Exercises for linear algebra

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Contents

1 Matrices

1.1 Small matrices

A good method for developing a feel for matrices is to work out problems on matrices of small order. For this reason we start out with some exercises on 2×2 and 3×3 matrices.

Example Find examples of 2×2 matrices A and B for which $A = A^t$, $B = B^t$ but $(AB)^t \neq AB$. (i.e. two symmetric matrices whose product is not symmetric).

Example Determine all 2×2 matrices A for which $A^2 = 0$.

Example Show that if C is a 2×2 matrix whose trace $c_{11} + c_{22}$ vanishes, then there are 2×2 matrices A,B with $C = AB - BA$.

Example Show that there is a 2×2 real matrix which is not similar to a triangular matrix.

Example Show that if A and B are invertible 2×2 matrices so that $ABA^{-1}B^{-1}$ commutes with A and B, then $AB = BA$ or $AB = -BA$.

Example Find all 3×3 matrices A with $A^2 = I$ (Consider separately the cases where the elements of A are in $\mathbf{Z}$, $\mathbf{R}$ or $\mathbf{C}$).

Example I.1.G Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

be the typical 2×2 matrix and define scalars a_n, b_n, c_n and d_n by putting

$$A^n = \begin{bmatrix} a_n & b_n \\ c_n & d_n \end{bmatrix}.$$

Derive recurrence relations for (a_n) etc. and use these to calculate the n-th powers of the matrices

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

1.2 Explicit matrices

The next exercises involve simple calculations with explicit matrices. The matrices chosen all play a role in applications:

Example Let

$$M_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

and

$$M_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix}$$

resp.

$$M_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

resp.

$L^+ = M_1 + iM_2$ and $L^- = M_1 - iM_3$. Show that

$$[M_1, M_2] = iM_3$$
$$M_1^2 + M_2^2 + M_3^2 = 2I$$
$$[M_2, L^+] = L^+, [L^+, L^-] = 2M_3.$$

Example Let C be the $n \times n$ matrix

$$\begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{bmatrix}.$$

Calculate CAC, CAC^t and CAC^{-1} where A is the general $n \times n$ matrix. If $A_p = C^p + C^{-p}$ ($p \in \mathbf{N}$), show that

$$\begin{aligned} A_p &= A_{n-p} \\ A_p A_q &= A_{p+q} + A_{p-q} \\ A_{p+1} &= A_1 A_p - A_{p-1}. \end{aligned}$$

Example Calculate the square of the Vandermonde matrix

$$\begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ x_1 & x_2 & x_3 & \dots & x_n \\ x_1^2 & x_2^2 & x_3^2 & \dots & x_n^2 \\ \vdots & & & & \vdots \\ x_1^{n-1} & x_2^{n-1} & x_3^{n-1} & \dots & x_n^{n-1} \end{bmatrix}.$$

Example Let $x_0, \dots x_n$ be distinct numbers and let $w_0, \dots, w_n$ be the solutions of the equation

$$\begin{bmatrix} 1 & 1 & \dots & 1 \\ x_0 & x_1 & \dots x_n & \\ \vdots & & & \vdots \\ x_0^{n-1} & x_1^{n-1} & \dots & x_n^{n-1} \end{bmatrix} \begin{bmatrix} w_0 \\ w_1 \\ \vdots \\ w_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ n! \end{bmatrix}.$$

Show that if f is a smooth function, then

$$f^{(n)}(t) = \lim_{h \to 0} \frac{1}{h^n} w_k f(t + a_k h)).$$

(This is a generalisation of such formulae as

$$f''(t) = \lim_{h \to 0} \frac{1}{h^2} \left(f(t+h) - 2f(t) + f(t-h) \right)$$

for higher derivatives of f).

Example Let A be the $n \times n$ matrix $[a_{ik}]$ where $a_{ij} = \binom{i}{j}$. Show that A^r is the matrix $[a_{ij} r^{i-j}]$.

Example Calculate the inverse of the matrix

$$A = \begin{bmatrix} 1 & -1 & 0 & \dots & 0 \\ 0 & 1 & -1 & \dots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \dots & -1 & 1 \end{bmatrix}$$

by considering the corresponding system of linear equations. Use this to calculate the inverse of the matrix

$$B = \begin{bmatrix} 2 & -1 & 0 & \dots & 0 \\ -1 & 2 & -1 & \dots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \dots & -2 & 1 \end{bmatrix}.$$

(Use the fact that $B = A^t A$). (We remark that matrices of the type B arise naturally in the discretisation of Sturm-Liouville operators).

1.3 General exercises

We conclude with some exercises on general properties of matrices

Example Let A be an invertible $n \times n$ matrix such that the row sums of A are constant. Show that this constant is non-zero and that the inverse of A also has non constant row sums. What is the common value of these sums?

Example Show that if an $m \times n$ matrix A is such that $A^t A = 0$, then $A = 0$.

Example Show that every invertible $n \times n$ matrix is a product of matrices of one of the following three forms

$$\begin{bmatrix} 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \\ \vdots & & & & \vdots \\ 0 & \dots & 1 & 0 & 0 \\ 0 & \dots & 0 & 1 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & k & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix},$$

and

$$\begin{bmatrix} \lambda & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

(Note that the last three are elementary matrices i.e. correspond to elementary row operations).

Example Recall that a square matrix A is **doubly stochastic** if its elements are non-negative and the sum of each of its row and columns is 1. Show that the latter condition is equivalent to the fact that $Ae = e = A^t e$ where e is the column matrix with all entries "1"s. If $y = (\eta_1, \dots \eta_n)$ and $x = (\xi_1, \dots, \xi_n)$, we say that y **dominates** x if

$$\xi_1 \leq \eta_1, \xi_1 + \xi_2 \leq \eta_1 + \eta_2, \dots, \xi_1 + \dots + \xi_{n-1} \leq \eta_1 + \dots + \eta_{n-1}$$

and

$$\xi_1 + \dots + \xi_n = \eta_1 + \dots \eta_n.$$

Show that this is equivalent to the fact that there is a doubly stochastic matrix A so that $X = AY$. (X and Y are the column matrices corresponding to x and y).

Example Let A be an $m \times n$ matrix, B and $n \times p$ one. Show that

$$r(AB) \geq r(A) + r(B) - n.$$

Deduce that if $r(A) = n$, $r(AB) = r(B)$.

Example Let $\mathcal{M}$ be a subset of the set M_n of $n \times n$ matrices with the properties

- if $A, B \in \mathcal{M}, \lambda, \mu \in \mathbf{R}$, then $\lambda A + \mu B \in \mathcal{M}$;
- if $P, Q \in M_n$, $A \in \mathcal{M}$, then $PAQ \in \mathcal{M}$.

Show that $\mathcal{M}$ is either $\{0\}$ or M_n.

2 Geometry

2.1 The plane as a vector space

We begin with a number of problems which illustrate the use of the vector space structure of $\mathbf{R}^2$ to obtain geometrical results:

II.1.A Calculate the barycentric coordinates of the point p with respect to A, B and C where the numbers in Figure 1 indicate the ratios in which the various lines are divided.

II.1.B Show that the in-centre of the triangle ABC is the point

$$\frac{1}{s}(ax_A + bx_B + cx_C)$$

where $a - |BC|$, $b = |CA|$, $c = |AB|$, $s = a+b+c$. What is the corresponding result for the out-centres?

II.1.C What are the barycentric coordinates of P,Q and R with respect to A, B and C? Use this to whose that the area of ABC is seven times that of PQR. (See Figure 2).

II.1.D What is the area of the convex n-gon with vertices $P_1 = (\xi_1^1, \xi_2^1), \dots, P_n = (\xi_1^n, \xi_2^n)$?

II.1.E Let X and Y be the midpoints of AC resp. BD and W the intersection of AD and BC. Show that the area of WXY is one quarter of that of the rectangle $ABCD$ (Figure 3).

II.1.F Show that the midpoints of the sides of a quadrilateral form the vertices of a parallelogram. What is the ratio of the area of this parallelogram to that of the originalquadrilateral? (Figure 4).

II.1.G Show that if M is the centroid of the triangle ABC, then the area of AC_1M is a sixth of that of ABC where C_1 is the midpoint of AB (Figure 5).

II.1.H Let A_1, A_2, A_3, A_4 be the vertices of a square and let B_1, B_2¡ B_3 and B_4 be as in diagram 6. Show that they also form the vertices of a square and that the area of the latter is one fifth of that of the original one.

II.1.I Show that a quadrilateral $ABCD$ is a parallelogram if and only if AC and BD cross at their midpoints. Show that if $ABCD$ and $APCQ$ are parallelograms (see Figure 7), then $BPDQ$ is also a parallelogram (which may be degenerate i.e. such that its vertices are collinear).

II.1.J Let $ABCD$ be a quadrilateral and choose B_1, D_1 as in diagram 8 so that CB_1 and AB (resp. CD_1 and AD) are equal and parallel. Show that

- BB_1DD_1 is a parallelogram;

- the lines BC, DC, D_1C and B_1C are equal and parallel to sides of the original quadrilateral and the angles between them are the angles of the original quadrilateral;
- the area of BB_1D_1D is twice that of $ABCD$.

(see Figure 9).

II.1.K Let ABC be a triangle and let the bisectors of the angles at A resp. B meet the opposite sides at A_1 and B_1. Show that

$$|BA_1|/|CA_1| = |AB|/|AC|$$

and that if $|CC_1| = |BB_1|$, then $|AB| = |AC|$. Show that if

$$x_{A_1} = x_B + (1-\lambda)x_C$$

then $|BA|/|AC| = \dfrac{\lambda}{1-\lambda}$.

II.1.L Show that the bisectors of the angles between the lines

$$(x - x_0|\mathbf{n}_0) = 0 \text{ and } (x - x_1|\mathbf{n}_1) = 0$$

have equations

$$(x - x_0|\mathbf{n}_1) = \pm(x - x_1|\mathbf{n}_1).$$

II.1.M Prove the following generalisation of Menelaus' theorem: $A_1, \ldots, A_5$ are the vertices of a pentagon and a line $\mathbf{L}$ meets the edges in the points $P_1, \ldots, P_5$ as in figure ?? Show that

$$\frac{|P_1A_1|}{|P_1A_1} \cdot \frac{|P_2A_2|}{|P_21A_3} \cdot \frac{|P_3A_3|}{|P_3A_4} \cdot \frac{|P_4A_4|}{|P_4A_5} \cdot \frac{|P_5A_5|}{|P_5A_6} = 1.$$

Can you find a similar generalisation of Ceva's theorem?

