 4 & -6 \\ 0 & -1 & 7 & -3 \end{bmatrix}$$

resp.

$$\begin{bmatrix} 1 & 0 & -1 & 3 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & -2 & 13 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 8 & -6 \end{bmatrix}$$

resp.

$$\begin{bmatrix} 1 & 0 & -1 & 3 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 13 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -6 \end{bmatrix}$$

resp.

$$\begin{bmatrix} 1 & 0 & -1 & 3 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and so the rank is 4.

EXAMPLES. We calculate the rank of the matrix

$$A = \begin{bmatrix} 3 & -1 & 4 & 6 \\ 2 & 0 & -2 & 6 \\ 0 & 3 & 1 & 4 \\ 0 & 1 & 1 & -3 \\ 2 & 2 & 2 & 0 \end{bmatrix}$$

.

Solution: We reduce to Hermitian form as follows:

$$\begin{bmatrix} 3 & -1 & 4 & 6 \\ 2 & 0 & -2 & 6 \\ 0 & 3 & 1 & 4 \\ 0 & 1 & 1 & -3 \\ 2 & 2 & 2 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -1 & 3 \\ 0 & 1 & 1 & -3 \\ 0 & 3 & 1 & 4 \\ 2 & 2 & 2 & 0 \\ 3 & -1 & 4 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -1 & 3 \\ 0 & 1 & 1 & -3 \\ 0 & 3 & 1 & 4 \\ 0 & 2 & 4 & -6 \\ 0 & -1 & 7 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -1 & 3 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & -2 & 13 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 8 & -6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -1 & 3 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 13 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -1 & 3 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Hence $r(A) = 4$.

With this characterisation of rank, the relationship between the Hermitian form and the solvability resp. uniqueness of solutions of the equation $AX = Y$ can be restated as follows:

Proposition 8 *Let A be an $m \times n$ matrix. Then*

- *the equation $AX = Y$ is always solvable if and only if $r(A) = m$ and this is equivalent to the fact that A has a right inverse;*
- *the equation $AX = 0$ has only the trivial solution if and only if $r(A) = n$ and this is equivalent to the fact that A has a left inverse.*

In particular, if A is square, then it has a left inverse if and only if it has a right inverse and in either case it is invertible (as we have already above).

Exercises: 1) Calculate the ranks of the following matrices:

$$\begin{bmatrix} 1 & 0 & 2 \\ 2 & 0 & 4 \\ 6 & 0 & 0 \\ 1 & 0 & 2 \end{bmatrix} \qquad \begin{bmatrix} 3 & -1 & 4 & 6 \\ 2 & 0 & -2 & 6 \\ 0 & 1 & 1 & -3 \\ 2 & 2 & 1 & 0 \end{bmatrix}.$$

2) If A is an $m \times n$ matrix of rank 1, then there exist $(\xi_i) \in \mathbf{R}^m$ and $(\eta_i) \in \mathbf{R}^n$ so that $A = [\xi_i \eta_j]$.
3) Show that if A and B are $m \times n$ matrices so that

$$r(A) = r(B) = r(A + B) = 1,$$

then there is an $1 \times n$ matrix X and scalars $\lambda_1, \dots, \lambda_m, \mu_1, \dots, \mu_m$ so that

$$A = \begin{bmatrix} \lambda_1 X \\ \vdots \\ \lambda_m X \end{bmatrix} \qquad B = \begin{bmatrix} \mu_1 X \\ \vdots \\ \mu_m X \end{bmatrix}$$

or the corresponding result holds for A^t and B^t.
4) Show that the equation $AX = Y$ is solvable if and only if $r(A) = r([A, Y])$.
5) Show that $r(A) \leq 2$ where A is the $n \times n$ matrix $[\sin(a_i + b_j)]_{i,j=1}^n$ ((a_i) and (b_j) are arbitrary sequences of real numbers).

1.5 Matrix equivalence

We have seen that we can reduce a matrix to Hermitian form by multiplying on the left by an invertible matrix. We now consider the simplifications which are possible if we are allowed also to multiply on the right by such matrices—equivalently, by multiplying on the right by the elementary matrices P_{ij}^{λ}, U_{ij} and M_k^{λ}. Now it is easy to check that these operations produce the following effects:

- right multiplication by P_{ji}^{λ} adds λ times the i-th column to the j-th column; right multiplication by U_{ij} exchanges column i and column j; right multiplication by M_i^{λ} multiplies column i by λ.

Let us consider what we can achieve by means of these operations. Starting with a matrix in Hermitian form we can, by exchanging columns, bring all of the leading 1's into the main diagonal. Now, by successively subtracting suitable multiples of the first column from the others we can arrange for all of the entries in the first row (with the exception of the initial 1 of course) to vanish. Similarly, by working with column 2 we can annihilate all of the entries of the second row (except for the 1 in the diagonal). Proceeding in the obvious way we obtain a matrix of the form

$$\begin{bmatrix} 1 & 0 & \dots & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & \dots & 0 \\ \vdots & & & & & \vdots \\ 0 & \dots & & 1 & \dots & 0 \\ 0 & & \dots & & & 0 \\ \vdots & & & & & \vdots \\ 0 & & \dots & & & 0 \end{bmatrix}$$

where the final 1 is in the r-th row. We denote this matrix symbolically by

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}.$$

We have thus proved the following:

Proposition 9 *If A is an $m \times n$ matrix, then there are invertible matrices P $(m \times m)$ and Q $(n \times n)$ so that*

$$PAQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

where r is the rank of A.

We introduce the following notation: We say that two $m \times n$ matrices A and B are **equivalent** when matrices P and Q as in the Proposition exist so that $PAQ = B$. Then this result states that A is equivalent to a matrix of the form

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

whereby $r = r(A)$.

It follows easily that two matrices are equivalent if and only if they have the same rank.

In order to calculate such P and Q for a specific matrix we proceed in a manner similar to that used to calculate inverses except that we keep track of the column operations by adding a second unit matrix, this time below the original one. We illustrate this with the example

$$A = \begin{bmatrix} 2 & 1 & 1 & 1 \\ -2 & 1 & 2 & 0 \\ -2 & 3 & 5 & 1 \end{bmatrix}.$$

The expanded matrix is

$$\begin{bmatrix} 2 & 1 & 1 & 1 & 1 & 0 & 0 \\ -2 & 1 & 2 & 0 & 0 & 1 & 0 \\ -2 & 3 & 5 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & & & \\ 0 & 1 & 0 & 0 & & & \\ 0 & 0 & 1 & 0 & & & \\ 0 & 0 & 0 & 1 & & & \end{bmatrix}.$$

We then compute as follows:

$$\begin{bmatrix} 2 & 1 & 1 & 1 & 1 & 0 & 0 \\ -2 & 1 & 2 & 0 & 0 & 1 & 0 \\ -2 & 3 & 5 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & & & \\ 0 & 1 & 0 & 0 & & & \\ 0 & 0 & 1 & 0 & & & \\ 0 & 0 & 0 & 1 & & & \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 3 & 1 & 1 & 1 & 0 \\ 0 & 4 & 6 & 2 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & & & \\ 0 & 1 & 0 & 0 & & & \\ 0 & 0 & 1 & 0 & & & \\ 0 & 0 & 0 & 1 & & & \end{bmatrix}$$

$$\left[\begin{array}{cccccccc} 2 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 3 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & -2 & 1 \\ 1 & 0 & 0 & 0 & & & \\ 0 & 1 & 0 & 0 & & & \\ 0 & 0 & 1 & 0 & & & \\ 0 & 0 & 0 & 1 & & & \end{array}\right]$$

$$\left[\begin{array}{ccccccc} 2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 3 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & -2 & 1 \\ 1 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & & & \\ 0 & 1 & 0 & 0 & & & \\ 0 & 0 & 1 & 0 & & & \\ 0 & 0 & 0 & 1 & & & \end{array}\right]$$

$$\left[\begin{array}{ccccccc} 2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & -2 & 1 \\ 1 & -\frac{1}{2} & \frac{1}{4} & -\frac{1}{4} & & & \\ 0 & 1 & -\frac{3}{2} & -\frac{1}{2} & & & \\ 0 & 0 & 1 & 0 & & & \\ 0 & 0 & 0 & 1 & & & \end{array}\right].$$

Hence if

$$Q = \frac{1}{2}\left[\begin{array}{cccc} 1 & -\frac{1}{2} & \frac{1}{4} & -\frac{1}{4} \\ 0 & 1 & -\frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

and

$$P = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & -2 & 1 \end{array}\right]$$

then we have

$$PAQ = \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

As an application of this representation we show that left or right multiplication by a matrix cannot increase the rank of A.

Proposition 10 *If A and B are matrices so that the product AB is defined, then the rank $r(AB)$ of the latter is less than or equal to both $r(A)$ and $r(B)$ (i.e. $r(AB) \leq \min(r(A), r(B))$).*

Proof. Note first that left or right multiplication by an *invertible* matrix does not change the rank. For example, left multiplication by A can be achieved by successive left multiplication by elementary matrices and it is clear that this does not affect the row rank. The same argument (with rows replaced by columns) works for right multiplication. We now return to the case of a product AB where A and B are not necessarily invertible. First suppose that A has the special form

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}.$$

Then AB is the matrix

$$\begin{bmatrix} B_r \\ 0 \end{bmatrix}$$

where B_r is the top $r \times p$ block of B. Clearly this has rank at most $r = r(A)$. For the general case, we choose invertible matrices P and Q so that PAQ has the form

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}.$$

Then $AB = P^{-1}(PAQ)(Q^{-1}B)$ and by the above result, the rank of $(PAQ)(Q^{-1}B)$ is at most $r(A)$. Hence the same is true of AB. Thus we have shown that $r(AB) \leq r(A)$. The fact that $r(AB) \leq r(B)$ follows from symmetry (e.g. by applying this result to the transposed matrices as follows:

$$r(AB) = r((AB)^t) = r(B^t A^t) \leq r(B^t) = r(B)).$$

This method of proof illustrates a principle which we shall use repeatedly in the course of this book. The proof of the above result was particularly simple in the case where one of the matrices (say A) had the special form

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}.$$

Since the properties involved in the statement (in this case the rank) were invariant under equivalence, we were able to reduce the general case to thie simple one. For further examples of this methods, see the Exercises 3) and 4) at the end of this section.

The example

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

where $AB = 0$ shows that we can have strict inequality in the above result.

Exercises: 1) Find invertible matrices P and Q so that PAQ has the form

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

where

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 0 & 0 & 0 \\ 6 & 2 & 2 \\ 3 & 1 & 1 \end{bmatrix}$$

resp.

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{bmatrix}.$$

2) Show that if A is an $m \times n$ matrix with rank r, then A can be expressed in the form BC where B is an $m \times r$ matrix and C is an $r \times n$ matrix (both necessarily of rank r).
3) Show that if A is an $m \times n$ matrix with $r(A) = r$, then A has a representation

$$A = A_1 + \cdots + A_r$$

where each A_i has rank 1. (In this and the previous example, prove it firstly for the case of a matrix of the form

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

and then use the principle mentioned in the text.)
4) Show that if A is an $m \times n$ matrix and B is $n \times p$, then

$$r(AB) \geq r(A) + r(B) - n.$$

Deduce that if $r(A) = n$, then $r(AB) = r(B)$.
5) Prove the inequality $r(AB) + r(BC) \leq r(B) + r(ABC)$ where A, B and C are $m \times n$ resp. $n \times p$ resp. $p \times q$ matrices (note in particular the case $B = I$).

1.6 Block matrices

We now discuss a topic which is useful theoretically and can be helpful in simplifying calculations in specific examples. This consists of partitioning a matrix into smaller units or **blocks**. For example the 4×4 matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 4 & 3 & 2 & 1 \\ 8 & 6 & 4 & 2 \end{bmatrix}$$

can be split into 4 2×2 blocks as follows:

$$A = \begin{bmatrix} 1 & 2 & & 3 & 4 \\ 2 & 4 & & 6 & 8 \\ & & & & \\ 4 & 3 & & 2 & 1 \\ 8 & 6 & & 4 & 2 \end{bmatrix}$$

which we write as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

where

$$A_{11} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

etc.

The general scheme is as follows: let A be an $m \times n$ matrix and suppose that

$$0 = m_0 \leq m_1 \leq \cdots \leq m_r = m$$

and

$$0 = n_0 \leq n_1 \leq \cdots \leq n_s = n.$$

We define $(m_i - m_{i-1}) \times (n_j - n_{j-1})$ matrices A_{ij} for $1 \leq i \leq r$ and $1 \leq j \leq s$ as follows:

$$A_{ij} = \begin{bmatrix} a_{m_{i-1}+1, n_{j-1}+1} & \dots & a_{m_{i-1}+1, n_j} \\ \vdots & & \vdots \\ a_{m_i, n_{j-1}+1} & \dots & a_{m_i, n_j} \end{bmatrix}.$$

Then we write

$$A = \begin{bmatrix} A_{11} & \dots & A_{1s} \\ \vdots & & \vdots \\ A_{r1} & \dots & A_{rs} \end{bmatrix}$$

and call this a **block representation** for A.

For our purposes it will suffice to note how to multiply matrices which have been suitably blocked. Suppose that we have block representations

$$A = \begin{bmatrix} A_{11} & \dots & A_{1s} \\ \vdots & & \vdots \\ A_{r1} & \dots & A_{rs} \end{bmatrix}$$

and

$$B = \begin{bmatrix} B_{11} & \dots & B_{1t} \\ \vdots & & \vdots \\ B_{s1} & \dots & B_{st} \end{bmatrix}$$

where the rows of B are partitioned in exactly the same way as the columns of A. Then the product AB has the block representation

$$\begin{bmatrix} C_{11} & \dots & C_{1t} \\ \vdots & & \vdots \\ C_{r1} & \dots & C_{rt} \end{bmatrix}$$

where $C_{ik} = \sum_{j=1}^{s} A_{ij} B_{jk}$.

PROOF. This is a rather tedious exercise in keeping track of indices. For the sake of simplicity we consider the special case of matrices blocked into groups of four i.e. we have

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \quad B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \quad C = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}$$

so that we have the equation

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

and claim that

$$C_{ik} = \sum_{j=1}^{2} A_{ij} B_{jk} \qquad (i, k = 1, 2).$$

We shall prove that $C_{11} = A_{11}B_{11} + A_{12}B_{21}$. The other equations are proved in the same way.

An element c_{ik} in the top left hand corner of C (i.e. with $i \leq m_1$ and $k \leq p_1$) has the form

$$\begin{aligned} c_{ik} &= a_{i1}b_{1k} + \dots + a_{in}b_{nk} && (18) \\ &= (a_{i1}b_{1k} + \dots + a_{i,n_1}b_{n_1,k}) + (a_{i,n_1+1}b_{n_1+1,k} + \dots + a_{in}b_{nk}) && (19) \end{aligned}$$

and the first bracket is the (i,k)-th element of $A_{11}B_{11}$ while the second is the corresponding element of $A_{12}B_{21}$.

As a simple but useful application we have the following result:

Proposition 11 *Let A be an $n \times n$ matrix with block representation*

$$\begin{bmatrix} B & C \\ 0 & D \end{bmatrix}$$

where B is a square matrix. Then A is invertible if and only if B and D are and its inverse is the matrix

$$\begin{bmatrix} B^{-1} & -B^{-1}CD^{-1} \\ 0 & D^{-1} \end{bmatrix}.$$

PROOF. Consider the matrix

$$\begin{bmatrix} E & F \\ G & H \end{bmatrix}$$

as a candidate for the inverse. Multiplying out we get

$$\begin{bmatrix} B & C \\ 0 & D \end{bmatrix} \begin{bmatrix} E & F \\ G & H \end{bmatrix} = \begin{bmatrix} BE+CG & BF+CH \\ DG & DH \end{bmatrix}.$$

If the right hand side is to be the unit matrix, we see immediately that we must have $DH = I$ i.e. D is invertible and $H = D^{-1}$. Since $DG = 0$ and D is invertible we see that $G = 0$. The equation now simplifies to

$$\begin{bmatrix} B & C \\ 0 & D \end{bmatrix} \begin{bmatrix} E & F \\ 0 & D^{-1} \end{bmatrix} = \begin{bmatrix} BE & BF+CD^{-1} \\ 0 & I \end{bmatrix}.$$

Now we see that $BE = I$ i.e. that B is invertible and $E = B^{-1}$. From the condition that $BF + CD^{-1} = 0$ it follows that $F = -B^{-1}CD^{-1}$. Thus we have shown that if A is invertible then so are B and D and the inverse has the above form. On the other hand, if B and D *are* invertible, a simple calculation shows that this matrix is, in fact, an inverse for A.

As an application, we calculate the inverse of the matrix:

$$\begin{bmatrix} \cos\alpha & -\sin\alpha & \cos\beta & -\sin\beta \\ \sin\alpha & \cos\alpha & \sin\beta & \cos\beta \\ 0 & 0 & \cos\gamma & -\sin\gamma \\ 0 & 0 & \sin\gamma & \cos\gamma \end{bmatrix}.$$

Since the inverse of

$$\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

is

$$\begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

it follows from the above formula and the result above on the product of matrices of this special form that the inverse is

$$\begin{bmatrix} \cos\alpha & \sin\alpha & -\cos(\alpha-\beta+\gamma) & -\sin(\alpha-\beta+\gamma) \\ -\sin\alpha & \cos\alpha & \sin(\alpha-\beta+\gamma) & -\cos(\alpha-\beta+\gamma) \\ 0 & 0 & \cos\gamma & \sin\gamma \\ 0 & 0 & -\sin\gamma & \cos\gamma \end{bmatrix}.$$

Exercises: 1) Show that if

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ a & b & -1 & 0 \\ c & d & 0 & -1 \end{bmatrix}$$

then $A^2 = I$.
2) Calculate the product

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 & 2 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 5 & 6 & 7 & 8 & 9 \\ 5 & 6 & 7 & 8 & 9 & 10 \\ 6 & 7 & 8 & 9 & 10 & 11 \end{bmatrix}.$$

3) Let C have block representation

$$\begin{bmatrix} A & -B \\ B & A \end{bmatrix}$$

where A and B are commuting $n \times n$ matrices. Calculate C^2 and

$$C \begin{bmatrix} A & B \\ -B & A \end{bmatrix}$$

. Show that C is invertible if and only if $A^2 + B^2$ is invertible and calculate its inverse.
4) Let the $n \times n$ matrix P have the block representation

$$\begin{bmatrix} A & B \\ 0 & D \end{bmatrix}$$

where A (and hence D) is a square matrix. Show that if A is invertible, then $r(P) = r(A) + r(D)$. Is the same true for general A (i.e. in the case where A is not necessarily invertible)?
5) Let A be an $n \times n$ matrix with block representation

$$\begin{bmatrix} A_1 & A_2 \\ alpha_3 & a \end{bmatrix}$$

where A_1 is an $(n-1) \times (n-1)$ matrix. Show that if A_1 is invertible and $a \neq A_3 A_1^{-1} A_2$, then A is invertible and its inverse is

$$\begin{bmatrix} \tilde{A}_1 & \tilde{A}_2 \\ \tilde{A}_3 & \tilde{a} \end{bmatrix}$$

where $\tilde{a} = (a - A_3 A_1^{-1} A_2)^{-1}$, $\tilde{A}_3 = -\tilde{a} A_3 A_1^{-1}$, $\tilde{A}_2 = -\tilde{a} A_1^{-1} A_2$ and $\tilde{A}_1 = A_1^{-1}(I - A_2 \tilde{A}_3)$. (This exercise provides an algorithm which reduces the calculation of the inverse of an $n \times n$ matrix to that of an $(n-1) \times (n-1)$ matrix).
6) Calculate the inverse of a matrix of the form

$$\begin{bmatrix} A & B & C \\ 0 & D & E \\ 0 & 0 & F \end{bmatrix}$$

where A, D and F are non-singular square matrices.

1.7 Generalised inverses

We now present a typical application of block representations. We shall use them, together with the reduction of an arbitrary matrix to the form

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

to construct so-called **generalised inverses.**

Recall that if A is an $n \times n$ matrix, the inverse A^{-1} (if it exists) can be used to solve the equation $AX = Y$—the unique solution being $X = A^{-1}Y$. More generally, if the $m \times n$ matrix A has a right inverse B, then $X = BY$ is a solution. On the other hand, if A has a left inverse C, the equation $AX = Y$ need not have a solution but if it *does* have one for a particular value of Y, then this solution is unique.

The generalised inverse acts as a substitute for the above inverses and allows one to determine whether a given equation $AX = Y$ has a solution and, if so, provides such a solution (in fact, *all* solutions). More precisely, it is an $n \times m$ matrix S with the property that if $AX = Y$ *has* a solution, then this solution is given by the vector $X = SY$. In addition, $AX = Y$ has a solution if and only if $ASY = Y$.

We shall use the principle mentioned in the above section, namely we begin with the transparent case where A has the form

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}.$$

(The general case will follow readily). This corresponds to the system:

$$\begin{bmatrix} x_1 & & & \dots & = y_1 \\ & x_2 & & \dots & = y_2 \\ & & & \vdots & \vdots \\ & & x_r & \dots & = y_r \end{bmatrix}$$

the remaining equations having the form $0 = y_k$ $(k = r+1, \dots, m)$.

Of course, this system has a solution if and only if $y_{r+1} = \cdots = y_m = 0$ and then a solution is $X = SY$ where S is the $n \times m$ matrix

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}.$$

Note that this matrix satisfies the two conditions:

1) $SAS = S$;

2) $ASA = A$.

We now turn to the case of a general $m \times n$ matrix A. An $n \times m$ matrix S with the above two properties is called a **generalised inverse** for A. The next result shows that every matrix has a generalised inverse.

Proposition 12 *Let A be an $m \times n$ matrix. Then there exists a generalised inverse S for A.*

PROOF. We choose invertible matrices P and Q so that $\tilde{A} = PAQ$ has the form

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

and let $\tilde{S}$ be a generalised inverse for $\tilde{A}$. Then one can check that $S = Q\tilde{S}P$ is a generalised inverse for A. For

$$SAS = Q\tilde{S}P(P^{-1}\tilde{A}Q^{-1})Q\tilde{S}P = Q\tilde{S}\tilde{A}\tilde{S}P = Q\tilde{S}P = S$$

and

$$ASA = (P^{-1}\tilde{A}Q^{-1})Q\tilde{S}P(P^{-1}\tilde{A}Q^{-1}) = A.$$

The relevance of generalised inverses for the solution of systems of equations which we discussed above is expressed formally in the next result:

Proposition 13 *Let A be an $m \times n$ matrix with generalised inverse S. Then the equation $AX = Y$ has a solution if and only if $ASY = Y$ and then a solution is given by the formula $X = SY$. In fact, the general solution has the form*

$$X = SY + (I - SA)Z$$

where Z is an arbitrary $n \times 1$ matrix.

PROOF. Clearly, if $AX = Y$, then $ASY = ASAX = AX = Y$ and so the above condition is fulfilled. On the other hand, if it *is* fulfilled i.e. if $ASY = Y$, then of course $X = SY$ is a solution. To prove the second part, we need only show that the general solution of the homogeneous equation $AX = 0$ has the form $(I - SA)Z$. But if $AX = 0$, then $SAX = 0$ and so $X = X - SAX = (I - SA)X$ and X has the required form. Of course a vector of type $(I - SA)Z$ is a solution of the homogeneous equation since

$$A(I - SA)Z = (A - ASA)Z = 0 \cdot Z = 0.$$

Exercises: 1) Let A be an $m \times n$ matrix with rank r. Recall that A has a factorisation BC where B is $m \times r$ and C is $r \times n$ (cf. Exercise I.5.3)). Show that $B^t B$ and CC^t are then invertible and that

$$S = C^t (CC^t)^{-1} (B^t B)^{-1} B^t$$

is a generalised inverse for A.
2) Let A be an $m \times n$ matrix and P and Q be invertible matrices so that PAQ has the form

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

. Show that a matrix S satisfies the condition $ASA = A$ if and only if S has the form

$$Q \begin{bmatrix} I & B \\ C & D \end{bmatrix} P$$

for suitable matrices B, C and D.
3) Let A be an $m \times n$ matrix and suppose that S_1 and S_2 are $n \times m$ matrices so that

$$AS_1A = A \quad \text{and} \quad AS_2A = A.$$

Then if $S_3 = S_1 A S_2$, show that S_3 is a generalised inverse.
4) Let S be an $n \times m$ matrix so that $ASA = A$. Show that $r(S) \geq r(A)$ and that $r(S) = r(A)$ if and only if S satisfies the additional condition $SAS = S$.
5) Show that an $n \times m$ matrix S satisfies the condition $ASA = A$ if and only if whenever the equation $AX = Y$ has a solution, then $X = SY$ *is* such a solution. 6) Consider the matrix equation $AX = B$ where A and B are given $n \times n$ matrices and X is to be found. Show that if S is a generalised inverse of A, then $X = SB$ is a solution, provided that one exists. Find a necessary and sufficient condition, analogous to the one of the text, for a solution to exist and describe all possible solutions.

1.8 Circulants, Vandermonde matrices

In this final section of Chapter I we introduce two special types of matrices which are of some importance.

Circulant matrices: These are matrices of the form

$$\begin{bmatrix} a_1 & a_2 & \dots & a_n \\ a_n & a_1 & \dots & a_{n-1} \\ \vdots & & & \vdots \\ a_2 & a_3 & \dots & a_1 \end{bmatrix}.$$

Such a matrix is determined by its first row and so it is convenient to denote it simply as

$$\operatorname{circ}(a_1, \dots, a_n).$$

For example, the identity matrix is $\operatorname{circ}(1, 0, \dots, 0)$. The next simplest circulant is $\operatorname{circ}(0, 1, \dots, 0)$. This is the matrix C which occurs in Exercise I.2.11). It plays a central role in the theory of circulants since the fact that a matrix A be a circulant can be characterised by the fact that A satisfies the equation $AC = CA$ (i.e. that A commutes with C). This can easily seen by comparing the expressions for AC and CA calculated in the exercise mentioned above. It follows easily from this characterisation that sums and products of circulants are themselves circulants.

