Introduction to Several Complex Variables

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1 Preliminaries

First we introduce some notational conventions.

If $w, z \in \mathbb{C}^n$, $w = (w_1, \dots, w_n)$, $z = (z_1, \dots, z_n)$ then

$$\langle w|z\rangle = \sum w_j \bar{z}_j,$$

$$|z| = \sqrt{\langle z|z\rangle} = \big(\sum |z_j|^2\big)^{1/2}.$$

A multi-index is any n-tuple $\alpha = (\alpha_1, \dots, \alpha_n)$ with $\alpha_j \in \mathbb{N}$ (we include 0 in $\mathbb{N}$). If α is a multi-index then

$$\alpha! = \alpha_1! \dots \alpha_n!,\ |\alpha| = \alpha_1 + \dots + \alpha_n,\ z^\alpha = z_1^{\alpha_1} \dots z_n^{\alpha_n}.$$

Exercise.

$$(z+w)^\alpha = \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!} z^\beta w^\gamma$$

where β, γ are multi-indices and the sum $\beta+\gamma$ is taken in the ordinary vector sense.

Exercise.

$$\Big(\sum_{j=1}^n a_j\Big)^k = \sum_{|\alpha|=k} \frac{k!}{\alpha!} a^\alpha$$

where $a = (a_1, \dots, a_n)$.

If $\boldsymbol{r} = (r_1, \dots, r_n)$, $r_j > 0$ then the polydisk centered at w with polyradius $\boldsymbol{r}$ is

$$D^n(w, \boldsymbol{r}) = \{z \in \mathbb{C}^n : |z_j - w_j| < r_j,\ \ j = 1, \dots, n\}.$$

A ball centered at w with radius ρ is

$$B(w, \rho) = \{z \in \mathbb{C}^n : |z - w| < \rho\}.$$

If $z = (z_1, \dots, z_n)$ then $z' = (z_1, \dots, z_{n-1})$ so that $z = (z', z_n)$.

Whenever it is convenient we will identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$. Thus a vector $w = (w_1, \dots, w_n) = (a_1 + ib_1, \dots, a_n + ib_n)$ is the same as $(a_1, \dots, a_n, b_1, \dots, b_n)_{\mathbb{R}}$. To get used to this identification let's realize how multiplication by i works on this $\mathbb{R}^{2n}$. We have

$$\begin{aligned} &i(a_1, \dots, a_n, b_1, \dots, b_n)_{\mathbb{R}} = iw = (iw_1, \dots, iw_n) \\ &= (-b_1 + ia_1, \dots, -b_n + ia_n) = (-b_1, \dots, -b_n, a_1, \dots, a_n)_{\mathbb{R}}. \end{aligned}$$

If D, Ω are subsets of $\mathbb{R}^m$ then we will write $D \subset\subset \Omega$ if $\overline{D} \subset \operatorname{int}\Omega$ and $\overline{D}$ is compact. Thus for instance $D \subset\subset \mathbb{R}^m$ means simply that D is bounded.

Whenever we say that Ω is a domain we mean that it is open and connected.

By ν we will always denote the Lebesgue measure of the underlying space, σ will be reserved for surface measures.

Uniform convergence on compact subsets of a given set plays a very important role in function theory of several complex variables and it is usually referred to as normal convergence. We will always use this term in this meaning.

2 Complex derivatives

2.1 Definitions

Let $\Omega \subset \mathbb{C}^n$ be open and $f\colon \Omega \to \mathbb{C}$ be differentiable in the real sense at a point $P \in \Omega$. We define complex derivatives of f at P as

$$\frac{\partial f}{\partial z_j}(P) = \frac{1}{2}\Big(\frac{\partial f}{\partial x_j}(P) - i\frac{\partial f}{\partial y_j}(P)\Big) \tag{2.1.1}$$

$$\frac{\partial f}{\partial \bar{z}_j}(P) = \frac{1}{2}\Big(\frac{\partial f}{\partial x_j}(P) + i\frac{\partial f}{\partial y_j}(P)\Big). \tag{2.1.2}$$

These definitions are extremely useful but they are purely formal and do not carry any particular geometrical meaning.

As a consequence of (2.1.1) and (2.1.2) we get

$$\frac{\partial f}{\partial x_j}(P) = \frac{\partial f}{\partial z_j}(P) + \frac{\partial f}{\partial \bar{z}_j}(P)$$

$$\frac{\partial f}{\partial y_j}(P) = i\Big(\frac{\partial f}{\partial z_j}(P) - \frac{\partial f}{\partial \bar{z}_j}(P)\Big).$$

2.2 Rules of complex differentiation

$$\frac{\partial}{\partial z_j}(f \circ g) = \sum_k \frac{\partial f}{\partial z_k}\frac{\partial g_k}{\partial z_j} + \sum_k \frac{\partial f}{\partial \bar{z}_k}\frac{\partial \bar{g}_k}{\partial z_j}$$

and similarly for $\frac{\partial}{\partial \bar{z}_j}$. We also have

$$\overline{\left(\frac{\partial f}{\partial z_j}\right)} = \frac{\partial \bar{f}}{\partial \bar{z}_j}.$$

We can express the action of the real global derivative of a function f on a vector $w \in \mathbb{C}^n$ as

$$D_{\mathbb{R}} f(P)(w) = \sum_{j=1}^{n} \frac{\partial f}{\partial z_j}(P) w_j + \sum_{j=1}^{n} \frac{\partial f}{\partial \bar{z}_j}(P)\overline{w}_j \tag{2.2.1}$$

To prove this let $w = (a_j + ib_j)_j$. Then

$$\begin{aligned} D_{\mathbb{R}} f(P)(w) &= \sum \frac{\partial f}{\partial x_j}(P) a_j + \sum \frac{\partial f}{\partial y_j}(P) b_j \\ &= \sum \Big(\frac{\partial f}{\partial z_j} + \frac{\partial f}{\partial \bar{z}_j} \Big) a_j + \sum i \Big(\frac{\partial f}{\partial z_j} - \frac{\partial f}{\partial \bar{z}_j} \Big) b_j \\ &= \sum \frac{\partial f}{\partial z_j}(a_j + ib_j) + \sum \frac{\partial f}{\partial \bar{z}_j}(a_j - ib_j) \\ &= \sum \frac{\partial f}{\partial z_j} w_j + \sum \frac{\partial f}{\partial \bar{z}_j} \overline{w}_j \end{aligned}$$

We define higher order complex derivatives in the obvious way. Of course these derivatives do not depend on the order of differentiation provided that f is smooth enough. Sometimes we will use shorthand notation like

$$\frac{\partial^{|\alpha|+|\beta|} f}{\partial z^\alpha \partial \bar{z}^\beta} = \frac{\partial^{\alpha_1}}{\partial z_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial z_n^{\alpha_n}} \frac{\partial^{\beta_1}}{\partial \bar{z}_1^{\beta_1}} \cdots \frac{\partial^{\beta_n}}{\partial \bar{z}_n^{\beta_n}} f.$$

Observe also that

$$\frac{\partial}{\partial z_j} \frac{\partial}{\partial \bar{z}_j} = \frac{1}{4} \left(\frac{\partial}{\partial x_j} - i \frac{\partial}{\partial y_j} \right) \left(\frac{\partial}{\partial x_j} + i \frac{\partial}{\partial y_j} \right) = \frac{1}{4} \left(\frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y_j^2} \right)$$

hence we get that

$$\sum_{j=1}^{n} \frac{\partial^2 f}{\partial z_j \partial \bar{z}_j} = \frac{1}{4} \Delta f. \tag{2.2.2}$$

To get a bit more feeling about complex derivatives let us derive the complex form of the Taylor expansion.

If $\Omega \subset \mathbb{R}^m$ and $f\colon \Omega \to \mathbb{R}^p$ is of class C^k in Ω then

$$\begin{aligned} f(x_0 + h) &= \sum_{j=0}^{k} \frac{1}{j!} D_{\mathbb{R}}^j f(x_0)(h^j) + o(\|h\|^k) \\ &= \sum_{j=0}^{k} \frac{1}{j!} \sum_{|\alpha|=j} \frac{|\alpha|!}{\alpha!} \frac{\partial^{|\alpha|} f}{\partial x^\alpha}(x_0) h^\alpha + o(\|h\|^k) \end{aligned}$$

The reason for the factor $|\alpha!|/\alpha!$ is that to each $\partial^{|\alpha|}/\partial x^\alpha$ correspond $|\alpha!|/\alpha!$ derivatives which have a ‘different ordering’ of variables.

If we consider the differential operator

$$\sum_{l=1}^{m} \frac{\partial}{\partial x_l}(x_0)h_l \tag{2.2.3}$$

corresponding to the global derivative evaluated on the vector h then the composition of j of such operators gives

$$\left(\sum_{l=1}^{m} \frac{\partial}{\partial x_l}(x_0)h_l\right)^j = \sum_{|\alpha|=j} \frac{|\alpha|!}{\alpha!}\frac{\partial^{|\alpha|}}{\partial x^\alpha}(x_0)h^\alpha$$

which is (modulo $1/j!$) exactly the j-th term of the Taylor expansion.

Now let us pass to $\mathbb{C}^n \approx \mathbb{R}^{2n}$ substituting P for x_0 and w for h. It directly follows from (2.2.1) that in this case operator (2.2.3) can be written as

$$\sum_{l=1}^{n} \frac{\partial}{\partial z_l}(P)w_l + \sum_{l=1}^{n} \frac{\partial}{\partial \bar{z}_l}(P)\overline{w}_l$$

hence the j-th term of the Taylor expansion will be:

$$\frac{1}{j!}\left(\sum_{l=1}^{n} \frac{\partial}{\partial z_l}(P)w_l + \sum_{l=1}^{n} \frac{\partial}{\partial \bar{z}_l}(P)\overline{w}_l\right)^j = \frac{1}{j!}\sum_{|\alpha|+|\beta|=j} \frac{(|\alpha|+|\beta|)!}{\alpha!\beta!}\frac{\partial^{|\alpha|+|\beta|}}{\partial z^\alpha \partial \bar{z}^\beta} w^\alpha \overline{w}^\beta.$$

Thus we get Taylor's formula

$$f(P+w) = \sum_{|\alpha|+|\beta|=j} \frac{1}{\alpha!\beta!}\frac{\partial^{|\alpha|+|\beta|}}{\partial z^\alpha \partial \bar{z}^\beta} f(P) w^\alpha \overline{w}^\beta + o(|w|^k). \tag{2.2.4}$$

Now let us use Tylor's formula to derive another useful identity

$$\sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial z_i \partial \bar{z}_j}(P)w_i\overline{w}_j = \frac{\partial^2 f}{\partial_{\mathbb{R}} w^2} + \frac{\partial^2 f}{\partial_{\mathbb{R}} (iw)^2} \tag{2.2.5}$$

On the right-hand side of (2.2.5) we have the sum of the second order terms in the Taylor expansion of $f(P+w)$ and $f(P+iw)$ respectively. When we apply (2.2.4) and sum the corresponding terms we get cancellation of terms where $|\alpha| = 2$ or $|\beta| = 2$ and only the terms with $|\alpha| = |\beta| = 1$ remain producing the left-hand side of (2.2.5).

3 Differential forms

3.1 Real forms

Let $\Omega \subset \mathbb{R}^m$ be open. We say that ω is a p-form in Ω if

$$\omega(x) = \sum_{1 \le j_1 < \ldots < j_p \le m} a_{j_1 \ldots j_p}(x)\, dx_{j_1} \wedge \ldots \wedge dx_{j_p}.$$

The formal way to define a p-form is to say that it is a function on Ω with values in exterior p-forms on $\mathbb{R}^m$ but a form can be also understood simply as a formal expression.

Since it is convenient to consider forms

$$\omega(x) = \sum_{1 \le j_1, \ldots, j_p \le m} a_{j_1 \ldots j_p}(x)\, dx_{j_1} \wedge \ldots \wedge dx_{j_p}.$$

where the indices are not necessarily in the increasing order we use the convention which says that interchanging two adjacent dx_j-s in the form $dx_{j_1} \wedge \ldots \wedge dx_{j_p}$ changes the sign of the form. Thus any form in which no two dx_j-s are identical can be reduced to one which has ordered indices by means of a number of transpositions. If two dx_j-s are identical than such a form is by definition zero.

If we are given two forms

$$\begin{aligned} \omega(x) &= \sum_{1 \le j_1, \ldots, j_p \le m} a_{j_1 \ldots j_p}(x)\, dx_{j_1} \wedge \ldots \wedge dx_{j_p}, \\ \alpha(x) &= \sum_{1 \le l_1, \ldots, l_q \le m} b_{l_1 \ldots l_q}(x)\, dx_{l_1} \wedge \ldots \wedge dx_{l_q} \end{aligned}$$

then we define their wedge product

$$\omega \wedge \alpha = \sum_{1 \le j_1, \ldots, j_p, l_1, \ldots, l_q \le m} a_{j_1 \ldots j_p}(x) b_{l_1 \ldots l_q}(x)\, dx_{j_1} \wedge \ldots \wedge dx_{j_p} \wedge dx_{l_1} \wedge \ldots \wedge dx_{l_q}.$$

The exterior derivative of a form ω is defined as

$$d\omega = \sum_{1 \le j_1, \ldots, j_p \le m} \sum_{l=1}^{m} \frac{\partial}{\partial x_l} a_{j_1 \ldots j_p}(x)\, dx_l \wedge dx_{j_1} \wedge \ldots \wedge dx_{j_p}.$$

One of the most important properties of this operator is that $dd\omega = 0$ for any form.

Any function f can be considered to be a 0-form and then

$$df = \sum \frac{\partial f}{\partial x_l} \, dx_l.$$

3.2 Complex forms

Define

$$\begin{aligned} dz_j &= dx_j + dy_j \\ d\bar{z}_j &= dx_j - dy_j \end{aligned}$$

then

$$\begin{aligned} dx_j &= \frac{1}{2}(dz_j + d\bar{z}_j) \\ dy_j &= \frac{1}{2i}(dz_j - d\bar{z}_j). \end{aligned}$$

As a consequence we get that every form on $\Omega \subset \mathbb{C}^n \approx \mathbb{R}^{2n}$ can be written in a unique way as a sum of terms of the form

$$a_{j_1,\ldots,j_p,l_1,\ldots,l_q} \, dz_{j_1} \wedge \ldots \wedge dz_{j_p} \wedge d\bar{z}_{l_1} \wedge \ldots \wedge d\bar{z}_{l_q} \tag{3.2.1}$$

The quickest way to see the uniqueness is to observe that forms like (3.2.1) with coefficient 1 span the space of forms and there is the sane number of then as of those composed by dx_j and dy_j. Since the latter form a basis of the space of forms the former must also be a basis.

A form which is a sum of components like (3.2.1) with the same p and q is called a form of type (p, q) or shortly a (p, q)-form. Of course forms of different types cannot be equal unless they are both zero.

Let $\gamma : [a, b] \to \mathbb{C} \approx \mathbb{R}^2$ be a C^1-curve then according to the standard definition

$$\int_\gamma f dz = \int_a^b f \circ \gamma(t) \; \gamma'(t) dt. \tag{3.2.2}$$

But if we look at $f dz$ as at a differential form then

$$\int_\gamma f dz \quad = \quad \int_\gamma (f_1 + i f_2)(dx + i dy)$$

$$\begin{aligned}
&= \int_\gamma (f_1 dx - f_2 dy) + i \int_\gamma (f_2 dx + f_1 dy) \\
&= \int_a^b (f_1 \circ \gamma)\gamma_1' - (f_2 \circ \gamma)\gamma_2' dt + i \int_a^b (f_2 \circ \gamma)\gamma_1' + (f_1 \circ \gamma)\gamma_2' dt \\
&= \int_a^b ((f_1 + i f_2) \circ \gamma)(\gamma_1' + i\gamma_2') dt \\
&= \int_a^b (f \circ \gamma)\gamma' dt.
\end{aligned}$$

It turns out that the seemingly completely arbitrary definition (3.2.2) given in the course of one complex variable is a simple reformulation of the standard curvilinear integral.