II.1.N Let tL be the line

$$\{x : (\mathbf{n}|x) = (\mathbf{n}|x_0)$$

where $\mathbf{n}$ is the unit normal. Show that the distance from y to $\mathbf{L}$ is $|(\mathbf{n}|y - x_0)$.

II.1.O Let tL_1, $\mathbf{L}_2$ and $\mathbf{L}_3$ through a fixed point P and let A_1 and B_1 (resp. A_2 and B_2 resp. A_3 and B_3) be points on $\mathbf{L}_1$ resp. $\mathbf{L}_2$ resp. $\mathbf{L}_3$. Show that if A_1A_2 is parallel to B_1B_2 and A_2A_3 is parallel to B_2B_3, then A_3A_1 is parallel to B_3B_1.

II.1.P Let A, B, C and D be non-collinear points in the plane such that $|AB| = |CD|$ and $|AD| = |BC|$. Show that $x_{AD} - x_{BC} \perp x_{AC}$ and $x_{Ad} - x_{BC} \perp x_{BD}$ and that $x_{AD} = x_{BC}$ or AC is parallel to BD.

II.1.Q Let $\mathbf{L} = \mathbf{L}_{a,b,c}$ and $\mathbf{L}_1 = \mathbf{L}_{a_1,b_1,c_1}$ be anon-parallel lines and put

$$\mathbf{L}_{\lambda,\mu} = \{(\xi_1, \xi_2) : \lambda(a\xi_1 + b\xi_2 + c) + \mu(a_1\xi_1 + b_1\xi_2 + c_1) = 0\}.$$

Show that $\{\mathbf{L}_{\lambda,\mu}; \lambda, \mu \in \mathbf{R}, \lambda + \mu = 1\}$ is the set of all lines through the point of intersection of $\mathbf{L}_1$ and $\mathbf{L}_2$.

II.1.R Let x, y and z be points in the plane. Show that x and y are parallel if and only if we have $\|x+y\| = \|x\| + \|y\|$ or $\|x-y\| = \|x\| + \|y\|$ and that both of these conditions are equivalent to the validity of the equality $|(x|y)| = \|x\|\|y\|$. Show that z is the midpoint of x and y if and only if we have the equalities

$$\|x-z\| = \|z-y\| = \frac{1}{2}\|x-y\|.$$

II.1.S Let ABC be a triangle and P, Q and S as in the diagram. Show that the area of the triangle PQR is one seventh that of ABC.

II.1.T Let x_1, x_2, x_3 and x_4 be vectors in the p lane. Show that there exists a triangle ABC with $x_{AB} = x_1$, $x_{BC} = x_2$ and $x_{CA} = x_3$ if and only if $x_1 + x_2 + x_3 = 0$. Show that there is a parallelogram $ABCD$ with $x_{AB} = x_1$ etc. if and only if $x_1 + x_2 + x_3 + x_4 = 0$ and $x_1 + x_3 = 0$.

II.1.U Let A, B, C and D be points in the plane and put

$$x_{A'} = \frac{1}{2}(x_B + x_C) \quad x_{B'} = \frac{1}{2}(x_C + x_A) \quad x_{C'} = \frac{1}{2}(x_A + x_B).$$

Show that

$$(x_{A'D}|x_{BC}) + (x_{B'D}|x_{CA}) + (x_{C'D}|x_{AB}) = 0.$$

II.1.V Let a, b and c be vectors in the plane. Does there always exist a triangle ABC so that $x_{A,\frac{1}{2}(B+C)} = a$ etc. How many such triangles exist?

II.1.W Let x, y, z be points in the plane. Show that

$$\|x + y - z\|^2 = \|x - z\|^2 + \|y - z\|^2 - \|x - y\|^2 + \|x\|^2 + \|y\|^2 - \|z\|^2.$$

2.2 Affine mappings of the plane

The essential reason for the success of applying methods of linear algebra to geometry is the fact that the interesting elementary transformations of the plane are affine and so essentially implemented by matrices. We bring some exercises on this them.

II.2.A Calculate the images resp. the pre-images of the line

$$a\xi_1 + b\xi_2 + c = 0$$

and the conic section

$$a\xi_1^2 + 2b\xi_1\xi_2 + c\xi_2^2 = 0$$

under the affine mapping

$$(\xi_1, \xi_2) \mapsto (a_{11}\xi_1 + a_{12}\xi_2 + c_1, a_{21}\xi_1 + a_{22}\xi_2 + c_2)$$

(where $a_{11}a_{22} - a_{12}a_{21} \neq 0$).

II.2.B More generally, find the image and pre-image of the conic section

$$(f(x)|x) + 2(b|x) + c = 0$$

(where f is an affine mapping with symmetric matrix and b is a vector in $\mathbf{R}^2$) under the affine mapping $x \mapsto g(x) + u$ (where g is linear and injective).

II.2.C Show that if f is an injective affine mapping and x, y, z are collinear and distinct, then

$$\frac{\|f(x) - f(y)\|}{\|x - y\|} = \frac{\|f(x) - f(z)\|}{\|x - z\|}.$$

II.2.D Let f be a linear, invertible mapping on $\mathbf{R}^2$. Show that there is an orthonormal basis (x_1, x_2) for $\mathbf{R}^2$ so that $f(x_1) \perp f(x_2)$. for which f is there precisely one such basis?

2.3 Circles

II.3.A Show that the line

$$a\xi_1 + b\xi_2 + c = 0$$

cuts the circle

$$\|x - x_0\|^2 = r^2$$

if and only if

$$(\xi_1^0 + b\xi_2^0 + c)^2 \leq r^2(a^2 + b^2).$$

II.3.B consider the circles

$$\xi_1^2 + \xi_2^2 + 2a\xi_1 + 2b\xi_2 + c = 0 \quad (1)$$
$$\xi_1^2 + \xi_2^2 + 2a\xi_1 + 2b\xi_2 + c_1 = 0 \quad (2)$$

with radii r resp. r_1. Put $d^2 = (a - a_1)^2 + (b - b_1)^2$. For which values of r,r_1 and d do the circles intersect? If they do intersect, show that the angle θ at which they cross each other is given by the formula

$$\cos\theta = \frac{r^2 + r_1^2 - d^2}{2rr_1}.$$

Deduce a criterium for the circles to cut at right angles.

II.3.C If **C** is the circle $\|x - a\|^2 = r^2$ in $\mathbf{R}^2$, the **power** of a point x with respect to **C** is the point

$$p(x) =$$

Interpret this geometrically.

II.3.D Show that the line

$$\{x_0 + tu : t \in \mathbf{R}\}$$

(where u is a unit vector) cuts the circle $\|x = a\|^2 = r^2$ at the points corresponding to the roots of the quadratic equation

in t. Show that the product of these roots is independent of u. Interpret this result geometrically (bear in mind the case where x_0 lies inside of the circle).

II.3.E Denote by $C_{a,b,c}$ the circle

$$\xi_1^2 + \xi_2^2 - 2a\xi_1 - 2b\xi_2 - c = 0.$$

- show that if $x = (\xi_1, \xi_2)$ is a point in $\mathbf{R}^2$, then

  $$S_C(x) = \xi_1^2 + \xi_2^2 - 2a\xi_1 - 2b\xi_2 + c$$

  is the square of the length of the tangent from x to $C = C_{a,b,c}$ provided that x is exterior to C;

- if $C_1 = C_{a_1,b_1,c_1}$, then C and C_1 are tangential to each other if and only if

  $$2aa_1 + 2bb_1 - (c + c_1 + 2rr_1) = 0$$

  where r resp. $r - 1$ are the radii of the circles;

- if C and C_1 are circles as above, then

  $$\{x \in \mathbf{R}^2 : S_C(x) = S_{C_1}(x)\}$$

  is a straight line. It is called the **radial axis** of C and C_1 (give a geometrical interpretation);

- show that if C, C_1 and C_2 are circles so that

  then the radial axes of the three circles are concurrent;

- show that if $\lambda \neq 1$, then

  $$C_\lambda = \{x \in \mathbf{R}^2 : S_C(x) = \lambda S_{C_1}(x)\}$$

  is a circle and that if C and C_1 intersect in two points, then it represents (with varying λ) the family of circles passing through these points;

- show that if a circle C_2 meets C and C_1 at right angles, then it also meets each C_λ at right angles (C, C_1 and C_λ as in (5)).

II.3.F Show that if C_1, C_2 and C_3 are three circles, no two of which are concentric, then the radial axes (cf. E above) are concurrent or parallel (the point of intersection in the former case is called the **power point** of the three circles).

II.3.G Show that if A_1, A_2, A_3, A_4, A_5 and A_6 are six points on a circle so that the lines

$$A_1A_4, A_2A_5, A_3A_6$$

meet in a point, then

$$|A_1A_2| \cdot |A_3A_4| \cdot |A_5A_6| = |A_2A_3| \cdot |A_4A_5| \cdot |A_6A_1|.$$

Can you generalise this result to $2n + 2$ points on a circle (whereby $n > 1$)?

II.3.G Show that if $A_1, \dots, A_{2n+1}$ are the vertices of a regular polygon inscribed in a circle of unit radius and L_i is the length of A_1A_{i+1}, then

$$\sum_{i=1}^{n} L_i^2 = \prod_{i=1}^{n} L_i.$$

II.3.H Let ABC be a triangle, A_1 a point on BC, B_1 on CA, C_1 on AB. Show that the circles through AB_1C_1, BC_1A_1 and CA_1B_1 are concurrent.

2.4 Conic sections

The next most complicated class of curves after the circles are the conic sections which were also studied by the Greeks. Their classification is one of the highpoints of plane geometry. Similar methods can be used to prove a number of elegant results on conic sections.