The Vandermonde matrices: These are matrices of the form

$$V = V(t_1, \dots, t_n) = \begin{bmatrix} 1 & 1 & \dots & 1 \\ t_1 & t_2 & \dots & t_n \\ t_1^2 & t_2^2 & \dots & t_n^2 \\ \vdots & & & \vdots \\ t_1^{n-1} & t_2^{n-1} & \dots & t_n^{n-1} \end{bmatrix}.$$

They arise in the treatment of so-called interpolation problems. Suppose that we are given a sequence $t_1, \dots, t_n$ of *distinct* points on the line. For values $y_1, \dots, y_n$ we seek a polynomial p of degree $n-1$ so that $p(t_i) = y_i$ for each i. If p has the form

$$p(t) = a_0 + a_1 t + \dots + a_{n-1} t^{n-1}$$

then we get the system

$$a_0 + a_1 t_i + \dots + a_{n-1} t_i^{n-1} = y_i$$

$(i = 1, \dots, n)$ of equations in the a's. The matrix of this system is just V^t, the transpose of V.

It is well-known that such interpolation problems always have a solution. This fact, which will be proved in VI.2, implies that the matrix V is invertible (under the assumption, of course, that the t_i are distinct).

Another situation where the Vandermonde matrix arises naturally is that of quadrature formulae. Let w be a positive, continuous function on the interval $[-1, 1]$ (a so-called **weight function**) and consider the problem of calculating numerically the weighted integral

$$\int_{-1}^{1} x(t)w(t)\, dt$$

of a continuous function x. A standard method of doing this is by discretisation i.e. by supposing that we are given points $t_1, \dots, t_n$ in the interval and positive numbers $w_1, \dots, w_n$. We consider an approximation of the form

$$I_n(x) = \sum_{j=1}^{n} w_j x(t_j)$$

for the integral. A suitable criterium for determining whether this provides a good approximation is to demand that it be correct for polynomials of degree at most $n - 1$. This leads to the equations

$$\sum_{j=1}^{n} w_j t_j^k = y_k$$

for the w's where $y_k = \int_{-1}^{1} w(t)t^k\, dt$. Once again, the matrix of this equation is V^t. Since V is invertible, these equations can be solved and provide suitable values for the weights $w_1, \dots, w_n$.

Exercises: 1) Show that each circulant matrix is a polynomial in the special circulant $C = \operatorname{circ}(0, 1, \dots, 0)$.
2) Calculate the square of the Vandermonde matrix $V(t_1, \dots, t_n)$. 3) Calculate the inverse of the special Vandermonde matrix

$$V(0, \frac{1}{n}, \frac{2}{n}, \dots, 1)$$

(this is an $(n + 1) \times (n + 1)$ matrix).
4) Describe the form of those circulant matrices A which satisfy one of the following conditions:

a) $A^t = A$;
b) $A^t = -A$;
c) A is diagonal.
5) Which $n \times n$ matrices A satisfy the equation $A = CAC$ where C is the circulant matrix $\mathrm{circ}(0, 1, 0, \dots, 0)$?
6) Which $n \times n$ matrices A satisfy the equation $AD = DA$ where D is the matrix

$$\begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -1 & 0 & 0 & \dots & 0 \end{bmatrix}?$$

7) Calculate the inverses of the Vandermonde matrices

$$\begin{bmatrix} 1 & 1 \\ s & t \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 1 \\ s & t & u \\ s^2 & t^2 & u^2 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 1 & 1 \\ s & t & u & v \\ s^2 & t^2 & u^2 & v^2 \\ s^3 & t^3 & u^3 & v^3 \end{bmatrix}.$$

Can you detect any pattern which would suggest a formula for the inverse of the general Vandermonde matrix?

2 ANALYTIC GEOMETRY IN 2 and 3 DIMENSIONS

2.1 Basic geometry in the plane

In this chapter we discuss classical analytic geometry. In addition to providing motivation and an intuitive basis for the concepts and constructions which will be introduced in the following chapters, this will enable us to apply the tool of linear algebra to the study of elementary geometry. We shall not give a systematic treatment but be content to pick out some selected themes which can be treated with these methods.

There are two possible approaches to euclidean geometry. The original one which was used by Euclid—so-called *synthetic geometry*—involves deducing the body of plane geometry from a small number of axioms. Unfortunately, the original axiom system employed by Euclid was insufficient to prove what he claimed to have done (the missing axioms were introduced surreptitiously in the course of the proofs). This has since been rectified by modern mathematicians but the resulting theory is too complicated to be of use in an introductory text on linear algebra. The second approach— *analytic geometry* is based on the idea of introducing coordinate systems and so expressing geometric concepts and relationships in numerical terms—thus reducing proofs to manipulations with numbers.

The basis for this approach is provided by the identification of the elements of $\mathbf{R}$, the set of real numbers, with the points of a line (hence the name "real line" which is sometimes used).

The arithmetical operations in $\mathbf{R}$ have then a natural geometric interpretation (addition or stretching of intervals). In a similar way, we can use the Cartesian product $\mathbf{R} \times \mathbf{R}$ (written $\mathbf{R}^2$) i.e. the set of ordered pairs (ξ_1, ξ_2) $(\xi_1, \xi_2 \in \mathbf{R})$ of real numbers as a model for the plane.

Once again we supply $\mathbf{R}^2$ with a simple arithmetic structure which has a natural geometric interpretation as follows. We define a **vector addition** by associating to two vectors $x = (\xi_1, \xi_2)$, $y = (\eta_1, \eta_2)$ their **sum** $x + y$ where $x + y$ is the vector $(\xi_1 + \eta_1, \xi_2 + \eta_2)$.

We define a **scalar multiplication** by assigning to a scalar λ (i.e. an element of $\mathbf{R}$) and a vector $x = (\xi_1, \xi_2)$ the vector λx with coordinates $(\lambda\xi_1, \lambda\xi_2)$ (i.e. the vector is stretched or shrunk by a factor of λ – if λ is negative, its direction is reversed). In manipulating vectors, the following simple rules are useful:

- $x + y = y + x$ (commutativity of addition);
- $(x + y) + z = x + (y + z)$ (associativity of addition);

- $x + \mathbf{0} = \mathbf{0} + x = x$ where $\mathbf{0}$ is the zero vector;
- $1.x = x$;
- $\lambda(x + y) = \lambda x + \lambda y$;
- $(\lambda + \mu)x = \lambda x + \mu x$;
- $\lambda(\mu x) = (\lambda\mu)x$.

Since we defined the operations in $\mathbf{R}^2$ by way of those in $\mathbf{R}$ it is trivial to prove the above formulae by reducing them to properties of the real line. For example, the commutativity law $x + y = y + x$ is proved as follows:

$$\begin{aligned} x + y &= (\xi_1, \xi_2) + (\eta_1, \eta_2) && (20)\\ &= (\xi_1 + \eta_1, \xi_2 + \eta_2) && (21)\\ &= (\eta_1 + \xi_1, \eta_2 + \xi_2) && (22)\\ &= y + x. && (23) \end{aligned}$$

We now discuss a less obvious property of $\mathbf{R}^2$. Intuitively, it is clear that the plane is two-dimensional in contrast to the space in which we live which is three-dimensional. We would like to express this property purely in terms of the algebraic structure of $\mathbf{R}^2$. For this we require the notions of linear dependence and independence (note the formal similarity between the following definitions and those of I.4). If x and y are vectors in $\mathbf{R}^2$, then a **linear combination** of x and y is a vector of the form $\lambda x + \mu y$. x and y are **linearly independent** if the only linear combination $\lambda x + \mu y$ which vanishes is the trivial one $0.x + 0.y$. x and y **span** $\mathbf{R}^2$ if every $z \in \mathbf{R}^2$ is expressible as a linear combination $\lambda x + \mu y$ of x and y. x and y form a **basis** for the plane if they span $\mathbf{R}^2$ and are linearly independent. Note that this condition means that every $y \in \mathbf{R}^2$ has a **unique** representation of the form $\lambda x + \mu y$. λ and μ are then called the **coordinates** of z with respect to the basis (x, y).

The simplest basis for $\mathbf{R}^2$ is the pair $e_1 = (1, 0)$, $e_2 = (0, 1)$. It is called the **canonical basis**.

If $x = (\xi_1, \xi_2)$ and $y = (\eta_1, \eta_2)$ are vectors in the plane, then the fact that they span $\mathbf{R}^2$ means simply that the system

$$\lambda\xi_1 + \mu\eta_1 = \zeta_1$$

$$\lambda\xi_2 + \mu\eta_2 = \zeta_2$$

(with unknowns λ,μ) always has a solution (i.e. for any choice of ζ_1, ζ_2). Similarly, the fact that they are linearly independent means that the corresponding homogeneous equations only have the trivial solution. Of course we know that either of these conditions is equivalent to the fact that $\xi_1\eta_2 - \xi_2\eta_1 \neq 0$. Notice that the negation of this condition (namely that $\xi_1\eta_2 - \xi_2\eta_1 = 0$) means that x and y are **proportional** i.e. there is a $t \in \mathbf{R}$ so that $x = ty$ or $y = tx$.

In the same way, we see that any collection of three vectors x_1, x_2, x_3 is linearly dependent i.e. there exist scalars $\lambda_1, \lambda_2, \lambda_3$, not all of which are zero, so that

$$\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 = 0.$$

For this is equivalent to a homogeneous system of two equations in three unknowns which, as we know, always has a non-trivial solution.

Having identified the plane with the set of points in $\mathbf{R}^2$, we now proceed to interpret certain geometric concepts and relationships in terms of the algebraic structure of the latter. Elements of $\mathbf{R}^2$ will sometimes be called **points** and, in order to conform with the traditional notation, will be denoted by capital letters such as A,B,C,P,Q etc. When emphasising the algebraic aspect they will often be denoted by lower case letters (x,y,z,… etc.) and called **vectors**. There is no logical reason for this dichotomy of notations which we use in order to comply with the traditional ones in bridging the gap between euclidean geometry and linear algebra.

If P is a point in space it will sometimes be convenient to denote the corresponding vector by x_P, with coordinates (ξ_1^P, ξ_2^P). The vector $x_{PQ} = x_P - x_Q$ is called the **arrow** from P to Q, also written PQ. Then we see that

- $x_{PQ} + x_{QR} = x_{PR}$ for any triple P,Q,R;
- if $P \in \mathbf{R}^2$ and x is a vector in $\mathbf{R}^2$, there is exactly one point Q with $x = x_{PQ}$.

We now introduce the concept of a **line**. If a,b,c are real numbers, where at least one of a and b is non-zero, then we write $L_{a,b,c}$ for the set of all $x = (\xi_1, \xi_2)$ in $\mathbf{R}^2$ so that

$$a\xi_1 + b\xi_2 + c = 0.$$

(In symbols: $L_{a,b,c} = \{(\xi_1, \xi_2) : a\xi_1 + b\xi_2 + c = 0\}$).

A subset of $\mathbf{R}^2$ of this form is called a **(straight) line**. The phrases "P lies on the line $L_{a,b,c}$" or "the line $L_{a,b,c}$ passes through P" just means that $x_P \in L_{a,b,c}$.

Note that two pairs (a, b, c) and (a', b', c') define the same line if and only if (a, b, c) and (a', b', c') are proportional i.e. there is a non-zero t so that

$$a = ta', b = tb', c = tc'.$$

If a, b and c are all non-zero, this condition can be rewritten in the more transparent form

$$\frac{a'}{a} = \frac{b'}{b} = \frac{c'}{c}.$$

In the classical geometry of Euclid, the proofs of the theorems were based on a scheme of "self-evident" truths or axioms which were satisfied by the primitive concepts such as line, point etc. In the following we shall show that the points and lines introduced here satisfy these axioms so that the geometry of the Cartesian plane can serve as a model for Euclidean geometry. Thus the choice of coordinate system supplies the link between the axiomatic approach and the analytic one. Both are based on self-evident properties of the two-dimensional planes of every day experience.

We begin by verifying that the so-called **incidence axioms** hold:

Axiom $\mathbf{I}_1$: if P and Q are distinct points in the plane, then there is precisely one straight line $L_{a,b,c}$ through P and Q;

Axiom $\mathbf{I}_2$: there are at least three points on a given line;

Axiom $\mathbf{I}_3$: there exist three points which do not lie on any line. (a collection of points all of which lie on the same line is called **collinear**).

We shall prove I_1 (I_2 and I_3 are obvious).

PROOF. If $L_{a,b,c}$ passes through $P = (\xi_1, \xi_2)$ and $Q = (\eta_1, \eta_2)$, then a, b and c must be solutions of the system

$$a\xi_1 + b\xi_2 + c = 0$$

$$a\eta_1 + b\eta_2 + c = 0.$$

It is a simple exercise to calculate explicit solutions, corresponding to the two cases a) $\xi_1\eta_2 - \xi_2\eta_1 \neq 0$ resp. b) $\xi_1\eta_2 - \xi_2\eta_1 = 0$.

Since we shall not require these explicit solutions we leave their calculation as an exercise for the reader.

A less computational proof of the result goes as follows: the condition that P and Q are distinct implies that the matrix

$$\begin{bmatrix} \xi_1 & \xi_2 & 1 \\ \eta_1 & \eta_2 & 1 \end{bmatrix}$$

has rank 2. Hence the homogeneous equation above has a one-parameter solution set and this is precisely the statement that we set out to prove.

■

Note that the distinction between the two cases in the proof is whether L passes through the origin or not. It will be convenient to call lines with the former property **one-dimensional subspaces** (we shall see the reason for this terminology in chapter III).

It follows from the above that if P and Q are distinct points, then the line through P and Q has the representation

$$L(P,Q) = \{tx_P + (1-t)x_Q : t \in \mathbf{T}\}.$$

This is called the **parametric representation** of the line.

We now turn to the concept of **parallelism**. Recall that two vectors $x = (\xi_1, \xi_2)$ and $y = (\eta_1, \eta_2)$ are **proportional** if there is a $t \in \mathbf{R}$ so that $x = ty$ or $y = tx$. If the points are distinct, then this means that the line through them passes through the origin. Then the following result is a simple exercise:

Proposition 14 *Consider the lines $L = L_{a,b,c}$ and $L_1 = L_{a_1,b_1,c_1}$. Then the following are equivalent:*

- *if P and Q (resp. P_1 and Q_1) are distinct points of L (resp. L_1), then x_{PQ} and $x_{P_1Q_1}$ are proportional;*

- *$ab_1 - a_1b = 0$.*

If either of these conditions is satisfied, then L and L_1 can be written in the form $L_{a,b,c}$ and L_{a,b,c_1}. They then either coincide (if $c = c_1$) or are disjoint (if $c \neq c_1$).

We then say that the two lines L and L_1 are **parallel** (in symbols $L||L_1$). The last statement of the above theorem can then be interpreted as follows: two distinct parallel lines do not meet.

Other simple properties of this relationship are:

- $L||L$ (i.e. each line is parallel to itself);

- if $L||L_1$ and $L_1||L_2$, then $L||L_2$.

The famous **parallel axiom** of Euclid which was to become the subject of so much controversy can be stated as follows: **Axiom P:** If L is a line and P a point, then there is precisely *one* line L_1 with $L||L_1$ which passes through P. (If $L = L_{a,b,c}$, then L_1 is the line L_{a,b,c_1} where c_1 is chosen so that $a\xi_1^P + b\xi_2^P + c_1 = 0$).

We have seen that the line $L(P, Q)$ through P and Q consists of the points with coordinates

$$tx_P + (1-t)x_Q = x_P + t(x_Q - x_P)$$

for some real number t. The pair $(t, 1-t)$ is uniquely determined by x and is called the pair of **barycentric coordinates** of x with respect to P and Q. We put $[P, Q] = \{tx_P + (1-t)x_Q : t \in [0,1]\}$ and call this set **the interval** from P to Q. If R lies in this interval we write $P|Q|R$ (read "R lies between P and Q"). (We exclude the cases where R coincides with P or Q in this notation).

It can then be checked that the three so-called **order axioms** of Euclid hold:
Axiom $\mathbf{O}_1$: $P|Q|R$ implies that $Q|R|P$;
Axiom $\mathbf{O}_2$: if P and Q are distinct, there exist R and S with $P|R|Q$ and $R|Q|S$;
Axiom $\mathbf{O}_3$: if P, Q and R are distinct collinear points, then one of the relationships $P|Q|R$, $Q|R|P$ or $R|P|Q$ holds.

Exercises: 1. a) Show that $(1,2)$ and $(2,1)$ are linearly independent. What are the coordinates of $(-1,1)$ with respect to these vectors? b) Give the parametric representation of the line through $(-1,1)$ and $(0,3)$. c) if $P = (1,3)$, $Q = (5,2)$, $R = (3,0)$, find a point S so that $PQ||SR$, $PR||QS$.
2. Let L_1, L_2 and L_3 be distinct lines, A_1, B_1 points on L_1, A_2, B_2 on L_2, A_3, B_3 on L_3. Show that if $A_1A_2||B_1B_2$ and $A_2A_3||B_2B_3$, then $A_1A_3||B_1B_2$.
3. Let $L_{a,b,c}$ and L_{a_1,b_1,c_1} be non-parallel lines. Show that the set of lines of the form

$$L_{\lambda a+\mu a_1, \lambda b+\mu b_1, \lambda c+\mu c_1}$$

as λ, μ vary in $\mathbf{R}$ represents the pencil of lines which pass through the intersection of the original ones. (What happens in the case where the latter are parallel?)
4. Let x_1, x_2, x_3 be vectors in $\mathbf{R}^2$. Show that they are the sides of a triangle (i.e. there is a triangle ABC with $x_1 = x_{AB}$, $x_2 = x_{BC}$, $x_3 = x_{CA}$) if and only if their sum is zero.

Use this to show that if a,b and c are given vectors in the plane, then there is a unique triangle which has them as medians.

2.2 Angle and length

In addition to the algebraic structure and the related geometric concepts that we have discussed up till now, euclidean geometry employs metric concepts such as length and angle. These can both be derived from the **scalar product** on $\mathbf{R}^2$ which we now introduce: let $x = (\xi_1, \xi_2)$ and $y = (\eta_1, \eta_2)$ be points in the plane. We define the scalar product of x and y to be the number

$$(x|y) = \xi_1\eta_1 + \xi_2\eta_2.$$

It has the following properties:

- $(\lambda x + \mu y|z) = \lambda(x|z) + \mu(y|z)$ $(\lambda, \mu \in \mathbf{R}, x, y, z \in \mathbf{R}^2)$ (i.e. it is linear in the first variable);
- $(x|y) = (y|x)$ $(x, y \in \mathbf{R}^2)$ (it is symmetric);
- $(x|x) \geq 0$ and $(x|x) = 0$ if and only if $x = 0$ (it is positive definite).

Of course it is then **bilinear** (i.e. linear in both variables), a fact which can be expressed in the following formula which we shall frequently use:

$$(\sum_{i=1}^{m} \lambda_i x_i | \sum_{j=1}^{n} \mu_j y_j) = \sum_{i,j=1}^{m,n} \lambda_i \mu_j (x_i|y_j).$$

With the help of the scalar product we can, as mentioned above, define the concept of the **length** $\|x\|$ of a vector x by the formula

$$\|x\| = \sqrt{(x|x)} = \sqrt{(\xi_1^2 + \xi_2^2)}.$$

An important property of these quantities is the following inequality, known as the **Cauchy-Schwarz inequality:**

$$|(x|y)| \leq \|x\|\|y\|$$

which we prove as follows: consider the quadratic function $t \mapsto (x+ty|x+ty)$. Since it is non-negative, its discriminant $4(x|y)^2 - 4\|x\|^2\|y\|^2$ is less than or equal to zero and this gives the inequality.

From this it follows that the norm satisfies the so-called **triangle inequality** i.e.

$$\|x + y\| \leq \|x\| + \|y\|.$$

For

$$\|x+y\|^2 = (x+y|x+y) \tag{24}$$
$$= (x|x) + 2(x|y) + (y|y) \tag{25}$$
$$\leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 \tag{26}$$
$$= (\|x\| + \|y\|)^2. \tag{27}$$

If P and Q are points in $\mathbf{R}^2$, then, in order to conform with classical notation, we denote by $|PQ|$ the distance from P to Q i.e. the norm $\|x_{PQ}\|$ of the arrow from P to Q. Thus

$$|PQ| = \sqrt{\{(\xi_1^P - \xi_1^Q)^2 + (\xi_2^P - \xi_2^Q)^2\}}.$$

Using the concept of distance, we can interpret the barycentric coordinates of a point as follows: let P and Q be distinct points and let R have barycentric coordinates $(t, 1-t)$ where $t \in]0,1[$ i.e.

$$x_R = tx_P + (1-t)x_Q.$$

Then $t = |RQ|/|PQ|$, $1-t = |PR|/|PQ|$. For

$$|RQ| = \|x_{RQ}\| = \|x_Q - x_R\| = \|x_Q - (tx_P + (1-t)x_Q)\| \tag{28}$$
$$= \|t(x_Q - x_P)\| = t|PQ|. \tag{29}$$

The reader may check that if x_R is as above where t is negative, then R is on the opposite side of Q from P and $t = -|RQ|/|PQ|$. Similarly, if $t > 1$, then R is on the opposite side of P from Q and $t = |RQ|/|PQ|$.

We now turn to the concept of **angle**. Firstly, we treat right angles or perpendicularity. If $x, y \in \mathbf{R}^2$, we say that x and y are **perpendicular** (written $x \perp y$) if $(x|y) = 0$. Similarly, we say that the lines $L = L_{a,b,c}$ and $L_1 = L_{a_1,b_1,c_1}$ are perpendicular, if $aa_1 + bb_1 = 0$. It is a simple exercise to show that this is equivalent to the fact that for each pair P, Q (resp. P_1, Q_1) of points on L (resp. L_1), $x_{PQ} \perp x_{P_1Q_1}$.

We can now show that the so-called **perpendicularity axioms** of Euclid are satisfied: **Axiom Pe$_1$:** If $L_1 \perp L_2$, then $L_2 \perp L_1$;
Axiom Pe$_2$: If P is a point, L a line, there is exactly one line through P which is perpendicular to L;
Axiom Pe$_3$: If $L_1 \perp L_2$, then L_1 and L_2 intersect.

Bases (x_1, x_2) with the property that $x_1 \perp x_2$ are particularly important. If, in addition, $\|x_1\| = \|x_2\| = 1$, the basis is said to be **orthonormal**. Of course, the canonical basis (e_1, e_2) has this property.

Using the concept of perpendicularity, we can give a more geometric description of a line. Consider $L_{a,b,c}$. We can write the defining condition

$$a\xi_1 + b\xi_2 + c = 0$$

in the form

$$(x|\mathbf{n}) = -\frac{c}{\sqrt{(a^2+b^2)}}$$

where $\mathbf{n}$ is the unit vector $(a,b)/\sqrt{(a^2+b^2)}$ This can, in turn, be rewritten in the form $(x - x_0|\mathbf{n}) = 0$ where x_0 is any point on L. This characterises the line as the set of all points x which are such that the vector from x to x_0 is perpendicular to a given unit vector $\mathbf{n}$ (the **unit normal** of L). (Note that $\mathbf{n}$ is determined uniquely by L up to the choice of sign i.e. $-\mathbf{n}$ is the only other possibility). The above equation is called the **Hessean form** of the equation of L.

We now turn to angles. It follows from the Cauchy-Schwarz inequality that if x and y are non-zero vectors, then

$$-1 \le \frac{(x|y)}{\|x\|\|y\|} \le 1.$$

Hence, by elementary trigonometry, there is a $\theta \in]-\frac{\pi}{2}, \frac{\pi}{2}]$ so that

$$\cos\theta = \frac{(x|y)}{\|x\|\|y\|}.$$

θ is called the **angle** between x and y—written $\angle(x,y)$. Similarly, if A, B and C are points (with $B \neq A, C \neq A$), then $\angle BAC$ denotes the angle $\angle(x_{AB}, x_{AC})$.

Note that the definition of θ leads immediately to the formula

$$(x|y) = \|x\|\|y\|\cos\theta$$

for the scalar product.

Example: If $x, y \in \mathbf{R}^2$, show that

$$\|x+y-z\|^2 = \|x-z\|^2 + \|y-z\|^2 - \|x-y\|^2 + \|x\|^2 + \|y\|^2 - \|z\|^2.$$

Solution: Expanding, we have the following expressions for the left-hand side:

$$(x+y-z|x+y-z) = (x|x) + (y|y) + (z|z) + 2(x|y) - 2(x|z) - 2(y|z).$$

We can express the right hand side in terms of scalar products in a similar manner and the reader can check that both sides simplify to the same expression.

Exercises: 1. If x, y are vectors in the plane, show that the following are equivalent:

a) x and y are proportional;

b) $\|x+y\| = \|x\| + \|y\|$ or $\|x-y\| = \|x\| + \|y\|$;

c) $|(x|y)| = \|x\|\|y\|$.

Show that for a vector z, the following are equivalent:

d) $z = \frac{1}{2}(x+y)$;

e) $\|z-x\| = \|z-y\| = \frac{1}{2}\|y-x\|$.

2. If L is the line $\{x : (\mathbf{n}|x) = (\mathbf{n}|x_0)\}$ where $\mathbf{n}$ is a unit vector, then the distance from a point x to L is given by the formula $d(x, L) = |(\mathbf{n}|x - x_0)|$. What is the nearest point to x on L?

3. Show that

$$[P, Q] = \{z \in \mathbf{R}^2 : \|z - x_P\| + \|z - x_Q\| = \|x_Q - x_P\|\}.$$

2.3 Three Propositions of Euclidean geometry

Having shown how to express the concepts of elementary geometry in terms of the algebraic structure of $\mathbf{R}^2$, we shall now attempt to demonstrate the effectiveness of this approach by proving the following three classical theorems with these methods (for further examples, see the exercises):

Proposition 15 *The altitudes of a triangle are concurrent.*

Proposition 16 *If ABC is a triangle in which two medians are equal (in length), then it is isosceles (i.e. two sides are of equal length).*

Proposition 17 *Let A, B, C and D resp. A_1, B_1, C_1 and D_1 be distinct collinear points so that the four lines AA_1, BB_1, CC_1 and DD_1 meet at a point. Then*

$$\frac{|A_1B_1|/|B_1D_1|}{|A_1C_1|/|C_1D_1|} = \frac{|AB|/|BD|}{|AC|/|CD|}.$$

PROOF. Proofs I. Let ABC be the triangle and let H be the points of intersection of the altitudes from C to AB resp. B to AC (Figure 1). Then we can express the condition $BH \perp AC$ as follows:

$$(x_B - x_H | x_C - x_A) = 0.$$

Similarly,

$$(x_C - x_H | x_B - x_A) = 0.$$

In order to finish the proof, we must show that $AH \perp BC$ i.e.

$$(x_A - x_H | x_B - x_A) = 0.$$

But we can rewrite the first two conditions in the form

$$(x_B - x_H | x_C - x_H) = -(x_B - x_H | x_H - x_A)$$

resp.

$$(x_C - x_H | x_B - x_H) = -(x_C - x_H | x_H - x_A).$$

Comparing these two, we see that

$$(x_B - x_H | x_H - x_A) = (x_C - x_H | x_H - x_A)$$

i.e.

$$(x_H - x_A | x_C - x_B) = 0$$

i.e. $BC \perp AH$

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II. Let P resp. Q be the midpoints of AC and AB (figure 2) and suppose that $|BP| = |QC|$. We shall show that $|AB| = |AC|$. We can write the midpoint condition in the form:

$$x_{AB} = 2x_{AQ} \qquad x_{AC} = 2x_{AP}.$$

Then $x_{CQ} = x_{AQ} - x_{AC} = x_{AQ} - 2x_{AP}$. Similarly, $x_{BP} = x_{AP} - 2x_{AQ}$. Since $\|x_{BP}\| - \|x_{CQ}\|$, we have

$$\|x_{AQ} - 2x_{AP}\|^2 = \|x_{AP} - 2x_{AQ}\|^2$$

or

$$\|x_{AQ}\|^2 + 4\|x_{AP}\|^2 - 4(x_{AQ}|x_{AP}) = \|x_{AP}\|^2 + 4\|x_{AQ}\|^2 - 4(x_{AP}|x_{AQ}).$$

Hence $|AB| = |AC|$.