Lemma 3.2.1 (Complex form of Green's theorem) *Let $\mathcal{D} \subset \mathbb{C}$ be C^1-bounded and let f be a C^1-function in a neighbourhood of $\overline{\mathcal{D}}$ then*

$$\frac{1}{2\pi i} \int_{\partial \mathcal{D}} f(\xi) d\xi = \frac{1}{\pi} \int_{\mathcal{D}} \frac{\partial f}{\partial \bar{z}}(\xi) d\nu(\xi).$$

Proof. Recall Green's theorem:

$$\int_{\partial \mathcal{D}} (g_1 dx + g_2 dy) = \int_{\mathcal{D}} \left(\frac{\partial g_2}{\partial x} - \frac{\partial g_1}{\partial y} \right) d\nu(x, y). \tag{3.2.3}$$

Then

$$\begin{aligned}
\int_{\partial \mathcal{D}} f d\xi &= \int_{\partial \mathcal{D}} (f_1 + i f_2)(dx + i dy) \\
&= \int_{\partial \mathcal{D}} (f_1 dx - f_2 dy) + i \int_{\partial \mathcal{D}} (f_2 dx + f_1 dy) \\
&= \int_{\mathcal{D}} - \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) d\nu + i \int_{\mathcal{D}} \left(\frac{\partial f_1}{\partial x} - \frac{\partial f_2}{\partial y} \right) d\nu \\
&= i \int_{\mathcal{D}} \left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial x} \right) + i \left(\frac{\partial f_1}{\partial y} + i \frac{\partial f_2}{\partial y} \right) d\nu \\
&= 2i \int_{\mathcal{D}} \frac{\partial f}{\partial \bar{\xi}} d\nu.
\end{aligned}$$

■

3.3 Operators ∂ and $\bar\partial$

If $f\colon \Omega \to \mathbb{C}$ is of class C^1 then

$$df = \sum \frac{\partial f}{\partial z_j} dz_j + \sum \frac{\partial f}{\partial \bar z_j} d\bar z_j.$$

Indeed

$$df = \sum \frac{\partial f}{\partial x_j} dx_j + \sum \frac{\partial f}{\partial y_j} dy_j = \sum \frac{\partial f}{\partial z_j}(dx_j + idy_j) + \sum \frac{\partial f}{\partial \bar z_j}(dx_j - idy_j)$$

(this is the same formal calculation as in the proof of (2.2.1)). It is quite natural to split the operator d into two parts denoted ∂ and $\bar\partial$ acting:

$$\begin{aligned} \partial f &= \sum \frac{\partial f}{\partial z_j} dz_j, \\ \bar\partial f &= \sum \frac{\partial f}{\partial \bar z_j} d\bar z_j. \end{aligned}$$

These definitions are extended to arbitrary forms by action on the coefficients. If ω is a (p,q)-form then $\partial\omega$ is a $(p+1,q)$-form and $\bar\partial\omega$ is a $(p,q+1)$-form. We have

$$dd\omega = (\partial + \bar\partial)(\partial + \bar\partial)\omega = \partial\partial\omega + \partial\bar\partial\omega + \bar\partial\partial\omega + \bar\partial\bar\partial\omega.$$

Since $\partial\partial\omega$ is a $(p+2,q)$-form, $\partial\bar\partial\omega$ and $\bar\partial\partial\omega$ are $(p+1,q+1)$-forms and $\bar\partial\bar\partial\omega$ is a $(p,q+2)$-form and $dd\omega = 0$ we get that $\partial\partial\omega = 0 = \bar\partial\bar\partial\omega$ and $\partial\bar\partial\omega = (-\bar\partial\partial\omega)$.

4 Holomorphic functions

4.1 One variable

Let $\Omega \subset \mathbb{C}$ be open. A function $f\colon \Omega \to \mathbb{C}$ is called *holomorphic* if it is differentiable in Ω in the complex sense. Its derivative is denoted by f' or $\frac{df}{dz}$. If we only assume that f is $\mathbb{R}$-differentiable then it will be holomorphic if and only if it satisfies the Cauchy-Riemann equations

$$\frac{\partial \mathrm{Re} f}{\partial x} = \frac{\partial \mathrm{Im} f}{\partial y}, \quad \frac{\partial \mathrm{Re} f}{\partial y} = -\frac{\partial \mathrm{Im} f}{\partial x}.$$

These can be reformulated to give:

$$\begin{aligned} 0 &= \frac{1}{2}\left(\frac{\partial \mathrm{Re} f}{\partial x} - \frac{\partial \mathrm{Im} f}{\partial y}\right) + \frac{1}{2} i \left(\frac{\partial \mathrm{Re} f}{\partial y} + \frac{\partial \mathrm{Im} f}{\partial x}\right) \\ &= \frac{1}{2}\left(\frac{\partial \mathrm{Re} f}{\partial x} + i\frac{\partial \mathrm{Im} f}{\partial x}\right) + \frac{1}{2} i \left(\frac{\partial \mathrm{Re} f}{\partial y} + i\frac{\partial \mathrm{Im} f}{\partial y}\right) \\ &= \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) \\ &= \frac{\partial f}{\partial \bar{z}}. \end{aligned}$$

Thus we obtained an equivalent formulation of the Cauchy-Riemann equations:

$$\frac{\partial f}{\partial \bar{z}} = 0.$$

Next observe that if $\frac{\partial f}{\partial \bar{z}} = 0$ then $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial z} + \frac{\partial f}{\partial \bar{z}} = \frac{\partial f}{\partial z}$. But $\frac{\partial f}{\partial x} = \frac{df}{dz}$ so we obtain that for holomorphic functions

$$\frac{\partial f}{\partial z} = \frac{df}{dz}. \tag{4.1.1}$$

In other words whenever the right-hand side of (4.1.1) makes sense we get the equality (4.1.1).

Another way to state that $\frac{\partial f}{\partial \bar{z}} = 0$ is that $\bar{\partial} f = 0$. This is the most compact form of Cauchy-Riemann equations.

4.2 Several variables

Now consider $\Omega \subset \mathbb{C}^n$. We want to define complex differentiability of a function $f\colon \Omega \to \mathbb{C}$. The most natural way is to follow the definition for the real case and say that f is $\mathbb{C}$-differentiable at a point $P \in \Omega$ if there exists a linear mapping $A\colon \mathbb{C}^n \to \mathbb{C}$ such that

$$\lim_{h\to 0} \frac{f(P+h) - f(P) - Ah}{|h|} = 0. \tag{4.2.1}$$

Suppose this condition is fulfilled. Than A can be considered to be a linear mapping from $\mathbb{R}^{2n}$ to $\mathbb{R}^2$ and (4.2.1) implies that f is differentiable in the real sense, A being its global derivative at P. If in turn we start with assuming that f is $\mathbb{R}$-differentiable at P then f needs not be $\mathbb{C}$-differentiable since $D_{\mathbb{R}} f(P)$ is not necessarily $\mathbb{C}$-linear. But if (and only then) this derivative happens to be $\mathbb{C}$-linear then we do get $\mathbb{C}$-differentiability. Since an $\mathbb{R}$-linear mapping A is $\mathbb{C}$-linear iff $A(iw) = iAw$ we get that f is $\mathbb{C}$-differentiable at P iff it is $\mathbb{R}$-differentiable and

$$D_{\mathbb{R}} f(P)(iw) = i D_{\mathbb{R}} f(P)(w). \tag{4.2.2}$$

So if we want $\mathbb{C}$-differentiability of f we must verify (4.2.2) and it is enough to do this on some basis of $\mathbb{C}^n$. If we take the basis

$$e_j = (0, \ldots, \underset{j}{1}, \ldots 0) \ \in \mathbb{C}^n$$

then

$$\begin{aligned} e_j &= (0, \ldots, \underset{j}{1}, \ldots, 0, 0, \ldots, 0)_{\mathbb{R}} \\ ie_j &= (0, \ldots, 0, 0, \ldots, \underset{n+j}{1}, \ldots, 0)_{\mathbb{R}} \end{aligned}$$

In particular

$$D_{\mathbb{R}} f(P)(e_j) = \frac{\partial f}{\partial x_j}$$

while

$$D_{\mathbb{R}} f(P)(ie_j) = \frac{\partial f}{\partial y_j}.$$

Thus we get the system of equations

$$\frac{\partial f}{\partial y_j} = i\frac{\partial f}{\partial x_j}$$

or equivalently

$$\begin{aligned} \frac{\partial \mathrm{Re} f}{\partial x_j} &= \frac{\partial \mathrm{Im} f}{\partial y_j} \\ \frac{\partial \mathrm{Re} f}{\partial y_j} &= -\frac{\partial \mathrm{Im} f}{\partial x_j}. \end{aligned}$$

These are the Cauchy-Riemann equations in several variables and as we have seen, together with $\mathbb{R}$-differentiability they are equivalent to the fact that f is $\mathbb{C}$-differentiable. Other equivalent formulations of the Cauchy-Riemann equations are:

$$\frac{\partial f}{\partial z_j} = 0, \;\; j = 1, \ldots, n$$

or

$$\bar{\partial} f = 0.$$

Definition 4.2.1 *A function $f\colon\Omega \to \mathbb{C}$ is holomorphic in Ω if it is $\mathbb{C}$-differentiable in Ω or equivalently if it is $\mathbb{R}$-differentiable and satisfies the Cauchy-Riemann equations in Ω. The collection of all holomorphic functions on Ω will be denoted by $H(\Omega)$.*

It is easily seen that $H(\Omega)$ is a linear space.

If f is holomorphic in $\Omega \subset \mathbb{C}^n$ then for every $z \in \Omega$ and $j = 1, \ldots, n$ the function

$$f_{z,j}(\lambda) = f(z_1, \ldots, \underset{j}{\lambda}, \ldots, z_n)$$

is holomorphic in the open set $\Omega_{z,j} = \{\lambda \in \mathbb{C} : (z_1, \ldots, \lambda, \ldots, z_n) \in \Omega\}$. This is completely obvious since

$$\frac{\partial f_{z,j}}{\partial \overline{\lambda}}(\lambda) = \frac{\partial f}{\partial \bar{z}_j}(z_1, \ldots, \lambda, \ldots, z_n) = 0.$$

It is not at all obvious (but true) that the opposite holds as well i.e., if all $f_{z,j}$-s are holomorphic than so is f. (The proof of this will be given in Section 6.1.)

We will introduce a temporary definition and say that a function $f\colon \Omega \to \mathbb{C}$ is coordinatewise holomorphic if all $f_{z,j}$-s are holomorphic.

Of course if f is C^1 and coordinatewise holomorphic then it is holomorphic.

4.3 Power series

A power series in $\mathbb{C}^n$ is a series

$$\sum_{\alpha} a_\alpha (z-P)^\alpha \tag{4.3.1}$$

where α ranges over all n-position multi-indices.

We will only consider absolute convergence of power series; then the sum does not depend on the order of summation. Whenever we say that a power series is convergent we mean that it is absolutely convergent.

Theorem 4.3.1 *If the power series* (4.3.1) *converges at a point* $z_0 \in \mathbb{C}^n$ *then it converges normally on the polydisk*

$$\{z \in \mathbb{C}^n : |z_j - P_j| < |z_{0,j} - P_j|\}.$$

PROOF. Exercise.

■

The above theorem tells us that if a power series converges pointwise in some open polydisk then it converges normally.

Definition 4.3.2 *A function* $f\colon \Omega \to \mathbb{C}$ *is analytic if for every* $P \in \Omega$ *there exists a neighbourhood* U_P *of* P *in* Ω *such that* f *can be represented in* U_P *as a sum of a power series* $\sum_\alpha a_\alpha (z-P)^\alpha$.

Every analytic function is of class C^∞ and holomorphic. If

$$f = \sum a_\alpha (z-P)^\alpha$$

then

$$\frac{\partial^{|\alpha|} f}{\partial z^\alpha}(P) = \alpha! a_\alpha. \tag{4.3.2}$$

4.4 Cauchy formula in the polydisk

Theorem 4.4.1 (Cauchy formula) *Let $\Omega \subset \mathbb{C}^n$ be open and f a coordinatewise holomorphic function in Ω. Let $P \in \Omega$ and $\boldsymbol{r} = (r_1, \ldots, r_n)$ be such that $D^n(P, \boldsymbol{r}) \subset\subset \Omega$. Then for every $z \in D^n(P, \boldsymbol{r})$*

$$f(z) = \frac{1}{(2\pi i)^n} \int\limits_{|\xi_n - P_n| = r_n} \cdots \int\limits_{|\xi_1 - P_1| = r_1} \frac{f(\xi)}{(\xi_1 - z_1) \ldots (\xi_n - z_n)} d\xi_1 \ldots d\xi_n \tag{4.4.1}$$

Proof. We proceed by induction. For $n = 1$, (4.4.1) is the usual one-dimensional Cauchy formula. Assume (4.4.1) is true for $n - 1$. Then for every z with $|z_j| \le r_j,\ j = 1, \ldots, n-1$ the function $f_{z,n}$ is holomorphic in a neighborhood of $\overline{D}(P_n, r_n)$ hence

$$f(z', z_n) = \frac{1}{2\pi i} \int\limits_{|\xi_n - P_n| = r_n} \frac{f_{z,n}(\xi_n)}{\xi_n - z_n} d\xi_n \tag{4.4.2}$$

Next consider the function $z \mapsto f_{z,n}(\xi_n) = f(z', \xi_n)$ where $\xi_n \in \overline{D}(P_n, r_n)$ is fixed. This function is coordinatewise holomorphic in a neighborhood of $\overline{D}^{n-1}(P', \boldsymbol{r}')$ hence, by the inductive hypothesis

$$f(z', \xi_n) = \frac{1}{(2\pi i)^{n-1}} \int\limits_{|\xi_{n-1} - P_{n-1}| = r_{n-1}} \cdots \int\limits_{|\xi_1 - P_1| = r_1} \frac{f(\xi)}{(\xi_1 - z_1) \ldots (\xi_{n-1} - z_{n-1})} d\xi_1 \ldots d\xi_{n-1}$$

Inserting this into (4.4.2) we get (4.4.1).

∎

The integral on the right-hand side of (4.4.1) has to be understood as an iterated integral. One cannot readily use Fubini's theorem at this point without some additional assumptions on f.

Corollary 4.4.2 *If f is coordinatewise holomorphic and locally bounded then f is analytic, in particular also holomorphic.*

Proof. The reader is advised to check that no global measurability assumptions on f are used in this proof.

Let $P \in \Omega$ and $\boldsymbol{r}$ be as in the proceeding theorem. Taking $\boldsymbol{r}$ small enough we may assume that f is bounded on $\overline{\mathcal{D}} = \overline{D}^n(P, \boldsymbol{r})$. We have

$$\xi_j - z_j = (\xi_j - P_j)\frac{\xi_j - z_j}{\xi_j - P_j} = (\xi_j - P_j)\left(1 - \frac{z_j - P_j}{\xi_j - P_j}\right).$$

So

$$\frac{1}{\xi_j - z_j} = \frac{1}{(\xi_j - P_j)}\frac{1}{(1 - (z_j - P_j)/(\xi_j - P_j))} = \frac{1}{\xi_j - P_j}\sum_{l=0}^{\infty}\left(\frac{z_j - P_j}{\xi_j - P_j}\right)^l$$

and

$$\frac{1}{(\xi_1 - z_1)\dots(\xi_n - z_n)} = \sum_\alpha b_\alpha(\xi)(z - P)^\alpha$$

where

$$b_\alpha(\xi) = \left(\frac{1}{\xi_1 - P_1}\right)^{\alpha_1+1}\dots\left(\frac{1}{\xi_n - P_n}\right)^{\alpha_n+1}.$$

It follows that

$$f(z) = \frac{1}{(2\pi i)^n}\int\dots\int\sum_\alpha b_\alpha(\xi)(z - P)^\alpha f(\xi)d\xi_1\dots d\xi_n$$

Since f is bounded, the series inside the integral converges absolutely and uniformly in ξ when z is fixed. Hence the summation can be taken outside of the integral giving:

$$f(z) = \frac{1}{(2\pi i)^n}\sum_\alpha\left(\int\dots\int b_\alpha(\xi)f(\xi)d\xi_1\dots d\xi_n\right)(z - P)^\alpha.$$

This series converges absolutely for each fixed $z \in \mathcal{D}$.

∎

Remark. It is important to note that if f is as above and $P \in \Omega$ then f can be written as a sum of a power series about P which converges in every polydisk centered at P and contained in Ω.

Corollary 4.4.3 *Every holomorphic function is analytic.*

PROOF. Since holomorphic functions are continuous they are also locally bounded.

■

Corollary 4.4.4 *If f is holomorphic then*

$$\frac{\partial^{|\alpha+\beta|} f}{\partial z^\alpha \partial \bar{z}^\beta} = 0$$

if $\beta \neq 0$. In particular $\Delta f = 0$.

PROOF. Since $\frac{\partial f}{\partial \bar{z}_j} = 0$ for every j, every other derivative will also vanish. The second part follows from (2.2.2)

■

Corollary 4.4.5 *If $\Omega \subset \mathbb{C}^n$ is open and connected, $f, g \in H(\Omega)$ and $\{z \in \Omega : f(z) = g(z)\}$ has non-empty interior then $f = g$ on Ω.*

PROOF. Let $h = (f - g)$. Let $V = \{z \in \Omega : \frac{\partial^{|\alpha|} h}{\partial z^\alpha}(z) = 0 \quad \text{for all} \quad \alpha\}$. Then V is closed in Ω and non-empty, but it is also open because if $P \in V$ then the series expansion of h about P has zero coefficients so h is zero in a neighbourhood of P.

■

Corollary 4.4.6 *If $\Omega \subset \mathbb{C}^n$ is open and connected and $f \in H(\Omega)$ is such that at some point $z_0 \in \Omega$, for all α, $\frac{\partial^{|\alpha|} f}{\partial z^\alpha}(z_0)$ then $f \equiv 0$ in Ω.*

PROOF. Apply the proof of the previous corollary.

■

Definition 4.4.7 *We say that φ is a homogeneous polynomial of degree s if*

$$\varphi(z) = \sum_{|\alpha|=s} a_\alpha z^\alpha.$$

This definition can be equivalently stated as $\varphi(lz) = l^s \varphi(z)$, for $z \in \mathbb{C}^n$, $l \in \mathbb{C}$.