II.4.A Show that if $f : \mathbf{R}^2 \to \mathbf{R}^2$ is an injective, affine mapping, then f maps conic sections onto conic sections. For which f is the image of a circle also a circle?

II.4.B Show that if $\mathbf{Q}$ is the conic section

$$\{x \in \mathbf{R}^2 : (f(x)|x) + 2(b|x) = c = 0\}$$

then the tangent to $\mathbf{Q}$ at x_0 is the line

$$\{x \in \mathbf{R}^2 : (f(x)|x_0) + (b|x + x_0) + c = 0\}.$$

What is the geometrical significance of this line if x_0 does not lie on $\mathbf{Q}$?

II.4.C Consider the conic section

$$a\xi_1^2 + 2b\xi_1\xi_2 + c\xi_2^2 d\xi_1 + e\xi_2 + f = 0.$$

Show that the diameter which bisects the chords of slope m has equation

$$2a\xi_1 + 2b\xi_2 + 2mb\xi_1 + 2m\xi_2 + d + em = 0.$$

II.4.D Express the eccentricity of the ellipse obtained by cutting the cylinder

$$\xi_1^2 + \xi_2^2 = 1$$

with a plane making an angle θ with the (x, y)-plane as a function of θ.

II.4.E Let C be a circle, $\mathbf{Q}$ an ellipse whose major axis coincides with a diameter of C. Choose two points Q_1 and Q_2 on the same side of this diameter so that the angle $QOQ_1 = 90^o$. (O is the centre of the circle). If P_1 and P_2 are as in figure ??, show that OP_1 and OP_2 lie on conjugate diameters and that their lengths are

$$a^2 \cos^2\theta + b^2 \sin^2\theta \text{ resp. } a^2 \sin^2\theta + b^2 \cos^2\theta$$

where a and b are the lengths of the semi=axes of the ellipse.

II.4.F Show that if L_1, L_2, L_3 and L_4 are four lines (where $L_1 = L_{a_1,b_1,c_1}$ etc.), then

$$a_1\xi_1 + b_1\xi_2 + c)(a_2\xi_1 + b_2\xi_2 + c) = k(a_3\xi_1 + b_3\xi_2 + c_3)(a_4\xi_1 + b_4\xi_2 + c_4)$$

is the equation of a conic with the four lines as tangents. For which values of k is it a circle?

II.4.G If $\mathbf{Q}$ is the conic

$$\{x \in \mathbf{R}^2 : a_{11}\xi_1^2 + 2a_{12}\xi_1\xi_2 + a_{22}\xi_2^2 + 2b_1\xi_1 + 2b_2\xi_2 + c = 0\}$$

and

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix} \qquad D = \begin{bmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ b_1 & b_2 & c \end{bmatrix}$$

, show that

- the conic has the form

$$\{x : X^t DX = 0\}$$

where

$$X = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix};$$

- the image of $\mathbf{Q}$ under the linear mapping with matrix U has the corresponding matrix

$$\begin{bmatrix} U^tAU & U^tB \\ B^tU & ? \end{bmatrix}$$

;

- that the following numerical functions of the conic section are invariant under such a transformation:

$$\operatorname{tr} A, \quad d_0 = \det A, \quad d = \det D;$$

- that $\mathbf{Q}$ is central (or empty) if and only if $d_0 \neq 0$ and $d \neq 0$;
- if $d_0 > 0$, then $\mathbf{Q}$ is an ellipse or empty;
- if $d_0 < 0$, then $\mathbf{Q}$ is a hyperbola;
- if $d_0 = 0$ and $d \neq 0$, then $\mathbf{Q}$ is a parabola;
- if $d = 0$ and $d_0 > 0$, then $\mathbf{Q}$ is a point;
- if $d = 0$, $d_0 < 0$, then $\mathbf{Q}$ is a pair of intersecting lines;
- if $d = d_0 = 0$, then $\mathbf{Q}$ is a line, a pair of parallel lines or empty.

II.4.G Calculate the locus of the foci of all conics which touch the four sides of a given parallelogram.

II.4.H Let A and B be fixed points in the plane and suppose that a point C moves on a fixed circle with centre A. Find the locus of the point P where P is the intersection of BC and the internal bisector of the angle ABC.

3 Isometries

One of the fundamental concepts of euclidean geometry is that of congruence. This corresponds to the modern notion of isometry. The basic result on the isometries of $\mathbf{R}^2$ is their classification into the following types

- translations
- reflections
- rotations
- glide reflections.

3.1 Concrete isometries

III.1.A Show that if f is a half-turn about the point P, then P is the midpoint of x and $f(x)$ for any x in $\mathbf{R}^2$.

III.1.B Show that if P, Q and R are points in the plane, then the relationship

$$x_{Q,\pi} \circ D_{x_P,\pi} = T_{x_{P_r}} = D_{x_R,\pi} \circ D_{x_Q,\pi}$$

holds if and only if fA is the midpoint of PR.

III.a.C Show that if f and g are isometries, then $G^{-1} \circ f \circ g$ is an isometry of the same geometrical type as f.

III.1.D Show that a glide reflection of the plane can be expressed as a product of three reflections in the sides of a triangle. (See figure ??).

III.1.E Show that a mapping of the form

$$D_{x_P,\pi} \circ D_{x_Q,\pi} \circ D_{x_P,\pi}$$

is a half-turn and calculate its axis. Carry out a similar analysis of the isometries

$$D_{x_P,\pi} \circ T_u \circ D_{x_P,\pi} \qquad T_u \circ D_{x_P,\pi} \circ T_{-u}.$$

III.1.F Show that if L_1, L_2 and L_3 are non-concurrent lines, then

$$(R_{L_1} \circ R_{L_2} \circ R_{L_3})^2$$

is a translation.

III.1.G Which isometries of the plane satisfy the condition $f^2 = \mathrm{Id}$?

III.1.H Let L_1 and L_2 be distinct one-dimensional subspaces of $\mathbf{R}^2$ and let x be a point on the bisector of the angle between them. Describe the successive images of x under the mappings

$$R_{L_1}, \quad R_{L_2} \circ R_{L_1}, \quad R_{L_1} \circ R_{L_2} \circ R_{L_1}, \ldots.$$

Distinguish between the cases where the angle between the lines has the form $2\pi\alpha$ where

- α is $\frac{1}{n}$ $(n \in \mathbf{N})$
- α is rational
- α is irrational.

III.1.I Characterise those matrices

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

for which the corresponding linear mapping

$$f : (\xi_1, \xi_2) \mapsto (a_{11}\xi_1 + a_{12}\xi_2, a_{21}\xi_1 + a_{22}\xi_2)$$

has the property that

$$(f(x)|f(y))_R = (x|y)_R$$

where $(x|y)_R = \xi_1\eta_1 - \xi_2\eta_2$.

3.2 Pseudo-squares

A pseudo-square is a rectangle $ABCD$ for which AC and BD are equal in length and perpendicular to each other. The use of the rotation operator allows an elegant approach to their theory.

III.2.A Show that $ABCD$ is a pseudo-square if and only if

$$x_{AC} = D_{\pm\frac{\pi}{2}}(x_{BD}).$$

III.2.B Show that the definition is equivalent to the existence of an F so that AFD and BFC are right-angled, isosceles triangles. Show that there then exists a G so that BGA and CGD are also right-angled and isosceles.

III.2.C Show that if M_1 (resp. M_2) is the midpoint of BD (resp. AC, then CGM_1M_2 is a square with centre the centroid of A,B,C and D (i.e. the point $\frac{1}{4}(x_A + x_B + x_C + x_D)$.

III.2.D Show that F and G lie on the bisectors of the angles between AC and BD.

III.2.E Show that

$$|AD|^2 + |BC|^2 = |AB|^2 + |CD|^2$$

and that the angles $\angle FAD$ and $\angle GAB$ are equal.

III.2.F Show that the midpoints of AB, BC, CD and DA form the vertices of a square.

III.2.G Show that if λ is a fixed real number and we define

$$x_P = \lambda x_A + (1-\lambda)x_B \quad x_Q = \lambda x_B + (1-\lambda)x_C \tag{3}$$

$$x_R = \lambda x_C + (1-\lambda)x_D \quad x_S = \lambda x_B + (1-\lambda)x_A \tag{4}$$

then $PQRS$ is also a pseudo-square.

III.2.H Show that if $A_1B_1C_1D_1$ is a second pseudo-square and λ is a fixed real number, then $A_2B_2C_2D_2$ is also a pseudo-square, where

$$x_{A_2} = \lambda x_A + (1-\lambda)x_{A_1} \text{ etc.}$$

3.3 Constructions

Another use of isometries is to provide elegant solutions of construction problems of the following type.

III.3.A Given two fixed lines L_1 and L_2 and a point AER as in the diagram, show how to find points B, C and D as in figure ?? with B on L_1 and D on L_2 so that $ABCD$ is a square.

III.3.B L, A, and B are given as in figure ?? Show how to construct a point P on L so that $LAP = LAB$.

III.3.C Given a circle and a point A outside of it, show how to construct a line through A which meets the circle in points P and Q so that P is the midpoint of AQ (figure ??).

III.3.D Given a triangle ABC and a segment p as in diagram ??, show how to find a line L, parallel to p, which meets the triangle at the endpoints of a segment which is equal to p.

III.3.E Given two circles C_1 and C_2 and a line L, find a line L_1 which is parallel to L and is such that the segment AB has a given length a where A and B are as in diagram ??

III.3.F Given a circle C, a line L and a point A, find a line L_1 through A so that $|PA| = |PB|$. (see figure ??

III.3.G Given intersecting circles C_1 and C_2 find a chord through a point of intersection A so that the difference $|PA| - |AQ|$ is equal to a given length a.

III.3.H Given a line L and curves C_1 and C_2 on opposite sides of L, construct a perpendicular to L which meets C_1 and C_2 at points equidistant to L (figure ??

III.3.I Given a line L and circles C_1 and C_2 on opposite sides of L, construct a square with two vertices on L and one each on C_1 and C_2.

3.4 Some geometrical results

In this section, we collect some attractive geometric results which can be proved elegantly by using isometries.