III. We shall determine the barycentric coordinates of B_1 and C_1 with respect to P, A and D in two distinct ways and compare coefficients. (P is the point of intersection of the four lines mentioned in the statement (figure 3)). We are tacitly assuming that P, A and D are affinely independent i.e. that P does not lie on the line $ABCD$. The case where this holds is simpler). Suppose that

$$x_{B_1} = rx_P + (1-r)x_B \tag{a}$$
$$x_B = sx_A + (1-s)x_D \tag{b}$$
$$x_{B_1} = tx_{A_1} + (1-t)x_{D_1} \tag{c}$$
$$x_{A_1} = ux_A + (1-u)x_P \tag{d}$$
$$x_{D_1} = vx_D + (1-v)x_P. \tag{e}$$

Then from (a) and (b) we get:

$$x_{B_1} = rx_P + (1-r)sx_A + (1-r)(1-s)x_D$$

and from (c) and (d) we get:

$$x_{B_1} = tux_A + t(1-u)x_P + (1-t)vx_D + (1-t)(1-v)x_P.$$

Comparing coefficients, we see that

$$\frac{1-r}{s} = \frac{(1-t)v}{tu}$$

i.e.

$$|AB|/|BD| = |A_1B_1|/|B_1D_1|\frac{v}{u}.$$

In exactly the same way, we can prove that

$$|AC|/|CD| = |A_1C_1|/|C_1D_1|\frac{v}{u}.$$

■

Exercises: 1) Let A, B, C, D be non-collinear points with $|AB| = |CD|$, $|AD| = |BC|$. Show that

$$x_{AD} - x_{BC} \perp x_{AC} \qquad x_{AD} - x_{BC} \perp x_{BD}$$

and so $x_{AD} = x_{BC}$ (i.e. $ABCD$ is a parallelogram) or $AC||BD$ (i.e. $ABCD$ is a trapezoid).
2) If A, B, C and D are four points in the plane and A_1 (resp. B_1, C_1, D_1) is the midpoint of BC (resp. CD, DA, AB), then

$$(x_{A_1 D}|x_{BC}) + (x_{B_1 D}|x_{CA}) + (x_{C_1 D}|x_{AB}) = 0.$$

3) Let ABC be a triangle, E and D as in figure 4. Then A is the centroid of DCE and the medians of DCE are equal and parallel to the sides of ABC.
4) If $ABCD$ is a (non-degenerate) parallelogram, E the intersection of BD and CM where M is the midpoint of AB, show that $|DE| = 2|EB|$ (figure 5).
5) let ABC be a non-degenerate triangle in the plane and suppose that P, Q and R are points on BC (resp. CA, AB) with coordinates

$$x_P = (1 - t_1)x_B + t_1 x_C \tag{30}$$
$$x_Q = (1 - t_2)x_C + t_2 x_A \tag{31}$$
$$x_R = (1 - t_3)x_A + t_3 x_B. \tag{32}$$

Show that P, Q and R are collinear if and only if

$$t_1 t_2 t_3 = -(1 - t_1)(1 - t_2)(1 - t_3)$$

and that AP, BQ and CR are concurrent if and only if

$$t_1 t_2 t_3 = (1 - t_1)(1 - t_2)(1 - t_3).$$

Interpret these result geometrically (they are the theorems of Menelaus and Ceva respectively).
6) Let ABC be a non-degenerate triangle and put

$$a = |BC| \quad b = |CA| \quad c = |AB|$$

so that $s = \frac{1}{2}(a + b + c)$ is the *semi-perimeter* of the triangle.. Show that if S is the centre of the inscribed circle of ABC, then

$$x_S = \frac{1}{s}(a x_A + b x_B + c x_C).$$

7) Let ABC be a non-degenerate triangle. Then

- a point P lies on the perpendicular bisector of AC if and only if

$$(x_P|x_C - x_A) = \frac{1}{2}(\|x_A\|^2 - \|x_C\|^2);$$

- the perpendicular bisectors meet at a point (which we denote by Q);

- if R is the point such that $x_R = -2x_Q + 3x_M$ where M is the centroid (i.e. $x_M = \frac{1}{3}(x_A + x_B + x_C)$, then

$$(x_A - x_R|x_B - x_C) = 0 = (x_B - x_R|x_C - x_A) = (x_C - x_R|x_A - x_B).$$

- if H is the intersection of the perpendiculars from the vertices of ABC to the opposite sides and A_1 (resp. B_1, C_1) is the midpoint of HA (resp. HB, HC) and A_2, B_2 C_2 are the midpoints of the sides BC, CA, AB, then

$$|A_1A_2| = |B_1B_2| = |C_1C_2|.$$

  Interpret these results geometrically, in particular concerning the properties of the line through R, M and Q and the circle with A_1A_2 as diameter (these are called the *Euler line* resp. the *nine-point circle* of the triangle).

2.4 Affine transformations

We now turn to the topic of transformations in geometry. Here the use of analytic methods is particularly appropriate as we shall now show. We will be interested in mappings which take lines into lines. Since the latter were defined as the zeros of affine functionals (i.e. mappings from $\mathbf{R}^2$ into $\mathbf{R}$ of the form $(\xi_1, \xi_2) \mapsto (a\xi_1 + b\xi_2 + c)$, the appropriate concept is the following: An **affine mapping** f on $\mathbf{R}^2$ is one of the form

$$(\xi_1, \xi_2) \mapsto (a_{11}\xi_1 + a_{21}\xi_2 + c_1, a_{21}\xi_1 + a_{22}\xi_2 + c_2)$$

where a_{11}, a_{12}, a_{21}, a_{22}, c_1 and c_2 are scalars. The 2×2 matrix

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

is called the **matrix of the transformation.**

Those affine mappings which map 0 into 0 are particularly important. They are characterised by the fact that $c_1 = c_2 = 0$ and are called **linear mappings**. f then maps the column vector

$$X = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}$$

into the product AX. Note that such mappings f satisfy the conditions

$$f(x + y) = f(x) + f(y) \qquad f(lx) + \lambda f(x)$$

(i.e. are linear in the terminology of I.4) and that any mapping which satisfies these condition is induced by a matrix, in fact the 2×2 matrix

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

where $f((1, 0)) = (a_{11}, a_{21})$, $f((0, 1)) = (a_{12}, a_{22})$ (that is, the columns of A are the images of the canonical basis). For

$$\begin{aligned} f(\xi_1, \xi_2) &= f(\xi_1(1, 0) + \xi_2(0, 1)) && (33) \\ &= \xi_1(a_{11}, a_{21}) + \xi_2(a_{12}, a_{22}) && (34) \\ &= (a_{11}\xi_1 + a_{12}\xi_2, a_{21}\xi_1 + a_{22}\xi_2). && (35) \end{aligned}$$

Note that the affine mapping with matrix

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

is injective if and only if $a_{11}a_{22}-a_{21}a_{12} \neq 0$ (this is essentially the contents of exercise 6) of section I.1). From now on, we shall tacitly assume that this is always the case.

Affine mappings with I_2 as matrix i.e. those of the form $(\xi_1, \xi_2) \mapsto (\xi_1 + c_1, \xi_2 + c_2)$ are called **translations**. We denote the above mapping by T_u where u is the **translation vector** (c_1, c_2).

The general affine mapping f can then be written in the form $T_u \circ \tilde{f}$ where $\tilde{f}$ is the linear mapping with the same matrix as f and $u = f(0)$.

In calculating with affine mappings it is useful to have a formula for their compositions. We begin with the case where f and g are linear. If f has matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

and g has matrix

$$B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

then by substituting and simplifying one calculates that $g \circ f$ maps $(\xi_1, \xi_2$ into

$$((b_{11}a_{11}+b_{12}a_{21})\xi_1+(b_{11}a_{12}+b_{12}a_{22})\xi_2, (b_{21}a_{11}+b_{22}a_{21})\xi_1+(b_{21}a_{12}+b_{22}a_{22})\xi_2)$$

i.e. the matrix of $g \circ f$ is BA. (Of course, this fact follows immediately from the interpretation of the operators as left multiplication of column vectors by their matrices).

Now consider the composition $g \circ f$ of two general affine mappings. If we write

$$f = T_u \circ \tilde{f} \qquad g = T_v \circ \tilde{g}$$

where $u = f(0)$, $v = g(0)$ and $\tilde{f}$ and $\tilde{g}$ are linear, then

$$\begin{aligned} g \circ f(x) &= T_v \circ \tilde{g} \circ T_u \circ \tilde{f}(x) && (36) \\ &= T_v(\tilde{g}(\tilde{f}(x) + u)) && (37) \\ &= \tilde{g} \circ \tilde{f}(x) + v + g(u) && (38) \\ &= T_{v+\tilde{g}(u)}\tilde{g} \circ \tilde{f}(x) && (39) \end{aligned}$$

i.e. $g \circ f = T_{v+\tilde{g}(u)}\tilde{g} \circ \tilde{f}$. In other words, it is the affine mapping whose matrix is the product of those of g and f and whose translation vector is $v + \tilde{g}(u)$. (For the reader who prefers a more computational proof, this result can be obtained by direct substitution).

Exercise: 1. Calculate the pre-images of the unit circle $\{(\xi_1, \xi_2) : \xi_1^2 + \xi_2^2 = 1\}$ with respect to the mappings

- $(\xi_1, \xi_2) \mapsto (\frac{1}{2}\xi_1 - \frac{\sqrt{3}}{2}\xi_2 + \frac{7}{2}, \frac{\sqrt{3}}{2}\xi_1 + \frac{1}{2}\xi_2 - 1)$
- $(\xi_1, \xi_2) \mapsto (4\xi_1 + 7\xi_2, \xi_1 + 2\xi_2)$.

2.5 Isometries and their classification

In Euclidean geometry and its applications, a particular role is played by those mappings which preserve distances. The formal definition is as follows: a mapping $f : \mathbf{R}^2 \to \mathbf{R}^2$ is an **isometry** if $\|f(x) - f(y)\| = \|x - y\|$ for each pair x, y of points in the plane. This corresponds to the elementary geometric concept of a *congruence*. If f is linear (and we shall see below that this is the case for any isometry which maps the origin into itself), then this is equivalent to the condition:

$$(f(x)|f(y)) = (x|y) \qquad (x, y \in \mathbf{R}^2).$$

PROOF. If the above equation holds, then we can put $x = y$ to get the equality $\|f(x)\| = \|x\|$. Hence, by linearity,

$$\|f(x) - f(y)\| = \|f(x - y)\| = \|x - y\|.$$

On the other hand, if $\|f(x)\| = \|x\|$ for each x, we use the fact that

$$(x|y) = \frac{1}{2}(\|x + y\|^2 - \|x\|^2 - \|y\|^2)$$

which can be verified by multiplying out. Then

$$\begin{aligned}
(f(x)|f(y)) &= \frac{1}{2}(\|f(x) + f(y)\|^2 - \|f(x)\|^2 - \|f(y)\|^2) && (40)\\
&= \frac{1}{2}(\|x + y\|^2 - \|x\|^2 - \|y\|^2) && (41)\\
&= (x|y). && (42)
\end{aligned}$$

■

We shall give a complete description of the isometries of the plane. We start with some concrete examples:

I. Translations: It is clear that these are isometries.

II. Reflections in one-dimensional subspaces: Let $L = \{x : (\mathbf{n}|x) = 0\}$ be such a subspace, with equation in Hessean form. Now if $x \in \mathbf{R}^2$, then it can be represented in the form

$$x = (\mathbf{n}|x)\mathbf{n} + (x - (x|\mathbf{n})\mathbf{n})$$

where $x - (x|\mathbf{n})\mathbf{n}$ is its component parallel to L and $(x|\mathbf{n})\mathbf{n}$ is it component perpendicular to L. The vector

$$R_L(x) = x - 2(x|\mathbf{n})\mathbf{n}$$

is the **reflection** (or **mirror image**) of x in L. We have thus defined a mapping from $\mathbf{R}^2$ into itself and it is easy to check that it is linear. We calculate its matrix as follows: suppose that $\mathbf{n} = (\cos\theta, \sin\theta)$. Then

$$x - 2(\mathbf{n}|x)\mathbf{n} = (\xi_1, \xi_2) - 2(\xi_1 \cos\theta + \xi_2 \sin\theta)(\cos\theta, \sin\theta) \tag{43}$$

$$= (\xi_1(1 - 2\cos^2\theta) - \xi_2(2\cos\theta\sin\theta), \xi_1(-2\cos\theta\sin\theta) + \xi_2(1 - 2\sin^2\theta)) \tag{44}$$

i.e. R_L has the matrix

$$\begin{bmatrix} 1 - 2\cos^2\theta & -2\sin\theta\cos\theta \\ -2\sin\theta\cos\theta & 1 - 2\sin^2\theta \end{bmatrix} = \begin{bmatrix} \cos 2\phi & \sin 2\phi \\ \sin 2\phi & -\cos 2\phi \end{bmatrix}$$

where $\phi = \theta - \frac{\pi}{2}$. (Note that ϕ is the angle between L and the x-axis (figure 2)).

R_L has the following properties:

- it is involutive i.e. $R_L \circ R_L = \mathrm{Id}$;
- $R_L(x) = x$ if and only if $x \in L$;
- R_L is an isometry i.e. $\|R_L(x) - R_L(y)\| = \|x - y\|$

which are clear from the geometrical interpretation. For the sake of completeness, we prove them analytically:
PROOF. (1) follows from the fact that the matrix A of R_L is I_2. (2) Since $R_L(x) = x - 2(\mathbf{n}|x)\mathbf{n}$, $R_L(x) = x$ if and only if $(\mathbf{n}|x) = 0$ i.e. $x \in L$. (3) Since $(\mathbf{n}|x)\mathbf{n}$ and $x - (\mathbf{n}|x)\mathbf{n}$ are perpendicular, we have

$$\|x\|^2 = |(\mathbf{n}|x)|^2 + \|x - (\mathbf{n}|x)\mathbf{n}\|^2 = \|R_L(x)\|^2.$$

III. Reflections in a line: If $L = L_{a,b,c}$ is a line, then for any $u \in L$, $L = T_u(L_1)$ where L_1 is the unique line which is parallel to L and passes through 0 (i.e. $L_1 = L_{a,b,0}$). Reflection in L is clearly described by the mapping

$$R_L = T_u \circ R_{L_1} \circ T_{-u} = T_{u - R_{L_1}(u)} \circ R_{L_1}$$

which is an isometric affine mapping. It also is involutive and the line L can be characterised as its fixed point set i.e. the $\{x : R_L(x) = x\}$.

All of these properties follow from the corresponding ones for R_{L_1} and trivial manipulations with mappings. For example, we show that $R_L^2 = \mathrm{Id}$:

$$R_L^2 = (T_u \circ R_{L_1} \circ T_{-u}) \circ (T_u \circ R_{L_1} \circ T_{-u}) \tag{45}$$

$$= T_u \circ R_{L_1} \circ T_{u-u} \circ R_{L_1} \circ T_{-u} \tag{46}$$

$$= T_u \circ R_{L_1} \circ R_{L_1} \circ T_{-u} = T_u \circ T_{-u} = \mathrm{Id}. \tag{47}$$

IV. Rotations: If $\theta \in \mathbf{R}$, the linear mapping

$$D_\theta : (\xi_1, \xi_2) \mapsto (\xi_1 \cos\theta - \xi_2 \sin\theta, \xi_1 \sin\theta + \xi_2 \cos\theta)$$

with matrix

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

is the **rotation through an angle** θ (figure 3). More generally, the mapping $D_{x,\theta} = T_x \circ D_\theta \circ T_{-x}$ is the corresponding rotation with the point x as axis. Since D_θ is linear, $D_{x,\theta}$ is affine. D_θ (and hence also $D_{x,\theta}$) is an isometry as we now check:

$$\begin{aligned} \|D_\theta(\xi_1,\xi_2)\|^2 &= \|(\xi_1\cos\theta - \xi_2\sin\theta, \xi_1\sin\theta + \xi_2\cos\theta)\|^2 && (48)\\ &= \xi_1^2\cos^2\theta + \xi_2^2\sin^2\theta + \xi_1^2\sin^2\theta + \xi_2^2\cos^2\theta && (49)\\ &= \|(\xi_1,\xi_2)\|^2. && (50)\end{aligned}$$

We can create new isometries of the plane by composing ones of the above types.

a) **A translation and a reflection:** We consider mappings of the form $T_u \circ R_L$. We claim that there is a line L_1 and a vector v, both parallel to L, so that the resulting map can be written in the form $T_v \circ R_{L_1}$ i.e. as a reflection in L_1 followed by a translation parallel to L_1. Such mappings are called **glide reflections** for obvious reasons.

PROOF. Consider figure 4. This suggest that we take $v = u - (\mathbf{n}|u)\mathbf{n}$ and $L_1 = T_y(L)$ were $y = \frac{1}{2}(u|\mathbf{n})\mathbf{n}$. Then we claim that $T_v \circ R_{L_1} = T_u \circ R_L$ or, equivalently, that $T_{v+y-R_L(y)} = T_u$. But

$$v + y - R_L(y) = u - (u|\mathbf{n})\mathbf{n} + \frac{1}{2}(u|\mathbf{n})\mathbf{n} = u.$$

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b) **A translation and a rotation:** i.e. mappings of the form $T_v \circ D_\theta$. We claim that this is also a rotation provided that the original one D_θ is non-trivial i.e. θ is not a whole-number multiple of 2π. More precisely there is a $u \in \mathbf{R}^2$ so that $D_{u,\theta} = T_v \circ D_\theta$.

PROOF. u must satisfy the condition

$$T_u \circ D_\theta \circ T_{-u} = T_v \circ D_\theta$$

i.e. $T_{u-D_\theta(u)} = T_v$. Thus we must show that for every $v = (v_1, v_2)$ there is a $u = (u_1, u_2)$ so that $u - D_\theta(u) = v$. This is equivalent to solving the system:

$$\begin{array}{ccccccc} (\cos\theta - 1)u_1 & - & \sin\theta u_2 & + & v_1 & = & 0 \\ (\sin\theta)u_1 & + & (\cos\theta - 1)u_2 & + & v_2 & = & 0. \end{array}$$

But this always has a solution since $(\cos\theta - 1)^2 + \sin^2\theta > 0$ if θ is not a multiple of 2π. On the other hand if u is a solution of the equation $u - D_\theta(u) = v$, then we can retrace the steps in the above argument to deduce that $D_{u,\theta} = T_v \circ D_\theta$.

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c) **Two rotations:** We consider operators of the form $D_{u,\theta} \circ D_{v,\phi}$. This simplifies to

$$\begin{aligned} T_u \circ D_\theta \circ T_{-u} \circ T_v \circ D_\phi \circ T_{-v} &= T_{u+D_\theta(-u+v)} \circ D_{\theta+\phi} \circ T_{-v} &(51)\\ &= T_{u+D_\theta(-u+v)-D_{\theta+\phi}(v)} \circ D_{\theta+\phi} &(52)\end{aligned}$$

and we distinguish between two cases:

- $\theta + \phi$ is not a multiple of 2π. Then the product is a rotation (by the above argument).
- $\theta + \phi$ is a multiple of 2π. Then we get a translation.

d) **Two reflections:** We first consider the product of two reflections in one-dimensional subspaces L and L_1. If the matrices of R_L and R_{L_1} are of the form

$$\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$$

resp.

$$\begin{bmatrix} \cos 2\phi & \sin 2\phi \\ \sin 2\phi & \cos 2\phi \end{bmatrix},$$

a simple calculation shows that the matrix of the product is

$$\begin{bmatrix} \cos 2(\theta-\phi) & -\sin 2(\theta-\phi) \\ \sin 2(\theta-\phi) & \cos 2(\theta-\phi) \end{bmatrix}$$

and this is the matrix of the rotation $D_{2(\theta-\phi)}$. If we recall the geometrical significance of the angles θ and π, then we see that $R_L \circ R_{L_1}$ is a rotation through twice the angle between L and L_1.

For the general case i.e. where L and L_1 do not necessarily pass through the origin, we write R_L and R_{L_1} in the forms $T_u \circ R_{\tilde{L}} \circ T_{-u}$ and $T_v \circ R_{\tilde{L}_1} \circ T_{-v}$ where $u \in L$, $v \in L_1$. Then

$$\begin{aligned} T_u \circ R_{\tilde{L}} \circ T_{-u} \circ T_v \circ R_{\tilde{L}_1} \circ T_{-v} &= T_{u-R_{\tilde{L}}(u-v)} \circ R_{\tilde{L}} \circ R_{\tilde{L}_1} \circ T_{-v} &(53)\\ &= T_{-u-R_{\tilde{L}}(u-v)-R_{\tilde{L}}\circ R_{\tilde{L}_1}(v)} \circ R_{\tilde{L}} \circ R_{\tilde{L}_1}. &(54)\end{aligned}$$

There are two possibilities.

- L and L_1 are parallel. Then $\tilde{L} = \tilde{L}_1$ and the product is the translation $T_{-u-v-R_{\tilde{L}}(u-v)}$.
- L and L_1 are not parallel and so intersect. Then we can choose $u = v$ and the product is $T_{-u-R_{\tilde{L}} \circ R_{\tilde{L}_1}(u)} \circ D_{2(\theta-\pi)}$ which is a rotation by the above.

With this information, we can carry out an analysis of the general type of isometry in $\mathbf{R}^2$. It turns out that we have already exhausted all of the possibilities: the only isometries of the plane are translations, rotations, reflections and glide reflections, a result of some consequence for euclidean geometry. We begin by noting two basic facts about isometries:
A. If M is a subset of the plane and $f : M \to \mathbf{R}^2$ is an isometry, then there is an isometry $\tilde{f} : \mathbf{R}^2 \to \mathbf{R}^2$ which extends f (i.e. is such that $\tilde{f}(x) = f(x)$ for $x \in M$). Since the congruences which are used in classical euclidean geometry do not in general act *a priori* on the whole plane but only on suitable subsets thereof (the geometrical figures that one happens to be examining in a particular problem) this shows that the following analysis also applies since any such isometry between geometrical figures is implemented by an isometry of the whole plane.

Although it is not particularly difficult to prove this result, we shall not do so here since it is not central to our argument (but see the exercises below for a sketch of the proof). Note that in the case where M doe not lie on a line (i.e. contains three points which are not collinear), then the extension $\tilde{f}$ of f is unique.
B). If $f : \mathbf{R}^2 \to \mathbf{R}^2$ is an isometry, then f is affine. It is this fact, which we shall now prove, which allows us to apply the methods of matrix theory to the problem.
PROOF. We first suppose that $f(0) = 0$ and show that f is linear. In this case we have the equality $\|f(x)\| = \|x\|$ for each x. We now use the identities:

- $\|x + y - z\|^2 = \|x - z\|^2 + \|y - z\|^2 - \|x - y\|^2 + \|x\|^2 + \|y\|^2 - \|z\|^2$;
- $\|\lambda x - y\|^2 = (1 - \lambda)(\|y\|^2 - \lambda\|x\|^2) + \lambda(\|x - y\|^2)$

which can be checked by multiplying out the expressions for the squares of the norms. Since we can replace x by $f(x)$ etc. on the right hand side of (1), we have the equality

$$\|x + y - z\|^2 = \|f(x) + f(y) - f(z)\|^2.$$

Hence if $z = x + y$, the left hand side vanishes and thus also the right hand side i.e. we have $f(z) = f(x) + f(y)$. One deduces from equality (2) that $f(\lambda x) = \lambda f(x)$ in a similar manner.

Now suppose that $f(0) = u$ is non-zero. Then it follows from the above that $\tilde{f} = T_{-u} \circ f$ is linear and hence $f = T_u \circ \tilde{f}$ is affine.

We now investigate under which conditions the matrix

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

is the matrix of an isometry. Since the columns of A are the images of the elements e_1 and e_2 (which form an orthonormal basis for $\mathbf{R}^2$), it follows from the condition $(f(x)|f(y)) = (x|y)$ that the former also form such a basis, a fact which can be expressed analytically in the equations

$$a_{11}^2 + a_{21}^2 = 1 = a_{12}^2 + a_{22}^2 \qquad a_{11}a_{12} + a_{21}a_{22} = 0$$

i.e. $A^t A = I$. From this it follows that A^t is the inverse of A and so we have the further set of equations arising from the matrix equality $AA^t = I$. We can proceed to solve these equations and so deduce the possible forms of the matrix. However, it is more natural to proceed geometrically as follows. We know that the first column of A is a unit vector and so has the form

$$\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

for some θ. Now the second column is a unit vector which is perpendicular to this one. Of course, there are only two possibilities –

$$\begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

or

$$\begin{bmatrix} \sin\theta \\ -\cos\theta \end{bmatrix}$$

and this provides the following two possible forms for A:

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$$

which we recognise as the matrices of a rotation respectively a reflection.

Hence we have proved the following result:

Proposition 18 *A linear isometry of* $\mathbf{R}^2$ *is either a rotation or a reflection in a one-dimensional subspace.*

With the information that we now possess, we can describe the possible isometries of the plane as follows:

Proposition 19 *An isometry of the plane has one of the following four forms:*

- *a rotation;*
- *a translation;*
- *a reflection;*
- *a glide reflection.*

PROOF. We write the isometry f in the form $T_u \circ \tilde{f}$ where $\tilde{f}$ is linear and so either a rotation or a reflection. In the former case, f itself is a rotation or a translation (if the rotation is the trivial one). In the latter case, f is a glide reflection or a reflection.