If f is a holomorphic function in a neighbourhood of 0 then f can be written as the Taylor series

$$f(z) = \sum_{j=0}^{\infty} \frac{1}{j!} D^j_{\mathbb{C}} f(0)(z^j),$$

the same effect can be obtained by grouping the power series

$$f(z) = \sum_{\alpha} a_\alpha z^\alpha = \sum_{j=0}^{\infty} \Big(\sum_{|\alpha|=j} a_\alpha z^\alpha \Big).$$

Both of these approaches give us the same expansion

$$f(z) = \sum_{j=0}^{\infty} \varphi_j(z)$$

where φ_j are homogeneous polynomials of degree j.

Another approach to such an expansion is by considering the function $\mathbb{C} \ni l \mapsto f(lz)$. Then

$$f(lz) = \sum_{j=0}^{\infty} l^j \varphi_j(z).$$

This is nothing else but the Taylor expansion of the function of one variable and $\varphi_j(z)$ serve as coefficients. This shows at once that the homogeneous expansion is unique and

$$\varphi_j(z) = \frac{1}{j!} \frac{d^j}{dl^j} f(lz)\Big|_{l=0} \tag{4.4.3}$$

$$\varphi_j(z) = \frac{1}{2\pi} \int_0^{2\pi} f(ze^{i\theta}) e^{-ij\theta} d\theta \tag{4.4.4}$$

We will later need one technical lemma about homogeneous polynomials:

Lemma 4.4.8 *If φ is a homogeneous polynomial of degree s then*

$$\varphi(z + o(z)) = \varphi(z) + o(|z|^s).$$

The proof is quite easy, it is enough to consider φ of the form z^α where $|\alpha| = s$, we leave the details to the reader.

Theorem 4.4.9 (Cauchy inequalities) *If f is coordinatewise holomorphic in Ω and $|f| \leq M$ in a polydisk $\overline{D}^n(P, \boldsymbol{r}) \subset \Omega$ then for every multi-index α*

$$\left| \frac{\partial^{|\alpha|} f}{\partial z^\alpha} \right| \leq M \alpha! \boldsymbol{r}^{-\alpha}.$$

Proof. We have to estimate the α-th coefficient in the series expansion of f about P which is equal to

$$a_\alpha = \frac{1}{(2\pi i)^n} \int\limits_{|\xi_n - P_n| = r_n} \cdots \int\limits_{|\xi_1 - P_1| = r_1} b_\alpha(\xi) f(\xi) d\xi_1 \dots d\xi_n.$$

So $|a_\alpha| \leq \sup_{\overline{D}^n(P,\boldsymbol{r})} |b_\alpha(\xi)| \sup_{\overline{D}^n(P,\boldsymbol{r})} |f(\xi)| r_1 \dots r_n$. But

$$|b_\alpha(\xi)| r_1 \dots r_n = \frac{1}{|\xi - P_1|^{\alpha_1 + 1}} \cdots \frac{1}{|\xi_n - P_n|^{\alpha_n + 1}} r_1 \dots r_n = r_1^{-\alpha_1} \dots r_n^{-\alpha_n} = \boldsymbol{r}^{-\alpha}$$

and $\sup |f| \leq M$.

■

Corollary 4.4.10 *If f and $\boldsymbol{r}$ are as in Theorem 4.4.9 and $\tilde{\boldsymbol{r}} = \boldsymbol{r}/2 := (r_1/2, \dots r_n/2)$ then*

$$\forall_{z \in D^n(P, \tilde{\boldsymbol{r}})} \forall_\alpha \ \left| \frac{\partial^{|\alpha|} f}{\partial z^\alpha}(z) \right| \leq M \alpha! 2^{|\alpha|} \boldsymbol{r}^{-\alpha}.$$

Corollary 4.4.11 *If $f_n \in H(\Omega)$ and $f_n \to f$ normally on Ω then $f \in H(\Omega)$ and for every multi-index α, $\frac{\partial^{|\alpha|} f_n}{\partial z^\alpha} \to \frac{\partial^{|\alpha|} f}{\partial z^\alpha}$ normally on Ω.*

Proof. It follows from Cor. 4.4.10 that $\frac{\partial f_n}{\partial z_j}$ converge normally on Ω. Since $\frac{\partial f_n}{\partial \bar{z}_j} = 0$ we get that also $\frac{\partial f_n}{\partial x_j}$ and $\frac{\partial f_n}{\partial y_j}$ converge normally. It follows that f is of class C^1 and $\frac{\partial f_n}{\partial x_j} \to \frac{\partial f}{\partial x_j}$ and $\frac{\partial f_n}{\partial y_j} \to \frac{\partial f}{\partial y_j}$. In particular $\frac{\partial f_n}{\partial \bar{z}_j} \to \frac{\partial f}{\partial \bar{z}_j}$ so $\bar{\partial} f = 0$ and $f \in H(\Omega)$. Convergence of higher order derivatives follows at once from Cor. 4.4.10 applied to $f_n - f$.

■

The topology of normal convergence on Ω is the most natural topology on $H(\Omega)$.

Exercise. Let K_n be a family of compact subsets of Ω such that $K_j \subset\subset K_{j+1}$ and $\bigcup_{j=1}^{\infty} K_j = \Omega$. For $f\colon\Omega \to \mathbb{C}$ let $\|f\|_j = \sup\{|f(z)| : z \in K_j\}$. Define

$$d(f,g) = \sum_{j=1}^{\infty} 2^{-j} \frac{\|f-g\|_j}{1+\|f-g\|_j}.$$

Show that d is a complete metric on the space of all continuous functions on Ω compatible with the topology of normal convergence on Ω.

Exercise. Show that $(H(\Omega), d)$ is a complete metric linear space.

Exercise. Show that the topology on $H(\Omega)$ induced by d is not given by any norm.

4.5 Holomorphic mappings

Similarly as holomorphic functions $f\colon\Omega \to \mathbb{C}$ we may consider holomorphic mappings $\Phi\colon\Omega \to \mathbb{C}^m$ defining them as $\mathbb{C}$-differentiable, which is equivalent to saying that all components Φ_j are holomorphic functions. A holomorphic mapping has a Jacobi matrix representing its complex global derivative. Sometimes we may want to compute integrals involving substitutions done with holomorphic mappings, to do this we must know the real-variable jacobian of such a mapping so it is important to realize how the complex and real Jacobi matrices are related. Schematically we can write the real Jacobi matrix as

$$M_j^{\mathbb{R}}\Phi = \begin{bmatrix} \frac{\partial \mathrm{Re}\Phi}{\partial x} & \frac{\partial \mathrm{Re}\Phi}{\partial y} \\ \frac{\partial \mathrm{Im}\Phi}{\partial x} & \frac{\partial \mathrm{Im}\Phi}{\partial y} \end{bmatrix}$$

where each entry represents an $m \times n$ – matrix. Using Cauchy-Riemann equations we get

$$M_J^{\mathbb{R}}\Phi = \begin{bmatrix} \frac{\partial \mathrm{Re}\Phi}{\partial x} & \frac{\partial \mathrm{Re}\Phi}{\partial y} \\ -\frac{\partial \mathrm{Re}\Phi}{\partial y} & \frac{\partial \mathrm{Re}\Phi}{\partial x} \end{bmatrix}.$$

Since Φ is holomorphic we have $\frac{\partial \Phi_k}{\partial z_j} = \frac{\partial \Phi_k}{\partial x_j}$ so

$$M_J^{\mathbb{C}} f = \left[\frac{\partial \Phi}{\partial x}\right] = \left[\frac{\partial \mathrm{Re}\Phi}{\partial x} + i\frac{\partial \mathrm{Im}\Phi}{\partial x}\right].$$

Now suppose $m = n$, then all matrices are square and using the fact that

$$\det \begin{bmatrix} A & B \\ -B & A \end{bmatrix} = |\det(A + iB)|^2$$

we get that

$$J^{\mathbb{R}}\Phi = |J^{\mathbb{C}}\Phi|^2.$$

Definition 4.5.1 *Let $\Omega_1, \Omega_2 \subset \mathbb{C}^n$ be open and $\Phi\colon \Omega_1 \to \Omega_2$. Φ is said to be biholomorphic if Φ is a holomorphic bijection of Ω_1 onto Ω_2 and Φ^{-1} is also holomorphic.*

It is a highly nontrivial fact that every holomorphic bijection is automatically biholomorphic.

Corollary 4.5.2 *If $\Phi\colon \Omega_1 \to \Omega_2$ is biholomorphic and $f\colon \Omega_2 \to \mathbb{C}$ is integrable then*

$$\int_{\Omega_2} f\, d\nu = \int_{\Omega_1} (f \circ \Phi)|J^{\mathbb{C}}\Phi|^2\, d\nu.$$

Theorem 4.5.3 (H. Cartan) *Suppose*

a) Ω is a bounded domain in $\mathbb{C}^n$,

b) $\Phi : \Omega \to \Omega$ is a holomorphic mapping,

c) for some $P \in \Omega$, $\Phi(P) = P$ and $D_{\mathbb{C}}\Phi(P) = I$.

Then $\Phi(z) = z$ for all $z \in \Omega$.

Proof. Without loss of generality we may assume that $P = 0$. Then there exists $r_1 > 0$, $r_2 < \infty$ such that $B(0, r_1) \subset \Omega \subset B(0, r_2)$. Every component of Φ has a homogeneous expansion

$$\Phi_k(z) = \sum_{j=0}^{\infty} f_{j,k}(z)$$

converging in $B(0, r_1)$. If we define

$$F_j := (f_{j,1}, \ldots, f_{j,k})$$

then

$$\Phi = \sum_{j=0}^{\infty} F_j(z).$$

The terms of this expansion correspond to the derivatives of Φ at 0 so $F_0(z) = \Phi(0) = 0$, $F_1(z) = D_{\mathbb{C}}\Phi(0)(z) = z$ and we have

$$\Phi(z) = z + \sum_{j=2}^{\infty} F_j(z).$$

It is our aim to show that $F_j = 0$ for $j \geq 2$. Suppose this is not true and let $m \geq j$ be the first index for which $F_j \not\equiv 0$. Then

$$\Phi(z) = z + F_m(z) + o(|z|^m).$$

Let Φ^s be the s-th iteration of Φ i.e., $\Phi^s = \underbrace{\Phi \circ \ldots \circ \Phi}_{\text{s times}}$ then

$$\Phi^s(z) = z + sF_m(z) + o(|z|^m) \tag{4.5.1}$$

We prove (4.5.1), again by induction, this time on s. For $s = 1$ the claim is obvious and for $s > 1$,

$$\begin{aligned}
\Phi^s(z) &= \Phi(\Phi^{s-1}(z)) = \Phi\Big(z + (s-1)F_m(z) + o(|z|^m)\Big) \\
&= z + (s-1)F_m(z) + o(|z|^m) \\
&\quad + F_m\Big(z + (s-1)F_m(z) + o(|z|^m)\Big) + o\Big(|\Phi^{s-1}(z)|^m\Big) \\
&= z + (s-1)F_m(z) + F_m\Big(z + o(|z|)\Big) + o(|z|^m) \\
&= z + (s-1)F_m(z) + F_m(z) + o(|z|^m) + o(|z|^m) \\
&= z + sF_m(z) + o(|z|^m)
\end{aligned}$$

It follows, by (4.4.4), that

$$sF_m(z) = \frac{1}{2\pi}\int_0^{2\pi} \Phi^s(ze^{i\theta})e^{-im\theta}d\theta.$$

Each Φ^s maps Ω into Ω hence $|\Phi^s(z)| < r_2$ and so we have

$$|sF_m(z)| \leq r_2 \quad \text{for all} \quad s \geq 1, |z| < r_1$$

hence $F_m(z) \equiv 0$ on $B(0, r_1)$ and so must be identically zero, which is a contradiction.

∎

Definition 4.5.4 *A set $E \subset \mathbb{C}^n$ is called circular if $lz \in E$ whenever $z \in E$ and $l \in \mathbb{C}$, $|l| = 1$.*

Theorem 4.5.5 (H. Cartan) *Suppose*

a) Ω_1, Ω_2 are circular domains in $\mathbb{C}^n$ with $0 \in \Omega_1$ and $0 \in \Omega_2$,

b) Φ is a biholomorphic map of Ω_1 onto Ω_2 with $\Phi(0) = 0$,

c) Ω_1 is bounded.

Then f is a linear.

Proof. Let, for $\theta \in [0, 2\pi]$

$$\Psi_\theta(z) := \Phi^{-1}(e^{-i\theta}\Phi(e^{i\theta}z)).$$

Ψ_θ is well defined since Ω_1, Ω_2 are circular and $\Psi_\theta(\Omega_1) = \Omega_1$. If we compute the derivative of Ψ_θ at 0 we get

$$D\Psi_\theta(0) = \Big(D(\Phi^{-1})(0)\Big) \circ \Big(e^{-i\theta}D\Phi(0)\Big) \circ \Big(e^{i\theta}I\Big) = I.$$

Since $\Psi_\theta(0) = 0$, we get by Theorem 4.5.3 that $\Psi_\theta = I$ so we have

$$z = \Phi^{-1}(e^{-i\theta}\Phi(e^{i\theta}z))$$

hence

$$e^{i\theta}\Phi(z) = \Phi(e^{i\theta}z).$$

But if

$$\Phi = \sum_{j=0}^{\infty} F_j$$

is the homogeneous expansion of Φ then, by (4.4.4)

$$F_m(z) = \frac{1}{2\pi}\int_0^{2\pi} \Phi(e^{i\theta}z)e^{-im\theta}d\theta = \Phi(z)\frac{1}{2\pi}\int_0^{2\pi} e^{-i(m-1)\theta}d\theta = 0$$

if $m \neq 1$ and thus $\Phi = F_1$ so Φ is linear.

∎

Theorem 4.5.6 *Let $U \subset \mathbb{C}$ be the unit disk. If $a \in U$ and*

$$B_a(\xi) := \frac{a-\xi}{1-\bar{a}\xi}$$

then B_a is a biholomorphic mapping of U onto itself and $B_a(a) = 0$, $B_a(0) = a$.

Proof. We have $B_a \circ B_a(\xi) = \xi$ so we only have to show that $|B_a(\xi)| < 1$ for $\xi \in U$ which is a trivial calculation.

■

The function $B_a(\xi)$ is called the *Blaschke factor* at a.

Corollary 4.5.7 *Let $\mathcal{D} = D^n(z, \boldsymbol{r})$. If $a \in \mathcal{D}$ then*

$$\Phi_a(w) = \left(r_1 \frac{w_1 - a_1}{1 - \bar{a}_1 w_1 / r_1^2}, \dots, r_n \frac{w_n - a_n}{1 - \bar{a}_n w_n / r_n^2} \right)$$

is a biholomorphic mapping of $\mathcal{D}$ onto itself with $\Phi_a(a) = 0$, $\Phi_a(0) = a$.

Proof. If $\boldsymbol{r} = (1, \dots, 1)$ then

$$\Phi_a(w) = \mathbf{B}_a(w) := (B_{a_1}(w_1), \dots, B_{a_n}(w_n))$$

so Φ_a acts into the polydisk and $\Phi_a \circ \Phi_a = I$ hence Φ_a is a biholomorphism of $\mathcal{D}$ onto $\mathcal{D}$ with $\Phi_a(a) = 0$, $\Phi_a(0) = a$.

In the general case we first map $\mathcal{D}$ onto the unit polydisk by the linear map $(w_1, \dots, w_n) \mapsto (w_1/r_1, \dots, w_n/r_n)$, then send the image of a onto 0 by a suitable $\mathbf{B}_b$ and finally map the unit disc onto $\mathcal{D}$ by the linear mapping $(w_1, \dots, w_n) \mapsto (r_1 w_1, \dots, r_n w_n)$. This shows that Φ_a is a composition of three biholomorphic mappings hence is biholomorphic.

■

We have just proved that a polydisk is an example of a homogeneous domain i.e., a domain in which any point can be mapped onto another by a biholomorphic transformation of the domain. (For the polydisk just take $\Phi_b \circ \Phi_a$.) A ball is another example of such a domain, see [R] sec. 2.2.

Corollary 4.5.8 *If $n > 1$ then there is no biholomorphic mapping of a polydisk onto a ball in $\mathbb{C}^n$.*

Proof. Without loss of generality we may assume that both the ball $\mathcal{B}$ and the polydisk $\mathcal{D}$ are centered at 0. If there were a biholomorphic mapping Ψ of $\mathcal{D}$ onto $\mathcal{B}$ then we would take $a = \Psi^{-1}(0)$ and define

$$\Phi = \Psi \circ \Phi_a.$$

Then $\Phi(0) = \Psi(\Phi_a(a)) = \Psi(a) = 0$ and Φ is also a biholomorphic map of $\mathcal{D}$ onto $\mathcal{B}$, hence by Theorem 4.5.5 Φ is linear, which is clearly impossible since a linear image of a polydisk is a rotated polydisk.

■

This corollary shows that there is no extension of the Riemann mapping theorem to several variables.

5 Harmonic and subharmonic functions

5.1 Harmonic functions

Even though harmonic and subharmonic functions are a real variable topic they are one of the most important tools of the theory of several complex variables.

Let $\Omega \subset \mathbb{R}^m$.