III.4.A Let $ABCD$ be a non-degenerate convex quadrilateral and denote by X the centre of the external square on AB. Y, Z and W are constructed similarly with respect to BC, CD and DA (see figure ??). Show that $XYZW$ is a pseudo-square (i.e. XZ and YW are equal and perpendicular—cf. III.4).

III.4.B Consider diagram ?? where $AC'B$ and $AB'C$ are similar isosceles triangles. Show that $AB'A''C$ and $AB''A'C''$ are parallelograms.

III.4.C Let $ABCD$ be a quadrilateral and P, Q, R and S be so that the triangles PDC, ARD AQB and BSC are equilateral (figure ??). Show that PQ and RS are equal and parallel (i.e. that) What can you say about $PQRS$ in the special case where $ABCD$ is a parallelogram?

III.4.D Consider diagram ?? where ABC is a triangle and P, Q and R are the centres of the external triangles which are equilateral. Show that PQR is equilateral (This is known as Napoleon's theorem and is sometimes attributed to him).

III.4.E Let A, B, C be a triangle, M the midpoint of the side BC. Show that if Z_1 and Z_2 are the centres of the exterior squares on the sides AB and BC resp., then Z_1MZ_2 is an isosceles, right-angled triangle (figure ??).

III.4.F Let A_1, A_2, A_3 and A_4 be points on a circle C and denote by H_1, H_2, H_3 and H_4 the orthocentres of the triangles, $A_2A_3A_4$, $A_3A_4A_1$, $A_4A_1A_2$ and $A_1A_2A_3$. Show that there is a point O so that $H_i = D_{x_O,\frac{\pi}{2}}(A_i)$ for each i.

III.4.G Let $ABCD$ be a quadrilateral, B_1 and D_1 as in figure ?? Show

- that BB_1DD_1 is a parallelogram;
- that the sides of $ABCD$ are equal and parallel to the segments from C to the vertices of BB_1D_1D (under a suitable pairing);
- that the angles described at C by these lines are the same as the angles of the quadrilateral $ABCD$;
- that the area of BB_1D_1D is twice that of $ABCD$.

3.5 Isometries in $\mathbf{R}^3$

Once again, one can, with the aid of the methods of linear algebra, catalogue the possible isometries of $\mathbf{R}^3$. They are

- translations;
- rotations;
- rotary translations;
- glide-reflections;

- rotary reflections;
- screw displacements.

III.5.A Describe the geometric form of the sets of successive images $\{f^n(x) : n \in \mathbf{N}\}$ of a point x in $\mathbf{R}^3$ under each of the above types of isometry.

III.5.B Let f be an isometry of $\mathbf{R}^3$. Show that if $x \in \mathbf{R}^3$, then

$$f(x) \times f(y) = (\det f)(x \times y)$$

for each $x, y \in \mathbf{R}^3$. Hence we have the equation $f(x) \times f(y) = x \times y$ if f is a proper motion (i.e. its determinant is positive and so has the value 1).

III.5.C A **rigid motion** of $\mathbf{R}^3$ is a continuous mapping T from $[0, 1]$ into the space of isometries of $\mathbf{R}^3$ so that $T(0) = \mathrm{Id}$. (Here continuity means that the elements of the matrices of the isometries $T(t)$ depend continuously on t). Show that an isometry f is proper (cf. III.5.B) if and only if there is a rigid motion T so that $T(1) = f$.

III.5.D Let U_t be the screw displacement

$$T_{(0,0,t)} \circ D_{[(0,0,1)],t}.$$

Calculate the matrix of U_t and show that $U_{t+s} = U_t \circ U_s$. Calculate the path

$$\{U_t(1, 0, 0) : t \in \mathbf{R}\}$$

of the point $(1, 0, 0)$ under this motion. What are the orthogonal projections of this path onto

- the (x, y)-plane;
- the (y, z)-plane;
- the (z, x)-plane;
- the plane $[(0, 1, 0), (\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})]$.

III.5.E Let f be a linear mapping on $\mathbf{R}^3$. Show that there is a rotation g so that $f \circ g$ has a diagonal matrix with respect to the canonical basis.

3.6 Directed angles

In this section we return to the plane. Using the scalar product, it is simple to define the angle between two vector and hence between two intersecting lines. However, in some situation, it is necessary to distinguish between the two possible directions (clockwise and anti-clockwise) along which an angle can be traversed. This requires a slightly more sophisticated approach which we now describe.

Firstly a definition: Let O and A be points in the plane. The **ray** OA (written r_{OA}) is the set

$$\{x_O + tx_{OA} : t > 0\}.$$

III.6.A Show that

$$r_{OA} = \{P : |AP| = |OA| - |OP|\}.$$

III.6.B Show that if $B \in r_{OA}$, then $r_{OB} = r_{OA}$.

III.6.C Show that two rays r_{OA} and r_{OB} are either disjoint or equal.

III.6.D Show that there is a unique $\theta \in [0, 2\pi[$ so that

$$D_{x_O,\theta}(r_{OA} = r_{OB}.$$

θ is then called the **directed angle** from OA to OB (written $\angle(r_{OA}, r_{OB})$).

III.6.E Show that if ABC is a triangle, then

$$\angle(r_{AB}, r_{AC}) + \angle(r_{BC}, r_{BA}) + \angle(r_{CA}, r_{CB}).$$

4 Solid geometry

IV.1.A Show that the lines

$$L_{a,b,c}, L_{a_1,b_1,c_1}, L_{a_2,b_2,c_2}$$

are concurrent or parallel if and only if the vectors (a, b, c), (a_1, b_1, c_1) and (a_2, b_2, c_2) are linearly dependent in $\mathbf{R}^3$. Use this to show that the bisectors of the angles of a triangle are concurrent (do the same for the altitudes and the perpendicular bisectors of the sides).

IV.2.A Find an explicit formula for the distance from the point x in space to the plane spanned by the three vectors y_1,, y_2 and y_3.

IV.2.B Let L_1 and L_2 be skew lines in space (i.e. they ;do not meet and are not parallel). Let P be a point in space which lies on neither of these lines. Show that there is precisely one line through P which meets L_1 and L_2. If L is $L_{(a,b,c,d)}$ and L_1 is L_{a_1,b_1,c_1,d_1}, calculate a formula for the distance between the lines.

IV.1.D Calculate the area of the projection of the ellipsoid

$$\left(\frac{\xi_1}{a}\right)^2 + \left(\frac{\xi_2}{b}\right)^2 + \left(\frac{\xi_3}{c}\right)^2 = 1$$

on the plane perpendicular to the unit vector

$$\mathbf{n} = (n_1, n_2, n_3).$$

IV.1.E Calculate the eccentricity of the ellipse shown in figure ??

IV.1.F Let V be the ellipsoid as in D above. Find an affine mapping f on $\mathbf{R}^3$ which maps V onto the unit ball and use this to calculate its volume.

4.1 Tetrahedra

IV.2.A Let $ABCD$ be a skew quadrilateral in $\mathbf{R}^3$ (i.e. the vertices are not coplanar). Show that the line joining the midpoints of AB and CD intersect and bisects the line joining the midpoint of AC and BD.

IV.2.B Let A, B, C and D be the vertices of a tetrahedron in space. Show that if AB and CD are perpendicular (resp. AC and BD are perpendicular), then AD and BC are also perpendicular.

IV.2.C Show that if one altitude of a tetrahedron intersects two others, then all four are concurrent.

IV.2.D Show that the lines joining the vertices of a tetrahedron $ABCD$ to the centroids of the opposite faces are concurrent and cut each other in the ratio $3:1$.

IV.2.D Let x, y and z be three vectors in space and denote by H (resp. K) the volume of the parallelotope spanned by them (res. the parallelotope with these vectors as altitudes). Show that

$$HK = \|x\|^2\|y\|^2\|z\|^2.$$

IV..2.E Let x, y and z be vectors in space. Show that

$$(x|x)(y|z) \geq (x|z)(x|y) = \|x \times z\|\|x \times y\|.$$

Deduce that if O, A, B and C are points in space as in diagram ?? then

$$\angle AOC \leq \angle AOB + \angle BOC.$$

IV.2.F If x, y and z are as above, show that

$$\|x - y\| + \|y - z\| + \|z - x\| \geq \|(x - y) \times (z - y)\|.$$

Deduce that if O, A, B and C are as in diagram ??, then

$$\triangle ABC \leq \triangle OAB + \triangle OBC + \triangle OCA.$$

4.2 The Platonic bodies

A Platonic body is a polyhedron which is regular in the sense that its faces consist of congruent regular polygons and its vertices are all similar. there are precisely five such bodies—the tetrahedron, hexahedron (cube), octahedron, dodecahedron and icosahedron. The next exercises are devoted to some of their properties.

IV.3.A Consider the points $A = (1, 0, 0)$, $B = (0, 1, 0)$ and $C = (0, 0, 1)$. Then there are two points O_1 and O_2 so that

$$|)_1 A| = |)_1 B| = |)_1 C| = \sqrt{2}$$

resp.

$$|O_2 A| = |O_2 B| = |O_2 C| = \sqrt{2}.$$

Chose one of these points and denote it by O. Then $OABC$ is a regular tetrahedron. Calculate

- its surface area;
- its volume;

- the angle between adjacent sides;
- the orthogonal projection of $OABC$ on the (x, y)-plane resp. the plane orthogonal to $(1, 1, 1)$.

Deduce from (1) and (2) an expression for the volume and surface area of a regular tetrahedron in terms of the length of a side.

IV.3.B The eight point $A = (1, 1, 1)$, $B = (-1, 1, 1)$, $C = (1, -1, 1)$, $D = (1, 1, -1)$, $F = (1, -1, 1)$, $G = (-1, 1, -1)$, $H = (-1, -1, 1)$ and $E = (-1, -1, -1)$ form the corners of a cube. Prove

- that A, H, F and G are the vertices of a regular tetrahedron;
- that the midpoints of the sides AB, BH, HE, EF, FD and DA lie on a plane and form the vertices of a regular hexagon. Calculate the latter's area;
- that the points A, B, G, D and H are the vertices of a pyramid. Calculate its volume and the angle between adjacent sides. Show that the cube can be dissected into three such congruent pyramids;
- that if $0 < \lambda < 1$, then the points A_1, B_1, C_1 and D_1 form the vertices of a square where

  $$A_1 = (-1, -1+\lambda, 1) \quad B_1 = (1-\lambda, 1, 1) \quad C_1 = (1, 1-\lambda, -1) \quad D_1 = (-1+\lambda, -1, -1).$$

  For which λ is the area of this square maximal?