■

In order to determine which of the above forms a given isometry has, we can proceed as follows. Firstly we calculate the value of the expression $a_{11}a_{22} - a_{12}a_{21}$ for its matrix. This must be either $+1$ or -1 as we see from the possible forms of the matrix. If the value is 1 then the mapping is either a rotation or a translation. If it is -1, then the isometry is a reflection or a glide reflection. These can be distinguished by the fact that a reflection has a fixed point (in fact, a whole line of them) whereas a genuine glide reflection has no fixed points.

Example: Describe the geometrical form of the mappings

$$(\xi_1, \xi_2) \mapsto (\frac{1}{2}\xi_1 - \frac{\sqrt{3}}{2}\xi_2 + 4, \frac{\sqrt{3}}{2}\xi_1 + \frac{1}{2}\xi_2 - 2) \tag{55}$$

$$(\xi_1, \xi_2) \mapsto (\frac{3}{5}\xi_1 + \frac{4}{5}\xi_2 + 1, \frac{4}{5}\xi_1 - \frac{3}{5}\xi_2 - 2). \tag{56}$$

Solution: The matrix

$$\begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

of the first mapping is that of the rotation $D_{\frac{\pi}{6}}$. Hence the mapping is $D_{x_0, \frac{\pi}{6}}$ where x_0 is the fixed point i.e. the solution of

$$\begin{array}{ccccccc} \frac{1}{2}\xi_1 & - & \frac{\sqrt{3}}{2}\xi_2 & + & 4 & = & \xi_1 \\ \frac{\sqrt{3}}{2}\xi_1 & - & \frac{1}{2}\xi_2 & - & 2 & = & \xi_2 \end{array}$$

i.e. $(2 + \sqrt{3}, 2\sqrt{3} - 1)$.

In the second case the matrix is

$$\begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{bmatrix}.$$

This is the matrix of a reflection and so the mapping is a glide reflection.

Example: We give an alternative calculation for the matrix of a reflection in a one-dimensional subspace. If the subspace makes an angle θ with the x-axis, then $R_L = D_\theta \circ R_{L_1} \circ D_{-\theta}$ where L_1 is the x-axis. Now reflection in the x-axis is the mapping

$$(\xi_1, \xi_2) \mapsto (\xi_1, -\xi_2)$$

with matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Thus the matrix of R_L is

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix},$$

i.e.

$$\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}.$$

Example: Let L_1 and L_2 be one dimensional subspaces of $\mathbf{R}^2$. Show that R_{L_1} and R_{L_2} commute if and only if $L_1 \perp L_2$ or $L_1 = L_2$.
Solution: Let the matrices of these reflections be

$$\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} \text{ resp. } \begin{bmatrix} \cos 2\phi & \sin 2\phi \\ \sin 2\phi & -\cos 2\phi \end{bmatrix}.$$

The commutativity condition means that

$$\begin{bmatrix} \cos 2(\theta-\phi) & -\sin 2(\theta-\phi) \\ \sin 2(\theta-\phi) & \cos 2(\theta-\phi) \end{bmatrix} = \begin{bmatrix} \cos 2(\phi-\theta) & -\sin 2(\phi-\theta) \\ \sin 2(\phi-\theta) & \cos 2(\phi-\theta) \end{bmatrix}.$$

Thus $\sin 2(\theta-\phi) = -\sin 2(\theta-\phi)$ i.e. $\sin 2(\phi-\theta) = 0$. Hence $\phi - \theta = \frac{n\pi}{2}$ for some $n = 0, \pm1, \pm2, \dots$ and so $L_1 = L_2$ or $L_1 \perp L_2$.

Exercises: 1) Construct a rotation $D_{x,\phi}$ which maps $(1,2)$ resp. $(4,6)$ onto $(5,2)$ resp. $(8,-2)$.
2) Give the coordinate representation of

- $D_{(1,6),\frac{\pi}{6}}$;
- the reflection in $L_{1,2,-1}$.

Find an x so that $D_{(3,2),\theta} = D_\theta \circ T_x$ where θ is the angle with $\cos\theta = \frac{3}{5}$ and $\sin\theta = \frac{4}{5}$.
3) Determine the geometric forms of the mappings

- $(\xi_1, \xi_2) \mapsto (\frac{8}{17}\xi_1 + \frac{15}{17}\xi_2 - 1, \frac{15}{17}\xi_1 - \frac{8}{17}\xi_2 + 3)$;
- $(\xi_1, \xi_2) \mapsto (\frac{3}{5}\xi_1 + \frac{4}{5}\xi_2 - 10, -\frac{4}{5}\xi_1 + \frac{3}{5}\xi_2 - 1)$.

4) Show that if ABC and PQR are triangles in $\mathbf{R}^2$ so that $|AB| = |PQ|$, $|BC| = |QR|$, $|CA| = |RP|$, then there is an isometry f on the plane which maps A,B,C onto P,Q,R respectively. When is such an f unique?
6)

- Simplify the expression $D_{x_1,\pi} \circ D_{x_2,\pi} \circ D_{x_3,\pi}$ and show that
$$D_{x_1,\pi} \circ D_{x_2,\pi} \circ D_{x_3,\pi} = D_{x_3,\pi} \circ D_{x_2,\pi} \circ D_{x_1,\pi}.$$
- Describe the geometrical form of the product
$$(T_u \circ R_L) \circ (T_{u_1} \circ R_{L_1})$$
of two glide reflections (Discuss the special cases where L and L_1 are parallel resp. perpendicular).
- Show that the product of three glide reflections is a glide reflection or a reflection. When is the latter the case? Discuss the possible forms of the product of three reflections.
- Show that $D_{x_A,\pi} \circ R_L = R_L \circ D_{x_B,\pi}$ if and only if L is the perpendicular bisector of AB.
- Show that $D_{x_A,\pi} \circ D_{x_B,\pi} = D_{x_B,\pi} \circ D_{x_C,\pi}$ if and only if B is the midpoint of AC.
- Show that P lies on the line L if and only if $R_L^{-1} \circ D_{x_P,\pi} \circ R_L = D_{x_P,\pi}$.

- Show that PQ and RS are equal and parallel if an only if

$$D_{x_P,\pi} \circ D_{x_Q,\pi} = D_{x_R,\pi} \circ D_{x_S,\pi}.$$

Deduce that if QQ_1 and SS_1 are also equal and parallel, then so are PQ_1 and RS_1.

- Show that the centre P of a square with side AB is given by the formula

$$x_P = \frac{x_A + x_B}{2} \pm D_{\frac{\pi}{2}}(\frac{x_A - x_B}{2}).$$

- Characterise those pairs L_1, L_2 of lines for which $R_{L_1} \circ R_{L_2}$ is a rotation of 180^0 about 0.

- If A, B, C are the vertices of an equilateral triangle, then

$$D_{x_A,\frac{\pi}{3}} \circ D_{x_B,\frac{\pi}{3}} \circ D_{x_C,\frac{\pi}{3}} = D_{x_B,\pi}$$

.

- Interpret the equation

$$R_L^{-1} \circ R_{L_1} \circ R_L = R_{L_2}$$

as a geometrical relation between the lines L, L_1 and L_2.

- Let $A_1, A_2, \ldots, A_n$ be points in space. Show that if $D_k = D_{x_{A_k},\pi}$, then

$$D_n \circ D_{n-1} \circ \cdots \circ D_1 \circ D_n \circ \cdots \circ D_1 = \mathrm{Id}$$

provided that n is odd. What happens if n is even?

7) Let f be an isometry of the plane. Describe the isometries

$$f^{-1} \circ R_L \circ f, \quad f^{-1} \circ D_{x,\theta} \circ f, \quad f^{-1} \circ (T_u \circ R_L) \circ f.$$

8) Show that every isometry f of $\mathbf{R}^2$ is the product of at most three reflections. If f has a fixed point, then two suffice.
9) Let f be an isometry on $\mathbf{R}^2$. Then

- if f has no fixed points, it is a translation or a glide reflection;

- if it has exactly one fixed point it is a rotation;

- if it has at least two fixed points it is a reflection or the identity;

- if it has no fixed points and $f(L)||L$ for each line L, then it is a translation;
- if there is a constant $M > 0$ so that $\|x - f(x)\| \leq M$ for each x, then it is a translation;
- if $f^2 = \mathrm{Id}$ then it is a rotation through 180^0, a reflection or the identity.

10) (Hjemslev's theorem) Let AB and PQ be intervals in the plane, f an isometry with $F(A) = P$, $f(B) = Q$. If for $x \in [A, B]$, Vx denotes the point $\frac{1}{2}(x + f(x))$ (i.e. $V(x)$ is the midpoint of the line between x and its image), then the image of $[A, B]$ under V is an interval or a point.
11) Let f be an isometry of the form

$$(\xi_1, \xi_2) \mapsto (a_{11}\xi_1 + a_{12}\xi_2 + b_1, a_{21}\xi_1 + a_{22}\xi_2 + b_2)$$

and denote by B (resp. C) the matrices

$$\begin{bmatrix} a_{11} - 1 & a_{12} \\ a_{21} & a_{22} - 1 \end{bmatrix} \text{ resp. } \begin{bmatrix} a_{11} - 1 & a_{12} & b_1 \\ a_{21} & a_{22} - 1 & b_2 \end{bmatrix}.$$

Show that f is

- the identity if and only if $r(B) = 0$, $r(C) = 0$;
- a translation if and only if $r(B) = 0$, $r(C) = 1$;
- a reflection if and only if $r(B) = 1$, $r(C) = 1$;
- a glide reflection if and only if $r(B) = 1$, $r(C) = 2$;
- a rotation if and only if $r(B) = 2$, $r(C) = 2$.

2.6 Conic sections

The description

$$L = \{x : a\xi_1 + b\xi_2 + c = 0\}$$

of a straight line in the plane means that they are defined as the zero sets of affine functions i.e. functions of the form

$$(\xi_1, \xi_2) \mapsto a\xi_1 + b\xi_2 + c.$$

We now investigate those curves in $\mathbf{R}^2$ which are defined as the zero sets of quadratic functions i.e. of the form

$$(\xi_1, \xi_2) \mapsto \sum_{i,j=1}^{2} a_{ij}\xi_1\xi_2 + 2(b_1\xi_1 + b_2\xi_2) + c$$

for suitable scalars a_{11}, a_{12}, a_{21}, a_{22}, b_1, b_2 and c. Such curves are called **conic sections**. Familiar examples are

$$\xi_1^2 + \xi_2^2 = 1 \qquad \text{the unit circle;}$$

$$\frac{\xi_1^2}{a^2} + \frac{\xi_2^2}{b^2} = 1 \qquad \text{an ellipse;}$$

and

$$\frac{\xi_1^2}{a^2} - \frac{\xi_2^2}{b^2} = 1 \qquad \text{a hyperbola.}$$

Before analysing the general conic section we introduce a more efficient notation. We can write the above equation in the form

$$(f(x)|x) + 2(b|x) + c = 0$$

where f is the linear mapping with matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

and b is the vector (b_1, b_2).

Further we can also suppose that A is symmetric i.e. that $A^t = A$ or $a_{12} = a_{21}$. For if we replace A by the matrix

$$\begin{bmatrix} a_{11} & \dfrac{a_{12} + a_{21}}{2} \\ \dfrac{a_{12} + a_{21}}{2} & a_{22} \end{bmatrix}$$

then the value of the left hand side of the equation is unchanged. Notice that the symmetry of A means that the mapping f defined above satisfies the condition

$$(f(x)|y) = (y|f(x)) \qquad (x, y \in \mathbf{R}^2)$$

as can be checked by expanding both sides. This fact will be used later.

We intend to provide a complete classification of conic sections in the plane. Before doing so, we recall the concept of an orthonormal basis. Two vectors x_1 and x_2 form such a basis if they are perpendicular to each other and both have norm 1. This condition can be conveniently expressed in the formula $(x_i|x_j) = \delta_{ij}$ where δ_{ij} is the Kronecker δ-function which was introduced in the first chapter. Examples of such bases are the images of the canonical basis under linear isometries i.e. reflections or rotations. Thus if we rotate the canonical basis through 45 degrees, we obtain the basis

$$x_1 = \frac{1}{\sqrt{2}}(1,1) \qquad x_2 = \frac{1}{\sqrt{2}}(1,-1).$$

Note that if $(x_, x_2)$ is an orthonormal basis, then the representation of x with respect to (x_1, x_2) is

$$x = (x|x_1)x_1 + (x|x_2)x_2.$$

and the norm can be expressed in terms of the coefficients as follows:

$$\|x\|^2 = |(x|x_1)|^2 + |(x|x_2)|^2.$$

This example shows the close connection between the concepts of isometries and orthonormal bases. In fact, linear isometries f can be characterised as those linear mappings which map orthonormal bases into orthonormal bases. In particular, if f is a linear mapping so that the vectors $f(e_1)$ and $f(e_2)$ form such a basis, then f is an isometry. For if $x = (\xi_1, \xi_2)$, then $f(x) = \xi_1 f(e_1) + \xi_2 f(e_2)$ and $\|f(x)\|^2 = \xi_1^2 + \xi_2^2 = \|x\|^2)$.

Our classification of conic sections is based on the following representation for quadratic forms:

Proposition 20 *Let*

$$Q : (\xi_1, \xi_2) \mapsto \sum_{i,j=1}^{2} a_{ij}\xi_1\xi_2 = (f(x)|x)$$

be a quadratic form on the plane (where we suppose that the matrix A is symmetric). Then there exists an orthonormal basis (x_1, x_2) and real numbers λ_1, λ_2 so that

$$Q(\eta_1 x_1 + \eta_2 x_2) = \lambda_1\eta_1^2 + \lambda_2\eta_2^2.$$

PROOF. The idea of the proof is as follows: if we can find an orthonormal basis (x_1, x_2) so that the vectors x_1 and x_2 do not have their directions changed by f (i.e. are such that $f(x_1) = \lambda_1 x_1$, $f(x_2) = \lambda_2 x_2$ for scalars λ_1, λ_2), then the quadratic form will be as required. For then

$$\begin{aligned} Q(\eta_1 x_1 + \eta_2 x_2) &= (f(\eta_1 x_1 + \eta_2 x_2)|\eta_1 x_1 + \eta_2 x_2) && (57) \\ &= (\lambda_1 \eta_1 x_1 + \lambda_2 \eta_2 x_2 | \eta_1 x_1 + \eta_2 x_2) && (58) \\ &= \lambda_1 \eta_1^2 + \lambda_2 \eta_2^2. && (59) \end{aligned}$$

We are therefore led to examine the equation $f(x) = \lambda x$ i.e. we are looking for a scalar λ so that the system

$$\begin{array}{ccccc} (a_{11} - \lambda)\xi_1 & + & a_{12}\xi_2 & = & 0 \\ a_{21}\xi_1 & + & (a_{22} - \lambda)\xi_2 & = & 0 \end{array}$$

has a non-trivial solution. This is our first meeting with a so-called **eigenvalue problem**. These play a central role in linear algebra and will be treated in some detail in Chapter VII. We know from the theory of the first chapter that the above system has a non-trivial solution if and only if λ is a solution of the quadratic equation

$$(a_{11} - \lambda)(a_{22} - \lambda) - a_{21}a_{12} = 0.$$

Now the discriminant of this quadratic is $(a_{11} - a_{22})^2 + 4a_{21}^2$ which is clearly non-negative and so there exists at least one real root λ_1. We now choose a corresponding solution x_1 of the system and we can clearly arrange for x_1 to have length 1. Now we put $x_2 = D_{\frac{\pi}{2}}(x_1)$ so that (x_1, x_2) forms an orthonormal basis. We shall complete the proof by showing that there is a second scalar λ_2 so that $f(x_2) = \lambda_2 x_2$. In order to do this it suffices to show that $f(x_2) \perp x_1$ and this follows from the chain of equalities:

$$(f(x_2)|x_1) = (x_2|f(x_1)) = (x_2|\lambda_1 x_1) = \lambda_1(x_2|x_1) = 0.$$

We now turn to the general conic $C = \{Q(x) = 0\}$ where $Q(x) = (f(x)|x) + 2(b|x) + c$ and choose a basis (x_1, x_2) as above. Then

$$\begin{aligned} Q(\eta_1 x_1 + \eta_2 x_2) &= \lambda_1 \eta_1^2 + \lambda_2 \eta_2^2 + 2(b|\eta_1 x_1 + \eta_2 x_2) + c && (60) \\ &= \lambda_1 \eta_1^2 + \lambda_2 \eta_2^2 + 2(\tilde{b}_1 \eta_1 + \tilde{b}_2 \eta_2) + c && (61) \end{aligned}$$

where $\tilde{b}_1 = (b|x_1)$, $\tilde{b}_2 = (b|x_2)$.

We now examine the various possible forms of this expression:
Case 1: $\lambda_1 \neq 0$, $\lambda_2 \neq 0$. Then by translating the axes and multiplying by a suitable factor, we can reduce the equation to one of the following forms:

- $\mu_1\eta_1^2 + \mu_2\eta_2^2 = 1$: this is an ellipse, the empty set or a hyperbola, depending on the signs of the μ's;
- $\mu_1\eta_1^2 + \mu_2\eta_2^2 = 0$: this is a point or a pair of intersecting lines.

Case 2: One of the λ's vanishes, say λ_2. By translating the x-axis, we can remove the linear term in η_1. This gives the following three types of equation:

- $\mu_1\eta_1^2 + \eta_2 = 0$: a parabola;
- $\mu_1\eta_1^2 - 1 = 0$: a pair of parallel lines if $\lambda_1 > 0$, otherwise the empty set;
- $\mu_1\eta_1^2 = 0$: a straight line.

Hence we have shown that the general conic section is either an ellipse, a hyperbola, a parabola, two parallel lines, two intersecting lines, a single line, a point or the empty set.

Example: Discuss the conic section:

$$3\xi_1^2 + 10\xi_1\xi_2 + 3\xi_2^2 + 18\xi_1 - 2\xi_2 + 10 = 0.$$

Solution: The matrix of the quadratic part is

$$\begin{bmatrix} 3 & 5 \\ 5 & 3 \end{bmatrix}.$$

The equation for λ is $(3-\lambda)^2 - 25 = 0$ i.e. $\lambda = -2$ or $\lambda = 8$. Then, solving the equations for the basis vectors x_1 and x_2 as above, we get

$$x_1 = \frac{1}{\sqrt{2}}(1,1) \qquad \xi_2 = \frac{1}{\sqrt{2}}(1,-1).$$

Then

$$Q(\eta_1 x_1 + \eta_2 x_2) = Q((\frac{1}{\sqrt{2}}(\eta_1+\eta_2), \frac{1}{\sqrt{2}}(\eta_1-\eta_2)) \tag{62}$$

$$= 8\eta_1^2 - 2\eta_2^2 + \frac{16}{\sqrt{2}}\eta_1 + \frac{20}{\sqrt{2}}\eta_2 + 10 \tag{63}$$

$$= 8(\eta_1 + \sqrt{2})^2 - 2(\eta_2 - \frac{5\sqrt{2}}{2})^2 + 19 \tag{64}$$

and the curve

$$8\zeta_1^2 - 2\zeta_2^2 + 19 = 0$$

is a hyperbola.

Exercises: 1) Describe the geometric form of the following curves:

- $\xi_1^2 + 6\xi_1\xi_2 + 9\xi_2^2 + 5\xi_1 + 2\xi_2 + 11 = 0$;
- $4\xi_1^2 + 4\xi_1\xi_2 - 10\xi_1 + 8\xi_2 + 15 = 0$;
- $\xi_1^2 + \xi_1\xi_2 + \xi_2^2 = 3$;
- $5\xi_1^2 + 6\xi_1\xi_2 + 5\xi_2^2 - 256 = 0$;
- $\xi_1^2 - 2\xi_1\xi_2 + \xi_2^2 = 9$.

2) Show that the following loci are conic sections and determine their form (i.e. ellipse, hyperbola etc.) In each case, L_1 and L_2 are fixed lines, x_1, x_2 fixed points, λ a fixed positive number and C a fixed circle:

- the set of points x so that $d(x, L_1) = \lambda d(x, L_2)$;
- the set of x so that $d(x, L_1)$ is constant;
- the set of x so that $d(x, x_1) = \lambda d(x, x_2)$;
- the set of x so that the lengths of the tangents from x to C are constant;
- the set of x so that $d(x, x_1) = \lambda d(x, L_1)$;
- the set of midpoints of lines joining x_1 to C;
- the set of x so that $d(x, x_1)^2 - d(x, x_1)^2 = \lambda$ (resp. $d(x, x_1)^2 + d(x, x_2)^2 = \lambda$).

2.7 Three dimensional space

We now turn to the space $\mathbf{R}^3$ of triples (ξ_1, ξ_2, ξ_3) of real numbers which plays a paticularly important role as a model of the three dimensional space of everyday experience. Just as in the case of $\mathbf{R}^2$ we can define an addition and multiplication by scalars as follows:

$$(\xi_1, \xi_2, \xi_3) + (\eta_1, \eta_2, \eta_3) = (\xi_1 + \eta_1, \xi_2\eta_2, \xi_3, \eta_3) \tag{65}$$

$$\lambda(\xi_1, \xi_2, \xi_3) = (\lambda\xi_1, \lambda\xi_2, \lambda\xi_3). \tag{66}$$

Then the relationships 1) - 7) from from II.1 for $\mathbf{R}^2$ hold and for exactly the same reasons. The three dimensionality of space means that there are triples (x_1, x_2, x_3) which form a basis i.e. are linearly independent and span $\mathbf{R}^3$. This means that each vector x has a unique representation of the form

$$\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3.$$

The λ_i are then called the **coordinates** of x with respect to the basis. The most natural such basis is of course the triple $((1, 0, 0), (0, 1, 0), (0, 0, 1))$ which is called the **canonical basis** and denoted by (e_1, e_2, e_3). Once again, questions about linear dependence resp. independence reduce to ones about solvability resp. uniqueness of solutions of systems of equations and so we can apply the theory of the first chapter to them. In particular, we have the following characterisation of bases:

Proposition 21 *Let $x_1 = (a_1, a_2, a_3)$, $x_2 = (b_1, b_2, b_3)$ and $x_3 = (c_1, c_2, c_3)$ be three vectors in $\mathbf{R}^3$. Then the following are equivalent:*
a) the vectors are linearly independent;
b) they span $\mathbf{R}^3$;
c) the matrix

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

is invertible.

The same line of thought shows that any set of four vectors in $\mathbf{R}^3$ is linearly dependent. For this corresponds to the fact that a homogeneous system of three equations in four unknowns always has a non-trivial solution.

On $\mathbf{R}^3$ we can define a scalar product $(x|y)$ by means of the three dimensional analogue

$$(x|y) = \xi_1\eta_1 + \xi_2\eta_2 + \xi_3\eta_3$$

of the formula for the scalar product in the plane. We can then define the length $\|x\|$ of a vector to be $\sqrt{(x|x)}$ and the distance $|PQ|$ between points P and Q to be

$$\|x_{PQ}\| = \sqrt{(x_Q - x_P|x_Q - x_P)}.$$

Just as in the two dimensional case one can show that the following properties hold:

- $(x|y) = (y|x)$
- $(\lambda x|y) = \lambda(x|y)$;
- $(x + y|z) = (x|z) + (y|z)$;
- $|(x|y)| \leq \|x\|\|y\|$;
- $\|x + y\| \leq \|x\| + \|y\|$.

If x and y are non-zero, then $\dfrac{(x|y)}{\|x\|\|y\|}$ lies between -1 and 1 (by 4) above) and so there exists $\theta \in]-\frac{\pi}{2}, \frac{\pi}{2}]$ with $\cos\theta = \dfrac{(x|y)}{\|x\|\|y\|}$. θ is called the **angle** between x and y (written $\angle(x, y)$) and x and y are said to be **perpendicular** (written $x \perp y$) if $(x|y) = 0$.

Bases whose elements are unit vectors which are perpendicular to each other are called **orthonormal**. We can conveniently express this condition in the form $(x_i|x_j) = \delta_{ij}$. The canonical basis is an example. Such bases have the following convenient properties:

- if $x \in \mathbf{R}^3$, then the representation of x with respect to (x_1, x_2, x_3) is $\sum_{i=1}^{3}(x|x_i)x_i$ (for if $x = \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3$, then $(x|x_1) = \lambda_1(x_1|x_1) = \lambda_1$ and so on);
- if $x = \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3$, $y = \mu_1 x_1, , +\mu_2 x_2 + \mu_3 x_3$, then

  $$(x|y) = \lambda_1\mu_1 + \lambda_2\mu_2 + \lambda_3\mu_3 \quad \text{and} \quad \|x\|^2 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$$

  i.e. we can calculate lengths and scalar coordinates with the coordinates with respect to *any* orthonormal basis exactly as we do with the natural coordinates (i.e. those with respect to the canonical basis).

We can also write the formulae of 2) above in the form

$$(x|y) = \sum_{i=1}^{3}(x|x_i)(y|x_i) \quad \text{and} \quad \|x\|^2 = \sum_{i=1}^{3}(x|x_i)^2.$$

There is a standard method to construct an orthonormal basis for $\mathbf{R}^2$ from a given basis (x_1, x_2, x_3). It is called the **Gram-Schmidt process** and has a very natural geometrical interpretation. We seek scalars λ, μ, ν so that the vectors

$$y_1 = x_1 \tag{67}$$
$$y_2 = x_2 + \lambda y_1 \tag{68}$$
$$y_3 = x_3 + \mu y_2 + \nu y_1 \tag{69}$$

are mutually perpendicular. For this to be the case, we must have

$$0 = (y_1|y_2) = (x_1|x_2) + \lambda(x_1|x_1) \quad \text{i.e.} \quad \lambda = -(x_1|x_2)/(x_1|x_1); \tag{70}$$
$$0 = (y_3|y_1) = (x_3|y_1) + \nu(y_1|y_1) \quad \text{i.e.} \quad \nu = -(x_3|y_1)/(x_1|x_1); \tag{71}$$
$$0 = (y_3|y_2) = (x_3|y_2) + \mu(y_2|y_2) \quad \text{i.e.} \quad \mu = -(x_3|y_2)/(y_2|y_2). \tag{72}$$

If we take these values of λ, μ, ν, we see that

$$z_1 = \frac{y_1}{\|y_1\|} \quad z_2 = \frac{y_2}{\|y_2\|} \quad z_3 = \frac{y_3}{\|y_3\|}$$

is an orthonormal basis.