Definition 5.1.1 *A function $h: \Omega \to \mathbb{R}$ is harmonic if h is of class C^2 in Ω and $\Delta h = 0$.*

It can be shown that every harmonic function is of class C^∞. We will often deal with complex-valued harmonic functions which are of course defined exactly as the real-valued ones. We have seen that holomorphic functions, hence also their real and imaginary parts are harmonic. If $\Omega \subset \mathbb{C}$ is simply connected then every real-valued harmonic function on Ω is a real part of a holomorphic function. In $\mathbb{C}^n$, $(n > 1)$, no matter what Ω is like, this is no longer true.

EXAMPLE. Let $h: \mathbb{C}^2 \to \mathbb{R}$, $h(z_1, z_2) = |z_1|^2 - |z_2|^2 = z_1\bar{z}_1 - z_2\bar{z}_2$. Then $\frac{\partial^2 h}{\partial \bar{z}_1 \partial z_1} = 1$, $\frac{\partial^2 h}{\partial \bar{z}_2 \partial z_2} = -1$ hence $\Delta h = 0$ so h is harmonic. But if $h = \mathrm{Re} f$ with f holomorphic then h would also be coordinatewise harmonic which it is not.

One of the most important properties of harmonic functions is the mean value property:

Theorem 5.1.2 (Mean value property) *Let $h: \Omega \to \mathbb{R}$ be harmonic and $x_0 \in \Omega$, $r > 0$ such that $B(x_0, r) \subset\subset \Omega$. Then*

$$h(x_0) = \frac{1}{\sigma(\partial B(x_0, r))} \int_{\partial B(x_0,r)} h(x)\, d\sigma(x)$$

$$h(x_0) = \frac{1}{\nu(B(x_0, r))} \int_{B(x_0,r)} h(x)\, d\nu(x)$$

where σ is the surface measure on the sphere $\partial B(x_0, r)$.

Corollary 5.1.3 (Maximum principle) *If $\Omega \subset \mathbb{R}^m$ is connected then every harmonic function $h: \Omega \to \mathbb{R}$ which attains its maximum in Ω is constant.*

PROOF. The set $\{x \in \Omega : h(x) = \max_h\}$ is non-empty and closed in Ω, but it follows from the mean value property that it is also open hence it must be equal to Ω.

■

Corollary 5.1.4 *If $h_1, h_2 \colon \overline{\Omega} \to \mathbb{C}$ are harmonic in a bounded set Ω and continuous on $\overline{\Omega}$ and $h_1 = h_2$ on $\partial\Omega$ then $h_1 = h_2$ on Ω.*

PROOF. Let $h = \mathrm{Re}(h_1 - h_2)$ then h is harmonic and $h = 0$ on $\partial\Omega$ hence $h \leq 0$ on Ω. We repeat the same with $(-h)$ and get $(-h) \leq 0$ thus $h = 0$. Next do the same with the imaginary part.

■

Theorem 5.1.5 (Solution to the Dirichlet problem in a ball) *If $f \colon \partial B(x_0, r) \to \mathbb{R}$ is continuous then there exists a unique continuous function $h \colon \overline{B}(x_0, r) \to \mathbb{R}$ which is harmonic in $B(x_0, r)$ and $h|_{\partial B(x_0,r)} = f$.*

5.2 Subharmonic functions

Definition 5.2.1 *A function $f \colon \Omega \to \mathbb{R} \cup \{-\infty\}$ is called subharmonic if*

a) f is upper semicontinuous, that is $\{x \in \Omega : f(x) < t\}$ is open for every real number t.

b) For every ball $B \subset\subset \Omega$ and every continuous function $h \colon \overline{B} \to \mathbb{R}$ which is harmonic in B and $h \geq f$ on ∂B we have $h \geq f$ on B.

A function f is called superharmonic if $(-f)$ is subharmonic.

One of the reasons to consider semicontinuous rather then continuous functions is that decreasing pointwise limits preserve subharmonicity but do not preserve continuity. This is also one of the reasons why value $\{-\infty\}$ is admitted. Another reason for this is that $\log|f|$ is subharmonic whenever f is holomorphic and this function is $(-\infty)$ at the zeroes of f.

Lemma 5.2.2 (Integration in spherical coordinates) *Let f be an integrable function on the ball $B(0, R)$ in $\mathbb{R}^m$. Then*

$$\int_{B(0,R)} f(x)\, d\nu(x) = \int_0^R r^{m-1} \int_{\partial B(0,1)} f(r\xi)\, d\sigma(\xi)\, dr$$

The proof is a simple observation that when one integrates the function on the left using classical spherical coordinates then integrating first in the 'angle' variables we get the integral over the unit sphere, the remaining 'radius' variable gives us the second integral.

Theorem 5.2.3 *Let $\Omega \subset \mathbb{R}^m$ be open and $f\colon \Omega \to \mathbb{R} \cup \{-\infty\}$ be upper semicontinuous. Then the following are equivalent:*

1) f is subharmonic in Ω.

2) If $x \in \Omega$ and $r > 0$ are such that $\overline{B}(x,r) \subset \Omega$ then

$$f(x) \le \frac{1}{\sigma(\partial B(x,r))} \int_{\partial B(x,r)} f(t)\, d\sigma(t). \tag{5.2.1}$$

2a) $\forall_{x\in\Omega} \exists_{r_0} \forall_{0<r<r_0}$ (5.2.1) holds.

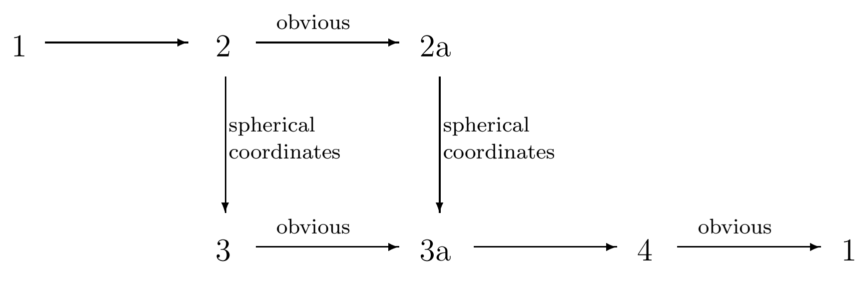
3) If $x \in \Omega$ and $r > 0$ are such that $B(x,r) \subset \Omega$ then

$$f(x) \le \frac{1}{\nu(B(x,r))} \int_{B(x,r)} f(t)\, d\nu(t). \tag{5.2.2}$$

3a) $\forall_{x\in\Omega} \exists_{r_0} \forall_{0<r<r_0}$ (5.2.2) holds.

4) If $K \subset \Omega$ is compact and $h\colon K \to \mathbb{R}$ is harmonic on int K *and $h \ge f$ on ∂K then $h \ge f$ on K.*

Remark. Upper semicontinuous functions are measurable and bounded above on compact sets hence the integrals in (5.2.1) and (5.2.2) make sense.
Proof.



(1 ⇒ 2) For $M \in \mathbb{R}$ let $f_M = \max\{f, M\}$. Then f_M is upper semicontinuous and $\mathbb{R}$-valued. Let x, r satisfy the hypothesis of 2) and h_M be the solution to the Dirichlet problem in $B(x,r)$ with boundary values f_M, i.e., h_M is continuous in $\overline{B}(x,r)$, harmonic in $B(x,r)$ and $h_M = f_M$ on $\partial B(x,r)$. It follows that $h_M \geq f_M$ on $B(x,r)$, in particular $h_M(x) \geq f_M(x)$. But then

$$f_M(x) \leq h_M(x) = \frac{1}{\sigma(\partial B(x,r))} \int_{\partial B(x,r)} h_M \, d\sigma = \frac{1}{\sigma(\partial B(x,r))} \int_{\partial B(x,r)} f_M \, d\sigma$$

and

$$\int_{\partial B} f_m \, d\sigma \xrightarrow{M \to -(\infty)} \int_{\partial B} f \, d\sigma. \tag{5.2.3}$$

In (5.2.3) we used the fact that there exists an $M_0 \in \mathbb{R}$ such that $f \leq M_0$ on $\overline{B}(x,r)$. Then for $M < M_0$ we have $M_0 - f_M \geq 0$ and $(M_0 - f_M) \nearrow (M_0 - f)$. Hence we can use Lebesgue's monotone convergence theorem to get (5.2.3). We end by observing that $f_M(x) \to f(x)$ as $M \to (-\infty)$.

(3a ⇒ 4). Let h be as in 4). Let G be a component of $\operatorname{int} K$. Then $(f-h)$ satisfies 3a) in G and $(f-h) \leq 0$ on ∂G ($\partial G \subset \partial K$). Let $M = \sup_{\overline{G}}(f(x) - h(x))$ and let $V = \{x \in G : f(x) - h(x) \geq M\}$. Then V is closed in G and it follows from 3a) that it is also open. Hence $V = \not f$ or $V = G$. In either case we get that $M \leq 0$ so $f \leq h$ on G. Since the choice of the component was arbitrary we get $f \leq h$ on $\operatorname{int} K$ thus on K.

■

Corollary 5.2.4 *Every function which is both super and subharmonic is harmonic.*

Proof. Let f be such a function in Ω. Then first of all f is continuous since it is both lower and upper simicontinuous. Let $B \subset\subset \Omega$ be a ball and let h be the solution of the Dirichlet problem in B with data f. Then by the definition $f \leq h$ in B. But $(-f)$ is also subharmonic and $(-f) = (-h)$ on ∂B hence $(-f) \leq (-h)$ in B and we get $f = h$ in B so f is harmonic in B.

■

Corollary 5.2.5 *If $f : \Omega \to \mathbb{C}$ is continuous and*

$$\forall_x \exists_{r_0} \forall_{0<r<r_0} \quad f(x) = \frac{1}{\nu(B(x,r))} \int_{B(x,r)} f \, d\nu \tag{5.2.4}$$

then f is harmonic.

PROOF. It is enough to consider real valued f. By condition 3a) of Theorem 5.2.3 f and $(-f)$ are subharmonic, hence f is harmonic.

■

EXERCISE. Let $f\colon \Omega \to \mathbb{C}$ be such that for every $K \subset\subset \Omega$ there exists an $r_K > 0$ such that

$$\forall_{x\in K}\forall_{0<r<r_K} \quad f(x)\frac{1}{\nu(B(x,r))}\int_{B(x,r)} f\, d\nu \tag{5.2.5}$$

then f is continuous (and by the previous corollary, harmonic). Show that if (5.2.5) is substituted by (5.2.4) then f need not be continuous.

Corollary 5.2.6 (Maximum principle for subharmonic functions) *If f is subharmonic in a connected open set Ω and attains its maximum in Ω then f is constant.*

PROOF. The set $\{x \in \Omega\colon f(x) \geq \max_f\}$ is closed in Ω since f is upper semicontinuous, and it follows from condition 2) of Theorem 5.2.3 that it is also open.

■

Corollary 5.2.7 *If $f_n \searrow f$ on Ω, when f_n are subharmonic then f is subharmonic.*

PROOF. Use condition 2) of Theorem 5.2.3.

■

Theorem 5.2.8 (Hartogs lemma) *Let f_j be a sequence of subharmonic functions in Ω which are uniformly bounded from above on every compact subset of Ω, and assume that there is a constant C such that*

$$\varlimsup_{j\to\infty} f_j(x) \leq C \quad \textit{for every} \quad x \in \Omega.$$

Then for every $\varepsilon > 0$ and every compact set $K \subset \Omega$ there is a j_0 such that

$$f_j(x) \leq C + \varepsilon \quad \textit{for} \quad x \in K, j > j_0.$$

PROOF. Fix a compact set $K \subset \Omega$. We may find an open set Ω_0 such that $K \subset \Omega_0 \subset\subset \Omega$, then f_j are bounded above, by say M, on Ω_0. Let $v_j = M - f_j$. Then $v_j \geq 0$ on Ω_0 and $\underline{\lim}_{j\to\infty} v_j(x) \geq C_1 := M - C$ for $x \in \Omega$. We are to show that $v_j(x) \geq C_1 - \varepsilon$ for $x \in K$, $j > j_0$. We may assume that $C_1 > 0$ otherwise there is nothing to prove. Choose $r > 0$ so that for every $x \in K$, $B(x, 2r) \subset \Omega_0$. By subharmonicity of f_j we have, for $x \in K$ and $0 < \varrho \leq 2r$,

$$f_j(x) \leq \frac{1}{\nu(B(x,\varrho))} \int_{B(x,\varrho)} f_j \, d\nu$$

hence

$$v_j(x) \geq \frac{1}{\nu(B(x,\varrho))} \int_{B(x,\varrho)} v_j \, d\nu.$$

By Fatou's lemma we also have

$$\underline{\lim}_{j\to 0}\Big(\frac{1}{\nu(B(x,r))} \int_{B(x,r)} v_j \, d\nu\Big) \geq \frac{1}{\nu(B(x,r))} \int_{B(x,r)} \underline{\lim}_{j\to 0} v_j \, d\nu \geq C_1.$$

Fix $\varepsilon > 0$, then for each $x \in K$ there is a $j(x)$ such that for $j > j(x)$

$$\frac{1}{\nu(B(x,r))} \int_{B(x,r)} v_j \, d\nu \geq C_1 - \frac{\varepsilon}{2}.$$

Now let $0 < \delta < r$ be such that

$$\Big(\frac{r}{r+\delta}\Big)^m \Big(C_1 - \frac{\varepsilon}{2}\Big) \geq C_1 - \varepsilon$$

and $y \in B(x, \delta)$ then $B(x, r) \subset B(y, r + \delta)$ and for $j > j(x)$

$$\begin{aligned} v_j(y) &\geq \frac{1}{\nu(B(y, r+\delta))} \int_{B(y,r+\delta)} v_j \, d\nu \\ &\geq \Big(\frac{r}{r+\delta}\Big)^m \frac{1}{\nu(B(x,r))} \int_{B(x,r)} v_j \, d\nu \\ &\geq \Big(\frac{r}{r+\delta}\Big)^m \Big(C_1 - \frac{\varepsilon}{2}\Big) \\ &\geq C_1 - \varepsilon. \end{aligned}$$

Hence we have shown that for each $x \in K$ there exists a $j(x)$ such that if $j > j(x)$ then for $y \in B(x, \delta)$, $v_j(y) \geq C_1 - \varepsilon$. Since K may be covered by a final number of balls $B(x_k, \delta)$ we can take $j_0 = \max\{j(x_k)\}$.

■

Proposition 5.2.9 *If $\Omega \subset \mathbb{C}^n$, $f \in H(\Omega)$ then* $\log|f|$ *is continuous and subharmonic in* Ω *and harmonic on the set where it is not* $(-\infty)$.

Proof. Let $x \in \Omega$ be such that $f(x) \neq 0$. Take $r_0 > 0$ such that $f(B(x, r_0)) \subset \mathbb{C} \setminus L$ where L is a ray starting from the origin. Let ℓ be a holomorphic branch of logarithm on $\mathbb{C} \setminus L$ then $\ell \circ f$ is holomorphic in $B(x, r_0)$ so $\log|f| = \operatorname{Re}(\ell \circ f)$ is harmonic. Thus $\log|f|$ is harmonic in $\Omega_1 = \Omega \setminus \big\{x : \log|f(x)| \neq -\infty\big\}$. Hence we get condition 2a) of Theorem 5.2.3 for $x \in \Omega_1$ and this condition is obviously fulfilled if $\log|f(x)| = -\infty$.

■

Theorem 5.2.10 (Jensen's inequality) *Let μ be a probability measure on A, $f: A \to \mathbb{R}$ a measurable function and $\varphi: \mathbb{R} \to \mathbb{R}$ a continuous convex function such that f and $\varphi \circ f$ are integrable on A. Then*

$$\varphi\Big(\int_A f\, d\mu\Big) \leq \int_A \varphi \circ f\, d\mu.$$

Remark. It is also possible to formulate Jensen's inequality when f and $\int_A f\, d\mu$ can take the value $(-\infty)$. We will use such a version below.

Corollary 5.2.11 *If f is subharmonic in Ω and $\varphi: \mathbb{R} \cup \{-\infty\} \to \mathbb{R} \cup \{-\infty\}$ is convex and increasing then $\varphi \circ f$ is subharmonic.*

Proof.

$$(\varphi \circ f)(x) \leq \varphi\Big(\frac{1}{\nu(B(x,r))} \int_{B(x,r)} f\, d\nu\Big) \leq \frac{1}{\nu\,(B(x,r))} \int_{B(x,r)} \varphi \circ f\, d\nu$$

The first inequality follows from the fact that φ is increasing and f is subharmonic, the second from Jensen's inequality applied to the measure $d\mu = \frac{1}{\nu\big(B(x,r)\big)} d\nu$ on $B(x,r)$. Thus we have verified condition 3) of Theorem 5.2.3 for $\varphi \circ f$.

Corollary 5.2.12 *If $f \in H(\Omega)$ then* $\log^+|f|$, $|f|^p$ $(p > 0)$ *are subharmonic.*

Proof. Functions $t \mapsto t^+$, $t \mapsto e^{pt}$ are convex and increasing and composed with $\log|f|$ give $\log^+|f|$ and $|f|^p$.