IV.3.C The centroids of the faces of the above cube form the vertices of a regular octahedron, as do the midpoints of the sides of the regular tetrahedron. Calculate the surface area and volume (as functions of the length of a side) and the angle between adjacent sides.

IV.3.D If $b > 0$, then the twelve points

$$(0, \pm 1, \pm b) \quad (\pm b, 0, \pm 1) \quad (\pm 1, \pm b, 0)$$

form the vertices of an icosahedron. Show that if $b = \tau$ (the golden mean $\frac{1}{2}(1 + \sqrt{5})$ (see ??), then this is a regular icosahedron. Calculate

- the length of its sides;
- the surface area;

- the volume;
- the angle between adjacent sides.

IV.3.E Calculate the centroids of the twenty faces of the regular icosahedron and show that they form the vertices of a regular dodecahedron. Calculate the quantities (1) - (4) of the last exercises for this solid.

IV.3.F Calculate the normal to one of the faces of the dodecahedron and the orthogonal projection of its vertices onto the plane of this face.

IV.3.G Prove the formulae

$$\frac{R}{a} = \frac{\sqrt{3}(\sqrt{5}+1)}{4} \qquad \frac{r}{a} = \frac{\sqrt{(25+\sqrt{15}}}{2\sqrt{10}}$$

for the regular dodecahedron and

$$\frac{R}{a} = \sqrt{\left(\frac{5+\sqrt{5}}{2\sqrt{2}}\right)} \qquad \frac{r}{a} + \frac{3+\sqrt{5}}{4\sqrt{3}}$$

for the regular icosahedron, where R is the radius of the circumscribed sphere, r is the radius of the inscribed sphere and a is the length of a side.

IV.3.H The rhombic dodecahedron is the polyhedron with vertices

$$(0,0,\pm 2) \quad (0,\pm 2,0) \quad (\pm 2,0,0) \quad (1,1,1) \quad (1,1,-1) \tag{5}$$
$$(1,-1,1) \quad (-1,1,1) \quad (-1,-1,1) \quad (1,-1,-1) \quad (-1,1,-1). \tag{6}$$

Calculate the angle between two faces, its surface area and the angles of the rhombi which constitute its faces.

IV.3.I Let A, B, C and D be the points $(1,0,0,0)$, $(0,1,0,0)$, $(0,0,1,0)$ and $(0,0,0,1)$ in $\mathbf{R}^4$. Show that there are two points E and F so that A, B, C, D and E resp. A, B¡ C, D and F are the corners of a regular polytope in $\mathbf{R}^4$. Calculate the angle between the sides of this polytope and its projection onto the x,y,z space resp. the x,y plane resp. on the space

$$\{x \in \mathbf{R}^4 : \xi_1 + \xi_2 + \xi_3 + \xi_3 = 0\}.$$

IV.3.J In this last exercise we consider the hypercube in $\mathbf{R}^4$. It has vertices the points with coordinates $(\pm 1, \pm 1, \pm 1, \pm 1)$. How many edges resp. sides does it have. What is the length of its diagonal?

5 Vector spaces and linear mappings

5.1 ?

V.1.A Show that two affine subspaces $T_u(M)$ and $T_v(M)$ of a vector space V coincide if and only if $u \in T_v(M)$ or equivalently $v - u \in M$.

V..1.B Show that if two affine subspaces of the form $T_u(M)$ and $T_v(N)$ coincide, then $M = N$.

V.1.C Show that the affine subspace spanned by the set
$\{x_1, \dots, x_n\}$ consists of those vectors of the form $\sum_{i=1}^n \lambda_i x_i$ where $\sum_{i=1}^n \lambda_i = 1$.

Show that this representation is unique if and only if the x_i are affinely independent.

5.2 ?

V.2.A If u is a vector in $\mathbf{R}^3$, then the mapping

$$x \mapsto u \times x$$

is linear. What is its matrix with respect to the canonical basis?

V.2.B Suppose that $\mathcal{M}$ is a subspace of $L(V)$ with the property that the only subspaces of V which are invariant under each $f \in \mathcal{M}$ are $\{0\}$ and V itself. Show that if $g \in L(V)$ commutes with each element of $\mathcal{M}$, then g is either the zero operator or invertible.

6 Determinants

6.1 The calculation of concrete determinants

The literature abounds with exercises involving the explicit calculation of determinants, starting with the Vandermonde determinant. We bring a selection of some of the more interesting ones:

VI.1.A Calculate the determinant of the following $n \times n$ matrices $A = [a_{ij}]$, whereby

- $a_{ij} -$ g.c.d. (i, j);
- $a_{ii} = 2\cos\theta$, $a_{i,i-1} = a_{i,i+1} = 1$, $a_{ik} = 0$ otherwise;
- $a_{ij} = ad_{ij} - b_i b_k$;
- $a_{ij} = \dfrac{1}{(i+j+1)!}$ $(0 \leq i, j \leq n-1)$;
- the a_{ij} are given recursively by the formulae:

$$a_{11} = 1, a_{i1} = a_{1,i-1} \quad (i > 1) \tag{7}$$
$$a_{ij} = a_{i+1,j-1} + a_{i,j-1} \quad (j > 1); \tag{8}$$

- $a_{ij} = 0$ if $i - j \neq 0, \pm 2$, $a_{ii} = \lambda_i + \lambda_{i-1}$ (where $\lambda_0 = \lambda_{n+1} = 0$), $a_{i+2,i} = 1$, $a_{i,i+2} = \lambda_1 \lambda_{i+1}$;
- a_{ij} is the number of common divisors of i and j $(2 \leq i, j \leq n)$;
- $a_{in} = 0$ if $i + j$ is odd, $a_{ij} = \binom{2k}{k}$ $(i + j = 2k)$;
- $a_{ij} = \frac{i+j-1}{j}$;
- $a_{ij} = p_j(x_i)$ where p_j is a polynomial of degree j with leading coefficient c_j and constant term 0;
- $a_{ij} = j^{i-1}$;
- $a_{ij} = x^{(i-1)(j-1)}$;
- $a_{ij} = \dfrac{p_{i+j}}{q_{i+j}}$ where (p_k) and $q_k)$ are arithmetic projections.

VI.1.B Calculate the following determinants:

6.2 General problems on determinants

We continue with some more theoretical results involving determinants:

VI.2.A Use the identity $\det AB = \det A \det B$ for the matrices

$$A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \qquad B = \begin{bmatrix} c & -d \\ d & c \end{bmatrix}$$

to prove the fact that if two integers are expressible as the sum of two squares (of integers), then the same is true of their product.

VI.2.B Let B be an $n \times n$ matrix. What is the determinant f the linear mapping

$$\Phi : A \mapsto AB - BA$$

on M_n?

VI.2.C Show that if A is an $(n+1) \times n$ matrix with integer entries all of whose row sums are 0, show that $\det(AA^t$ has the form nk^2 for some $k \in \mathbf{N}$.

VI.2.D Let

$$A = \begin{bmatrix} A_1 & A_2 & \dots & A_n \end{bmatrix}$$

be an $n \times n$ matrix with columns $A_1, \dots, A_n$. Show that

$$\det \begin{bmatrix} A_2 + \dots + A_n & A_1 + A_3 + \dots + A_n & \dots & A_1 + \dots A_{n-1} \end{bmatrix} = (-1)^{n-1}(n-1) \det A.]$$

VI.2.E Show that if a matrix A is such that $A^2 = I$ and B is the matrix $[A_{ij}]$, then $B^2 = I$.

VI.2.F Let $S^n_{x_1,\dots,x_k} = \sum_{\alpha_1 + \dots + \alpha_k = n} x_1^{\alpha_1} \dots x_k^{\alpha_k}$. Show that

$$S^n_{x_1,\dots,x_k} = \frac{\det \begin{bmatrix} a \end{bmatrix}}{\det \begin{bmatrix} a \end{bmatrix}}.$$

VI.2.G Show that if A is an $n \times n$ matrix, then

$$|\det A| \leq \prod_j \sum_k a_{jk}.$$

Show that if for each i, $a_{ii} \geq \frac{1}{2} \sum_{k=1}^n a_{ik}$, then

$$\det A \geq a_{11} \prod_{i=2}^n (a_{ii} - \sum_{k=1}^n |a_{ik} A).$$

VI.2.H Let (σ_n) be a sequence of real umbers and define

$$v(t) = 1 - \sigma_1 t + \sum_{n=2}^{\infty} \det \left[\ a \ \right] \frac{(-t)^n}{n!}$$

resp.

$$w(t) = 1 + \sigma_1 t + \sum_{n=2}^{\infty} \det \left[\ a \ \right] \frac{t^n}{n!}.$$

Show that $v(t)w(t) = 1$.

VI.2.I Show that if the elements of $p + 2$ columns of a matrix form an arithmetic progression of length p, then the determinant vanishes.

VI.2.J Suppose that $a_{11}, \dots, a_{1n}$ are integers. Show that one can find integers a_{ij} (for $2 \leq i \leq n$, $1 \leq j \leq n$) so that $\det[a_{ij}] = 1$ if and only if th e greatest common divisor of $a_{11}, \dots, a_{1n}$ is 1.

VI.2.K Let $\xi_1, \dots, \xi_n$ be real numbers so that

$$\sum_i \xi_i = 0 \quad \sum_i \xi_i^2 = 0, \dots, \sum_i \xi_i^n = 0.$$

Show that the ξ_i vanish.