Planes and lines in space: If a, b and c are real numbers, not all of which are zero, then the set

$$M = M_{a,b,c} = \{(\xi_1, \xi_2, \xi_3) \in \mathbf{R}^3; a\xi_1 + b\xi_2 + c\xi_3 = 0\}$$

is called a **two dimensional subspace of $\mathbf{R}^3$**. If we denote by $\mathbf{n}_1$ the unit vector $\dfrac{(a,b,c)}{\sqrt{a^2+b^2+c^2}}$, then $M_{a,b,c}$ is just the set

$$\{x \in \mathbf{R}^3; (\mathbf{n}_1|x) = 0\}$$

of vectors which are perpendicular to $\mathbf{n}_1$. If we choose $\mathbf{n}_2$ and $\mathbf{n}_3$ so that $(\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3)$ is an orthonormal basis for $\mathbf{R}^3$, then M consists of those vectors of the form

$$\{\lambda_2\mathbf{n}_2 + \lambda_3\mathbf{n}_3 : \lambda_2, \lambda_3 \in \mathbf{R}\}$$

(this is the parameter representation of the subspace). A **one-dimensional subspace** is a set of the form

$$\{\lambda x : \lambda \in \mathbf{R}\}$$

where x is a non-zero element of $\mathbf{R}^3$. Once again, if we choose x to be a unit vector and extend to an orthonormal basis, we can represent such spaces in the form

$$\{y \in \mathbf{R}^3 : (y|\mathbf{n}_2) = (y|\mathbf{n}_3) = 0\}$$

where $\mathbf{n}_2$ and $\mathbf{n}_3$ are perpendicular unit vectors.

In order to introduce planes and lines we consider the translation mapping

$$T_u : x \mapsto x + u.$$

A **line** in $\mathbf{R}^3$ is a set of the form $T_u(L)$ where L is a one-dimensional subspace and a **plane** has the form $T_u(M)$ where M is a two-dimensional subspace. In coordinates, they have the form

$$\{(\xi_1, \xi_2, \xi_3) : a_1\xi_1 + a_2\xi_2 + a_3\xi_3 + a_4 = 0 = b_1\xi_1 + b_2\xi_2 + b_3\xi_3 + b_4\}$$

where (a_1, a_2, a_3) and (b_1, b_2, b_3) are linearly independent vectors, respectively

$$\{(\xi_1, \xi_2, \xi_3) : a_1\xi_1 + a_2\xi_2 + a_3\xi_3 + a_4 = 0\}$$

where $(a_1, a_2, a_3) \neq 0$.

Exercises: 1) Calculate the angle between the planes

$$3\xi_1 - 2\xi_2 + 4\xi_3 - 10 = 0$$

and

$$-2\xi_1 + 3\xi_2 - 7\xi_3 + 5 = 0.$$

2) Let A, B and C be non-collinear points in $\mathbf{R}^3$. For which values of λ_1, λ_2 and λ_3 does

$$x = \lambda_1 x_A + \lambda_2 x_B + \lambda_3 x_C$$

lie on the line joining A to the midpoint of BC?
3) If A, B, C and D are points in $\mathbf{R}^3$, then the lines joining the vertices to the centroids of the opposite triangles of the tetrahedron $ABCD$ are collinear and the point of intersection divides each of them in the ratio $3:1$.
4) Find an explicit formula for the distance from x to the plane through y_1, y_2 and y_3.
5) Let L_1 and L_2 be lines in $\mathbf{R}^3$ which do no lie on parallel planes. Then for each $x \in \mathbf{R}^3$, not on L_1 or L_2, there is a unique line L through x which meets L_1 and L_2.
6) If A, B, C and D are points in $\mathbf{R}^3$ and E resp. F are the midpoints of AC resp. BD, then

$$|AC|^2 + |BD|^2 = |AB|^2 + |BC|^2 + |CD|^2 + |DA|^2 - 4|EF|^2.$$

7) Show that three lines $L_{a,b,c}$, L_{a_1,b_1,c_1} and L_{a_2,b_2,c_2} in the plane have a common point if and only if (a, b, c), (a_1, b_1, c_1) and (a_2, b_2, c_2) are linearly dependent in $\mathbf{R}^3$. Use this to give new proofs that the following lines are concurrent:

a) the bisectors of the angles of a triangle;

b) the perpendicularbisectors of the sides;

c) the perpendiculars from the vertices to the opposite sides.

8) Show that if an altitude of a tetrahedron intersects two others, then all four intersect.

9) Let A, B, C and D be the vertices of a tetrahedron. Show that if AB is perpendicular to CD and AC is perpendicular to BD, then AD is perpendicular to BD. 10) Let A, B, C and D be points in $\mathbf{R}^3$, no three of which are collinear. Then the line joining the midpoints of AB and CD bisects the line joining the midpoints of AC and BD.

2.8 Vector products, triple products, 3×3 determinants

We now introduce two special structures which are peculiar to $\mathbf{R}^3$ (i.e. have no immediate analogues in the plane or in higher dimensions) and which play an important role in classical physics—the vector product and the triple product:

Vector products: If $x = (\xi_1, \xi_2, \xi_3)$ and $y = (\eta_1, \eta_2, \eta_3)$ are vectors in $\mathbf{R}^3$, their vector product is the vector $x \times y$ with components

$$(\xi_2\eta_3 - \xi_3\eta_2, \xi_3\eta_1 - \xi_1\eta_3, \xi_1\eta_2 - \xi_2\eta_1).$$

A simple calculation shows that

$$x \perp x \times y \quad y \perp x \times y \quad \|x \times y\| = \|x\|\|y\| \sin\theta$$

where θ is the angle between x and y i.e. the length of $x \times y$ is equal to the area of the parallelogram spanned by x and y. These properties determine $x \times y$ up to direction. The direction is determined by the so-called right hand screw rule.

We list some other simple properties of the vector product:

a) $x \times y = -y \times x$;

b) $(\lambda x) \times y = x \times (\lambda y) = \lambda(x \times y)$;

c) $x \times (y \times z) = (x|z)y - (x|y)z$;

d) $x \times (y \times z) + y \times (z \times x) + z \times (x \times y) = 0$. Note that, in particular, the vector product is neither commutative or associative.

Proof. a) and b) are trivial calculations.

c) We calculate the first coordinate of both sides and get

$$\xi_2(\eta_1\zeta_2 - \eta_2\zeta_1) - \xi_3(\eta_3\zeta_1 - \eta_1\zeta_3)$$

resp.

$$(\xi_1\zeta_1 + \xi_2\zeta_2 + \xi_3\zeta_3)\eta_1 - (\xi_1\eta_1 + \xi_2\eta_2 + \xi_3\eta_3)\zeta_1$$

which are equal. (Equality in the other coordinates follows by symmetry).

d) Substituting the values obtained in c) we get

$$x \times (y \times z) + y \times (z \times x) + z \times (x \times y)$$

which is equal to

$$(x|z)y - (x|y)z + (y|z)x - (y|x)z + (z|y)x - (z|x)y = 0.$$

∎

The triple product: This is defined by the formula

$$[x, y, z] = (x|y \times z).$$

Using the geometric interpretations of the scalar and vector product, we see that up to sign this is the quantity

$$\|x\|\|y\|\|z\| \sin\theta \sin\psi$$

where θ and ψ are the angles in the diagram (figure ??) This is just the volume of the parallelotope spanned by x, y and z. Simple properties of the triple product are:

a) $[x, y, z] = [y, z, x] = [z, x, y] = -[y, x, z] = -[x, z, y] = -[z, y, x]$;

b) $[x_1 + x, y, z] = [x_1, y, z] + [x, y, z]$;

c) $[\lambda x, y, z] = \lambda[x, y, z]$;

d) $[x+\lambda y+\nu z, y, z] = [x, y, z]$. Note that if x_1, x_2, x_3 is an arbitrary basis for $\mathbf{R}^3$ we can calculate the triple product of the vectors

$$x = \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 \tag{73}$$

$$y = \mu_1 x_1 + \mu_2 x_2 + \mu_3 x_3 \tag{74}$$

$$z = \nu_1 x_1 + \nu_2 x_2 + \nu_3 x_3 \tag{75}$$

in terms of their coordinates with respect to this basis as follows:

$$[x, y, z] = (\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 |(\mu_1 x_1 + \mu_2 x_2 + \mu_3 x_3) \times (\nu_1 x_1 + \nu_2 x_2 + \nu_3 x_3)) \tag{76}$$

$$= [\lambda_1(\mu_2\nu_3 - \mu_3\nu_2) - \lambda_2(\mu_1\nu_3 - \mu_3\nu_1) + \lambda_3(\mu_1\nu_2 - \mu_2\nu_1)]\,[x_1, x_2, x_3]. \tag{77}$$

We call the expression in the bracket above the **determinant** of the matrix

$$A = \begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & \mu_2 & \mu_3 \\ \nu_1 & \nu_2 & \nu_3 \end{bmatrix}$$

(in symbols $\det A$). If we denote the rows of A by A_1, A_2 and A_3, then we can state the following simple properties of the determinant:

a)

$$\det \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} = \det \begin{bmatrix} A_1 + \lambda A_2 \\ A_2 \\ A_3 \end{bmatrix}$$

etc. i.e. if we add a multiple of one row of the matrix to another one, the determinant remains unchanged;

b)

$$\det \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} = -\det \begin{bmatrix} A_2 \\ A_1 \\ A_3 \end{bmatrix}$$

etc. i.e. if we exchange two rows of the matrix we alter the sign of the determinant.
c)

$$\det A = \det \begin{bmatrix} \lambda_1 & \mu_1 & \nu_1 \\ \lambda_2 & \mu_2 & \nu_2 \\ \lambda_3 & \mu_3 & \nu_3 \end{bmatrix}$$

i.e. $\det A = \det A^t$. d) $\det A \neq 0$ if and only if the vectors $(\lambda_1, \lambda_2, \lambda_3)$, (μ_1, μ_2, μ_3) and (ν_1, ν_2, ν_3) (or equivalently, the original nvectors x, y, z) are linearly independent.
e) Consider the system of equations:

$$b_{11}\xi_1 + b_{12}\xi_2 + b_{13}\xi_3 = \eta_1 \tag{78}$$
$$b_{21}\xi_1 + b_{22}\xi_2 + b_{23}\xi_3 = \eta_2 \tag{79}$$
$$b_{31}\xi_1 + b_{32}\xi_2 + b_{33}\xi_3 = \eta_3. \tag{80}$$

Then it has a unique solution for each right hand side if and only if the determinant of its matrix B is non-zero. In this case the solution is

$$\xi_1 = \det \begin{bmatrix} \eta_1 & b_{12} & b_{13} \\ \eta_2 & b_{22} & b_{23} \\ \eta_3 & b_{32} & b_{33} \end{bmatrix} \div \det B$$

etc.
PROOF. a) and b) follow from the equations

$$[x + \lambda y, y, z] = [x, y, z] + \lambda[y, y, z]] = [x, y, z] \tag{81}$$
$$[x, y, z] = -[y, x, z]. \tag{82}$$

c) is a routine calculation.
e) is a rather tedious computation and d) follows from e) and the results of the first chapter.

■

We remark that we shall prove the general version of e) in chapter V

Example: Show that
a) $[x+y, y+z, z+x] = 2[x,y,z]$;
b) $(x \times y|z \times u) = (x|z)(y|u) - (x|u)(y|z)$;
c) $(x \times y) \times (y \times u) = [x,z,u]y - [y,z,u]x$.
Solution: a)

$$\begin{aligned}
[x+y, y+z, z+x] &= [x+y|(y+z)\times(z+x)] && (83)\\
&= [x+y|y\times z + z\times z + y\times x + z\times x] && (84)\\
&= [x,y,z]+[x,y,x]+[x,z,x]+[y,y,z]+[y,y,x]+[y,z,x] && (85)\\
&= 2[x,y,z]. && (86)
\end{aligned}$$

b)

$$\begin{aligned}
(x\times y|z\times u) &= [x\times y, z, u] && (87)\\
&= [u, x\times y, z] && (88)\\
&= (u|(x\times)\times z) && (89)\\
&= -(u|z\times(x\times y)) && (90)\\
&= -(u|(z|y)x-(z|x)y) && (91)\\
&= (x|z)(y|u) = (x|u)(y|z). && (92)
\end{aligned}$$

c)

$$\begin{aligned}
(x\times y)\times(z\times u) &= -(z\times u)\times(x\times y) && (93)\\
&= -(z\times u|y)x + (z\times u|x)y && (94)\\
&= [x,z,u]y - [y,z,u]x. && (95)
\end{aligned}$$

Exercises: 1) Show that if x, y, z, x_1, y_1, z_1 are points in $\mathbf{R}^3$, then

$$[x,y,z][x_1,y_1,z_1] = \det \begin{bmatrix} (x|x_1) & (x|y_1) & (x|z_1) \\ (y|x_1) & (y|y_1) & (y|z_1) \\ (z|x_1) & (z|y_1) & (z|z_1) \end{bmatrix}.$$

2) Show that the line through the distinct points A and B in $\mathbf{R}^3$ has equation

$$x \times x_A + x_A \times x_B + x_B \times x = 0.$$

3) Calculate the determinants of the following 3×3 matrices:

$$\begin{bmatrix} x+1 & 1 & 1 \\ 1 & x+1 & 1 \\ 1 & 1 & x+1 \end{bmatrix} \quad \begin{bmatrix} x & -1 & 0 \\ 0 & x & -1 \\ c & b & x+a \end{bmatrix} \quad \begin{bmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{bmatrix}$$

$$\begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{bmatrix}.$$

4) Show that $(x \times y) \times (z \times w) = 0$ if x, y, z, w lie on a plane. For which x, y, z in $\mathbf{R}^3$ do we have the equality

$$(x \times y) \times z = x \times (y \times z)?$$

2.9 Covariant and contravariant vectors

Physicists often make a distinction between covariant and contravariant vectors. We shall discuss this in a more abstract setting in Chapter IX. Here we shall deal with the three dimensional case. In order to conform with the standard notation we shall denote by (e_1, e_2, e_3) a basis for $\mathbf{R}^3$ (which, we emphasise, need *not* be the canonical basis). Every vector x has a unique representation $\lambda_1 e_1 + \lambda^2 e_2 + \lambda^3 e_3$ (where we write the indices of the coefficients as superscripts for traditional reasons connected with the Einstein summation convention). If we define new vectors

$$e^1 = \frac{e_2 \times e_3}{[e_1, e_2, e_3]} \quad e^2 = \frac{e_3 \times e_1}{[e_1, e_2, e_3]} \quad e^3 = \frac{e_1 \times e_2}{[e_1, e_2, e_3]}$$

then as is easily checked we have the relationships

$$(e_1|e^1) = (e_2|e^2) = (e_3|e^3) = 1$$

and $(e_i|e^j) = 0$ for $i \neq j$. From this it follows that the coefficients λ^i in the expansion of x is just $(x|e^i)$ i.e.

$$x = \sum_{i=1}^{3} (x|e^i) e_i.$$

On the other hand, (e^i) also forms a basis for $\mathbf{R}^3$ and we have the formula

$$x = \sum_{i=1}^{3} (x|e_i) e^i.$$

For if we put

$$y = x - \sum_{i=1}^{3} (x|e_i) e^i,$$

then a simple calculation shows that $(y|e_i) = 0$ for each i and so $y = 0$.

In order to distinguish between the two representations

$$x = \sum_{k=1}^{3} \lambda^i e_i = \sum_{i=1}^{3} \lambda_i e^i$$

we call the first one the **contravariant representation** and the second one the **covariant representation.** (Note that the two bases—and hence the two representation—coincide exactly when the basis (e_i) is orthonormal).

If we define the coefficients g_{ij} and g^{ij} by the formulae

$$g_{ij} = (e_i|e_j) \qquad g^{ij} = (e^i|e^j),$$

then we have the relationships:

$$e_i = \sum_{j=1}^{3}(e_i|e_j)e^j = \sum_{j=1}^{3} g_{ij}e^j \tag{96}$$

$$e^i = \sum_{j=1}^{3}(e^i|e^j)e_j = \sum_{j=1}^{3} g^{ij}e_j \tag{97}$$

$$(e_i|e^k) = \sum_{j=1}^{3} g_{ij}g^{jk}. \tag{98}$$

Since the left hand side of the last equation is 1 or 0 according as i is or is not equal to k, we can rewrite it in the form

$$A \cdot \tilde{A} = I_3 \quad \text{where} \quad A = [g_{ij}], \quad \tilde{A} = [g^{ij}]$$

i.e. $\tilde{A} = A^{-1}$. Further formulae which can be calculated directly without difficulty are:

a) $\lambda_i = \sum_{j=1}^{3} g_{ij}\lambda^j, \lambda^j = \sum_{i=1}^{3} g^{ij}\lambda_i$;

b) $(x|y) = \sum_{i,j=1}^{3} g^{ij}\lambda_i\mu_j = \sum_{i,j=1}^{3} g_{ij}\lambda^i\mu^j$ (where $y = \sum_{j=1}^{3} \mu_j e^j = \sum_{j=1}^{3} \mu^j e_j$);

c) $\|x\|^2 = \sum_{i=1}^{3} \lambda_i\lambda^i = \sum_{i,j=1}^{3} g_{ij}\lambda^i\lambda^j = \sum_{i,j=1}^{3} g^{ij}\lambda_i\lambda_j$.

In order to obtain a relationship between the g's we consider the formulae:

$$\sum_{j=1}^{3} g_{ij}g^{jk} = \delta_{jk}.$$

For fixed k we can regard this as a system of equations with unknowns G^{jk} ($j = 1, 2, 3$). If we solve these by using the formulae given at the end of the previous section we get:

$$g^{11} = \frac{(g_{22}g_{33} - g_{23}^2)}{G} \tag{99}$$

$$g^{12} = \frac{(g_{31}g_{32} - g_{12}g_{33})}{G} = g^{21} \tag{100}$$

$$g^{13} = \frac{(g_{12}g_{23} - g_{22}g_{13})}{G} = g^{31} \tag{101}$$

and so on (where G is the determinant of the matrix

$$\left[\begin{array}{ccc} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{array}\right]).$$

Exercises: 1) Calculate the dual bases for

a) $((1,1,1),(1,1,-1),(1,-1,1))$;

b) $((0,1,1),(1,0,1),(1,1,0))$;

c) $((1,1,1),(0,1,1),(0,0,1))$ and the corresponding matrices $[g_{ij}]$ and $[g^{ij}]$.

2) Show that if

$$x = \lambda^1 e_1 + \lambda^2 e_2 + \lambda^3 e_3 \quad y = \mu^1 e_1 + \mu^2 e_2 + \mu^3 e_3$$

then the k-th coefficient of $x \times y$ with respect to the basis is

$$\sum_{i,j=1}^{3} \lambda^i \mu^j \epsilon_{ijm}$$

where $\epsilon_{ijk} = (e_i \times e_j | e_k)$.

3) If $x = 2e_1 - e_2 + 4e_3$ and $y = 2e_1 + 3e_2 - e_3$, calculate the coordinates of $x \times y$ with respect to (e^1, e^2, e^3).

4) If (e^1, e^2, e^3) is the dual basis to (e_1, e_2, e_3), then

$$[e_1, e_2, e_3][e^1, e^2, e^3] = 1.$$

5) If (e_1, e_2, e_3) is a basis, then the set

$$\{x \in \mathbf{R}^3; |(x|e^i)| \leq \|e_i\| \quad \text{for each} \quad i\}$$

is called the **Brouillon zone** of the unit cell (i.e. the set

$$\{\lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 : -1 \leq \lambda_i \leq 1 \quad \text{for each} \quad i\}).$$

Calculate it for the following bases

a) $((1,1,1),(1,1,-1),(1,-1,-1))$;

b) $((1,1,0),(0,1,1),(1,0,1))$;

c) $((0,0,1),(3,-1,0),(0,-1,0))$.

2.10 Isometries of $\mathbf{R}^3$

We now analyse the isometries of $\mathbf{R}^3$. We begin by extending some definitions and results on $\mathbf{R}^2$ to $\mathbf{R}^3$ in the natural way. A mapping $f : \mathbf{R}^3 \to \mathbf{R}^3$ is **linear** if $f(x+y) = f(x)+f(y)$ and $f(\lambda x) = \lambda f(x)$ for $x, y \in \mathbf{R}^3$ and $\lambda \in \mathbf{R}$. Then f naps the vector (ξ_1, ξ_2, ξ_3) onto

$$(a_{11}\xi_1 + a_{12}\xi_2 + a_{13}\xi_3, a_{21}\xi_1 + a_{22}\xi_2 + a_{23}\xi_3, a_{31}\xi_1 + a_{32}\xi_2 + a_{33}\xi_3)$$

where $f(e_1) = (a_{11}, a_{21}, a_{31})$ etc. The 3×3 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

is called the **matrix of** f.

If $u \in \mathbf{R}^3$, T_u denotes the translation operator $x \mapsto x + u$. An **affine mapping** f is one of the form $t_u \circ \tilde{f}$ where $\tilde{f}$ is linear. Hence in coordinates, f takes the vector (ξ_1, ξ_2, ξ_3) into the vector

$$(a_{11}\xi_1 + a_{12}\xi_2 + a_{13}\xi_3 + u_1, a_{21}\xi_1 + a_{22}\xi_2 + a_{23}\xi_3 + u_2, a_{31}\xi_1 + a_{32}\xi_2 + a_{33}\xi_3 + u_3).$$

A mapping f is an **isometry** if

$$\|f(x) - \|f(y)\| = \|x - y\| \quad (x, y \in \mathbf{R}^3).$$

If f is linear this is equivalent to the fact that $(f(x)|f(y)) = (x|y)$ for each x, y. An isometry is automatically affine and if it is linear it maps orthonormal bases onto orthonormal bases. (This follows immediately from the fact that $(f(x_i)|f(x_j)) = (x_i|x_j)$). On the other hand, if f is a linear mapping which maps *one* orthonormal basis (x_1, x_2, x_3) onto an orthonormal basis (which we denote by (y_1, y_2, y_3)), then it is an isometry. For if

$$y = \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 \quad y = \mu_1 x_1 + \mu_2 x_2 + \mu_3 x_3,$$

then $(x|y) + \lambda_1\mu_1 + \lambda_2\mu_2 + \lambda_3\mu_3$. On the other hand,

$$f(x) = \lambda_1 y_1 + \lambda_2 y_2 + \lambda_3 y_3 \quad f(y) = \mu_1 y_1 + \mu_2 y_2 + \mu_3 y_3$$

and so

$$(f(x)|f(y)) = \lambda_1\mu_1 + \lambda_2\mu_2 + \lambda_3\mu_3 = (x|y).$$

In terms of matrices, we see that f is an isometry if and only if the column vectors of its matrix A for an orthonormal basis (since they are the images of the canonical basis) and this just means that

$$A^t A = I \quad \text{i.e.} \quad A^t = A^{-1}.$$

Such matrices are called **orthonormal.**

With these preliminaries behind us we proceed to classify such isometries. The naive approach mentioned in the case of the plane i.e. that of writing down the equations involved in the orthogonality conditions on A and solving them is now hopelessly impractical (there are eighteen equations) and we require a more sophisticated one which allows us to reduce to the two-dimensional case. This is done by means of the following lemma:

Lemma 2 *Let $r : \mathbf{R}^3 \to \mathbf{R}^3$ be a linear isometry. Then there is an $x_1 \in \mathbf{R}^3$ with $\|x_1\| = 1$ so that $f(x_1) = \pm x_1$. If x_2 and x_3 are then chosen so that (x_1, x_2, x_3) is an orthonormal basis, then the matrix of f with respect to this basis has block representation*

$$\begin{bmatrix} \pm 1 & 0 \\ 0 & A \end{bmatrix}$$

where A is the matrix of an isometry of the plane.

PROOF. First of all we consider an eigenvalue problem as in the proof of the result used in section II.6 to classify the conic sections i.e. we look for a non-zero vector x and a $\lambda \in \mathbf{R}$ so that $f(x) = \lambda x$. Once again, this means that $x = (\xi_1, \xi_2, \xi_3)$ must be a solution of the homogeneous system:

$$\begin{array}{ccccccc} (a_{11}-\lambda)\xi_1 & + & a_{12}\xi_2 & + & a_{13}\xi_3 & = 0 \\ a_{21}\xi_2 & + & (a_{22}-\lambda)\xi_2 & + & a_{23}\xi_3 & = 0 \\ a_{31}\xi_1 & + & a_{32}\xi_2 & + & (a_{33}-\lambda)\xi_3 & = 0. \end{array}$$

However, we know that such an x exists if and only if the determinant of the associated matrix

$$\begin{bmatrix} (a_{11}-\lambda) & a_{12} & a_{13} \\ a_{21} & (a_{22}-\lambda) & a_{23} \\ a_{31} & a_{32} & (a_{33}-\lambda) \end{bmatrix}$$

vanishes. But this is an equation of the third degree in λ and so has a real solution.

Hence we can fond a unit vector x_1 so that $f(x_1) = \lambda x_1$. From the fact that $\|f(x_1)\| = \|x_1\|$ (f is an isometry) it follows that $\lambda = \pm 1$. Thus we have proved the first part of the Lemma. If we choose an orthonormal basis (x_1, x_2, x_3) as in the formulation, then the matrix of f with respect to this basis has the form

$$\begin{bmatrix} \pm 1 & 0 & 0 \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{bmatrix}.$$

For the equation $f(x_1) = \pm x_1$ means that the first column is

$$\left[\ \pm 1//0//0\ \right].$$

For the second column note that a vector y is in the span of x_2 and x_3 if and only if $(y|x_1) = 0$. But then $(f(y)|x_1) = 0$ and the same holds for $f(y)$ (for $(z|x_1) = 0$ implies $0 = (f(y)|f(x_1)) = \pm(f(y)|x_1))$. This implies that the remaining two columns of the matrix has the required form. That

$$\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$$

is the matrix of an isometry follows from the fact that $\|f(y)\| = \|y\|$ for each y of the form $\eta_2 x_2 + \eta_3 x_3$).

We can now classify the linear isometries f on $\mathbf{R}^3$. By the above we can choose an orthonormal basis so that the matrix of f has the form

$$\begin{bmatrix} \pm 1 & 0 & 0 \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{bmatrix}$$

where

$$\tilde{A} = \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$$

is the matrix of an isometry in $\mathbf{R}^2$. There are four possibilities:
1) $f(x_1) = x_1$ and $\tilde{A}$ is the matrix of a rotation. Then f is called a **rotation about the axis** x_1;
2) $f(x_1) = x_1$ and $\tilde{A}$ is the matrix of a reflection in a line L in the x_2, x_3-plane. Then f is a **reflection in the plane through** x_1 **and** L. If we arrange for x_2 to be on L then the matrix of f is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

and f is reflection in the (x_1, x_2)-plane;
3) $f(x_1) = -x_1$ and $\tilde{A}$ is the matrix of a rotation. Then f is a rotation about the x_1-axis, followed by a reflection in the plane perpendicular to x_1. Such mappings are called **rotary reflections**;
4) $f(x_1) = -x_1$ and $\tilde{A}$ is the matrix of a reflection. In this case we can choose x_2 and x_3 so that the matrix of f is

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

This is a rotation of 180^o about the x_3-axis.