■

Definition 5.2.13 *For f defined on $B(x,r_0) \subset \mathbb{R}^m$ and $0 < r < r_0$ let*

$$\varrho_{f,x}(r) := \frac{1}{\omega_{m-1}} \int_{\partial B(0,1)} f(x + r\xi)\, d\sigma(\xi) = \frac{1}{\sigma(\partial B(x,r))} \int_{\partial B(x,r)} f(\xi)\, d\sigma(\xi)$$

where $\omega_{m-1} = \sigma(\partial B(0,1))$.

Lemma 5.2.14 *Let $f\colon \Omega \to \mathbb{R} \cup \{-\infty\}$ be continuous and subharmonic in Ω and $B(x,r_0) \subset \Omega$. Then $\varrho_{f,x}(r)$ is an increasing function of r.*

Proof. Assume $f \neq (-\infty)$ on Ω and consider $r_1 < r_2 < r_0$. Let h be a continuous function on $\overline{B}(x,r_2)$ with $h = f$ on $\partial B(x,r_2)$. Then $h \geq f$ on $B(x,r_2)$ and, by the mean value property for h we have

$$\begin{aligned} \varrho_{f,x}(r_2) &= h(x) = \frac{1}{\sigma(\partial B(x,r_1))} \int_{\partial B(x,r_1)} h\, d\sigma \\ &\geq \frac{1}{\sigma(\partial B(x,r_1))} \int_{\partial B(x,r_1)} f\, d\sigma = \varrho_{f,x}(r_1). \end{aligned}$$

If we allow that f attains the value $(-\infty)$ then we must first consider $f_M := \max\{f, M\}$ for which the lemma has already been proved and then let $M \to (-\infty)$.

■

Corollary 5.2.15 *Let $f \in H(B_{\mathbb{C}^m}(0,1))$ and*

$$\begin{aligned} Nf(r) &:= \frac{1}{\omega_{2N-1}} \int_{\partial B(0,1)} \log^+ |f(r,\xi)|\, d\sigma(\xi) \\ M_p f(r) &:= \Big(\frac{1}{\omega_{2N-1}} \int_{\partial B(0,1)} |f(r\xi)|^p\, d\sigma(\xi) \Big)^{\frac{1}{p}}. \end{aligned}$$

Then Nf and $M_p f$ are increasing functions of r.

Lemma 5.2.16 *Let $f \in C(\Omega)$ then $\lim_{r \to 0} \varrho_{f,x}(r) = f(x)$ and if in addition $f \in C^2(\Omega)$ then*

$$\frac{d\varrho_{f,x}}{dr}(r) = \frac{1}{\omega_{m-1} r^{m-1}} \int_{B(x,r)} \Delta f\, d\nu.$$

PROOF. The first part is obvious. For the second we have

$$\begin{aligned}
\omega_{m-1}\frac{d\varrho}{dr} &= \frac{d}{dr}\int_{\partial B(0,1)} f(x_0+r\xi)\,d\sigma(\xi)\\
&= \int_{\partial B(0,1)} \sum \frac{\partial f}{\partial x_i}(x_0+r\xi)\xi_i\,d\sigma(\xi)\\
&= \int_{\partial B(x,r)} \sum \frac{\partial f}{\partial x_i}(\xi)\Big(\frac{1}{r}(\xi_i-x_i)\Big)\frac{1}{r^{m-1}}\,d\sigma(\xi)\\
&= \frac{1}{r^{m-1}}\int_{\partial B(x,r)} \frac{\partial f}{\partial \mathbf{n}}(\xi)\,d\sigma(\xi)\\
&= \frac{1}{r^{m-1}}\int_{B(x,r)} \Delta f(\xi)\,d\nu(\xi)
\end{aligned}$$

where $\mathbf{n}$ is the unit normal vector and we use a version of Green's theorem. ∎

Theorem 5.2.17 *Let $f\colon\Omega\to\mathbb{R}$ be C^2 in Ω. Then f is subharmonic iff $\Delta f\geq 0$ on Ω.*

PROOF.

$$\begin{aligned}
\Delta f\geq 0 &\Leftrightarrow \forall_x\exists_{r_0}\forall_{r<r_0}\int_{B(x,r)}\Delta f\geq 0\\
&\Leftrightarrow \forall_x\exists_{r_0}\,(\varrho'(r)\geq 0 \quad\text{for}\quad 0\leq r\leq r_0)\\
&\Leftrightarrow \forall_x\exists_{r_0}\,(\varrho \quad\text{is increasing on}\quad (0,r_0))
\end{aligned}$$

Thus we have reduced the study of the laplacian to the study of $\varrho_{f,x}$. But we have (f is subharmonic) $\Rightarrow$ ($\forall_x\,\varrho_{f,x}$ increasing) $\Rightarrow$ ($\forall_x\exists_{r_0}\forall_{r<r_0}\,\varrho_{f,x}(r)\geq f(x)$) $\Rightarrow$ (f is subharmonic) (by condition 2a) of Theorem 5.2.3).

Definition 5.2.18 *A function f is called strictly subharmonic if f is C^2 and $\Delta f>0$.*

5.3 Pluriharmonic and plurisubharmonic functions

We will fix a open set $\Omega\subset\mathbb{C}^n$.

Definition 5.3.1 *An $f\in C^2(\Omega)$ is pluriharmonic if for every $z\in\Omega$ and every $w\in\mathbb{C}^n$ the function $\xi\mapsto f(z+w\xi)$ is harmonic where defined.*

Exercise. Every pluriharmonic function is harmonic.

Observe that if f is pluriharmonic then so are $\bar{f}$, Ref, Imf.

Theorem 5.3.2 *f is pluriharmonic in Ω iff*

$$\forall_{i,j} \quad \frac{\partial^2 f}{\partial z_i \partial \bar{z}_j}(z) = 0 \quad on \quad \Omega. \tag{5.3.1}$$

Proof. We have

$$\begin{aligned}
&\frac{\partial^2}{\partial\xi\partial\bar{\xi}} f(z+w\xi) \\
&= \frac{\partial}{\partial\xi}\Big(\sum_{j=1}^{n}\frac{\partial f}{\partial z_j}(z+w\xi)\frac{\partial(z+w\xi)_j}{\partial\bar{\xi}} + \sum_{j=1}^{n}\frac{\partial f}{\partial \bar{z}_j}(z+w\xi)\frac{\partial\overline{(z+w\xi)}_j}{\partial\bar{\xi}}\Big) \\
&= \frac{\partial}{\partial\xi}\Big(\sum_{j=1}^{n}\frac{\partial f}{\partial \bar{z}_j}(z+w\xi)\overline{w}_j\Big) \\
&= \sum_{j=1}^{n}\left(\sum_{i=1}^{n}\frac{\partial}{\partial\bar{z}_i}\Big(\frac{\partial f}{\partial\bar{z}_j}\Big)(z+w\xi)\frac{\partial(z+w\xi_i}{\partial\xi}\overline{w}_j\right. \\
&\quad \left. + \sum_{i=1}^{n}\frac{\partial}{\partial\bar{z}_i}\Big(\frac{\partial f}{\partial\bar{z}_j}\Big)(z+w\xi)\frac{\partial\overline{(z+w\xi)}_i}{\partial\xi}\overline{w}_j\right) \\
&= \sum_{j=1}^{n}\sum_{i=1}^{n}\frac{\partial^2 f}{\partial z_i\partial\bar{z}_j}(z+w\xi)w_i\overline{w}_j
\end{aligned} \tag{5.3.2}$$

Evaluating (5.3.2) at $\xi = 0$. We get

$$\frac{1}{4}\Delta_\xi f(z+w\xi)\Big|_{\xi=0} = \sum_{i,j=1}^{n}\frac{\partial^2 f}{\partial z_i\partial\bar{z}_j}(z)w_i\overline{w}_j.$$

Thus (5.3.1) implies that f is pluriharmonic. It is enough to prove the reverse implication for real f. Then we have

$$\frac{\partial^2 f}{\partial z_i\partial\bar{z}_j} = \overline{\left(\frac{\partial^2 f}{\partial\bar{z}_i\partial z_j}\right)}$$

so the matrix $H = \left(\frac{\partial^2 f}{\partial z_i\partial\bar{z}_j}\right)_{i,j}$ is hermitian and $\langle Hw|w\rangle = 0$ for every $w \in \mathbb{C}^n$. It follows that $H = 0$ so all its entries are zero.

∎

Corollary 5.3.3 *A holomorphic function and its real and imaginary part are pluriharmonic.*

FACT. Every pluriharmonic function is locally a real part of a holomorphic function.

EXERCISE. $|z|^{-2(n-1)}$ is harmonic but not pluriharmonic.

Definition 5.3.4 *A function $f\colon \Omega \to \mathbb{R} \cup \{-\infty\}$ is plurisubharmonic (plush) if f is upper semicontinuous and for every $z \in \Omega$ and $w \in \mathbb{C}$ the function $\xi \mapsto f(z + \xi w)$ is subharmonic (where defined).*

The three following propositions are immediat consequences of corresponfong facts about subharmonic functions.

Proposition 5.3.5 *If $f_j \searrow f$ and f_j are plush then f is also plush.*

Proposition 5.3.6 *If f is plush and continuous in Ω and $\varphi\colon \mathbb{R} \cup \{-\infty\} \to \mathbb{R} \cup \{-\infty\}$ is convex and increasing then $\varphi \circ f$ is plush.*

Proposition 5.3.7 *If $f\colon \Omega \to \mathbb{C}$ is holomorphic then $\log|f|$, $\log^+|f|$, $|f|^p$ are plush and continuous in Ω.*

Proposition 5.3.8 *Let $f\colon \Omega \to \mathbb{R} \cup \{-\infty\}$ be continuous. If f is plush and $z \in \Omega$, $w \in \mathbb{C}^n$, $r > 0$ are such that $z + rw\xi \in \Omega$ when ξ is in a neighbourhood of the closed unit disc $\overline{U} \subset \mathbb{C}$ then*

$$f(z) \le \frac{1}{2\pi} \int_0^{2\pi} f(z + rwe^{i\theta})\, d\theta. \tag{5.3.3}$$

Conversely if

$$\forall_{z\in\Omega} \forall_{w\in\mathbb{C}^n} \exists_{r_0} \forall_{0<r<r_0}\, f(z) \le \frac{1}{2\pi} \int_0^{2\pi} f(z + rwe^{i\theta})\, d\theta$$

then f is plush.

PROOF. The first part is obvious as it simply uses the submean value property of the subharmonic function $\xi \mapsto f(z + rw\xi)$. The second uses condition 2a) of Theorem 5.2.3 for this function to show that it is subharmonic (it still requires a little work — exercise).

■

Corollary 5.3.9 *If f is plush then f is subharmonic.*

PROOF. We will show that condition 2) of Theorem 5.2.3 is satisfied for f. We are to show that

$$f(z) \le \frac{1}{\nu(B(z,r))} \int_{B(z,r)} f(w)\, d\nu(w).$$

But observe that $w \mapsto z + e^{i\theta}(w - z)$ is a measure-preserving transformation on $B(z,r)$ (it is a rotation about the center of the ball). Hence for each fixed θ we have

$$\int_{B(z,r)} f(w)\, d\nu(w) = \int_{B(z,r)} f(z + e^{i\theta}(w - z))\, d\nu(w) \tag{5.3.4}$$

Integrating both sides of (5.3.4) in θ over $[0, 2\pi]$ we get

$$\begin{aligned} 2\pi \int_{B(z,r)} f(w)\, d\nu(w) &= \int_0^{2\pi} \int_{B(z,r)} f(z + e^{i\theta}(w - z))\, d\nu(w)\, d\theta \\ &= \int_{B(z,r)} \int_0^{2\pi} f(z + e^{i\theta}(w - z))\, d\theta\, d\nu(w) \end{aligned}$$

Using (5.3.3) with $r = 1$ we get

$$f(z) \le \frac{1}{2\pi} \int_0^{2\pi} f(z + e^{i\theta}(w - z))\, d\theta.$$

Hence

$$\begin{aligned} \int_{B(z,r)} f(w)\, d\nu(w) &= \frac{1}{2\pi} \int_{B(z,r)} \int_0^{2\pi} f(z + e^{i\theta}(w - z))\, d\theta\, d\nu(w) \\ &\ge \int_{B(z,r)} f(z)\, d\nu(w) = f(z) \end{aligned}$$

■

REMARK. The trick used in this proof based on circular invariance of the ball is very commonly used in SCV. The same argument proves that whenever $\mathcal{D}$ is circular with respect to z, i.e., $\forall_{w \in \mathcal{D}} \forall_{\lambda \in \mathbb{C}} (|\lambda| = 1 \Rightarrow z + \lambda w \in \mathcal{D})$ and f is plush in $\mathcal{D}$ we have

$$f(z) \le \int_{\mathcal{D}} f(\xi)\, d\nu(\xi)$$

Theorem 5.3.10 *Let $f\colon \Omega \to \mathbb{R}$ be C^2 in Ω. Then f is plush in Ω iff*

$$\forall_{z \in \Omega} \forall_{w \in \mathbb{C}^n} \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial z_i \partial \bar{z}_j}(z) w_i \overline{w}_j \ge 0 \tag{5.3.5}$$

Proof. We know that $\xi \mapsto f(z+w\xi)$ is subharmonic iff $\Delta_\xi f(z+w\xi) \geq 0$. But

$$\frac{1}{4}\Delta_\xi f(z+w\xi)|_{\xi=0} = \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial z_i \partial \bar{z}_j}(z) w_i \overline{w}_j$$

and the claim follows.

■

Definition 5.3.11 *A C^2 function $f:\Omega \to \mathbb{R}$ is called strictly plurisubharmonic if*

$$\forall_{z\in\Omega}\forall_{\substack{w\in\mathbb{C}\\ w\neq 0}} \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial z_i \partial \bar{z}_j}(z) w_i \overline{w}_j > 0.$$

The matrix $\left(\frac{\partial^2 f}{\partial z_i \partial \bar{z}_j}(z)\right)_{i,j}$ is called the complex Hessian of f at z. Thus we can say that a function is strictly plush iff its Hessian is positive definite on Ω.

Proposition 5.3.12 *If $f:\Omega \to \mathbb{R}$ is strictly plush and $K \subset\subset \Omega$ then there are constants $C, c > 0$ such that*

$$C|w|^2 \geq \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial z_i \partial \bar{z}_j}(P) w_i \overline{w}_j \geq c|w|^2 \quad \textit{for} \quad P \in K, w \in \mathbb{C}^n.$$

Proof. Consider the function

$$\Omega \times \mathbb{C}^n \ni (P,w) \overset{\Phi}{\longmapsto} \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial z_i \partial \bar{z}_j}(P) w_i \overline{w}_j.$$

Obviously Φ is continuous and $\Phi(P,w) > 0$ if $w \neq 0$. Let

$$\begin{aligned} c &:= \min\{\Phi(P,w) : P \in K, |w| = 1\} > 0 \\ C &:= \max\{\Phi(P,w) : P \in K, |w| = 1\} < \infty \end{aligned}$$

Since $\Phi(P,w) = |w|^2 \Phi\Big(P, \frac{w}{|w|}\Big)$ we get

$$C|w|^2 \geq \Phi(P,w) \geq c|w|^2.$$

■

5.4 Smooth approximation

Lemma 5.4.1 *Let φ be a C^∞ function on $\mathbb{R}^m$ which is non-negative, supported on the unit ball and $\int_{\mathbb{R}^m} \varphi = 1$. Let $\varphi_\varepsilon(x) = \varepsilon^{-m}\varphi(x/\varepsilon)$. If $f\colon \Omega \to \mathbb{R}$ is locally integrable then the function $f_\varepsilon(x) = \int_{\mathbb{R}^m} f(y)\varphi_\varepsilon(x-y)\, d\nu(y)$ is of class C^∞ on $\Omega_\varepsilon = \{x \in \Omega : \delta_\Omega(x) > \varepsilon\}$ and if f is continuous then $f_\varepsilon \to f$ uniformly on compact subsets of Ω.*

The family $\{\varphi_\varepsilon\}_{\varepsilon>0}$ is an example of a so called approximating identity. The integral defining f_ε is called the convolution of f and φ_ε and denoted $f * \varphi_\varepsilon$. Convolutions with smooth approximating identities are very often used to approximate a given function by smooth functions.

Proof. Since integrals defininf f_ε are taken only over balls $B(x,\varepsilon) \subset\subset \Omega$, we have that f is integrable on thes balls so the integrals are well defined. The fact that f_ε is C^∞ follows at once by differentiating integral on the parameter.

Since $\int \varphi_\varepsilon = 1$ we get

$$f(x) = f(x)\int \varphi_\varepsilon(y)\, d\nu(y) = f(x)\int \varphi_\varepsilon(x-y)\, d\nu(y) = \int f(x)\varphi_\varepsilon(x-y)\, d\nu(y).$$

Now observe that φ_ε is supported on $B(0,\varepsilon)$ so

$$\begin{aligned} |f_\varepsilon(x) - f(x)| &= \Big|\int (f(x) - f(y))\varphi_\varepsilon(x-y)\, d\nu(y)\Big| \\ &\le \int_{B(x,\varepsilon)} |f(x) - f(y)|\varphi_\varepsilon(x-y)\, d\nu(y) \end{aligned}$$

If $K \subset \Omega$ is compact then for small δ we have $K \subset \Omega_\delta$. For such δ let $K^\delta = \{x : \mathrm{dist}(x,K) \le \delta\}$ then $K^\delta \subset \Omega$ and K^δ is compact hence if we fix an $\eta > 0$ then there is an $\varepsilon_0 < \delta$ such that for $x, y \in K^\delta$ satisfying $||x-y|| < \varepsilon_0$ we have

$$|f(x) - f(y)| < \eta. \tag{5.4.1}$$

If we now use (5.4.1) for $x \in K$, $\varepsilon < \varepsilon_0$ then for $y \in B(x,\varepsilon)$ we get $|f(x) - f(y)| < \eta$ so $|f_\varepsilon(x) - f(x)| < \eta$.