VI.2.L Let $A = [a_{ij}]$ be an $n \times n$ matrix. Show that

$$\det(I+A) = 1+\sum_{i=1}^{n} a_{ii}+\frac{1}{2!}\det \begin{bmatrix} a_{i_1,i_1} & \_i_1, i_2 \\ a_{i_2,i_1} & a_{i_2,i_2} \end{bmatrix}+\cdots+\frac{1}{n!}\det \begin{bmatrix} a_{i_1,i_1} & \dots & a_{i_1,i_n} \\ \vdots & & \vdots \\ a_{i_n,i_1} & \dots & a_{i_n,i_n} \end{bmatrix}.$$

VI.2.M Show that if the last element a_{nn} of an $n \times n$ matrix is non-zero, then

$$(a_{nn})^{n-1} \det A = \det B$$

where B is the $(n-1) \times (n-1)$ matrix with

$$b_{ij} = a_{ij} a_{nn} - a_{in} a_{nj}.$$

(This provides an algorithm for calculating determinants by reducing the dimension step by step).

VI.2.N Let $\Phi : L(V) \to \mathbf{C}$ be a linear mapping so that

- $\Phi(f \circ g) = \Phi(f)\Phi(g)$;
- $\Phi(\lambda \mathrm{Id}) = \lambda^n$

where V is a complex vector space of dimension n. Show that $\Phi(f) = \det f$. Show that the corresponding result for real vector spaces hold if n is odd but that if n is even we require the additional condition that $\Phi(f) = -1$ for some reflection in $L(V)$ to conclude that Φ is the determinant.

VI.2.O (Sylvester's criterium)

6.3 Determinants and smooth functions

In this section, we gather some exercises on determinants involving derivatives of smooth functions. In particular, they can be used to give a particularly elegant formulation of the chain rule for higher derivatives.

IV.3.A Let x and y be continuously differentiable functions on the interval $[a, b]$. What well known result of analysis can one deduce by applying Rolle's theorem to the function

$$t \mapsto \det \begin{bmatrix} 1 & 1 & 1 \\ x(t) & x(a) & x(b) \\ y(t) & y(a) & y(b) \end{bmatrix}$$

VI.3.B Let

$$x_1 = rc_1c_2 \dots c_{n-2}c_{n-1} \tag{9}$$
$$x_2 = rc_1 \dots c_{n-2}s_{n-1} \tag{10}$$
$$vdots \tag{11}$$
$$x_j = rc_1 \dots c_{n-j}s_{n-j+1} \tag{12}$$
$$\vdots \tag{13}$$
$$x_n = rs_1 \tag{14}$$

where $c_i = \cos\theta_i$, $s_i = \sin\theta_i$ (these are the equations of the transformation to polar coordinates in n dimensions). Calculate the Jacobi matrix

$$\frac{\partial(x_1, \dots, x_n)}{\partial(r, \theta_1, \dots, \theta_{n-1})}$$

and show that its determinant is $r^{n-1}c_1^{n-2}c_2^{n-3} \dots c_{n-2}$.

VI.3.C Suppose that x is a smooth function on $[a, b]$ and that

$$a = t_0 < t_1 < \cdots < t_n = b.$$

Show tht there is an $s \in]a, b[$ so that

$$\det \left[\; a \; \right] = ??$$

VI.3.D Show that if $x_1, \dots, x_n$ and $y_1, \dots, y_n$ are continuous functions on $[a, b]$, then

$$\det \left[\int_a^b x_i(t) y_j(t) \, dt \right] = \frac{1}{n!} \int \dots \int \det[x_i(t_j)] \, dt_1 \dots dt_n.$$

VI.3.E Let $t \mapsto A(t)$ be a continuous function from $\mathbf{R}$ into the set M_n of $n \times n$ matrices and let $X_1, \dots, X_n$ be solutions of the equation

$$X'(t) = A(t) X(t).$$

Show that if

$$W(t) = \det[X_1(t) \dots X_n(t)]$$

then

$$W(t) = W(t_0) \exp \int_{t_0}^t \operatorname{tr} A(t) \, dt.$$

Show that if the function is continuously differentiable, then

$$\frac{d}{dt} \det A(t) = \operatorname{tr}(\operatorname{adj} A(t)) A'(t)$$

and thus if each $A(t)$ is invertible, then this can be simplified to the form

$$\operatorname{tr}(A^{-1}(t) A'(t)) \det A(t).$$

VI.3.F Let $x_1, \dots, x_n$ be linearly independent smooth functions on an interval, $k \in \mathbf{N}$. Calculate

$$lim_{h \to 0} h^{-k} \det \begin{bmatrix} x_1(t) & \dots & x_n(t) \\ x_1(t+h) & \dots & x_n(t+h) \\ \vdots & & \vdots \\ x_1(t+(n-1)h) & \dots & x_n(t+(n-1)h) \end{bmatrix}.$$

VI.3.G If $x_1, \ldots, x_n$ are smooth functions on $[0,1]$, the Wronskian $W(x_1, \ldots, x_n)$ is the function

$$t \mapsto \det \begin{bmatrix} x_1(t) & \ldots & x_n(t) \\ x_1'(t) & \ldots & x_n'(t) \\ \vdots & & \vdots \\ x_1^{(n)}(t) & \ldots & x_n^{(n)}(t) \end{bmatrix}.$$

Show that this either has no zeroooos or is identically zero. Show that if the x_i are analytic, then the latter is the case f and only if the x_i are linearly independent. Show that if x is a further function, then

$$W(xx_1, \ldots, xx_n) = x^n W(x_1, \ldots, x_n)$$

and use this to evaluate the determinant

$$\det \begin{bmatrix} 1 & \frac{1}{2!} & \ldots & \frac{1}{n!} \\ \frac{1}{2!} & \frac{1}{3!} & \ldots & \frac{1}{(n+1)!} \\ \vdots & & & \vdots \\ \frac{1}{n!} & \frac{1}{(n+1)!} & \ldots & \frac{1}{(2n-1)!} \end{bmatrix}.$$

VI.3.H Let x be a smooth function. Show that if

$$D_n = \det \begin{bmatrix} x' & x & 0 & \ldots & 0 \\ \frac{x''}{2} & x' & x & \ldots & 0 \\ \vdots & & & & \vdots \\ \frac{x^{(n)}}{n!} & \frac{x^{(n-1)}}{(n-1)!} & \frac{x^{(n-2)}}{(n-2)!} & \ldots & x' \end{bmatrix},$$

then

$$D_{n+1} = x'D_n - \frac{1}{n+1} x \cdot D_n'.$$

Show that if $d_r = \frac{1}{(r-1)!} \frac{d^r \ln x}{dx}$, then

$$x^{(n)} = \det \begin{bmatrix} d_1 & -1 & 0 & \ldots & 0 \\ d_2 & d_1 & -2 & \ldots & 0 \\ \vdots & & & & \vdots \\ d_n & d_{n-1} & d_{n-2} & \ldots & s_1 \end{bmatrix} x^n.$$

VI.3.I Show that if z is the composition of the smooth function x and y (i.e. $z = x \circ y$), then

$$z^{(n)} = \det \left[\; a \; \right].$$

VI.3.J Let x be a function which is smooth in the neighbourhood of a point a and define

$$T_{n,a}x(t) = x(a) + x'(a)(t-a) + \cdots + \frac{x^{(n)}(a)}{n!}(t-a)^n$$

(i.e. the Taylor approximation to x of degree n). Show that $T_{n,a}x(t)$ is equal to

$$\det \left[\ a \ \right]$$

and that

$$f(x) - T_{n,a}f(x) = \det \left[\ a \ \right].$$

7 Complex numbers

7.1 Complex numbers and geometry

VII.1.A Show that if z_1 and z_2 are distinct complex numbers, then

$$\det \begin{bmatrix} z & \bar{z} & 1 \\ z_1 & \overline{z_1} & 1 \\ z_2 & \overline{z_2} & 1 \end{bmatrix} = 0$$

is the equation of the straight line through $z-1$ and z_2.

VII.1.B Let z_1, z_2 and z_3 be complex numbers all with absolute value 1 and put

$$s = z_1 + z_2 + z_3 \tag{15}$$
$$t = z_1z_2 + z_2z_3 + z_3z_1 \tag{16}$$
$$p = z_1z_2z_3. \tag{17}$$

Show that

- $\frac{s}{3}$ is the centroid of the triangle with vertices at z_1,z_2 and z_3;
- orthocentre
- circumcentre

Show that the Euler line of the trianlge is

$$\{z : z = \frac{\bar{s}}{s}z\} = \{z : \Im s\bar{z} = 0\}$$

and that the nine-0nt circle has equation

$$z = \frac{1}{4}(z - \frac{s}{2})^{-1} + \frac{\bar{s}}{2}.$$

VII.1.C Let z_0, z_1 and z_2 be distinct complex numbers and define

$$R(z_0, z_1, z_2) = \frac{z_2 - z_0}{z_1 - z_0}.$$

Show that the numbers are collinear if and only if this is real. Similarly, we define $CR(z_0, z_1; z_2, z_3)$ (the **cross-ratio**) of four distinct points to be

$$\frac{R(y_0, y_1, y_2)}{R(z_0, z_1, z_3)}.$$

Show that this is real if andonly if the points lie on a circle or a straight line.

VII.1.D Show that if $\mathbf{C}$ is the circle

$$az\bar{b} + bz + \bar{b}\bar{z} + c = 0$$

then $\frac{c}{a}$ is the power of 0 with respect to the circle. More generally, the power of w is

$$w\bar{w} + \frac{b}{a}w - \frac{\bar{b}}{a}\bar{w} + \frac{c}{a}.$$

Deduce that the locus of the set of points with constant power with respect to a given circle is itself a circle. If $\mathbf{C}$ and $\mathbf{C}_1$ are two circles, the locus of the points which have the same power with respect to $\mathbf{C}$ and $\mathbf{C}_1$ is a straight line which passes through the intersection of $\mathbf{C}$ and $\mathbf{C}_1$ (if they intersect).