Summarising, we have proved the following result:

Proposition 22 *A linear isometry of $\mathbf{R}^3$ has one of the following three forms:*

- *rotation;*
- *a reflection;*
- *a rotary reflection.*

Using this result, we can classify all of the isometries of $\mathbf{R}^3$ (including those which are not necessarily linear) as follows:
Case a): The isometry f has a fixed point x i.e. is such that $f(x) = x$. Then if $\tilde{f} = T_{-x} \circ f \circ T_x$, $\tilde{f}$ is a linear isometry. Hence $\tilde{f}$ and so also f is one the above three types.
Case b): As in the two-dimensional case, we put $x = f(0)$ with the result that $f = T_x \circ \tilde{f}$ where $\tilde{f}$ is a linear isometry and so of one of the above types. Hence it suffices to analyses the possible form of such compositions. We relegate this to the following set of exercises and note the results here. There are two new possibilities:

a screw displacement i.e. a mapping of the form $T_u \circ D_{L,\theta}$ where u is parallel to the line L;

a glide reflection i.e. a mapping of the form $T_u \circ R_M$ where u is parallel to the plane M.

Exercises: 1) Give the equation of the following isometries

- a rotation of 60^o about the axis $(1, -1, 1)$; item a rotation of 45^o about the axis $(1, 2, 3)$;
- a reflection in the two dimensional subspace spanned by the vectors $(2, 3, 1)$ and $(1, -1, 1)$.

Analyse the linear isometries with matrices

$$\begin{bmatrix} 0 & 0 & 1 \\ \frac{3}{5} & -\frac{4}{5} & 0 \\ \frac{4}{5} & \frac{3}{5} & 0 \end{bmatrix} \quad \text{resp.} \quad \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{4}{3\sqrt{2}} & \frac{1}{3\sqrt{2}} & \frac{1}{3\sqrt{2}} \end{bmatrix}.$$

2) Show that the product $R_M \circ R_{M_1}$ of two reflections is

- a rotation about L if the planes meet in the plane L;

- a translation (if M and M_1 are parallel).

Note the particular case of (1) where M and M_1 are perpendicular. The corresponding operator is then a **reflection in** L and denoted by R_L.
3) Show that the product $R_L \circ R_{L_1}$ of two reflections in lines (cf. Exercise 2)) is

- a translation if the lines are parallel;
- a rotation if the lines intersect in a point;
- a screw rotation if the lines are skew.

In the latter case, describe the geometrical significance of the axis of rotation.
4) Consider the product $R_M \circ R_{M_1} \circ R_{M_2}$ of three reflections in planes. Show that this is

- a point inversion if the planes are mutually perpendicular;
- a translation if the planes are parallel;
- a reflection in a line if the planes meet in a single line;
- a glide reflection if they do not intersect, but there is a line which is parallel to each of the planes.
- a screw reflection if the planes meet at a point.

5) Consider an isometry of the form $T_x \circ D_{L,\theta}$. Show that there is a vector u and a line L_1 (both parallel to L) so that the isometry can be represented as $T_u \circ D_{L_1,\theta}$ (i.e. it is a screw-displacement). (Split the vector x into its components parallel and perpendicular to L).
6) Consider an isometry of the form $T_x \circ R_M$. Show that there is a vector u and a plane M_1 (both parallel to M) so that the isometry can be represented as $T_u \circ R_{M_1}$ (i.e. it is a glide reflection).
7) Show that the product of two screw displacements is also a screw displacement (possibly degenerate i.e. so that the translation or rotation component vanishes).
8) In three dimensional space, we have three types of reflections—point reflections, reflections in lines and reflections in planes. For the sake of uniformity of notation we write R_A for the operation of reflection in A i.e. the mapping $T_{x_A} \circ (-\mathrm{Id}) \circ T_{-x_A}$). The reader is invited to translate the following operator equations into geometrical statements:

- $R_A \circ R_M \circ R_B \circ R_M = \mathrm{Id}$;

- $(R_L \circ R_L)^2 = \mathrm{Id}$;
- $(R_A \circ R_M)^2 = \mathrm{Id}$;
- $(R_L \circ R_M)^2 = \mathrm{Id}$;
- $R_{L_1} \circ R_{L_2} \circ R_{L_3} = \mathrm{Id}$;
- $R_A \circ R_B \circ R_C \circ R_B = \mathrm{Id}$;
- $(R_{M_1} \circ R_{M_2} \circ R_{M_3})^2 = \mathrm{Id}$;
- $R_{M_1} \circ R_{M_2} \circ R_{M_3} \circ R_{M_4} = \mathrm{Id}$.

9) Let f be the isometry which takes the vector (ξ_1, ξ_2, ξ_3) onto

$(a_{11}\xi_1 + a_{12}\xi_2 + a_{13}\xi_3 + b_1, a_{21}\xi_1 + a_{22}\xi_2 + a_{23}\xi_3 + b_2, a_{31}\xi_1 + a_{32}\xi_2 + a_{33}\xi_3 + b_3)$.

Denote by B the matrix

$$\begin{bmatrix} a_{11}-1 & a_{12} & a_{13} \\ a_{21} & a_{22}-1 & a_{23} \\ a_{31} & a_{32} & a_{33}-1 \end{bmatrix}$$

and by C the matrix

$$\begin{bmatrix} & b_1 \\ B & b_2 \\ & b_3 \end{bmatrix}.$$

Show that f is

- the identity if and only if $r(B) = r(C) = 0$;
- a translation if and only if $r(B) = 0, r(C) = 1$;
- a reflection if and only if $r(B) = 1, r(C) = 1$;
- a glide reflection if and only if $r(B) = 1, r(C) = 2$;
- a rotation if and only if $r(B) = 2, r(C) = 3$;
- a screw displacement if and only if $r(B) = 2, r(C) = 3$;
- a rotary reflection if and only if $r(B) = 3, r(C) = 3$.

3 VECTOR SPACES

3.1 The axiomatic definition, linear dependence and linear combinations

In chapter II we considered the spaces $\mathbf{R}^2$ and $\mathbf{R}^3$ as models for the two and three dimensional spaces of our everyday experience and showed how we could express the concepts of geometry in terms of their various structures. We now go on to higher dimensional spaces. In the spirit of the previous chapter we could simply consider the space $\mathbf{R}^n$ of n-tuples of real numbers, provided with the obvious natural operations of addition and scalar multiplication. However, we prefer now to take a more abstract point of view and use the axiomatic approach whereby we shall introduce vector spaces as sets with structures which satisfy a variety of properties based on those of $\mathbf{R}^2$ and $\mathbf{R}^3$.

We shall see that by introducing suitable coordinate systems we can always reduce to the case of the space $\mathbf{R}^n$ (and we usually must do this if we want to calculate concrete examples.) In spite of this fact, the axiomatic or coordinate-free approach brings several important advantages, not the least of which are conciseness and elegance of notation and a flexibility in the choice of coordinate system which can lead to considerable simplification in concrete calculations.

We begin by introducing the concept of a vector space and develop the linear part of the programme of the preceding chapter i.e. that part which does not involve the concepts of length, angle and volume. The latter will be treated in a later chapter.

Definition: A **vector space** (over $\mathbf{R}$) (or simply a **real vector space**) is a set V together with an addition and a scalar multiplication i.e. mappings from $V \times V$ into V resp. $\mathbf{R} \times V$ into V written

$$(x, y) \mapsto x + y \text{ resp. } (\lambda, x) \mapsto \lambda x$$

so that

- $(x + y) + z = x + (y + z) \quad (x, y, z \in V)$;
- $x + y = y + x \quad (x, y \in V)$;
- there exists a zero element i.e. $0 \in V$ so that $x + 0 = 0 + x = x$ for $x \in V$;
- for each $x \in V$ there is an element $z \in V$ so that $x + z = z + x = 0$;

- $(\lambda + \mu)x = \lambda x + \mu x \quad (\lambda, \mu \in \mathbf{R}, x \in V)$;
- $\lambda(x + y) = \lambda x + \lambda y \quad (\lambda \in \mathbf{R}, x, y \in V)$;
- $\lambda(\mu x) = (\lambda\mu)x \quad (\lambda, \mu \in \mathbf{R}, x \in V)$;
- $1 \cdot x = x$.

We make some simple remarks on these axioms. Firstly, note that the zero element is unique. More precisely, if $y \in V$ is such that $x + y = x$ for *some* $x \in V$, then $y = 0$. For if $z \in V$ is such that $x + z = 0$ (such a z exists by (4)), then $z + (x + y) = z + x = 0$ and so $(z + x) + y = 0$ i.e. $0 + y = 0$. Hence $y = 0$.

Secondly, we have $0 \cdot x = 0$ for each $x \in V$. (Here we are guilty of an abuse of notation by using the symbol "0" for the zero element of $\mathbf{R}$ and of the vector space V). For $x = 1 \cdot x = (1 + 0)x = 1 \cdot x + 0 \cdot x = x + 0 \cdot x$ and so $0 \cdot x = 0$ by the above. Thirdly, the element $(-1) \cdot x$ has the property of z in (4) i.e. is such that $x + (-1) \cdot x = 0$. Hence we denote it by $-x$ and write $x - y$ instead of $x + (-1) \cdot y$. Note that $(-1) \cdot x$ is the *only* element with this property. For if $x + y = 0$, then

$$y = 0 + y = ((-x) + x) + y = (-x) + (x + y) = (-x) + 0 = -x.$$

Examples of vector spaces: I. We have already seen that $\mathbf{R}^2$ and $\mathbf{R}^3$ are vector spaces. In exactly the same way we can regard $\mathbf{R}^n$ (the set of n-tuples $(\xi_1, \dots, \xi_n)$ of real numbers) as a vector space by defining

$$x + y = (\xi_1 + \eta_1, \dots, \xi_n + \eta_n) \qquad \lambda x = (\lambda\xi_1, \dots, \lambda\xi_n)$$

where $x = (\xi_1, \dots, \xi_n)$ and $y = (\eta_1, \dots, \eta_n)$.
II. The set of all $n \times 1$ column vectors i.e. $M_{n,1}$ is a vector space with the operations defined in the first chapter.
III. More generally, the set $M_{m,n}$ of $m \times n$ matrices is a vector space.
IV. $\mathrm{Pol}\,(n)$, the set of all polynomials of degree at most n, is a vector space under the usual arithmetic operations on functions.
V. The space $C([0, 1])$ of continuous real-valued functions on $[0, 1]$ is also a vector space. We will be interested in subsets of vector spaces which are vector spaces in their own right. Of course, this means that they must be closed under the algebraic operations on V. This leads to the following definition: a subset V_1 of a vector space V is a **(vector) subspace** if whenever $x, y \in V_1, \lambda \in \mathbf{R}$, then $x + y \in V_1$ and $\lambda x \in V_1$ (a useful reformulation of this condition is as follows: whenever $x, y \in V_1, \lambda, \mu \in \mathbf{R}$, then $\lambda x + \mu y \in V_1$).

Examples: The following two subsets

$$\{(\xi_1, \xi_2, \xi_3) \in \mathbf{R}^3 : \xi_1 + \xi_2 + \xi_3 = 0\}$$

$$\{f \in C([0,1]) : f(0) = f(1)\}$$

are subspaces of $\mathbf{R}^3$ and $C([0,1])$ respectively whereas

$$\{(\xi_1, \xi_2, \xi_3) \in \mathbf{R}^3 : \xi_1 - \xi_2 = 5\}$$

$$\{f \in C([0,1]) : \int_0^1 f^2(x)\, dx = 1\}$$

are not. The notion of dimension played a crucial role in pinpointing the difference between the plane and space and we shall extend it to general vector spaces. It is a precise formulation of what is often loosely referred to under the name of the "number of degrees of freedom". We shall develop an algebraic characterisation of it, just as we did in $\mathbf{R}^2$ and $\mathbf{R}^3$.

Definition: A **linear combination** of the vectors $x_1, \dots, x_m$ from V is a vector of the form $\lambda_1 x_1 + \cdots + \lambda_m x_m$ for scalars $\lambda_1, \dots, \lambda_m$. The set of all such vectors is denoted by $[x_1, \dots, x_m]$ and is called the **space spanned** by $\{x_1, \dots, x_m\}$, a name which is justified by the following fact.

Proposition 23 *$[x_1, \dots, x_m]$ is a subspace of V – in fact it is the smallest subspace containing the vectors $x_1, \dots, x_m$.*

PROOF. If $x = \lambda_1 x_1 + \cdots + \lambda_m x_m$ and $y = \mu_1 x_1 + \cdots + \mu_m x_m$ are in $[x_1, \dots, x_m]$, $\lambda \in \mathbf{R}$, then

$$x + y = (\lambda_1 + \mu_1)x_1 + \cdots + (\lambda_m + \mu_m)x_m$$

and

$$\lambda x = (\lambda\lambda_1)x_1 + \cdots + (\lambda\lambda_m)x_m$$

are also in $[x_1, \dots, x_m]$ and so the latter is a subspace. On the other hand, if V is a subspace which contains all of the x_i, then it clearly contains each linear combination thereof and so contains $[x_1, \dots, x_m]$.

■

Examples: In Pol (5), $[1, t, t^2]$ is the set of polynomials of degree at most 2 (in other words, Pol (2)) and $[t, t^2, t^3, t^4, t^5]$ is the set of polynomials which vanish at zero.

Continuing with our definitions, we say that a set $\{x_1, \dots, x_m\}$ of elements is **linearly independent** if only the trivial linear combination $0 \cdot x_1 + \dots + 0 \cdot x_m$ vanishes (in symbols: $\lambda_1 x_1 + \dots + \lambda_m x_m = 0$ implies $\lambda_1 = \dots = \lambda_m = 0$). As one sees immediately, elements of a linearly independent set must be distinct and non-zero. Linear dependence (which, of course, is the negation of linear independence) can be characterised thus: the set $\{x_1, \dots, x_m\}$ is linearly dependent if and only if at least one of the x_i is a linear combination of the others (for if $\lambda_1 x_1 + \dots + \lambda_m x_m = 0$ with say $\lambda_i \neq 0$, then

$$x_i = -\frac{1}{\lambda_i}(\lambda_1 x_1 + \dots + \lambda_{i-1} x_{i-1} + \lambda_{i+1} x_{i+1} + \dots \lambda_m x_m).$$

We shall sometimes use this fact in the slightly sharper form that if $\{x_1, \dots, x_m\}$ is a linearly dependent set, then there is a smallest i so that x_i is a linear combination of $\{x_1, \dots, x_{i-1}\}$. To see this simply check the sets $\{x_1\}$, $\{x_1, x_2\}$, $\{x_1, x_2, x_3\}$,... for linear dependence and stop at the first one which is linearly dependent.

Examples: The set $\{1, t, t^2, \dots, t^n\}$ is linearly independent in $C([0, 1])$ for any n. This is just a restatement of the fact that if a polynomial vanishes on $[0, 1]$ (or, indeed, on any infinite set), then it is the zero polynomial. The set $\{(1, 0, \dots, 0), (0, 1, 0, \dots, 0), \dots, (0, \dots, 0, 1)\}$ is linearly independent in $\mathbf{R}^n$ as can be seen immediately. We have already encountered the concepts of linear combination resp. linear independence in the first two chapters (in the contexts of linear equations and geometry). The latter intuitive approach will be useful in giving later propositions geometrical content. In actual calculations, questions about the linear dependence or independence of concrete vectors can be restated in terms of the solvability of systems of linear equations and these can be resolved with the techniques of the first chapter as we shall see below.

3.2 Bases (Steinitz' Lemma)

We are now in a position to define the concept of a **basis** for a vector space. This is a sequence $(x_1, \dots, x_n)$ in V so that

- $[x_1, \dots, x_n] = V$ i.e. the set **spans** V;
- $\{x_1, \dots, x_n\}$ is linearly independent.

These two conditions mean precisely that every vector $x \in V$ has a *unique* representation of the form $\lambda_1 x_1 + \cdots + \lambda_n x_n$. For the first condition ensures the existence of such an expansion while the second one guarantees that it is unique.

The λ_i are then called the **coordinates** of x with respect to the basis.

Examples: I. The set $((1, 0, \dots, 0), (0, 1, 0, \dots), \dots, (0, \dots, 0, 1))$ is clearly a basis for $\mathbf{R}^n$. Owing to its importance it is called the **canonical basis** and its elements are denoted by $e_1, \dots, e_n$.
II. More generally, if $x_1, \dots, x_n$ are elements of $\mathbf{R}^n$ and we construct the matrix X whose columns are the vectors $x_1, \dots, x_n$ (regarded as column matrices), then the results of the first chapter can be interpreted as stating that $(x_1, \dots, x_n)$ is a basis for $\mathbf{R}^n$ if and only if X is invertible.
III. The functions $(1, t, \dots, t^n)$ form a basis for $\mathrm{Pol}\,(n)$.
IV. If we denote by E_{ij} the $m \times n$ matrix with a "1" in the (i, j)-th place and zeroes elsewhere, then the matrices $(E_{ij} : 1 \leq i \leq m, 1 \leq j \leq n)$ are a basis for $M_{m,n}$.

Despite these examples, there *do* exits vector spaces without bases, one example being $C([0, 1])$. For suppose that this space does have a basis, say $f_1, \dots, f_n$. We choose $m > n$ and express the polynomials $t^j \quad (j = 1, \dots, m)$ as linear combinations of the basis, say

$$t^j = \sum_{i=1}^{n} a_{ij} f_i \qquad (j = 1, \dots, m).$$

Consider now the equation $\sum_{j=1}^{m} \lambda_j t^j = 0$. This is equivalent to the system

$$\sum_{j=1}^{m} a_{ij} \lambda_j = 0 \qquad (i = 1, \dots, n).$$

Since this is a homogeneous system of n equations in m unknowns, it has a non-trivial solution—which contradicts the linear independence of $(1, t, \dots, t^m)$ in $C([0, 1])$.

Notice that we have actually proved that if a vector space V has an infinite subset, all of whose finite subsets are linearly independent, then V cannot have a basis. Such vector spaces are said to be **infinite dimensional**. Since we shall not be further concerned with such spaces we will tacitly assume that all spaces are finite dimensional i.e. have (finite) bases.

We shall now prove a number of results on bases which, while being intuitively rather obvious, are not completely trivial to demonstrate. The key to the proofs is the following Lemma which says roughly that a linearly independent set spans more space than a linearly dependent one with the same number of elements.

Proposition 24 *Let $(x_1, \dots, x_m)$ and $(y_1, \dots, y_n)$ be finite subsets of a vector space V. Suppose that $\{x_1, \dots, x_m\}$ is linearly independent and that $[x_1, \dots, x_m] = [y_1, \dots, y_n]$. Then $m \leq n$ and we can relabel $(y_1, \dots, y_n)$ as $(z_1, \dots, z_n)$ so that*

$$[y_1, \dots, y_n] = [z_1, \dots, z_n] = [x_1, \dots, x_m, z_{m+1}, \dots, z_n]$$

i.e. we can successively replace the elements of $\{z_1, \dots, z_n\}$ without affecting the linear span.

PROOF. It is convenient to prove the following slight extension of the above result: we show that if $1 \leq r \leq m$, there is a rearrangement $(z_1, \dots, z_n)$ (which depends on r) so that

$$[z_1, \dots, z_n] = [x_1, \dots, x_r, z_{r+1}, \dots, z_n].$$

We shall use induction on r.
The case $r = 1$. Since x_1 is in the linear span of the y_i's, we have a non-trivial representation

$$x_1 = \lambda_1 y_1 + \cdots + \lambda_n y_n.$$

Of course, the first coefficient λ_1 can vanish. However, by interchanging two of the y_i's if necessary, we can find a rearrangement $(z_1, \dots, z_n)$ so that

$$x_1 = \mu_1 z_1 + \cdots + \mu_n z_n$$

with $\mu_1 \neq 0$. Then

$$[z_1, \dots, z_n] = [x_1, z_2, \dots, z_n]$$

since x_1 is a linear combination of $\{x_1, z_2, \dots, z_n\}$.
The step from r to $r + 1$. There is a rearrangement $(z_1, \dots, z_n)$ so that

$$[x_1, \dots, x_r, z_{r+1}, \dots, z_n] = [z_1, \dots, z_n].$$

Now x_{r+1} has a representation of the form

$$\mu_1 x_1 + \cdots + \mu_r x_r + \mu_{r+1} z_{r+1} + \cdots + \mu_n z_n.$$

Once again, we can arrange for the coefficient μ_{r+1} to be non-zero (by re-ordering the z's if necessary). A similar argument now shows that

$$[x_1, \ldots, x_{r+1}, z_{r+2}, \ldots, z_n] = [z_1, \ldots, z_n].$$

■

Corollar 1 *If a vector space V has two bases $(x_1, \ldots, x_m)$ and $(y_1, \ldots, y_n)$, then $m = n$.*

For it follows immediately from the Lemma that $m \leq n$ and $n \leq m$.

Hence if a vector space has a basis, then the number of basis elements is independent of its choice. We call this number the **dimension** of V (written $\dim V$). Of course, the dimension of $\mathbf{R}^n$ is n and we have achieved our first aim of giving an abstract description of dimension in terms of the vector space structure.

Corollar 2 *If $(x_1, \ldots, x_r)$ is a linearly independent sequence in an n-dimensional vector space, then there is a basis of V of the form $(x_1, \ldots, x_n)$ (i.e. we can extend the sequence to a basis for V).*

PROOF. Let $(y_1, \ldots, y_n)$ be a basis for V. Applying the Lemma, we get a relabelling $(z_1, \ldots, z_n)$ of the y's so that $(x_1, \ldots, x_r, z_{r+1}, \ldots, z_n)$ spans V. This is the required basis.

■

Corollar 3 *If a vector space is spanned by n elements, then it is finite dimensional and $\dim V \leq n$.*

Corollar 4 *Let $x_1, \ldots, x_n$ be elements of a vector space. Then $(x_1, \ldots, x_n)$ is a basis if and only if any two of the following three conditions holds:*

- $n = \dim V$;
- $\{x_1, \ldots, x_n\}$ *is linearly independent;*
- $\{x_1, \ldots, x_n\}$ *spans* V.

In concrete situations, questions about linear dependence and independence can be settled with the techniques of the first chapter. Suppose that we have a sequence $(x_1, \dots, x_n)$ of vectors in an n-dimensional vector space V and wish to determine, for example, the dimension of their linear span $[x_1, \dots, x_n]$. We proceed as follows: via a basis we can identify them with row vectors $X_1, \dots, X_m$ in $\mathbf{R}^n$ and so define an $m \times n$ matrix

$$A = \left[\begin{array}{c} X_1 \\ \vdots \\ X_n \end{array} \right].$$

We now reduce A to hermitian form $\tilde{A}$ and the required dimension is number of non-vanishing rows of $\tilde{A}$ i.e. the rank of A.

This method provides the following geometric interpretation of Gaussian elimination: the elementary row operations on the rows of A correspond to the following operators on the vectors of $(x_1, \dots, x_n)$:

- exchanging 2 vectors;
- multiplication of x_i by a non-zero scalar λ;
- addition of λ times x_j to x_i.

It is clear that none of these operations affects the linear hull of the set of vectors. Thus the method of Gauß can be described as follows: a given sequence $(x_1, \dots, x_n)$ can be reduced to a linearly independent set (whereby we ignore vanishing vectors) with the same linear span by successive applications of the above three elementary operations.

Example: Show that the functions $(1+t, 1-t, 1+t+t^2)$ form a basis for $\mathrm{Pol}\,(2)$. What are the coordinates of the polynomial $\alpha_0 + \alpha_1 t + \alpha_2 t^2$ with respect to this basis?

Solution: First we check the linear independence. The condition

$$\lambda(1+t) + \mu(1-t) + \nu(1+t+t^2) = 0$$

is equivalent to the system

$$\lambda + \mu + \nu = 0 \qquad (102)$$

$$\lambda - \mu + \nu = 0 \qquad (103)$$

$$\nu = 0 \qquad (104)$$

which has only the trivial solution.

Similarly, the fact that this linear combination is equal to $\alpha_0 + \alpha_1 t + \alpha_2 t^2$ reduces to the system

$$\lambda + \mu + \nu = \alpha_0 \tag{105}$$

$$\lambda - \mu + \nu = \alpha_1 \tag{106}$$

$$\nu = \alpha_2 \tag{107}$$

with solution $\lambda = \frac{1}{2}(\alpha_0 + \alpha_1 - 2\alpha_2)$, $\mu = \frac{1}{2}(\alpha_0 - \alpha_1)$, $\nu = \alpha_2$.

3.3 Complementary subspaces

A useful technique in the theory of vector spaces is that of reducing dimension by splitting a space into two parts as follows:

Definition: Two subspaces V_1 and V_2 of a vector space V are said to be **complementary** if

- $V_1 \cap V_2 = \{0\}$;
- $V = V_1 + V_2$ where the latter denotes the subspace of V consisting of those vectors which can be written in the form $y + z$ where $y \in V_1$, $z \in V_2$.

Then if both of the above conditions hold, each $x \in V$ has a *unique* representation $y + z$ of the above type (for if $y + z = y_1 + z_1$ with $y, y_1 \in V_1$, $z, z_1 \in V_2$, then $y - y_1 = z_1 - z$ and so both sides belong to the intersection $V_1 \cap V_2$ and must vanish).

V is then said to be split into the two subspaces V_1 and V_2, a fact which we represent symbolically by the equation $V = V_1 \oplus V_2$.