∎

Lemma 5.4.2 *Let φ be as before and in addition radial (i.e., $\varphi(x) = \psi(||x||)$). If f is a subharmonic function on Ω then f_ε is also subharmonic and for $\varepsilon_1 > \varepsilon_2 > 0$ we have $f_{\varepsilon_1} \ge f_{\varepsilon_2} \ge f$.*

Proof. We have

$$\begin{aligned} f_\varepsilon(x) &= \int_{B(0,\varepsilon)} f(x-y)\varphi_\varepsilon(y)\,d\nu(y) \\ &= \int_0^\varepsilon r^{m-1}\int_{\partial B(0,1)} f(x-r\xi)\varphi_\varepsilon(r\xi)\,d\sigma(\xi)\,dr \\ &= \int_0^\varepsilon r^{m-1}\varepsilon^{-m}\psi(r/\varepsilon)\int_{\partial B(0,1)} f(x-r\xi)\,d\sigma(\xi)\,dr \\ &= \int_0^\varepsilon r^{m-1}\varepsilon^{-m}\psi(r/\varepsilon)\varrho_{f,x}(r)\,dr \\ &= \int_0^1 t^{m-1}\psi(t)\varrho_{f,x}(\varepsilon t)dt. \end{aligned}$$

Since $\varrho_{f,x}(r) \geq f(x)$ and $\int_0^1 t^{m-1}\psi(t)\,dt = \int_{B(0,1)}\varphi(y)\,d\nu(y) = 1$ we get that $f_\varepsilon(x) \geq f(x)$ and since $\varrho_{f,x}(\varepsilon t)$ decreases together with ε we get that the whole integral decreases.

Now we establish subharmonicity of f_ε. Let $x \in \Omega$ and $r > 0$. Then

$$\begin{aligned} \int_{B(x,r)} f_\varepsilon(x)\,d\nu(x) &= \int_{B(x,r)}\int_{\mathbb{R}^m} f(x-y)\varphi_\varepsilon(y)d\nu(y)\,d\nu(x) \\ &= \int_{\mathbb{R}^m}\int_{B(x,r)} f(x-y)\,d\nu(x)\,\varphi_\varepsilon(y)\,d\nu(y) \\ &= \int_{\mathbb{R}^m}\int_{B(x-y,r)} f(w)\,d\nu(w)\,\varphi_\varepsilon(y)\,d\nu(y) \\ &\geq \int_{\mathbb{R}^m} \nu(B(x-y,r))f(x-y)\varphi_\varepsilon(y)\,d\nu(y) \\ &= \nu(B(x,r))f_\varepsilon(x) \end{aligned}$$

■

Remark. If $\Omega \subset \mathbb{C}^n$ and f is plush in Ω then we get that under assumptions of Lemma 5.4.2 f_ε is also plush, we simply have to repeat the above proof integrating over circles instead of balls.

Lemma 5.4.3 *If $\Omega_1 \subset \mathbb{C}^n$, $\Omega_2 \subset \mathbb{C}^m$ are open, $\Phi\colon \Omega_1 \to \Omega_2$ is holomorphic and $f\colon \Omega_2 \to \mathbb{C}$ is of class $\mathbb{C}^2$ then*

$$\sum_{i,j=1}^n \frac{\partial^2}{\partial z_i \partial \bar{z}_j}(f\circ\Phi)(P)w_i\overline{w}_j = \sum_{k,l=1}^m \frac{\partial^2 f}{\partial \xi_k \partial \overline{\xi}_l}(\Phi(P))v_k\overline{v}_l$$

where

$$v := \sum_{i=1}^{n} \frac{\partial \Phi}{\partial z_i}(P) w_i = D\Phi(P)(w)$$

Proof. It is an exercise in complex differentiation, one has to remember that if φ is a holomorphic function then $\frac{\partial \varphi}{\partial \bar{z}_k} = 0$ and $\frac{\partial \overline{\varphi}}{\partial z_k} = \overline{\left(\frac{\partial \varphi}{\partial \bar{z}_k}\right)} = 0$.

■

Theorem 5.4.4 *If $\Omega_1 \subset \mathbb{C}^n$, $\Omega_2 \subset \mathbb{C}^n$ are open, $\Phi \colon \Omega_1 \to \Omega_2$ is holomorphic and f is a continuous plush function in Ω_2 then $f \circ \Phi$ is plush in Ω_1. If in addition Φ is biholomorphic and f is strictly plush then $f \circ \Phi$ is also strictly plush.*

Proof. Assume that f is C^2. Then it immediately follows form Lemma 5.4.3 that f is plush. If Φ is biholomorphic then $D\Phi(P)$ is invertible hence if $w \neq 0$ then $v = D\Phi(P)(w) \neq 0$,and we get that if f is strictly plush then so is $f \circ \Phi$. If f is not $\mathbb{C}^2$ then we consider approximating functions f_ε, which are plush, C^∞ and $f_\varepsilon \searrow f$ as $\varepsilon \to 0$. It follows that $f_\varepsilon \circ \Phi$ are plush and $f_\varepsilon \circ \Phi \searrow f_\varepsilon \circ \Phi$ hence $f_\varepsilon \circ \Phi$ is plush.

■

6 Some applications of subharmonicity to SCV

6.1 The Bergman space

As we have seen in Section 4.4, $H(\Omega)$ can be considered in a natural way as a complete metric linear space. Unfortunately this space cannot be normed so we cannot use the tools developed for Banach spaces. It is however possible to introduce a large Hilbert space of holomorphic functions and then use the rich theory of Hilbert spaces.

Definition 6.1.1 *Let $\Omega \subset \mathbb{C}^n$ be open and*

$$L^2H(\Omega) = L^2(\Omega, d\nu) \cap H(\Omega).$$

$L^2H(\Omega)$ equipped with the scalar product from $L^2(\Omega, d\nu)$ is called the Bergman space on Ω.

Theorem 6.1.2 *If $\Omega \subset \mathbb{C}^n$ is open then $L^2H(\Omega)$ is a Hilbert space.*

Lemma 6.1.3 *Let $f \in H(B(z,r))$ then*

$$\sup_{B(z,r/2)} |f| \le 2^n \left(\frac{1}{\nu(B(z,r)))} \int_{B(z,r)} |f(w)|^2 \, d\nu(w) \right)^{\frac{1}{2}}$$

Proof. Let $\xi \in B(z,r/2)$. Since $|f|^2$ is subharmonic in $B(z,r)$ and $B(\xi, r/2) \subset B(z,r)$ we have

$$\begin{aligned} |f(\xi)|^2 &\le \frac{1}{\nu(B(\xi,r/2))} \int_{B(\xi,r/2)} |f(w)|^2 \, d\nu(w) \\ &\le \frac{1}{\nu(B(\xi,r/2))} \int_{B(z,r)} |f(w)|^2 \, d\nu(w) \\ &= \frac{2^{2n}}{\nu(B(z,r))} \int_{B(z,r)} |f(w)|^2 \, d\nu(w). \end{aligned}$$

■

PROOF OF THE THEOREM. We only have to prove that $L^2H(\Omega)$ is closed in $L^2(\Omega)$ so let $f_n \in L^2H(\Omega)$, $f_n \xrightarrow{L^2} f \in L^2(\Omega)$. Then $\{f_n\}$ forms a Cauchy sequence in $L^2(\Omega)$. Fix $x \in \Omega$ and $r > 0$ such that $B(z,r) \subset \Omega$ then it trivially follows from above lemma that $\{f_n\}$ form a Cauchy sequence on $B(z,r/2)$ in the topology of uniform convergence so f_n converge uniformly on $B(z,r/2)$, of course to f, hence f is holomorphic in $B(z,r/2)$.

■

Proposition 6.1.4 *Let $\Phi\colon \Omega_1 \to \Omega_2$ be biholomorphic then the operator*

$$T := (f \circ \Phi) J^{\mathbb{C}} \Phi$$

is an isometry of $L^2H(\Omega_2)$ onto $L^2H(\Omega_1)$.

PROOF. – Exercise.

■

Of course any two non-zero L^2H spaces are isometric since they are separable Hilbert spaces but one usually needs an explicit and natural isomorphism so this proposition is quite useful.

Let $\Omega \subset \mathbb{C}^n$ be open and $\varphi : \Omega \to \mathbb{R}$ a positive, continuous function then define

$$L^2H(\Omega, \varphi) = \{f \in H(\Omega) : \int_\Omega |f|^2 \varphi \, d\nu < \infty\}.$$

We equip $L^2H(\Omega, \varphi)$ with the natural scalar product

$$\langle f|g\rangle_\varphi = \int_\Omega f \bar{g} \, \varphi \, d\nu.$$

It is completely trivial to extend results of this section to $L^2H(\Omega, \varphi)$ and prove

Theorem 6.1.5 *$L^2H(\Omega, \varphi)$ is a Hilbert space.*

6.2 Hartogs theorem on separate analyticity

This section is devoted to the proof of

Theorem 6.2.1 (Hartogs) *Every coordinatewise holomorphic function $f\colon \Omega \to \mathbb{C}$ is holomorphic.*

Recall that according to Cor. 4.4.2 every locally bounded and coordinatewise holomorphic function is holomorphic.

We will proceed by induction and assume that Theorem 6.2.1 has already been proved for dimensions smaller then n (for $n = 1$ there is nothing to prove).

Lemma 6.2.2 *Let f be coordinatewise holomorphic in Ω and let $\mathcal{D} = \prod_{j=1}^n D_j \subset\subset \Omega$, where $D_j \subset \mathbb{C}$ are discs. Then there exist discs $U_j \subset D_j$ with $U_n = D_n$ such that f is bounded in $\overline{\mathcal{U}} = \prod_{j=1}^n \overline{U}_j$, hence holomorphic in $\mathcal{U}$.*

Proof. Denote by $\mathcal{D}' = \prod_{j=1}^{n-1} D_j$. For $M \in \mathbb{N}$ let

$$E_M = \left\{ z' \in \overline{\mathcal{D}}' : |f(z', z_n)| \le M \quad \text{when} \quad z_n \in \overline{D}_n \right\}.$$

By inductive hypothesis for a fixed z_n the function $z' \mapsto f(z', z_n)$ is holomorphic, in particular continuous so E_M is closed. We also have $\bigcup_{M=1}^\infty E_M = \overline{\mathcal{D}}'$ hence, by Baire's category theorem there is an M such that E_M has nonempty interior and we may find $\mathcal{U}' = \prod_{j=1}^{n-1} U_j \subset E_M$. Then $\mathcal{U} = \mathcal{U}' \times D_n$ is the desired polydisk (f is bounded by M in $\mathcal{U}$).

■

This lemma does not yet prove our theorem as we have no control over the polydisk $\mathcal{U}$.

Lemma 6.2.3 *Let f be a complex valued function in a neighbourhood of the polydisk $\overline{\mathcal{D}} = \{z : |z_j - z_j^0| \le R,\ j = 1, \dots, n\}$, assume that f is holomorphic in z' if z_n is fixed and that f is holomorphic in a neighbourhood of*

$$\overline{\mathcal{U}} = \{z : |z_j - z_j^0| \le r,\ j = 1, \dots, n-1\,,\ |z_n - z_n^0| \le R\}$$

for some $r > 0$. Then f is holomorphic in $\mathcal{D}$.

Proof. We may assume that $z^0 = 0$. Choose R_1 and R_2 with $0 < R_1 < R_2 < R$. Since f is holomorphic in z' we have the power series expansion

$$f(z) = \sum_\alpha a_\alpha(z_n) z'^\alpha \quad \text{for} \quad z \in \mathcal{D} \tag{6.2.1}$$

where α only runs trough multi-indices with $n-1$ places. Since by (4.3.2)

$$a_\alpha(z_n) = \frac{1}{\alpha!}\frac{\partial^\alpha f}{\partial z'^\alpha}(0, z_n)$$

and f is holomorphic in $\mathcal{U}$, we get that a_α is a holomorphic function of z_n when $|z_n| < R$. Hence, the terms of the series (6.2.1) are holomorphic in $\mathcal{D}$ and it is our aim to show that the convergence is normal which will prove that the sum of the series is holomorphic.

Since for every $|z_n| < R$ the series (6.2.1) converges in $\mathcal{D}'$ and $R_2 < R$ we get

$$|a_\alpha(z_n)|R_2^{|\alpha|} \to 0 \quad \text{as} \quad |\alpha| \to \infty. \tag{6.2.2}$$

f is bounded in $\overline{\mathcal{U}}$, by say M, so applying the Cauchy inequalities to f in $\mathcal{U}'$ we get

$$|a_\alpha(z_n)|r^{|\alpha|} \le M \quad \text{when} \quad |z_n| < R. \tag{6.2.3}$$

Now consider the subharmonic functions

$$z_n \mapsto \frac{1}{|\alpha|}\log|a_\alpha(z_n)|.$$

By (6.2.3) these functions are bounded from above when $|z_n| < R$ and, by (6.2.2),

$$|\alpha|\log R_2 + \log|a_\alpha(z_n)| \xrightarrow{|\alpha|\to\infty} -\infty,$$

so

$$\overline{\lim_\alpha}\left(\frac{1}{|\alpha|}\log|a_\alpha(z_n)|\right) \le \log(1/R_2).$$

Now we can use Th. 5.2.8 to get that for large $|\alpha|$ and $|z_n| < R_1$

$$\frac{1}{|\alpha|}\log|a_\alpha(z_n)| \le \log(1/R_1)$$

or equivalently

$$|a_\alpha(z_n)|R_1^{|\alpha|} \le 1 \quad \text{for large } |\alpha| \text{ and } |z_n| < R_1.$$

It follows that the series (6.2.1) converges uniformly on every polydisk $\{z : |z_j| < R_3,\ 1, \dots, n\}$ if $R_3 < R_1$. Since R_1 can be arbitrarily close to R we get normal convergence on $\mathcal{D}$.

■

Now we can end the proof of the theorem.

Choose a point $w \in \Omega$ and let R_0 be such that $\{z : |z_j - w_j| \leq 2R_0\} \subset \Omega$. Apply Lemma 6.2.2 to the polydisk $\mathcal{D}_0 = \{z : |z_j - w_j| < R_0\}$. Thus we get a new polydisk $\mathcal{U}_0 \subset \mathcal{D}_0$, centered at $z^0 \in \mathcal{D}_0$.

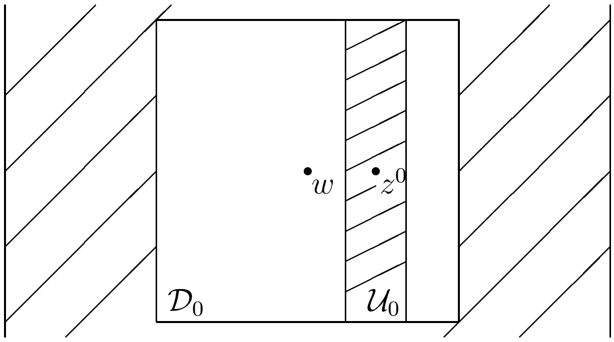


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Figure 1.

f is bounded in $\mathcal{U}_0$. Now let $\mathcal{D}_1 = \{z : |z_j - z_j^0| < R_0\}$. Then $\mathcal{D}_1 \subset\subset \Omega$ and $w \in \mathcal{D}_1$.

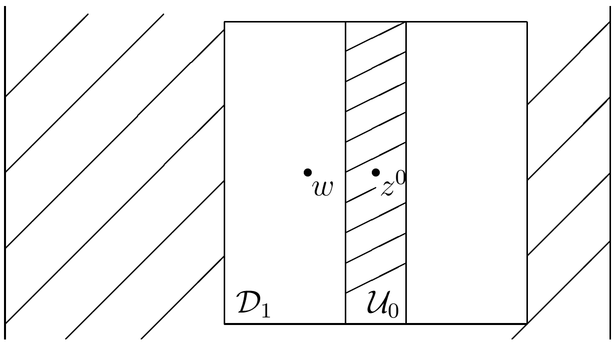

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Figure 2.

Take R slightly smaller then R_0 so that $w \in \mathcal{D} = \{z : |z_j - z_j^0| < R\}$ and let $\mathcal{U} = \mathcal{U}_0 \cup \mathcal{D}$. Then the hypothesis of Lemma 6.2.3 is fulfilled and we get that f is holomorphic in $\mathcal{D}$ hence at w.