VII.1.E Let $\mathbf{C}$ be the circle

$$(z - a)\overline{(z - a)} = r^2$$

and define the mapping ϕ by

$$\phi(z) = \frac{r^2}{\bar{z} - \bar{a}}$$

i.e. $\phi(z)$ is the inverse of z in $\mathbf{C}$. Show that

- $\mathbf{C} = \{z \in \mathbf{C} : \phi(z) = z\}$;
- if z_1, z_2, z_3, z_4 are points which are collinear with a, then

$$CR(z_1, z_2, z_3, z_4) = CR(\phi(z_1), \phi(z_2), \phi(z_3), \phi(z_4));$$

- the mapping ϕ maps circles and straight line onto circles and straight lines;

- a circle $\mathbf{C}_1$ cuts $\mathbf{C}$ at right angles if and only if $\phi(\mathbf{C}_1) = \mathbf{C}_1$;
- if $\mathbf{C}_1 = \phi(\mathbf{C}_2)$ for two circles $\mathbf{C}_1$ and $\mathbf{C}_2$, then a is a centre of similitude for $\mathbf{C}_1$ and $\mathbf{C}_2$ (i.e.);
- if $\mathbf{C}_1$ and $\mathbf{C}_2$ are two circles which are not concentric and have distinct radii, then there is a circle $\mathbf{C}$ as above so that $\mathbf{C}_2 = \phi(\mathbf{C}_2)$.

VII.1.F Show that if z_1,z_2, z_3 and z_4 are complex numbers, then

$$(z_1 - z_4)(z_2 - z_3) + (z_2 - z_4)(z_3 - z_1) + (z_3 - z_4)(z_1 - z_2) = 0.$$

Deduce that if A, B, C and D are four points in the plane, then

$$|AD||BC| \leq |BD||CA| + |CD||AB.$$

VII.1.G Let $z_1, \dots, z_n$ be points on the unit circle. Show that

$$\prod_{i \neq j} |z_i \overline{z_j} - 1| \leq n^n$$

with equality if and only if the z_i are the vertices of a regular n-gon.

7.2 Complex numbers and quaternions

complex numbers also allow an elegant approach to the theory of quaternions as the next examples show:

VII.2.A Consider the set $\tilde{\mathbb{Q}}$ of ordered pairs (z, w) of complex numbers with the natural addition and multiplication defied as follows:

$$(z_0, w_0)(z_1, w_1) = (z_0 z_1 - w_0 w_1, z_0 w_1 + z_1 w_0).$$

Show that they satisfy all of the axioms of a field with the exception of the commutativity of multiplication (such structures are called **skew fields**).

VII.2.B Show that there is a natural isomorphism from the set $\mathbb{Q}$ of quaternions and the set $\tilde{\mathbb{Q}}$ above whereby we map $\mathbf{i}$ onto $(i, 0)$, $\mathbf{j}$ onto $(0, i)$ and $\mathbf{k}k$ onto ??

VII.2.C A quaternion of the form $(x, 0)$ $(x \in \mathbf{R})$ is said to be real. Show that a quaternion is real if and only if it commutes with all quaternions.

VII.2.D A **double number** is a number $z = x + ey$ whereby $e^2 = 1$. More formally, the set of double nnumbers is $\mathbf{R}^2$ provided with the multiplication

Then we define

$$\Re x = x \quad \Im z = y \quad \bar{z} = x - ey \quad z \cdot \bar{z} = x^2 - y^2.$$

Show that

7.3 Polynomials

VII.3.A Suppose that λ_1, λ_2 and λ_3 are the roots of the cubic

$$z^3 + pz^2 + qz + r = 0.$$

Show that

Similarly, show that if $\lambda_1, \ldots \lambda_4$ are the roots of

$$z^4 + pz^3 + qz^2 + rz + s = 0$$

then

VII.3.B Show that the roots of the equation

$$a_0 z^n + a_1 z^{n-1} + \cdots + \_n = 0$$

are the vertices of a regular n-gon if

$$n^k a_k a_0^{k-1} = \binom{n}{k} a_1^k$$

for ???? What are the special conditions for the case $n = 1$?

VII.3.C Show that if p and q are polynomials with leading coefficients 1 where f is of degree with distinct zero $\lambda_1, \ldots, \lambda_n$ and g has degree $n - 1$, then

$$\sum g(\lambda_i) f'(\lambda_j) = 1.$$

VII.3.D Consider the polynomial

$$p(z) = z^n + a_1 z^{n-1} + \cdots + a_n$$

with zeroes $\lambda_1, \ldots, \lambda_n$ and

$$p^*(z) = z^n + a_1^* z^{n-1} + \cdots + a_n^*.$$

Shwo that if z is a zero of p, then

$$|z| \leq \max\{1, \sum |a_k|\}.$$

Show that if z^* is a root of p of multiplicity m and if $\epsilon > 0$, then there is a $\delta > 0$ so that if $\max |a_n - a_n^*| < \delta$, then p has m roots within ϵ of z^*. (Suppose that $p_k \to p$ and that p_k has less than m roots within ϵ of z^*. Use the fact taht the roots are bounded and so have a convergent subsequence to get a contradiction.

8 Eigenvalues, diagonalisation

8.1 Eigenvalues of concrete matrices

VIII.1.A Let A be a complex $n \times n$ matrix of the form

$$\begin{bmatrix} a_1 & 0 & \ldots & 0 & b_1 \\ 0 & a_2 & \ldots & b_2 & 0 \\ \vdots & & & & \vdots \\ b_n & 0 & \ldots & 07\_n & \end{bmatrix}$$

i.e. where $a_{ij} = 0$ if $i + j \neq n + 1$ and $i - j \neq n + 1$. Calculate the eigenvalues of A. For which values of the a's and b's do its eigenvectors span $\mathbf{C}^n$?

VIII.1.B What are the eigenvalues of the matrix

$$\begin{bmatrix} 2a & b & 0 & 0 & 7\ldots & 0 \\ b & 2a & b & 0 & \ldots & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & \ldots & b & 2a \end{bmatrix}$$

VIII.1.C Find the eigenvalues of the matrix $A = [a_{ij}]$ where $a_{ij} = \dfrac{k_i}{k_j}$ for a suitable sequence $k_1, \ldots, k_n$ of non-zero numbers.

8.2 Difference equations

The method of diagonalising a matrix can be used to give an elegant treatment of some problems involving the difference equations—for example, questions involving the Fibonacci numbers. We bring some examples of a similar nature.

VIII.2.A Show that there is precisely one sequence (a_n) of non-negative numbers which satisfies the recursion relation

$$a_1 = 1 \text{ and } a_{n+2} = a_n - a_{n+1}.$$

VIII.2.B Let (x_n) be a given sequence. For $k > o$, define a new sequence by the relations

$$y_1 = x_1 \tag{18}$$
$$y_n = kx_n + x_{n-2} \quad (n \geq 2). \tag{19}$$

for which k do we have the following: the sequence (y_n) is convergent if an only if (k_n) is?

VIII.2.C Show that if f_n is the n-th Fibonacci number, then

$$f_{n+1}^3 + f_n^3 + f_{n-1}^4 = f_{3n} \tag{20}$$
$$f_{n+2}^3 - 3f_n^3 + f_{n-2}^3 = ?f_{3n}. \tag{21}$$

VIII.2.D Outline a method for solving the difference equation

$$x_n = a_1 x_{n-1} + a_2 x_{n-2} + \cdots + a_k x_{n-k}.$$

Apply it to the equation

$$x_{n+1} = x_n + x_{n-1} + \cdots + x_{n-k+1}.$$

VIII.2.E Define a sequence (t_n) by putting

$$t_1 = 2, t_2 = 3, t_{2n} = ??$$

Show that $t_{2n} = \frac{\beta_n - \alpha_n}{2}$, $t_{2n+1} = 2\beta_n + \frac{3\alpha_n}{2}$ where the α_n and the β_n are define by the relationships

$$\alpha_n I + \beta_n A = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}$$

where $A = ??$.

VIII.2.F A sequence (x_n) of functions is defined by the recurrence relation:

$$x_n(t) = tx_{n-1}(t) + x_{n-2}(t).$$

Show that

$$x_n^2(t) \leq (t^2+1)^2(t^2+2)^{n-3}.$$

8.3 Location of eigenvalues

For some applications it is important to have *a priori* estimates for the eigenvalues of a matrix (i.e. estimates involves the elements of the matrix). We bring two such estimates:

VIII.3.A Show that if A is an $n \times n$ complex matrix, then its eigenvalues lie in the set:

$$\bigcup_i \{\lambda \in \mathbf{C}; |\lambda - a_{ii}| \leq \sum_{j \neq i} |a_{ij}|.$$

VIII.3.B Show that if A is a real $n \times n$ matrix, then its eigenvalue λ satisfy the inequality

$$|\Im \lambda| \leq \frac{n(n-1)^{1/2}}{2} \max_{1 \leq i,j \leq n} |a_{ij} - a_{ji}|^{1/2}.$$

(This is a quantitative version of the fact that the eigenvalues of a symmetric matrix are real).

8.4 General

We close this Chapter with some general properties of eigenvalues:

VIII.4.A Show that if A is a complex $N \times n$ matrix and λ is an eigenvalue, then it is also an eigenvalue of $\operatorname{adj} A$. Show that

$$\chi_{\operatorname{adj} A}(\lambda) = (-1)^n \frac{\lambda^n}{\det A} \chi_A(\frac{\det A}{\lambda})$$

and that if λ is not an eigenvalue of A, then

$$\operatorname{tr}(\lambda I - A)^{-1} = \frac{\chi_A'(\lambda)}{\chi_A(\lambda)}.$$

VIII.4.B Show that if A is an $n \times m$ matrix and B is an $m \times n$ matrix, then

$$\lambda^m \chi_{AB}(\lambda) = \lambda^n \chi_{BA}(\lambda).$$

What can you deduce about the relation between the eigenvalues of AB and those of BA?

VIII.4.C Show that if A has a block representation

$$\begin{bmatrix} B & C \\ 0 & D \end{bmatrix}$$

where B and C are square, then λ is an eigenvalue of A if and only if it is an eigenvalue of B or of D.

VIII.4.D Let A be an $n \times n$ matrix, X a $1 \times n$ and Y an $n \times 1$ matrix. Show that the matrix

$$B = \begin{bmatrix} A & -AY \\ -XA & XAY \end{bmatrix}$$

is singular.

Show that if $\det A = 0$, then 0 is an eigenvalue of multiplicity at least two of the above matrix.