If we have such a splitting, we can construct a basis for V as follows: take bases $(x_1, \dots, x_r)$ resp. $(y_1, \dots, y_s)$ for V_1 resp. V_2. Then the combined sequence $(x_1, \dots, x_r, y_1, \dots, y_s)$ is obviously a basis for $V_1 \oplus V_2$. The same idea shows how to construct subspaces which are complementary to a given subspace V_1. We simply choose a basis $(x_1, \dots, x_r)$ for V_1 and extend it to a basis $(x_1, \dots, x_n)$ for V. Then $V_2 = [x_{r+1}, \dots, x_n]$ is a complementary subspace. This shows that in general (i.e. except for the trivial case where $V_1 = V$), a subspace has many complementary spaces. For example, if V_1 is a plane in $\mathbf{R}^3$ (passing through 0), then any line through 0 which does not lie on V_1 is complementary. However, regardless of the choice of V_2, its dimension is always $n - r$ and so is determined by V_1. This is a special case of the following result which is useful in dimension counting arguments:

Proposition 25 *Let V_1 and V_2 be subspaces of a vector space V which together span V (i.e. are such that $V = V_1 + V_2$). Then*

$$dim\, V = dim\, V_1 + dim\, V_2 - dim\,(V_1 \cap V_2).$$

PROOF. Note first that $V_1 \cap V_2$ is a subspace of V. Suppose that it has dimension m and that $\dim V_1 = r$, $\dim V_2 = s$, $\dim V = n$. We begin with a basis $(x_1, \dots, x_m)$ for the intersection and extend it to basis

$$(x_1, \dots, x_m, x_{m+1}, \dots, x_r) \qquad \text{for } V_1$$

and

$$(x_1, \ldots, x_m, x'_{m+1}, \ldots, x'_s) \qquad \text{for } V_2.$$

We claim that $(x_1, \ldots, x_m, \ldots, x_r, x'_{m+1}, \ldots, x'_s)$ is a basis for V. This suffices to complete the proof since we then merely have to count the number of elements in this basis and set it equal to n.

First note that the set spans V since parts of it span V_1 and V_2 which in turn span V. Now we shall show that they are linearly independent. For suppose that

$$\lambda_1 x_1 + \cdots + \lambda_r x_r + \lambda_{m+1} x'_{m+1} + \cdots + \lambda_s x'_s = 0.$$

Then

$$\lambda_1 x_1 + \ldots \lambda_r x_r = -(\lambda_{m+1} x'_{m+1} + \cdots + \lambda_s x"_s).$$

Now both sides belong to the intersection and so have a representation of the form $\mu_1 x_1 + \cdots + \mu_m x_m$. Equating this to the right hand side of the last equation above, we get one of the form

$$\mu_1 x_1 + \cdots + \mu_m x_m + \lambda_{m+1} x'_{m+1} + \cdots + \lambda_s x'_s = 0$$

and so $\lambda_{m+1} = \cdots = \lambda_s = 0$. This means that the left hand side vanishes and so all of the λ are zero.

3.4 Isomorphisms, transfer matrices

Isomorphisms: We have just seen that every vector space V has a basis and this allows us to associate to each vector $x \in V$ a set of coordinates (λ_i). These behave exactly as the ordinary coordinates in $\mathbf{R}^n$ with respect to addition and scalar multiplication. Hence in a certain sense a vector space with a basis "is" $\mathbf{R}^n$. In order to make this concept precise, we use the following definition:

Definition: An **isomorphism** f between vector spaces V and V_1 is a bijection f from V onto V_1 which preserves the algebraic operations i.e. is such that

$$f(x+y) = f(x) + f(y) \qquad f(\lambda x) = \lambda f(x) \quad (x, y \in V, \lambda \in \mathbf{R}).$$

V and V_1 are then said to be **isomorphic.** For example, the following spaces are isomorphic: $\mathbf{R}^n$, $M_{n,1}$, $\mathrm{Pol}\,(n-1)$. Suitable isomorphism are the mappings

$$(\xi_1, \dots, \xi_n) \mapsto \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix} \mapsto \xi_1 + \xi_2 t + \dots + \xi_n t^{n-1}.$$

We note some simple properties of isomorphisms:

- the identity is an isomorphism and the composition $g \circ f$ of two isomorphism is an isomorphism as is the inverse f^{-1} of an isomorphism;
- isomorphisms respect bases i.e. if $f : V \to V_1$ is an isomorphism and $(x_1, \dots, x_n)$ is a basis for V, then $(f(x_1), \dots, f(x_n))$ is a basis for V_1. In particular, V and V_1 have the same dimension.

In the light of the above remarks and examples, the next result is rather natural:

Proposition 26 *Two vector spaces V and V_1 are isomorphic if and only if they have the same dimension.*

PROOF. Suppose that V and V_1 have dimension n and choose bases $(x_1, \dots, x_n)$ resp. $(y_1, \dots, y_n)$ resp. Then we define a mapping f from V into V_1 by setting

$$f(\lambda_1 x_1 + \dots + \lambda_n x_n) = \lambda_1 y_1 + \dots + \lambda_n y_n.$$

Then f is well-defined because of the uniqueness of the representation of x, injective since the (y_i) are linearly independent and surjective since they span V_1. It is clearly linear.

■

In particular, any n-dimensional vector space is isomorphic to $\mathbf{R}^n$, any choice of basis inducing an isomorphism. Owing to the arbitrariness of such a choice, we shall require formulae relating the coordinates with respect to two distinct bases. For example, in $\mathbf{R}^2$, the coordinates (λ_1, λ_2) of the vector (ξ_1, ξ_2) with respect to the basis $((1,1),(,1-1))$ are the solutions of the equations

$$\lambda_1 + \lambda_2 = \xi_1 \tag{108}$$

$$\lambda_1 - \lambda_2 = \xi_2 \tag{109}$$

i.e. $\lambda_1 = \frac{1}{2}(\xi_1 + \xi_2)$, $\lambda_2 = \frac{1}{2}(\xi_1 - \xi_2)$. Here we recognise the role of the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

whose columns are the coordinates of the new basis elements. In the general situation of bases $(x_1, \dots, x_n)$ resp. $(x_1', \dots, x_n')$ for V, we define the **transfer matrix** from (x_i) to (x_j') to be the $n \times n$ matrix $T = [t_{ij}]$ where the t_{ij} are defined by the equations $x_j' = \sum_{i=1}^n t_{ij} x_i$ i.e. the columns of T are formed by the coordinates of the x_j' with respect to (x_i). If $x \in V$ has the representation $\sum_{j=1}^n \lambda_j x_j'$ then, by substituting the above expression for x_j' we get

$$x = \sum_{j=1}^n \lambda_j' x_j' = \sum_{j=1}^n \lambda_j' (\sum_{i=1}^n t_{ij} x_i) \tag{110}$$

$$= \sum_{i=1}^n (\sum_{j=1}^n t_{ij} \lambda_j') x_i. \tag{111}$$

Thus we have proved the following result:

Proposition 27 *Let $T = [t_{ij}]$ be the transfer matrix from (x_i) to (x_j'). Then the following relationship holds between the coordinates (λ_j') and (λ_i) of $x \in V$ with respect to (x_j'):*

$$\lambda_i = \sum_{j=1}^n t_{ij} \lambda_j'$$

(i.e. the column matrix of the λ is obtained from that of the λ'-s by multiplying by the transfer matrix).

If we have to carry our more than one coordinate transformation in the course of a calculation, the following result is useful:

Proposition 28 *Let T (resp. T') be the transformation matrix from (x_i) to (x_j') resp. from (x_j') to (x_k''). Then the transfer matrix from (x_i) to (x_k'') is $T \cdot T'$. (Note the order of the factors).*

PROOF. By definition we have

$$x'_j = \sum_{i=1}^{n} t_{ij} x_i \qquad x''_k = \sum_{j=1} t'_{jk} x'_k.$$

Hence

$$x''_k = \sum_{j=1}^{n} t'_{jk} (\sum_{i=1}^{n} t_{ij} x_i) \tag{112}$$

$$= \sum_{i=1}^{n} (\sum_{j=1}^{n} t_{ij} t'_{jk}) x_i \tag{113}$$

i.e. $t''_{ik} = \sum_{j=1}^{n} t_{ij} t'_{jk}$ or $T'' = T \cdot T'$ where $T'' = [t''_{ik}]$ is the transfer matrix from (x_i) to (x''_k).

■

Corollar 5 *If T is the transfer matrix from $(x_1, \ldots, x_n)$ to $(x'_1, \ldots, x'_n)$, then its inverse T^{-1} is the transfer matrix from $(x'_1, \ldots, x'_n)$ to $(x_1, \ldots, x_n)$.*

3.5 Affine spaces

We conclude this chapter with some remarks on so-called affine spaces. The reason for introducing these is that the identification of say three dimensional space with $\mathbf{R}^3$ is based on two arbitrary elements. The first is the choice of a basis and we have developed methods to deal with this using the transfer matrix between two bases. The second is the choice of the origin or zero element. This lies rather deeper since it directly involves the algebraic structure of the spaces. This has already manifested itself in the rather inelegant separate treatment that we were forced to give to one (resp. two) dimensional subspaces and to lines resp. plane. The way out of this difficulty lies in the concept of an affine space which we now discuss very briefly.

Definition: An **affine space** is a set M and a mapping which associates to each pair (P, Q) in $M \times M$ a vector x_{PQ} in a given vector space V in such a way that the following conditions hold

- for each $P \in M$, the mapping $Q \mapsto x_{PQ}$ is a bijection from M onto V;
- for three points P, Q, R in M we have $x_{PQ} + x_{QR} = x_{PR}$.

It follows easily from the above conditions that
$x_{PP} = 0$ for each P;
$x_{PQ} = -x_{QP}$;
$x_{PQ} = x_{RS}$ implies that $x_{PR} = x_{QS}$.

Of course, there is a very close connection between vector spaces and affine spaces. Every vector space V is an affine space when we define $x_{PQ} = x_Q - x_P$ as we did for $\mathbf{R}^3$. On the other hand, if M is an affine space and we choose some point P_0 as a zero, then the mapping $Q \mapsto x_{P_0Q}$ is a bijection from M onto V. We can use it to transfer the vector structure of V to M by defining the sum of two points Q and Q_1 to be that point Q_2 for which

$$x_{P_0Q_2} = x_{P_0Q} + x_{P_0Q_1}$$

and λQ to be that point Q_3 for which $x_{P_0Q_3} = \lambda x_{P_0Q}$. (In other words, we can make an affine space into a vector space by arbitrarily promoting one particular point to become the zero).

A subset M_1 of M is an **affine subspace** if the set $\{x_{PQ} : P, Q \in M_1\}$ is a subspace of V. The dimension of this subspace is then defined to be the (affine) dimension. We shall give more intrinsic descriptions of these concepts shortly. In $\mathbf{R}^3$, the zero-dimensional subspaces are the points, the one-dimensional ones are the lines and the two-dimensional ones are planes.

Note that a subset M_1 of a vector space V (regarded as an affine space as above) is an affine subspace if and only if it is the translate $T_u(V_1)$ of some vector subspace V_1 i.e. the set $\{u + x : x \in V_1\}$ for some $u \in V$.

If $x_0, \dots, x_n$ are points in a vector space V then

$$M_1 = \{\lambda_0 x_0 + \cdots + \lambda_n x_n : \lambda_0 \dots, \lambda_n \in \mathbf{R}, \lambda_0 + \cdots + \lambda_n = 1\}$$

is an affine subspace. In fact, it is the smallest such subspace containing the points and hence is called the **affine subspace generated by them.** For three non-collinear points in space, this is the plane through these points. One can characterise the above space as $T_{x_0}([x_1 - x_0, \dots, x_n - x_0])$ as one sees immediately. From this representation, one sees that the following conditions are equivalent:

- $\dim M_1 = n$;
- the vectors $\{x_1 - x_0, \dots, x_n - x_0\}$ are linearly independent.

In this case, the vectors $\{x_0, \dots, x_n\}$ are said to be **affinely independent** (note that the apparent special role of x_0 is spurious as the equivalence to condition (1) shows). In this case the λ's in the representation

$$\lambda_0 x_0 + \cdots + \lambda_n x_n$$

(where $\lambda_0 + \cdots + \lambda_n = 1$) of a typical element in the affine subspace are uniquely determined. They are called its **barycentric coordinates** with respect to $x_0, \dots, x_n$. As an example, two points in $\mathbf{R}^3$ are affinely independent if they are distinct, three if they are not collinear. In these cases the barycentric coordinates coincide with those introduced in Chapter 2.

The signs of the barycentric coordinates determine the position of x with respect to the faces of the polyhedron with $x_0, \dots, x_n$ as vertices. Figure 1 illustrates the situation for three points in the plane.

Example: a) Is the triple $\{(1, 3, -1), (3, 0, 1), (1, -1, 1)\}$ linearly independent? b) Is

$$\begin{bmatrix} 3 & 1 \\ 1 & -1 \end{bmatrix}$$

linearly dependent on

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 2 \\ 0 & -1 \end{bmatrix}?$$

Solution: a) is equivalent to the following: does the system

$$\lambda(1,3,-1)+\mu(3,0,1)+\nu(1,-1,1)=0$$

i.e.

$$\begin{array}{rcrcrcr} \lambda & + & 3\mu & + & \nu & = & 0 \\ 3\lambda & & & - & \nu & = & 0 \\ -\lambda & + & \mu & + & \nu & = & 0 \end{array}$$

have a non-trivial solution? b) is equivalent to the following: does the system

$$\lambda\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}+\mu\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}+\nu\begin{bmatrix} 0 & 2 \\ 0 & -1 \end{bmatrix}=\begin{bmatrix} 3 & 1 \\ 1 & -1 \end{bmatrix}$$

i.e.

$$\begin{array}{rcrcrcr} \lambda & + & \mu & & & = & 3 \\ \lambda & + & \mu & + & 2\nu & = & 1 \\ \lambda & + & \mu & & & = & 1 \\ & & \mu & - & \nu & = & -1 \end{array}$$

have a solution?

We leave to the reader the task of answering these questions with the techniques of Chapter I.

Exercises: 1)

- Which of the following sets of vectors are linearly independent?

$$(1,2,-1),(2,0,1),(1,-1,1) \tag{114}$$

$$(1,-2,5,1),(3,2,1,-2),(1,6,-5,-4) \tag{115}$$

$$(1,\cos t,\sin t). \tag{116}$$

- For which α are the vectors $(\alpha,1,0),(1,\alpha,0),(0,1,\alpha)$ linearly independent?

- Show that the following sets form bases of the appropriate spaces and calculate the coordinates of the element in brackets with respect to this basis:

$$(1,1,\dots,1),(0,1,\dots,1),\dots,(0,\dots,0,1) \text{ in } \mathbf{R}^n \ ((\xi_1,\dots,\xi_n)) \tag{117}$$

$$1,(t-1),\dots,(t-1)^n \text{ in } \mathrm{Pol}\,(n) \ (a_0+a_1t+\dots a_nt^n)). \tag{118}$$

- Calculate the transfer matrix between the following bases:

$$(x_1,x_2,x_3) \text{ and } (x_1+x_2,x_2+x_3,x_3+x_1); \tag{119}$$

$$(1,t,t^2,\dots,t^n) \text{ and } (1,(t-a),\dots,(t-a)^n) \text{ in } \mathrm{Pol}\,(n). \tag{120}$$

- Which of the following subsets of the appropriate vector spaces are subspaces?

$$\{(\xi_1, \xi_2, \xi_3) \in \mathbf{R}^3 : \xi_2 = 0\} \tag{121}$$
$$\{(\xi_1, \xi_2, \xi_3) \in \mathbf{R}^3 : \xi_2 > 0\} \tag{122}$$
$$\{(\xi_1, \xi_2, \xi_3) \in \mathbf{R}^3 : \xi_1^2 + \xi_2^2 + \xi_3^2 = 0\} \tag{123}$$
$$\{p \in \mathrm{Pol}\,(n) : p(3) = 0\} \tag{124}$$
$$\{p \in \mathrm{Pol}\,(n) : p \text{ contains only even powers of } t\} \tag{125}$$
$$\{A \in M_2 : a_{ij} = a_{ji} \text{ for each } i, j\}. \tag{126}$$

- Calculate the dimensions of the following spaces:

$$[(1, -2, 1), (-1, 1, 3), (2, 4, 6)] \tag{127}$$
$$\{(\xi_1, \xi_2, \xi_3) \in \mathbf{R}^3 : \begin{bmatrix} 1 & 0 & 2 \\ 3 & 1 & -1 \\ 4 & 1 & 1 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} = 0\} \tag{128}$$
$$\{A \in M_n : A = A^t\}. \tag{129}$$

- Show that the following vector space decompositions hold:

$$\mathbf{R}^4 = [(1, 0, 1, 0), (1, 1, 0, 0, 0] \oplus [(0, 1, 0, 1), (0, 0, 1, 1)] \tag{130}$$
$$\mathbf{R}^{2n} = \{(\xi_1, \dots, \xi_n, 0, \dots, 0) : (\xi_1, \dots, \xi_n) \in \mathbf{R}^n\} \tag{131}$$
$$\oplus \{(\xi_1, \dots, \xi_n, \xi_1, \dots, \xi_n) : (\xi_1, \dots, \xi_n) \in \mathbf{R}^n\}. \tag{132}$$

- What are the dimensions of the following subspaces of M_n?

$$\{A : A \text{ is diagonal}\} \tag{133}$$
$$\{A : A \text{ is upper triangular}\} \tag{134}$$
$$\{A : \mathrm{tr}\, A = 0\}. \tag{135}$$

2) Show that if V_1 and V_2 are subspaces of V with $V_1 \cup V_2 = V$, then $V_1 = V$ or $V_2 = V$.
3) Let $(x_1, \dots, x_m)$ be a sequence in V which spans V. Show that it is a basis if there *one* vector x whose representation is unique. 4) Let V_1 be a subspace of V with $\dim V_1 = \dim V$. Then $V = V_1$.
5) Show that if V, V_1 and V_2 are subspaces of a given vector space, then

$$V \cap V_1 + V \cap V_2 \subset (V_1 + V_2) \cap V \tag{136}$$
$$V \cap (V_1 + (V \cap V_2)) = (V \cap V_1) + (V \cap V_1). \tag{137}$$

Show by way of a counterexample that equality does not hold in the first equation in general.
6) Let V_1 be a subset of the vector space V. Show that V_1 is an affine subspace of and only if the following condition holds: if $x, y \in V_1$, $\lambda \in \mathbf{R}$, then $\lambda x + (1 - \lambda)y \in V_1$.
7) Show that if V_1 and V_2 are subspaces of a given vector space V, then so are $V_1 \cap V_2$ and $V_1 + V_2$ where

$$V_1 + V_2 = \{x + y : x \in V_1, y \in V_2\}.$$

8) Show that the solution space of a system $AX = Y$ is an affine subspace of $\mathbf{R}^n$ and that every affine subspaces can be represented in this form. 9) Show that if V_1 and V_2 are k-dimensional subspaces of the n dimensional vector space V, then they have simultaneous complements i.e. there is a subspace W of dimension $n - k$ so that

$$V_1 \oplus W = V_2 \oplus W = V.$$

4 LINEAR MAPPINGS

4.1 Definitions and examples

In this chapter we introduce the concept of a linear mapping between vector spaces. We shall see that they are coordinate free versions of matrices, a fact which will be useful for two reasons. Firstly, the machinery developed in chapter I will provide us with a method for computing with linear mappings and secondly this interpretation will provide a useful conceptual framework for our treatment of some more advanced topics in matrix theory. Note that a finite dimensional linear mappings play a central role in mathematics for the very reason that, while they are simple enough to allow a fairly thorough analysis of their properties, they are general enough to have applications in many branches of mathematics. For while most phenomena which occur in nature are described by mappings which are neither finite dimensional nor linear, the standard approach is to approximate such mappings by linear ones (this is the main task of differential calculus) and then these linear ones by finite dimensional operators (this is an important theme of functional analysis). The solution of the corresponding simplified problem is an important first step in solving the original one (which can be regarded as a perturbed version of the former).

In fact, we have already met several examples of linear mappings. In chapter 1 we saw that an $m \times n$ matrix A defines a linear mapping between spaces of column vectors and in the second chapter we studied linear mappings in $\mathbf{R}^2$ and $\mathbf{R}^3$, in particular isometries.

Definition: A mapping $f : V \to V_1$ between vector spaces is **linear** if it satisfies the following conditions:

- $f(x + y) = f(x) + f(y) \qquad (x, y \in V)$;
- $f(\lambda x) = \lambda f(x) \qquad (\lambda \in \mathbf{R}, x \in V)$.

A simple induction argument shows then that

$$f(\lambda_1 x_1 + \cdots + \lambda_n x_n) = \lambda_1 f(x_1) + \cdots + \lambda_n f(x_n)$$

for any n-tuple $x_1, \dots, x_n$ in V and scalars $\lambda_1, \dots, \lambda_n$.
Examples of such mappings are
I. Evaluation resp. integration of continuous mappings e.g. the mappings

$$f \mapsto f(0) \tag{138}$$

$$f \mapsto \int_0^1 f(t)\,dt \tag{139}$$

on $C([0,1])$. These can also be regarded as mappings on the spaces $\mathrm{Pol}\,(n)$ of polynomials.
II. Formal differentiation and integration of polynomials i.e. the mappings

$$a_0 + a_1 t + \cdots + a_n t^n \mapsto a_1 + 2a_2 + \cdots + n a_n t^{n-1} \tag{140}$$

$$a_0 + a_1 t + \cdots + a_n t^n \mapsto a_0 t + \frac{1}{2} t^2 + \cdots + \frac{1}{n+1} a_n t^{n+1} \tag{141}$$

from $\mathrm{Pol}\,(n)$ into $\mathrm{Pol}\,(n+1)$ resp. $\mathrm{Pol}\,(n)$ into $\mathrm{Pol}\,(n+1)$. (We use the adjective formal to indicate that we are using a purely algebraic definition of these operations—in contrast to the general definition, no limiting processes are required in this context).

An important property of linear mappings is the fact that in order to specify them it suffices to do so on the elements of some bases (x_i). For then the value of f at $x = \lambda_1 x_1 + \cdots + \lambda_n x_n$ is automatically $\lambda_1 f(x_1) + \cdots + \lambda_n f(x_n)$.

Before studying linear mappings in more detail, we note the fact that if we denote by $L(V, V_1)$ the set of linear mapping from V into V_1, then this space has a natural linear structure if we define the algebraic operations in the obvious way i.e.

$$(f_1 + f_2)(x) = f_1(x) + f_2(x) \tag{142}$$

$$(\lambda f)(x) = \lambda f(x). \tag{143}$$

We shall see later that the dimension of this space is the product of the dimensions of V and V_1. In addition, if $f : V \to V_1$ and $g : V_1 \to V_2$ are linear, then so is the composition $g \circ f : V \to V_2$.

4.2 Linear mappings and matrices

Just as in the case of mappings on $\mathbf{R}^2$ and $\mathbf{R}^3$, matrices in higher dimensions generate linear mappings as follows: if $A = [a_{ij}]$ is an $m \times n$ matrix, we denote by f_A the linear mapping

$$(\xi_1, \dots, \xi_n) \mapsto (\sum_{j=1}^{n} a_{ij}\xi_j)_{i=1}^{m}$$

from $\mathbf{R}^n$ into $\mathbf{R}^m$ (this coincides with the f_A introduced in the first chapter up to the identification of the set of column vectors with $\mathbf{R}^n$. Hence we already know that f_A is linear). Note that the columns $A_1, \dots, A_n$ of A are precisely the images of the basis elements $e_1, \dots, e_n$ of $\mathbf{R}^n$ (regarded as column vectors).

In fact, using coordinate systems, any linear mapping can be represented by a matrix, as we now show. First we suppose that f maps $\mathbf{R}^n$ into $\mathbf{R}^m$ and we define an $m \times n$ matrix as above i.e.

$$A = [a_{ij}] \quad \text{where} \quad f(e_j) = \sum_{i=1}^{m} a_{ij} e_i.$$

Then f coincides with f_A since both agree on the elements of the canonical basis). Of course, this construction does not depend on the particular form of the bases and so we can make the following definition:

Definition: Let V and V_1 be vector spaces with bases $(x_1, \dots, x_n)$ resp. $(y_1, \dots, y_m)$. If $f : V \to V_1$ is linear, we define its matrix A with respect to (x_i) and (y_j) to be the $m \times n$ matrix $[a_{ij}]$ where the a_{ij} are defined by the equations

$$f(x_j) = \sum_{i=1}^{m} a_{ij} y_i.$$

Of course the matrix A depends on the choice of bases and later we shall examine in detail the effect that a change of basis has on the representing matrix.

We compute the matrices of some simple operators:

I. The matrix of the identity mapping on vector space with respect to any basis is the unit matrix (of course, we are using the same basis in the range and domain space).

$(x_1, \dots, x_n)$ and $(x'_1, \dots, x'_n)$ are bases for V, then the matrix of the identity with respect to (x'_j) and (x_i) is just the transfer matrix T from (x_i) to (x'_j)

as we see by comparing the above definition with the one of the previous chapter.

III. The matrix of the differentiation operator from Pol (n) to Pol $(n-1)$ with respect to the natural basis is the $n \times (n+1)$-matrix

$$\begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 2 & \dots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \dots & n \end{bmatrix}.$$

IV. The matrix of the integral operator from Pol (n) into Pol $(n+1)$ is the $(n+2) \times (n+1)$ matrix

$$\begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ 0 & \frac{1}{2} & 0 & \dots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \dots & \frac{1}{n+1} \end{bmatrix}.$$

As the following result shows, the algebraic operations for matrices correspond to those for linear mappings:

Proposition 29 *Let f and f_1 be linear mappings from V into V_1 resp. g a linear mapping from V_1 into V_2 where V resp. V_1 resp. V_2 have bases $(x_1, \dots, x_p)$ resp. $(y_1, \dots, y_n)$ resp. $(z_1, \dots, z_m)$. Denote the corresponding matrices by A^f, A^{f_1} etc. Then*

$$A^{\lambda f} = \lambda A^f \quad A^{f+f_1} = A^f + A^{f_1} \quad A^{g \circ f} = A^g A^f.$$

PROOF. We prove only the last part. Suppose that $A^f = [a_{jk}]$ and $A^g = [b_{ij}]$ i.e.

$$f(x_k) = \sum_{j=1}^{n} a_{jk} y_j \quad g(y_j) = \sum_{i=1}^{m} b_{ij} z_i.$$

Then

$$(g \circ f)(x_k) = g(\sum_{j=1}^{m} a_{jk} y_j) \tag{144}$$

$$= \sum a_{jk} g(y_j) \tag{145}$$

$$= \sum_{j=1}^{n} a_{jk} (\sum_{i=1}^{m} b_{ij} z_i) \tag{146}$$

$$= \sum_{i=1}^{m} (\sum_{j=1}^{n} b_{ij} a_{jk}) z_i \tag{147}$$

i.e. $A^{g\circ f} = [\sum_{j=1}^n b_{ij}a_{jk}]_{i,k} = A^g A^f$.

■

Of course, this provides further justification for our choice of the definition of matrix multiplication.