$\blacksquare$

7 Pseudoconvexity and domains of holomorphy

7.1 Introduction

We will now start studying subsets of $\mathbb{C}^n$ serving as domains of definition of holomorphic functions. It turns out that there is a lot of difference between one and several dimensions. One example we have already seen, there is no extension of the Riemann mapping theorem, another is even more striking: if $\Omega \subset \mathbb{C}$ then one can always find a holomorphic function on Ω which cannot be extended to a holomorphic (or even continuous) function on any larger set; this is not true in several dimensions, there exist pairs of domains $\Omega_1 \subset \Omega_2 \subset \mathbb{C}^n$ such that any holomorphic function on Ω_1 is a restriction of a holomorphic function defined on Ω_2. It was one of the central points of interest to describe domains which are 'maximal' i.e., they have functions which cannot be extended.

Even if certain results do extend to several variables then usually one needs to put some additional restrictions on generality. For these and other reasons one must decide which domains are especially important or simply most convenient to study.

The conditions that we will meet can seem quite unnatural if seen without any additional background hence the purpose of the next two sections will be to supply this background by studying convex subdomains of $\mathbb{R}^m$ and giving various descriptions of this well known geometrical notion. It will be seen later that nearly identical expressions will appear in our study of $\mathbb{C}^n$. The contents of these two sections is not a part of a standard course in SCV but it is very useful for deeper understanding of the material that follows. For this reason the presentation in the script is rather extensive even though large parts, mainly the proofs, will be excluded from the lectures.

7.2 Smoothly bounded domains

In many cases one reduces the study to domains whose boundary forms a smooth hypher-surface. We will now introduce elements of the theory of such sets in $\mathbb{R}^m$.

By a C^k-bounded domain in $\mathbb{R}^m$ we will understand a bounded domain Ω whose topological boundary $\partial\Omega$ is an $(m-1)$-dimensional surface (manifold) of class C^k i.e., the parameterizations which define it are required to be of class C^k.

Theorem 7.2.1 *If $\Omega \subset \mathbb{R}^m$ is a C^k-bounded domain then there exists a real valued function $\varrho \in C^k(\mathbb{R}^m)$ with*

$$\begin{aligned} \Omega &= \{x \in \mathbb{R}^m : \varrho(x) < 0\} \\ \overline{\Omega}^c &= \{x \in \mathbb{R}^m : \varrho(x) > 0\} \\ \nabla\varrho &\neq 0 \quad \text{on} \quad \partial\Omega. \end{aligned}$$

We will often use directional derivatives as they seem to be most intuitive. It is important to remember that if $P, w \in \mathbb{R}^m$ then the directional derivative of f in the direction of w may be obtained in the following way: let γ be any smooth curve such that $\gamma(0) = P$, $\gamma'(0) = w$, then

$$\frac{\partial f}{\partial w}(P) = \frac{d}{dt}(f \circ \gamma)\Big|_{t=0}.$$

We also have expressions

$$\begin{aligned} \frac{\partial f}{\partial w}(P) &= Df(P)(w) \\ \frac{\partial f}{\partial w}(P) &= \sum_{j=1}^{m} \frac{\partial f}{\partial w}(P) w_j. \end{aligned}$$

Definition 7.2.2 *A function ϱ satisfying conditions of Theorem 7.2.1 is called a defining function for Ω.*

It is obvious that if $\varrho \in C^k(\mathbb{R}^m)$ is such that $\nabla\varrho \neq 0$ on the set $\{x : \varrho(x) = 0\}$ then each component of the set $\{x : \varrho(x) < 0\}$ is a C^k-bounded domain.

Proposition 7.2.3 *Let $\Omega \subset \mathbb{R}^m$ be a C^k-bounded domain $(k \geq 2)$ and let ϱ be a defining function for Ω. Let $V \subset \mathbb{R}^m$ be open, $V \cap \partial\Omega \neq \emptyset$. Suppose $\tilde{\varrho}\colon V \to \mathbb{R}$ is a function of class C^j, $(2 \leq j \leq k)$ such that $\tilde{\varrho} = 0$ on $V \cap \partial\Omega$. Then there is a $C^{(j-1)}$-function ψ on V such that $\tilde{\varrho} = \psi\varrho$. We have then*

$$\nabla\tilde{\varrho} = \psi\nabla\varrho \quad \text{on} \quad V \cap \partial\Omega \tag{7.2.1}$$

and if $w, v \in \mathbb{R}^n$ are tangent to $\partial\Omega$ at $P \in \partial\Omega$ (i.e., $\langle\nabla\varrho(P)|w\rangle = \langle\nabla\varrho(P)|v\rangle = 0$) then

$$\frac{\partial^2\tilde{\varrho}}{\partial v\partial w} = \psi(P)\frac{\partial^2\varrho}{\partial v\partial w}. \tag{7.2.2}$$

Sketch of the proof. One has to define ψ on $V\setminus\partial\Omega$ by $\psi = \tilde{\varrho}/\varrho$. Using a suitable change of variables we see that ψ extends to a $C^{(j-1)}$-function on V. Then

$$\nabla\tilde{\varrho} = \nabla(\psi\varrho) = \varrho\nabla\psi + \psi\nabla\varrho.$$

On $\partial\Omega$ this equals to $\psi\nabla\varrho$ hence

$$\frac{\partial\tilde{\varrho}}{\partial w}(x) = \psi\frac{\partial\varrho}{\partial w}(x) \quad \text{for} \quad w \in \mathbb{R}^m, x \in \partial\Omega. \tag{7.2.3}$$

Let $v \in T_P\partial\Omega$. If we take a curve in $\partial\Omega$ defining this vector then we have (7.2.3) satisfied all the time on this curve and differentiation of $\frac{\partial\tilde{\varrho}}{\partial w}$ along it produces

$$\frac{\partial}{\partial v}\frac{\partial\tilde{\varrho}}{\partial w} = \frac{\partial\psi}{\partial v}\frac{\partial\rho}{\partial w} + \psi\frac{\partial^2\varrho}{\partial v\partial w}.$$

If in addition $w \in T_P\partial\Omega$ then the first term of the right-hand side vanishes.

■

Definition 7.2.4 *For $x \in \mathbb{R}^m$ let $\delta_\Omega(x) = \operatorname{dist}(x, \partial\Omega)$, and for $A \subset \mathbb{R}^m$, $\delta_\Omega(A) = \inf_{x\in A}\delta_\Omega(x)$. Define also*

$$\delta^*_\Omega(x) = \begin{cases} -\delta_\Omega(x) & x \in \overline{\Omega} \\ \delta_\Omega(x) & x \notin \Omega \end{cases}$$

Theorem 7.2.5 *If Ω is C^k-bounded ($k \geq 2$) then δ^*_Ω is of class C^k in some neighbourhood of $\partial\Omega$ and $\nabla\delta^*_\Omega \neq 0$ on $\partial\Omega$.*

If $k = 1$ then δ^*_Ω need not be differentiable, also even if $k \geq 2$ δ^*_Ω need not be differentiable in all of Ω.

Hints for the proof of this theorem are given in [KR], Exercise 4 in Chapter 3.

Since δ^*_Ω need not be of class C^2 in all of Ω we cannot take it as a defining function but it is trivial to find a defining function ϱ which is equal to δ^*_Ω on a neighbourhood of $\partial\Omega$. Just take any defining function $\tilde{\varrho}$ and a C^∞-function $\psi \geq 0$ such that $\psi = 1$ on some neighbourhood of $\partial\Omega$ and $\operatorname{supp}(\psi)$ is contained in the set where δ^*_Ω is smooth. Then $\varrho := \psi\tilde{\varrho} + (1-\psi)\delta^*_\Omega$ has all desired properties.

7.3 Geometric convexity

The purpose of this section is to present some characterizations of convexity in order to explain, later on, what is the origin of certain definitions used in the theory of SCV. We also introduce some important definitions.

Fix an open connected set $\Omega \subset \mathbb{R}^m$.

Definition 7.3.1 *Let $\varphi\colon \Omega \to \mathbb{R}$. We say that φ is convex in Ω if whenever $[x, y] \subset \Omega$ and $t \in (0, 1)$ we have $\varphi(tx + (1-t)y) \leq t\varphi(x) + (1-t)\varphi(y)$, and that it is strictly convex if the inequality is strict.*

Proposition 7.3.2 *If $\varphi\colon \Omega \to \mathbb{R}$ is of class C^2 then it is convex iff*

$$\forall_{P\in\Omega}\forall_{w\in\mathbb{R}^m} \sum_{i,j=1}^{m} \frac{\partial^2\varphi}{\partial x_i \partial x_j}(P) w_i w_j \geq 0$$

and strictly convex iff

$$\forall_{P\in\Omega}\forall_{\substack{w\in\mathbb{R}^m \\ w\neq 0}} \sum_{i,j=1}^{m} \frac{\partial^2\varphi}{\partial x_i \partial x_j}(P) w_i w_j > 0.$$

Proof. Both parts become obvious if one realizes that φ is (strictly) convex iff every function $g(t) = \varphi(P + tw)$ is (strictly) convex in a neighbourhood of zero and that in turn can be expressed as $g'' \geq 0$ ($g'' > 0$). But

$$g''(0) = \frac{\partial^2\varphi}{\partial w^2} = \sum_{i,j=1}^{m} \frac{\partial^2\varphi}{\partial x_i \partial x_j}(P) w_i w_j. \tag{7.3.1}$$

so our conditions seem to say the right thing but only at 0, still since we can vary the point P we can get that they imply g'' being (strictly) positive in a neighbourhood of zero.

■

Corollary 7.3.3 *Every C^2 (strictly) convex is (strictly) subharmonic.*

Proof. It follows from the above proposition that the real Hessian (i.e., the matrix of the second derivative) of a (strictly) convex function f is (strictly) positive definite hence has (strictly) positive trace so Δf is (strictly) positive. Now use Theorem 5.2.17.

■

Corollary 7.3.4 *Every C^2 (strictly) convex function on a domain $\Omega \subset \mathbb{C}^n$ is (strictly) plush.*

Proof. According to (2.2.5) a function is plush if the sum of second directional derivatives in directions w and iw is non-negative. If a function is convex then by Proposition 7.3.2 and (7.3.1) both of these derivatives are non-negative hence also their sum.

Lemma 7.3.5 *Let φ be as in Lemma 5.4.1. If f is a convex function on Ω then f_ε is convex on Ω_ε.*

Proof. If $[x, z] \subset \Omega_\varepsilon$ and $t \in (0, 1)$ then we have

$$\begin{aligned} f_\varepsilon\Big(tx + (1-t)z\Big) &= \int_{\mathbb{R}^m} f\Big(tx + (1-t)z - y\Big)\varphi_\varepsilon(y)\, d\nu(y) \\ &= \int_{B(0,\varepsilon)} f\Big(t(x-y) + (1-t)(z-y)\Big)\varphi_\varepsilon(y)\, d\nu(y) \\ &\leq \int_{B(0,\varepsilon)} \Big(tf(x-y) + (1-t)f(z-y)\Big)\varphi_\varepsilon(y)\, d\nu(y) \\ &\leq tf_\varepsilon(x) + (1-t)f_\varepsilon(z) \end{aligned}$$

■

Corollary 7.3.6 *Every convex function is subharmonic.*

Proof. Let $f\colon \Omega \to \mathbb{R}$ be convex. Since convex functions are continuous, it follows from Lemmas 5.4.1 and 7.3.5 that f can be approximated uniformly on compact subsets of Ω by C^∞ convex functions. Such functions are subharmonic and it is easily seen from integral conditions on subharmonicity that normal convergence preserves subharmonicity so f is indeed subharmonic.

Theorem 7.3.7 *Let Ω be a C^2-bounded domain in $\mathbb{R}^m$ then Ω is convex iff*

$$\forall_{p\in\partial\Omega}\forall_{w\in T_p(\partial\Omega)} \sum_{i,j=1}^{m} \frac{\partial^2 \varrho}{\partial x_i \partial x_j}(P) w_i w_j \geq 0. \tag{7.3.2}$$

The condition in (7.3.2) can be stated equivalently as $\frac{\partial^2\varrho}{\partial w^2}(P) \geq 0$ or $D^2\varrho(P)(w^2) \geq 0$ it does not depend on the choice of ϱ (this follows from (7.2.2) after observing that (7.2.1) implies that $\psi > 0$ on $\partial\Omega$).

PROOF. We will only show that condition (7.3.2) is necessary.

Let $P \in \partial\Omega$ and $H + P$ be the hypher-plane separating P from Ω. Let $w \in H$. Then $\forall_t\ \varrho(P + tw) \geq 0$ and $\varrho(P) = 0$. It follows that the function $g(t) := \varrho(P + tw)$ has a minimum at 0 so its derivative vanishes and thus $\frac{\partial \varrho}{\partial w}(P) = 0$, so $w \in T_p\partial\Omega$. Hence $H \subset T_p(\partial\Omega)$ and considering the dimensions we see that they must be equal. Considering the minimum of g again we see that $g''(0) \geq 0$ so $\frac{\partial^2 \varrho}{\partial w^2}(P) \geq 0$. This condition holds for every $w \in H$ so for every $w \in T_p\partial\Omega$.

■

Definition 7.3.8 *Suppose $P \in \partial\Omega$ is such that $\partial\Omega$ is of class C^2 in a neighbourhood of this point. P is said to be a point of strict convexity if*

$$\forall_{\substack{w \in T_P\partial\Omega \\ w \neq 0}} \quad \sum_{i,j=1}^{m} \frac{\partial^2 \varrho}{\partial x_i \partial x_j}(P) w_i w_j > 0$$

and a point of strict concavity if the strict reverse inequality holds.

Definition 7.3.9 *A C^2-bounded domain Ω is called strictly convex if all its boundary points are points of strict convexity.*

Proposition 7.3.10 *A strictly convex set is convex.*

Obviously this proposition follows at once from Theorem 7.3.7 but from the implication which we did not prove. Since we will use this proposition in the proof of Theorem 7.3.13 we give an independent proof.

PROOF. Let

$$S = \{(x, y) \in \Omega \times \Omega : [x, y] \subset \Omega\}$$

then S is a non-empty open subset of $\Omega \times \Omega$. Let $(x_j, y_j) \in \Omega \times \Omega$, $(x_j, y_j) \to (x_0, y_0)$. Suppose $(x_0, y_0) \notin S$ then $[x_0, y_0] \not\subset \Omega$ but certainly $[x_0, y_0] \subset \overline{\Omega}$ so let $P \in\,]x_0, y_0[\cap \partial\Omega$, $w = x_0 - y_0$ and $g(t) := \varrho(P + tw)$. Then $g(0) = \varrho(P) = 0$ and $g(t) \leq 0$ in a neighbourhood of 0 since $[x_0, y_0] \subset \overline{\Omega}$ and $\varrho \leq 0$ on $\overline{\Omega}$. It follows that g has a local maximum at 0 so $g'(0) = 0$ and $g''(0) \leq 0$. But $g'(0) = \frac{\partial \varrho}{\partial w}(P)$ so $w \in T_p\partial\Omega$ and $g''(0) = \frac{\partial^2 \varrho}{\partial w^2}(P) > 0$ which is a contradiction.

■

Definition 7.3.11 *Let $\mathcal{F}$ be a family of real or complex valued functions on Ω. If $K \subset \Omega$ then we define the $\mathcal{F}$-convex hull of K to be*

$$\hat{K}_{\mathcal{F}} = \Big\{x \in \Omega : |f(x)| \leq \sup_{t \in K} |f(t)| \quad \forall_{f \in \mathcal{F}}\Big\}.$$

Ω is said to be $\mathcal{F}$-convex (or convex with respect to $\mathcal{F}$) if

$$K \subset\subset \Omega \quad \Rightarrow \quad \hat{K}_{\mathcal{F}} \subset\subset \Omega.$$

Example. If $\mathcal{A}(\Omega)$ is the family of all affine functions on Ω then $\hat{K}_{\mathcal{A}(\Omega)} = \overline{\mathrm{conv}(K)} \cap \Omega$. Indeed

$$\hat{K}_{\mathcal{A}} = \Big(\bigcap_{\varphi \in \mathcal{A}} \{x : \varphi(x) \leq \max_K \varphi\}\Big) \cap \Omega.$$

The first set is an intersection of closed half-spaces containing K hence contains $\overline{\mathrm{conv}(K)}$. It also follows from the separation theorem for convex sets that it is the smallest closed convex set containing K.
Example. $\hat{K}_{C(\Omega)} = \overline{K} \cap \Omega$.

Definition 7.3.12 *A function $l\colon \Omega \to \mathbb{R}$ is said to be an exhaustion function for Ω if*

$$\forall_{t \in \mathbb{R}} \Big(\{x \in \Omega : l(x) < t\} \subset\subset \Omega\Big)$$

Example. $l(x) = \max\{(-\log \delta_\Omega), \|x\|\}$ is always an exhaustion function for Ω.

Theorem 7.3.13 *The following conditions are equivalent*

a) Ω is convex.

b) Ω is convex with respect to the family of all affine functions on Ω.

c) $(-\log \delta_\Omega)$ is a convex function on Ω.

d) Ω has a convex exhaustion function.

e) Ω has a C^∞ convex exhaustion function.