VIII.4.E Let P and Q be $n \times n$ matrices, where we assume that P is non-singular and that A has distinct eigenvalues. Show that the polynomials $\det(\lambda I - Q)$ and $\det(\lambda P - PQ)$ have the same non-zeroes. Hence give a new proof of the fact that

$$\det(PQ) = \det Q \cdot \det Q.$$

Show that the latter equation can be extended to general P and Q by using a continuity argument.

VIII.4.F Suppose that A is an $n \times n$ matrix with rank r. Then as we know, A can be expressed as a product BC where

$$B = \begin{bmatrix} B_1 & 0 \end{bmatrix}$$

,

$$C = \begin{bmatrix} C_1 \\ 0 \end{bmatrix}$$

where B_1 is an $n \times r$ matrix and C_1 is $r \times n$. Show that

$$\chi_A(\lambda) = \lambda^{n-r} \chi_D(\lambda)$$

where $D = C_1 B_1$. Use this to calculate χ_A where $A = [b_i c_j]$.

VIII.4.G Let

$$X = \left[\begin{array}{ccc} X_1 & \dots & X_n \end{array} \right]$$

be a non-singular $n \times n$ complex matrix (i.e. the X are the columns of X) and put

$$Y = \left[\begin{array}{ccccc} X_2 & X_3 & \dots & X_n & 0 \end{array} \right]$$

Show that ${}^{Y}X^{-1}$ has rank $n-1$ and that 0 is its only eigenvalue. Show that if A is a matrix with rank $n-1$ and 0 as its only eigenvalue, then it has the above form.

VIII.4.H Let A and B be fixed $n \times n$ matrices. Show that there is a matrix X so that

$$X^2 - 2AX + B = 0$$

provided that the matrix

$$\left[\begin{array}{cc} A & I \\ A^2 - B & A \end{array} \right]$$

has $2n$ distinct eigenvalues.

VIII.4.I Let A and B be fixed $n \times n$ matrices. Show that the operator

$$\Phi : X \mapsto AX + XB$$

is a linear operator on M_n with eigenvalues $\{\lambda_i + \mu_j\}$ where $\{\lambda_i\}$ resp. $\{\mu_j\}$ are the eigenvalues of A resp. B. Deduce a criterium for Φ to be invertible.

VIII.4.J Show that if A and B are commuting $n \times n$ matrices and p is a polynomial in two variables, then the eigenvalues of $p(A, B)$ are the numbers $p(\lambda_j, \mu_j)$ for some numbering $\lambda_1, \dots, \lambda_n$ resp., $\mu_1, \dots, \mu_n$ of the eigenvalues of A and B.

VIII.4.K Let A and B be complex $n \times n$ matrices so that $AB = \omega BA$ where ω is the primitive q-th root of unity. Show that if λ is an eigenvalue of A with eigenvector X so that $BX \neq 0$, then $\omega\lambda$ is an eigenvalue of B.

Show that the eigenvalues of A resp. B can be numbered as $\lambda_1, \dots, \lambda_n$ resp. $\mu_1, \dots, \mu_n$ in such a way that the eigenvalues of $A + B$ resp. AB are

$$\{(\lambda_i^q + \mu_i^q)^{1/q} : i = 1, \dots, n\}$$

resp.

$$\{\omega^{\frac{q-1}{2}} \lambda_i \mu_i : i = 1, \dots, n\}.$$

9 The Jordan form

9.1 The functional calculus

One of the most fundamental consequences of the existence of the Jordan form is the fact that every operator f has a unique representation $f = d + n$ where d is diagonalisable, n is nilpotent and d and n commute. The if x is a function which is such that ???? we define

$$x(f) = \sum_k \sum_{r=0}^{n} \frac{(f - \lambda_k \mathrm{Id})^r}{r!} x^{(r)}(\lambda_k) E(\lambda_k) = \sum_{r=1}^{n} \int_{\sigma(T)} x^{(r)}(\lambda) E \, d\lambda.$$

IX.A.1 Show that if the eigenvalues of f are distinct, then

$$x(f) = \sum_k x(\lambda_k) \prod_{k \neq j} \frac{f - \lambda_j \mathrm{Id}}{\lambda_k - \lambda_j}.$$

IX.A.? A particularly important case is that where x is the exponential function, when we write e^f resp. e^A for the corresponding operator or matrix. Show that if A has precisely one eigenvalue λ, then

$$e^A = e^\lambda \sum_{k=0}^{n} \frac{1}{k!} (A - \lambda I)^k.$$

Show that if A has n distinct eigenvalues $\lambda_1, \dots, \lambda_n$, then

$$e^A = \sum e^{\lambda_k} L_k(A)$$

where

$$L_k(t) = \prod_{j \neq k} \frac{t - \lambda_j}{\lambda_k - \lambda_j}.$$

Use this to calculate e^A where

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

resp.

$$A = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$$

resp. e^D where D is the differentiation operator on $\mathrm{Pol}\,(n)$.

IX.A.? Verify the following identities:

- $e^{A+B} = e^A \cdot e^B$ when A and B commute;
- $\dfrac{d}{dt} e^{tA} = A \cdot e^{tA}$;
- $e^A = \lim_{n\to\infty} (I + \frac{A}{n})^n$;
- $\det(\exp A) = \exp(\operatorname{tr} A)$.

IX.A.4 Show that if all of the entries of A (with the possible exception of the diagonal ones) are positive, then all of the entries of e^A are positive.

IX.A.5 Show that if A is an $n \times n$ matrix such that $A^2 = -I$, then

$$e^{tA} = I \cos t + A \in t.$$

Use this to solve the equation

$$y'' + y = 0.$$

IX.A.6 Show that the general solution of the equation

$$\frac{dX}{dt} = AX + B$$

where B is a continuous mapping form $\mathbf{R}$ into $M_{n,1}$ with initial condition $X(0) = x_0$ is given by the equation

$$X(t) = \int_0^t e^{(t-s)A} B(s)\, ds + e^{tA} X_0.$$

use this to solve the system

$$\frac{dx}{dt} = \quad y- \quad 12 + \quad e^{-3t} \tag{22}$$

$$\frac{dy}{dt} = \quad -x+ \quad 7y- \quad 20z \tag{23}$$

$$\frac{dz}{dt} = \quad x- \quad -5z \quad +\cos t. \tag{24}$$

IX.A.7 If A is an $n \times n$ matrix, define functions a_{ij} by the equation

$$e^{tA} = [a_{ij}(t)].$$

Show that

$$\chi_A(D) a_{ij}(t) = 0$$

where D is the differentiation operator.

(This reduces the calculation of e^{tA} to the solution of the differential equations

$$\chi_A(D) G(t) = 0$$

with initial conditions

$$G(0) =), \quad G'(0) = A, \ldots, G^{(n-1}(0) = A^{n-1}.$$

Use this to calculate e^{tA} where

$$A = \begin{bmatrix} -2 & -1 & -1 \\ 6 & 3 & 2 \\ 471 & 3 & \end{bmatrix}.$$

9.2 Miscellaneous

We conclude with a number of theoretical exercises where the Jordan form can be used to advantage:

IX.C.1 Let A be an $n \times n$ matrix with characteristic polynomial

$$(-1)^n \chi_A(\lambda) = \lambda^n + a_1 \lambda^{n-1} + \cdots + a_n.$$

Show that

$$a_k = -\frac{1}{k}(a_{k-1} S_1 + a_{k-2} S_2 + \cdots + S_k)$$

where $S_k = \mathrm{tr}\,(A^k)$. (This provides an algorithm for calculating the coefficients of the characteristic polynomial without computing determinants).

IX.C.2 If p is a polynomial of degree n, say

$$p(\lambda) = \lambda^n + a_1 \lambda^{n-1} + \cdots + a_n$$

we put

$$A = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & & & & \vdots \\ -a_n n & -a_{n-1} & -a_{n-2} & \ldots & -a_1 \end{bmatrix}.$$

A is called the **companion matrix** of p. Show that its characteristic polynomial is p and calculate a Jordan form for A. (consider first the special cases where the eigenvalues of A (i.e. the zeroes of p) are distinct, respectively where they all coincide. Show how to use this result to solve the differential equation $p(D)x = 0$.

IX.C.3 Let A be a fixed $n \times n$ matrix. What is the characteristic polynomial of the mapping

$$B \mapsto AB$$

on M_n?

IX.C.4 Characterise the set of all matrices which commute with the Jordan matrix $J_n(\lambda)$ resp. those matrices which commute with all matrices which commute with $J_n(\lambda)$. Use this to show that a matrix B is of the form $p(A)$ where A is a fixed $n \times n$ matrix if and only if B commutes with every matrix which commutes with A.

IX.C.5 Let λ be an eigenvalue of the operator f on V. Define

$$E_\lambda = \bigcup_k \operatorname{Ker}(f - \lambda I)^k.$$

We say that the non-zero vector x in V is a p-eigenvector of f if $(f - \lambda I)^p(x) = 0$ but $(f - \lambda I)^{p-1}(x) \neq 0$. Show that x is then a p eigenvector for f^{-1} provided that the latter exists.

IX.C.6 Let A be an $n \times n$ matrix with distinct

10 Euclidean spaces

X.1.A Show that if x, y, z are vector in a euclidean space, then

$$\|x + y + z\| + \|x\| + \|y\| + \|z\| \geq \|x + y\| + \|y + z\| + \|z + x\|.$$

X.1.B Show that if $(a_1, \dots, a_n)$ is a sequence of non-negative numbers, then

$$\sum_{i=1}^{n} \left(\frac{a_i}{i}\right)^2 \leq \sum_{i,j=1}^{n} \frac{a_i a_j}{i + j}.$$

X.1.C Show that if $a_1 \geq 0 \cdots \geq a_n \geq 0$, then $\sum_{j=1}^{k} \sum_{j=1}^{k} b_j$ implies that $\sum_{j=1}^{n} a_j^2 j = 1^n b_j^2$.

X.1.D Show that if (ξ_i) and (η_i) are vectors in $\mathbf{R}^n$, then

$$(\sum \xi_i^2)(\sum \eta_i^2) - (\sum \xi_i \eta_i)^2 = \sum_{i \leq i < j \leq n} (\xi_i \eta_j - \xi_j \eta_i)^2.$$

(this is a quantitative version of the Cauchy-Schwarz inequality).