If we apply this result to the case of an operator $f : V \to V_1$ where V has bases $(x_1, \dots, x_n)$ and $(x'_1, \dots, x'_n)$ resp. V_1 has bases $(y_1, \dots, y_m)$ and $(y'_1, \dots, y'_m)$, then we get the following result:

Proposition 30 *Let f have matrix A with respect to $(x_1, \dots, x_n)$ and $(y_1, \dots, y_m)$ resp. A' with respect to $(x'_1, \dots, x'_n)$ and $(y'_1, \dots, y'_m)$. Then we have the relation:*

$$A' = S^{-1}AT$$

(where S and T are the transfer matrices between the y-bases and the x-bases respectively).

PROOF. We use the fact noted above that T is the matrix of the identity operator on V with respect to the basis (x'_i) into V with the basis (x_i). We now express f as the product

$$\mathrm{Id}_{V_1} \circ f \circ \mathrm{Id}_V$$

and read off the corresponding matrices.

■

Of course, in the case where we are deal with an operator f on a space V and the x- and y-bases coincide, then the appropriate formula for a change in coordinates is

$$A' = T^{-1}AT.$$

These facts are the motivation for the following definitions (the first one of which we have already encountered in chapter I):

Definition: Two $m \times n$ matrices A and A' are **equivalent** if there exist invertible matrices S and T where S is $m \times m$, T is $n \times n$ and $A' = S^{-1}AT$. This just means that A and A' are the matrices of the same linear operator with respect to different bases.

Two $n \times n$ matrices A and A' are **similar** if there is an invertible $n \times n$ so that $A' = S^{-1}AS$. This has a similar interpretation in terms of representations of operators, now with $f \in L(V)$ represented by a single basis.

One of the main tasks of linear algebra is to choose a basis (resp. bases) so that the matrix representing a given operator f has a particularly form (which of the two concepts of equivalence above is appropriate depends on the nature of the problem). For example, the result on matrix equivalence in I.5 can be formulated as follows: every $m \times n$ matrix A is equivalent to one of the form

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

where $r = r(A)$. As a result about linear mappings this says that if $f \in L(V, V_1)$, then there are bases (x_j) resp. (y_i) for V resp. V_1 so that the matrix of f with respect to these bases has the above form (where $r = \dim f(V)$).

It is instructive to give a coordinate free proof of this result. First we introduce some notation. We consider the subspaces

$$\mathrm{Ker} f = \{x \in V; f(x) = 0\} \quad \mathrm{Im} f = f(V).$$

Note that if f is the mapping f_A defined by a matrix A then $\mathrm{Ker} f$ is the set of solutions of the homogeneous equation $AX = 0$ and $f(V)$ is the set of those Y for which the equation $AX = Y$ is solvable. In particular, the rank of A is the dimension of $\mathrm{Im} f$. Hence we define the **rank** $r(f)$ of a general linear mapping f to be the dimension of the range $\mathrm{Im} f$ of f.

Before continuing, we note some simple facts:

- $r(f) \leq \dim V$ and $r(f) \leq \dim V_1$ (for if $(x_1, \dots, x_n)$ is a basis for V, then the elements $((f(x_1), \dots, f(x_n))$ span $\mathrm{Im} f$. Hence $r(f) \leq \dim V$. It is trivial that $r(f) \leq \dim (V_1)$.

- if $f : V \to V_1$, $g : V_1 \to V_2$, then

  $$r(g \circ f) \leq r(g) \quad \text{and} \quad r(g \circ f) \leq f r(f).$$

  This follows immediately from (1) since

  $$\dim g(f(V)) \leq \dim f(V) \quad \text{and} \quad \dim g(f(V)) \leq \dim g(V_1).$$

Note that these correspond to the inequalities $r(AB) \leq r(A)$ and $r(AB) \leq r(B)$ for matrices which we have already proved.

We now turn to the proof of the representation of a mapping f by a matrix of the form

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}.$$

First we choose a basis for $\mathrm{Ker} f$ which, for reasons which will soon be apparent, we number backwards as $(x_{r+1}, \dots, x_n)$. We extend this to a basis $(x_1, \dots, x_r, x_{r+1}, \dots, x_n)$ for V. Let $V_1 = [x_1, \dots, x_r]$ so that $\mathrm{Ker} f$ and V_1 are complementary. Then we claim that $f|_{V_1}$ is an isometry from V_1 onto $\mathrm{Im} f$.

Proof. It is clear that the mapping is surjective. Suppose that $x, y \in V_1$ are such that $f(x) = f(y)$. Then $x - y$ is both in V_1 and V_2 and so vanishes since they are complementary. Hence f is injective on V_1.

■

It then follows that if $y_1 = f(x_1), \dots, y_r = f(x_r)$, then $(y_1, \dots, y_r)$ is a basis for $\mathrm{Im} f$. We extend it to a basis $(y_1, \dots, y_m)$ for W. Then we see that the matrix of f with respect to these bases is of the required form.

From this analysis it is clear that r is just the rank $r(f)$ of f and that

- $\dim(\mathrm{Ker} f) = n - r$;
- if $\dim V = \dim W$, then the properties of being surjective, injective or an isomorphism for f are equivalent.

Also we have the following splittings:

$$V = V_1 \oplus V_2 \quad W = W_1 \oplus W_2$$

where $V_2 = \mathrm{Ker}\ \mathrm{f}, W_1 = f(V)$, and $f|_{V_1}$ is an isomorphism onto W_1 resp. $f|_{V_2}$ vanishes.

We remark that it is often useful to be able to describe the null-space of a linear mapping f induced by a matrix A explicitly e.g. by specifying a basis. This can be done with the aid of Gaußian elimination as follows: consider the transposed matrix A^t and suppose that left multiplication by the invertible $n \times n$ matrix B reduces A^t to Hermitian form $\tilde{A^t}$. Then the last $(n - r)$ rows of B are a basis for $\mathrm{Ker} f$. This is clear from the equation

$$BA^t = \tilde{A^t} \quad \text{i.e.} \quad AB^t = \tilde{A}$$

which shows that the last $(n - r)$ columns of B^t are annihilated by f. In other words, they are in the kernel. Of course they are linearly independent (as row of the invertible matrix B) and $(n-r)$ is the dimension of the latter.

If A has block form

$$\begin{bmatrix} A_1 & C \end{bmatrix}$$

where A_1 is a square invertible matrix, then this can be simplified by noting that the columns of the matrix

$$\begin{bmatrix} -A_1^{-1}C \\ I \end{bmatrix}$$

form a basis for the kernel of A.

If we require a basis for the range of an $m \times n$ matrix A then we can simply take those of its columns which correspond to the pivot elements in the hermitian form.

4.3 Projections and splittings

Projections: An important class of linear mappings are the so-called projections which are defined as follows: let V_1 be a subspace of V with complementary subspace V_2. Thus every vector x in V has a unique representation of the form $y + z$ ($y \in V_1$, $z \in V_2$) and we call y the **projection of x onto V_1 along V_2**. This defines a mapping $P : x \mapsto y$ form V into V_1 which is clearly linear. Any mapping of this form is called a **projection**.

The following facts follow immediately from the definition:

- $P^2 = P$;
- if we construct a basis (x_i) for V by combining bases $(x_1, \dots, x_r)$ for V_1 and $(x_{r+1}, \dots, x_n)$ for V_2, then the matrix of P is
$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix};$$
- $\mathrm{Id} - P$ is the projection of V onto V_2 along V_1.

In fact, either of the first two properties above characterises projections as the following result shows:

Proposition 31 *If f is a linear operator on the space V then the following are equivalent:*

- *f is a projection;*
- *f is idempotent i.e. $f^2 = f$;*
- *there is a basis (x_i) for V with respect to which the matrix of f has the form*
$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}.$$

Then $r = r(f)$ is the dimension of the range of f.

Proof. We have already seen that (1) implies (2) and (3). It is clear that (3) implies (2). Hence it remains only to show that (2) implies (1). To do this we define V_1 and V_2 to be $\operatorname{Im} f$ and $\operatorname{Im}(\mathrm{Id} - f)$ respectively. Then if $x \in V$ it has a representation $f(x) + (x - f(x)$ as the sum of an element in V_1 and one in V_2. We shall show that $V_1 \cap V_2 = \{0\}$ and so that V is the direct sum of V_1 and V_2. It is then clear that f is the projection onto V_1, the image of f along V_2 (the image of $\mathrm{Id} - f$ and, simultaneously, the kernel of f).

Suppose then that $x \in V_1 \cap V_2$. Then $x = f(y)$ for some $y \in V$ and $x = z - f(z)$ for some $z \in V$. Then

$$f(x) = f(z) - f^2(z) = 0$$

since f is idempotent and

$$x = f(y) = f^2(y) = f(x) = 0$$

i.e. $x = 0$.

■

The reader should compare the third of the above conditions with the result that *every* linear mapping between vector spaces has a matrix of this form with respect to some choice of bases. Of course, the above condition is much more restrictive since we are taking the matrix of the operator with respect to a *single* basis.

The definition of the splitting of a vector space into a direct sum of two vector subspaces can be generalised to decomposition

$$V + V_1 \oplus V_2 \oplus \cdots \oplus V_r$$

of V into such sums of more than two subspaces. This is the case, by definition, if each $x \in V$ has a unique representation of the form $x_1 + \cdots + x_r$ where $x_i \in V_i$. Then we can define operators P_i where $P_i(x) = x_i$ $(i = 1, \dots, x_r)$. Each P_i is a projection and they satisfy the conditions

- $P_1 + \cdots + P_r = \mathrm{Id}$;
- $P_i P_j = 0 (i \neq j)$.

Such a sequence of projections is called a **partition of unity** for V. They correspond to decompositions of a basis $(x_1, \dots, x_n)$ for V into r subbases:

$$\xi_1, \dots, x_{i_1} \quad x_{i_1+1}, \dots, x_{i_2} \quad \dots \quad x_{i_{r-1}+1}, \dots, x_n.$$

Decompositions of the underlying space induce decompositions of linear mappings which correspond to the block representations of matrices discussed in chapter I. For simplicity, we consider the following situation: V has a decomposition $V = V_1 \oplus V_2$, $W = W_1 \oplus W_2$ and f is a linear mapping from V into W. We denote by P_1, P_2, Q_1 and Q_2 the projections from V onto V_1 and V_2 resp. from W onto W_1 and W_2 and define mappings $f_{11} : V_1 \to W_1$ etc. where

$$f_{11}x = Q_1 f(x) \quad (x \in V_1); \tag{148}$$
$$f_{12}(x) = Q_1 f(x) \quad (x \in V_2); \tag{149}$$
$$f_{21}(x) = Q_2 f(x) \quad (x \in V_1); \tag{150}$$
$$f_{22}(x) = Q_2 f(x) \quad (x \in V_2). \tag{151}$$

Then if $x = x_1 + x_2$ $(x_1 \in V_1, x_2 \in V_2)$ we see that

$$f(x) = f(x_1) + f(x_2) \tag{152}$$
$$= (f_{11}(x_1) + f_{12}(x_2)) + (f_{21}(x_1) + f_{22}(x_2)) \tag{153}$$

a fact which can be written symbolically in the form

$$\begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ xi_2 \end{bmatrix} = \begin{bmatrix} f_{11}(x_1) + f_{21}(x_2) \\ f_{21}(x_1) + f_{22}(x_2) \end{bmatrix}.$$

If we choose bases (x_i) and (y_j) for V and W so that $(x_1, \dots, x_s)$ is a basis for V_1, $(x_{s+1}, \dots, x_n)$ is a basis for V_2, $(y_1, \dots, y_r)$ is a basis for W_1 and $(y_{r+1}, \dots, y_m)$ is a basis for W_2, then the matrix of f has the block form

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

where A_{ij} is the matrix of f_{ij}.

We have assembled this apparatus in order to be able to reduce the dimensions in particular computations. This is especially useful in the following situations:

- where $f(V_1) \subset W_1$. Then $f_{21} = 0$ and the block representation has the form
$$\begin{bmatrix} A_{11} & A_{21} \\ 0 & A_{22} \end{bmatrix};$$
- where $f(V_1) \subset W_1$ and $f(V_2) \subset W_2$. Then the block representation has the form
$$\begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix}.$$

In most applications $V = W$, $V_1 = W_1$ and $V_2 = W_2$. In this case the first condition reduces to $f(V_1) \subset V_1$ in which case we say that V_1 is f**-invariant.** In the second case we have $f(V_1) \subset V_1$ and $f(V_2) \subset V_2$ and we say that the splitting $V = V_1 \oplus V_2$ **reduces** f.

In this latter case we can reduce questions about f to ones about f_{11} and f_{22} which generally brings about great advantages in computations.

4.4 Generalised inverses of linear mappings

As we have seen, properties of linear mappings are mirrored in those of matrices and *vice versa*. At this point we shall give a coordinate-free approach to the topic of generalised inverses (cf. I.7). Suppose that $f : V \to W$ is a linear operator. We are seeking a linear operator $g : W \to V$ so that *if* the equation $f(x) = y$ has a solution, then $x = g(x)$ is such a solution. We can do this very simply by using the splittings

$$V = V_1 \oplus V_2 \qquad W = W_1 \oplus W_2$$

constructed in the previous paragraph. Recall that V_2 is the kernel of f and W_1 is its range. V_1 and W_2 are arbitrary complementary subspaces. We have seen that the restriction $\tilde{f}$ of f to V_1 is an isomorphism from V_1 onto W_1. We denote its inverse by $\tilde{g}$ and define g to be $P \circ \tilde{g}$ where P is the projection of W onto W_1 along W_2. Then it is easy to check that g is a generalised inverse for f i.e. that the equations

$$g \circ f \circ g = g \qquad f \circ g \circ f = f$$

are satisfied. Note also that $g \circ f$ is the projection onto V_1 along V_2 while $f \circ g$ is the projection onto W_1 along W_2.

The following two special cases are worth noting separately:

- f is injective. Then $\operatorname{Ker} f = V_2 = \{0\}$ and so $V = V_1$. In this case, it is easy to see that $g = \tilde{g} \circ P$ is a left inverse for f i.e. satisfies the equation $g \circ f = \mathrm{Id}$. On the other hand, the injectivity of f is clearly necessary for f to possess a left inverse (for if f is not injective, then neither is any composition $g \circ f$).

- f is surjective. Then $W_2 = \{0\}$. In this case we can argue in a similar fashion to show that f has a right inverse.

In terms of matrices, these results are restatement of the facts which were discussed in the first chapter.

4.5 Norms and convergence in vector spaces

The theme of convergence of vectors is not properly a part of linear algebra. However, for large systems of equations, the algebraic methods of solution described in the first chapter are often hopelessly impractical and can be replaced by iteration methods which provide approximate solutions with much less effort. The theoretical basis for such methods lies in some simple properties of vector convergence which we now develop.

Definition: A sequence (x_n) in $\mathbf{R}^k$ converges to x if it converges coordinatewise i.e. if $\xi_r^n \to \xi_r$ for each r where $x_n = (\xi_r^n)_{r=1}^k$ and $x = (\xi_n)$.

For practical purposes, it is convenient to specify convergence quantitatively by means of a so-called norm which is defined as follows:

Definition: A **norm** on a vector space V is a mapping $\| \; \| : V \to \mathbf{R}_+$ so that

- $\|x + y\| \leq \|x\| + \|y\| (x, y \in V)$;
- $\|\lambda x\| = |\lambda| \|x\| (\lambda \in \mathbf{R}, x \in V)$;
- $\|x\| = 0$ implies that $x = 0$.

The three most commonly used examples of a norm on $\mathbf{R}^n$ are as follows: the ℓ_1-norm: $\|x\|_1 = \sum_i |\xi_i|$; the ℓ^2-norm: $\|x\|_2 = \sqrt{\sum_i |\xi_i|^2}$; the maximum norm: $\|x\|_\infty = \max_i |\xi_i|$.

Of course, the ℓ^2-norm is exactly the usual euclidean length of the vector in $\mathbf{R}^n$. All three of these norms induce the convergence described above in the sense that the sequence (x_n) converges to x if and only if the sequence of norms $\|x_n - x\|$ converges to zero. In fact, it is a general result that *all* norms on $\mathbf{R}^n$ induce this convergence.

Matrix convergence: In an analogous way, we can define a notion of convergence for matrices. A sequence (A_k) of $m \times n$ matrix, where $A_k = [a_{ij}^k]$, converges to A if $a_{ij}^k \to a_{ij}$ for each i and j. It follows easily from this definition that if $A_k \to A$ and $B_k \to B$, then $A_k + B_k \to A + B$ and $A_k B_k \to AB$.

In practice, convergence of matrices is described by norms of the following type. V and V_1 are vector spaces with norms $\| \; \|$ and $\| \; \|_1$ respectively. Then we define a norm on $L(V, V_1)$ as follows:

$$\|f\| = \sup\{\|f(x)\|_1 : x \in V, \|x\| \leq 1\}.$$

This clearly a norm and we have the estimate

$$\|f(x)\| \leq \|f\|\|x\| \quad (x \in V).$$

Note that if $f \in L(V, V_1)$ and $g \in L(V_1, V_2)$ then the norm of the composition $g \circ f$ is at most the product $\|g\|\|f\|$. For

$$\begin{aligned}
\|g \circ f\| &= \sup\{\|g(f(x))\| : \|x\| \leq 1\} && (154)\\
&\leq \|g\| \sup\{\|f(x)\| : \|x\| \leq 1\} && (155)\\
&\leq \|g\|\|f\|. && (156)
\end{aligned}$$

In the language of matrices, this means that we have the inequality $\|AB\| \leq \|A\|\|B\|$. We shall use this estimate to prove the following simple fact which is the basis for many iteration methods for obtaining approximate solutions for systems of equations. Suppose that A is an $n \times n$ matrix whose norm $\|A\|$ (with respect to some norm on $\mathbf{R}^n$) is strictly less than 1. Then the series

$$I - A + A^2 - A^3 + \cdots + (-1)^n A^n + \ldots$$

converges and its sum is an inverse for the matrix $I + A$. (In particular, we are claiming that the latter is invertible). In order to prove the convergence one apes the proof of the fact that the geometric series converges if $|a| < 1$. One then multiplies out the partial sums to get the equations

$$(I + A)S_n = S_n(I + A) = I - A^{n+1}$$

where $S_n = \sum_{r=0}^{n}(-1)^r A^r$. Now $\|A^{n+1}\| \leq \|A\|^{n+1}$ by a repeated used of the inequality for the norm of a product and so the right hand side converges to I from which it follows that the limit of the sequence (S_n) is an inverse for $I + A$.

This can be used to give an iterative method for solving the equation $BX = Y$ where B is a matrix of the form $I + A$ with $\|A\| < 1$ for some norm on $\mathbf{R}^n$. By the above, the solution is

$$X = (I + A)^{-1}Y = Y - AY + A^2Y - A^3Y + \ldots.$$

This can be most conveniently implemented with the following scheme. One chooses an initial value X_0 for the solution and defines a sequence (X_n) of vectors iteratively by means of the formula $X_{n+1} = Y - AX_n$. Then the vectors X_n converge to the solution of the equation. For if we substitute successively in the defining equations for X_n we see that

$$\begin{aligned}
X_1 &= Y - AX_0 && (157)\\
X_2 &= Y - AY + A^2X_0 && (158)\\
&\vdots && (159)\\
xi_n &= (I - A + A^2 - \cdots + (-1)^{n-1}A^{n-1})Y \pm A^n X_0. && (160)
\end{aligned}$$

In order to be able to apply this method, we require estimates for the norms of matrices with respect to suitable norms on $\mathbf{R}^n$. We note here that for the cases of $\| \ \|_1$ and $\| \ \|_\infty$, we have the exact formulae:

$$\|A\| = \max_j \sum_i |a_{ij}| \tag{161}$$

$$\|A\|_\infty = \max_i \sum_j |a_{ij}| \tag{162}$$

(see the exercises below).

Example: Consider the equation $BX = Y$ where B is an $n \times n$ matrix which is dominated by its diagonal in the sense that

$$|b_{ii}| > \sum_{j \neq i} |b_{ij}|$$

for each i. Then we can split B into the sum $D + B_1$ where $D = \operatorname{diag}(b_{11}, \dots, b_{nn})$ and

$$B_1 = \begin{bmatrix} 0 & b_{12} & \dots & b_{1n} \\ b_{21} & 0 & \dots & b_{2n} \\ \vdots & & & \vdots \\ b_{n1} & b_{n2} & \dots & 0 \end{bmatrix}.$$

The equation can now be written in the form

$$D^{-1}BX = D^{-1}Y$$

and the matrix $D^{-1}B$ has the form $I + A$ where the norm of A is less than 1. Hence we can apply the above to obtain the following method of solution. We choose a suitable approximation X_0 to the solution and use the formula

$$X_{n+1} = D^{-1}(Y - B_1 X_n)$$

to define recursively a sequence of vectors which converges to the exact solution.

Similar considerations lead to the following iterative method for determining approximations to the inverse of a matrix A. We start off with a reasonable approximation D to the inverse. How far away this is from being an inverse is described by the difference $C = I - DA$. We assume that this is small in the sense that has norm less than one for some norm on $\mathbf{R}^n$. Then we use the iteration scheme

$$D_1 = D(I + C) \tag{163}$$
$$C_1 = I - AD_1 \tag{164}$$
$$D_2 = D_1(I + C_1) \tag{165}$$
$$C_2 = I - AD_2 \tag{166}$$

and so on. The resulting sequence (D_n) converges to the inverse of A.

Exercises 1)

- Consider the mapping

  $$(\xi_1, \xi_2) \mapsto (\xi_1 + 3\xi_2, 3\xi_1 + \xi_2).$$

  Calculate its matrix with respect to the following pairs of bases:

  $$((1,0), 0, 1)) \text{ and } ((1,1), (1,-1)); \tag{167}$$
  $$((1,1), (1,-1)) \text{ and } ((1,0), (0,1)); \tag{168}$$
  $$((1,1), (1,-1)) \text{ and } ((1,1), (1,-1)). \tag{169}$$

- Calculate the matrix of the mapping

  $$(\xi_1, \xi_2, \xi_3) \mapsto (2\xi_2 + \xi_3, \xi_1 + \xi_2 + \xi_3, \xi_1 - \xi_2).$$

- Calculate the matrix of the mappings

  $$p \mapsto (t \mapsto (p(t-1) - p(t)) \tag{170}$$
  $$p \mapsto (t \mapsto p(t+1)) \tag{171}$$

  with respect to the canonical basis of $\mathrm{Pol}\,(n)$.

- Calculate the kernel and the range of the mapping f which takes $(\xi_1, \xi_2, \xi_3, \xi_4)$ onto the vector

  $$(\xi_1 + \xi_2 - \xi_3 + \xi_4, \xi_2 - \xi_3, 6\xi_1 + 2\xi_2 - 3\xi_3 + 4\xi_4).$$

- Calculate the matrix of the projection onto $[(1,0,1), (0,-1,0)]$ along $[(2,2,2)]$ with respect to the canonical basis.

- Calculate the image of the corners $(\pm 1, \pm 1, \pm 1)$ of the cube under the projection onto the plane

  $$\{(\xi_1, \xi_2, \xi_3); \xi_1 + \xi_2 + \xi_3 = 0\}$$

  along $(1,1,1)$.

- Find the matrix of the mapping $A \mapsto AB$ (where B is a fixed 2×2 matrix) with respect to the basis

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

2) Calculate the matrix of the operator $\sum_{i=1}^n a_i D^n$ on $\mathrm{Pol}\,(n)$. For which values of the a_i is it invertible? Calculate the inverse in the case where $a_i = \dfrac{1}{i!}$.
3) For which $f \in L(V)$ does there exist a $g \in L(V)$ wo that $g \circ f = 0$ (with g non-zero).
4) Let V_1 (resp. W_1) be a subspace of V (resp. W) so that

$$\dim V_1 + \dim W_1 = \dim V.$$

Show that there is a linear mapping f from V into W so that $V_1 = \mathrm{Ker}\, f$, $W_1 = f(V_1)$.
5) Show that if $f \in L(V)$, then there is a polynomial p so that $p(f) = 0$ (where if $p(t) = a_0 + a_1 t + \dots + a_r t^r$, then $p(f) = a_0 \mathrm{Id} + a_1 f + \dots + a_r f^r$). Deduce that if f is invertible, then there is a polynomial p so that $f^{-1} = p(f)$.
6) Let p_1, p_2 be projections in $L(V)$. Show that
$p_1 + p_2$ is a projection if and only if $p_1 p_2 = p_2 p_1 = 0$;
$p_1 - p_2$ is a projection if and only if $p_1 p_2 = p_2 p_1 = p_2$; indent $p_1 p_2$ is a projection if and only if $p_1 p_2 = p_2 p_1$.
Interpret these statements geometrically and describe the ranges of the corresponding projections in terms of those of p_1 and p_2.
If p is a projection onto the subspace V_1 of V and $f \in L(V)$, show that $f(V_1) \subset V_1$ if and only if $pfp = fp$. For which f is the condition $fp = pf$ satisfied?
7) Calculate the matrix of the differentiation operator on the space of continuous functions spanned by

$$\{e^{\lambda t}, te^{\lambda t}, \dots, t^n e^{\lambda t}\}$$

(with these functions as basis).
8) Verify the above formulae for the norm of A with respect to $\|\ \|_1$ respectively $\|\ \|_\infty$.
9) A norm $\|\ \|$ on $\mathbf{R}^n$ is **monotonic** if $\|x\| \leq \|y\|$ whenever $|\xi_i| \leq |\eta_i|$ for each i. Show that if D is the diagonal matrix $\mathrm{diag}\,(\lambda_1, \dots, \lambda_n)$, then the associated norm of D for a monotonic norm is

$$\max(|\lambda_1|, \dots, |\lambda_n|).$$

10) (The following is a particular example of a method for calculating certain indefinite integrals of classical functions). Denote by V the two dimensional vector space generated by the functions

$$x(t) = e^{at} \cos bt, \quad y(t) = e^{at} \sin bt.$$

Note that the operation of differentiation maps V into itself and calculate its matrix with respect to the basis (x, y). Use the inverse of this matrix to calculate the indefinite integrals of x and y. (This works for any finite dimensional vector spaces of suitable differentiable functions which do not contain the constants and are invariant under differentiation. The reader is invited to construct further examples).

11) Suppose that $\mathcal{M}$ is a subset of the space $L(V)$ of linear operators on the vector space V. Show that if the only subspaces of V which are invariant under each f in $\mathcal{M}$ are $\{0\}$ and V itself, then an operator g which commutes with each element of $\mathcal{M}$ is either 0 or is invertible.