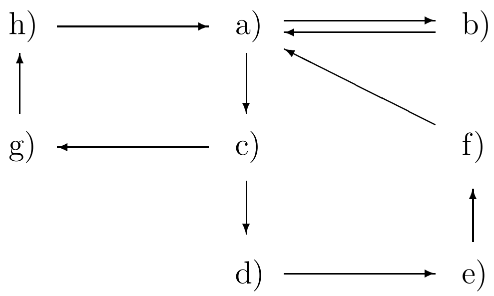
f) $\Omega = \bigcup_j \Omega_j$ where $\Omega_j \subset\subset \Omega_{j+1}$ and Ω_j are strictly convex.

g) For every affine function $h\colon [a,b] \to \Omega$

$$\delta_\Omega\Big(h([a,b])\Big) = \delta_\Omega\Big(\{h(a), h(b)\}\Big).$$

h) If $\{I_\alpha\}$ *is a family of intervals in* Ω *such that* $\bigcup_\alpha \partial I_\alpha \subset\subset \Omega$ *then* $\bigcup_\alpha I_\alpha \subset\subset \Omega$ *where* ∂I *denotes the set of endpoints of* I*.*

PROOF.



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(a ⇒ b) Let $\mathcal{A}(\Omega)$ be the family of all affine functions on Ω then, as we have seen $\hat{K}_{\mathcal{A}(\Omega)} = \overline{\text{conv}(K)} \cap \Omega$. Now if Ω is convex and $K \subset\subset \Omega$ then $\overline{K} \subset \Omega$ and is compact so $\text{conv}(\overline{K})$ is compact and of course $\text{conv}(\overline{K}) \subset \Omega$. We also have $\overline{\text{conv}(K)} = \text{conv}(\overline{K})$ thus $\hat{K}_{\mathcal{A}(\Omega)} \subset\subset \Omega$.

(b ⇒ a) Take $K = \{x, y\} \subset \Omega$ then by b), $[x, y] = \overline{\text{conv}(K)} = \hat{K}_{\mathcal{F}} \subset \Omega$ so Ω is convex.

(a ⇒ c) Suppose $(-\log \delta_\Omega)$ is not convex. Then there are points $x, y \in \Omega$ and a $t \in (0,1)$ such that

$$\log\Big(\delta_\Omega(tx + (1-t)y)\Big) < t \log \delta_\Omega(x) + (1-t) \log \delta_\Omega(y)$$

hence

$$\delta_\Omega(tx + (1-t)y) < e^{t \log \delta_\Omega(x) + (1-t) \log \delta_\Omega(y)} \le t\delta_\Omega(x) + (1-t)\delta_\Omega(y).$$

Let $P = tx + (1-t)y$ and let $P' \in \mathbb{R}^m \setminus \Omega$ be such that $||P' - P|| < t\delta_\Omega(x) + (1-t)\delta_\Omega(y)$. There exist $a, b > 0$ such that $a < \delta_\Omega(x)$, $b < \delta_\Omega(y)$ and $ta + (1-t)b = ||P' - P||$. Finally let $w = (P' - P)/||P' - P||$, $x' = x + aw$,

$y' = y + bw$. Then $x', y' \in \Omega$ and $P' = tx' + (1-t)y'$. Since Ω is convex we get $P' \in \Omega$ which is contradiction.

(c $\Rightarrow$ d) Define $l(x) = \max\{-\log \delta_\Omega(x), ||x||\}$.

(d $\Rightarrow$ e $\Rightarrow$ f) These proofs are identical to the proofs of Theorem 7.4.8 ((3)$\Rightarrow$(4)$\Rightarrow$(5)) after substituting 'convex' for 'plush'. Since these proofs are quite complicated and this is, after all, a course in SCV we refer the reader there.

(f $\Rightarrow$ a) Since each of Ω_j is convex Ω is also convex.

(c $\Rightarrow$ g) If $t \in [0,1]$ then

$$\begin{aligned} \log \delta_\Omega\Big(h(ta + (1-t)b)\Big) &= \log \delta_\Omega(th(a) + (1-t)h(b)) \\ &\geq t \log \delta_\Omega(h(a)) + (1-t) \log \delta_\Omega(h(b)) \\ &\geq \min\Big\{\log \delta_\Omega(h(a)), \log \delta_\Omega(h(b)) \\ &= \log \delta_\Omega\Big(\{h(a), h(b)\}\Big) \end{aligned}$$

and the claim follows.

(g $\Rightarrow$ h) What g) really says is that the distance from an interval in Ω to the boundary of Ω is attained at the endpoints of that interval so for each interval $I \subset \Omega$ we have $\delta_\Omega(I) = \delta_\Omega(\partial I)$. The fact that $K \subset\subset \Omega$ is equivalent to the fact that K is bounded and $\delta_\Omega(K) > 0$. Now let $\{I_\alpha\}$ satisfy the hypothesis of h). Then

$$0 < \delta_\Omega(\bigcup_\alpha \partial I_\alpha) = \inf_\alpha \delta_\Omega(\partial I_\alpha) = \inf_\alpha \delta_\Omega(I_\alpha) = \delta_\Omega(\bigcup_\alpha I_\alpha)$$

and since $\bigcup_\alpha \partial I_\alpha$ is bounded we get that $\bigcup_\alpha I_\alpha$ is bounded so $\bigcup_\alpha I_\alpha \subset\subset \Omega$.

(h $\Rightarrow$ a) Let

$$S = \{(z, y) \in \Omega \times \Omega : [x, y] \subset \Omega\}$$

then S is non-empty open subset of $\Omega \times \Omega$. Now let $(x_j, y_j) \in S$, $(x_j, y_j) \to (x_0, y_0) \in \Omega \times \Omega$ and let $I_j = [x_j, y_j]$, $j \geq 0$. Then $\bigcup_{j\geq 1} \partial I_j \subset\subset \Omega$ so $\bigcup_{j\geq 1} I_j \subset\subset \Omega$ but it is easily seen that $I_0 \subset \overline{\bigcup_{j\geq 1} I_j}$ so $I_0 \subset \Omega$ and thus $(x_0, y_0) \in S$ which shows that S is also closed and as $\Omega \times \Omega$ is connected we get that $S = \Omega \times \Omega$ so Ω is convex.

■

7.4 Pseudoconvexity

We will now consider an extension of the notion of convexity to one which is more suitable for the study of holomorphic functions. One of the main objectives is that the class we define be closed under biholomorphic transformations. The class of all convex domains does not satisfy this condition but it can be a starting point. Suppose $\Phi\colon \Omega_1 \to \Omega_2$ is biholomorphic and Ω_2 is convex. What can be said about Ω_1? One approach is to use exhaustion functions. If $l\colon \Omega_2 \to \mathbb{R}$ is a convex exhaustion function then $l \circ \Phi$ is an exhaustion function for Ω_1 but it need not be convex, however it must be plush (as l is plush and this is preserved). Another approach is to look at C^2-bounded domains and assume that Φ is also a C^2-diffeomorphism defined on a neighbourhood of $\overline{\Omega}_1$ onto a neighbourhood of $\overline{\Omega}_2$. In this case if ϱ is a defining function for Ω_2 then $\varrho \circ \Phi$ is a defining function for Ω_1. ϱ satisfies condition (7.3.2) which can be stated as $\frac{\partial^2 \varrho}{\partial v^2}(Q) \geq 0$ for $Q \in \partial\Omega_2$, $v \in T_Q\partial\Omega_2$. This condition need not be preserved by holomorphic mappings but we can derive one that will be. Namely if $v, iv \in T_Q\partial\Omega_2$ then

$$\sum_{i,j=1}^{n} \frac{\partial^2 \varrho}{\partial z_i \partial \bar{z}_j}(Q) v_i \overline{v}_j = \frac{\partial^2 \varrho}{\partial v^2}(Q) + \frac{\partial^2 \varrho}{\partial (iv)^2}(Q) \geq 0$$

but if $Q = \Phi(P)$ and $v = D\Phi(P)(w)$ then , by Lemma 5.4.3

$$\sum_{i,j=1}^{n} \frac{\partial^2 \varrho \circ \Phi}{\partial z_i \partial \bar{z}_j}(P) w_i \overline{w}_j = \sum_{i,j=1}^{n} \frac{\partial^2 \varrho}{\partial z_i \partial \bar{z}_j}(Q) v_i \overline{v}_j.$$

Since $(v \in T_Q\partial\Omega_2) \Leftrightarrow (w \in T_P\partial\Omega_1)$ and $D\Phi(P)(iw) = iv$ we see that condition

$$\forall_{Q\in\partial\Omega_2} \forall_v \; v, iv \in T_Q\partial\Omega_2 \Rightarrow \frac{\partial^2 \varrho}{\partial v^2}(Q) + \frac{\partial^2 \varrho}{\partial (iv)^2}(Q) \geq 0 \tag{7.4.1}$$

is preserved.

Definition 7.4.1 *If Ω is a C^1 bounded domain then the complex tangent space to $\partial\Omega$ at $P \in \partial\Omega$ is defined as*

$$T_P^{\mathbb{C}}\partial\Omega = \{w \in T_P\partial\Omega : iw \in T_P\partial\Omega\}$$

Proposition 7.4.2 *Let ϱ be a defining function for Ω then $w \in T_P^{\mathbb{C}}\partial\Omega$ iff $\sum_{j=1}^{n} \frac{\partial \varrho}{\partial z_j}(P) w_j = 0$.*

PROOF. Let $H = \left\{ w : \sum_{j=1}^{n} \frac{\partial \varrho}{\partial z_j} w_j = 0 \right\}$. Then

$$\frac{\partial \varrho}{\partial z_j} w_j = \frac{1}{2}\left(\frac{\partial \varrho}{\partial x_j} a_j + \frac{\partial \varrho}{\partial y_j} b_j \right) + \frac{1}{2} i \left(\frac{\partial \varrho}{\partial x_j} b_j - \frac{\partial \varrho}{\partial y_j} a_j \right).$$

Hence $w \in H \Rightarrow \sum_{j=1}^{n} \frac{\partial \varrho}{\partial x_j} a_j + \sum_{j=1}^{n} \frac{\partial \varrho}{\partial y_j} b_j = 0$ so $w \in T_P \partial\Omega$. Now we have $H \subset T_P \partial\Omega$, and H is closed under multiplication by i so $H \subset T_P^{\mathbb{C}} \partial\Omega$, but the complex dimension of H is $(n-1)$ so $H = T_P^{\mathbb{C}} \partial\Omega$.

■

We can now state our first definition

Definition 7.4.3 *A C^2 bounded domain Ω is called Levi pseudoconvex if*

$$\sum_{i,j=1}^{n} \frac{\partial^2 \varrho}{\partial z_i \partial \bar{z}_j}(P) w_i \overline{w}_j \geq 0$$

whenever $P \in \partial\Omega$ and $\sum_{j=0}^{n} \frac{\partial \varrho}{\partial z_j}(P) w_j = 0$, where ϱ is a C^2 defining function for Ω.

Definition 7.4.4 *The form*

$$\sum_{i,j=1}^{n} \frac{\partial^2 \varrho}{\partial z_i \partial \bar{z}_j}(P) w_i \overline{w}_j$$

is called the Levi form of ϱ.

Some as in the case of convexity we define a strictly pseudoconvex domain:

Definition 7.4.5 *A C^2 bounded domain Ω is strictly pseudoconvex if*

$$\sum_{i,j=1}^{n} \frac{\partial^2 \varrho}{\partial z_i \partial \bar{z}_j}(P) w_i \overline{w}_j > 0$$

whenever $P \in \partial\Omega$, $w \neq 0$, $\sum_{j=0}^{n} \frac{\partial \varrho}{\partial z_j}(P) w_j = 0$.

It is easily seen that C^2 bounded convex domains are pseudoconvex and strictly convex domains are strictly pseudoconvex.

Remark. It follows from Proposition 7.2.3 and Theorem 7.2.5 that the definitions of Levi pseudoconvexity and strict pseudoconvexity do not depend on the choice of the defining function. It is in fact enough to check condition (7.4.1) for δ^*_Ω as only the behaviour close to $\partial\Omega$ is important. It also follows from considerations at the beginning of this section that Levi pseudoconvexity and strict pseudoconvexity is preserved by biholomorphic mappings which smoothly extend onto a neighbourhood of the closure of the domain.

It should be apparent by now that if we want conditions which are preserved by biholomorphic mappings then convex functions must be substituted by plush functions, so the next definition should not be surprising.

Definition 7.4.6 *A domain $\Omega \subset \mathbb{C}^n$ is called Hartogs pseudoconvex if the function $(-\log \delta_\Omega)$ is plush in Ω.*

We will now state a theorem which will introduce some order to this new topic. Before that we will need one more definition:

Definition 7.4.7 *Let U be the unit disk in $\mathbb{C}$. An analytic disk in $\mathbb{C}^n$ is a non-constant holomorphic mapping $\Delta\colon U \to \mathbb{C}^n$. If Δ extends continuously to $\overline{U}$ then we call it a closed analytic disk and denote $\partial\Delta = \Delta(\partial U)$ to be the boundary of the analytic disk.*

We will often identify an analytic disc with its image, so depending on the context Δ can be either a function or a set.

Theorem 7.4.8 *The following conditions are equivalent*

1) *Ω is convex with respect to the family of all plurisubharmonic functions in Ω.*

2) *Ω is Hartogs pseudoconvex.*

3) *Ω has a continuous plush exhaustion function.*

4) *Ω has a C^∞ strictly plush exhaustion function.*

5) *$\Omega = \bigcup_j \Omega_j$ where $\Omega_j \subset\subset \Omega_{j+1}$ and Ω_j are strictly pseudoconvex.*

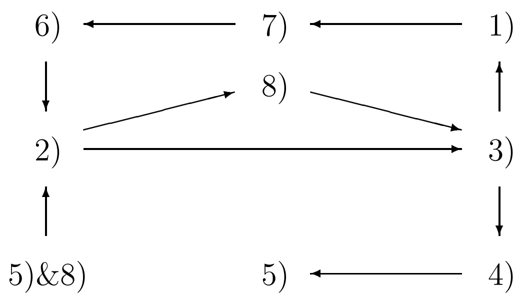
6) *If $\Delta \subset \Omega$ is a closed analytic disk then $\delta_\Omega(\partial\Delta) = \delta_\Omega(\Delta)$.*

7) Let (Δ_α) be a family of closed analytic disks in Ω. If $\bigcup_\alpha \partial\Delta_\alpha \subset\subset \Omega$ then $\bigcup_\alpha \Delta_\alpha \subset\subset \Omega$.

Under the additional assumption that Ω is C^2-bounded all the above are equivalent to

8) Ω is Levi pseudoconvex.

PROOF.



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1) $\Rightarrow$ 7) Let $P(\Omega)$ denote the family of plush functions in Ω. Let Δ be a closed analytic disk and $f \in P(\Omega)$ then $f \circ \Delta$ is subharmonic, hence

$$\forall_{|t|<1} \quad |f \circ \Delta(t)| \le \sup_{\partial U} |f \circ \Delta|$$

so

$$\forall_{z\in\Delta} \quad |f(z)| \le \sup_{\partial\Delta} |f|$$

and hence we have that $\Delta \subset \widehat{\partial\Delta}_{P(\Omega)}$. It follows that $\bigcup_\alpha \Delta_\alpha \subset \bigcup_\alpha \widehat{\partial\Delta}_\alpha \subset \bigcup_\alpha \widehat{\partial\Delta}_\alpha \subset\subset \Omega$.

7) $\Rightarrow$ 6) Suppose that there is a closed analytic disk $\Delta \subset \Omega$ such that $\delta_\Omega(\Delta) < \delta_\Omega(\partial\Delta)$. Let $P_0 \in \Delta$, $z_0 \in \partial\Omega$ be such that $|z_0 - P_0| = \delta_\Omega(\Delta)$ and let $w = z_0 - P_0$.

Define $\Delta_j = \Delta + (1 - \frac{1}{j})w$. If $P \in \Delta$, $z \in \partial\Omega$ then

$$\Big|\Big(P + \Big(1 - \frac{1}{j}\Big)w\Big) - z\Big| \ge |P - z| - \Big(1 - \frac{1}{j}\Big)|w| \ge |w| - \Big(1 - \frac{1}{j}\Big)|w| = \frac{1}{j}|w|$$

hence $\delta_\Omega(\Delta_j) \geq \frac{1}{j}|w| > 0$ so $\Delta_j \subset \Omega$. Next let $\varepsilon > 0$ be such that $\delta_\Omega(\partial\Delta) > \delta_\Omega(\Delta) + \varepsilon$. Then for $P \in \partial\Delta$, $z \in \partial\Delta$ we get

$$\Big|\Big(P + \Big(1 - \frac{1}{j}\Big)w\Big) - z\Big| \geq \delta_\Omega(\partial\Delta) + \Big(1 - \frac{1}{j}\Big)|w| \geq \frac{1}{j}\delta_\Omega(\Delta) + \varepsilon.$$

Hence $\forall_j\ \delta_\Omega(\partial\Delta_j) \geq \varepsilon$ so $\bigcup_j(\partial\Delta_j) \subset\subset \Omega$ and by the assumption $\bigcup_j \Delta_j \subset\subset \Omega$. But $P_0 + (1 - \frac{1}{j})w \in \Delta_j$ and $P_0 + (1 - \frac{1}{j})w = (z_0 - \frac{1}{j}w) \to z_0 \in \partial\Omega$ which is a contradiction.

6) $\Rightarrow$ **2)** We have to show that for a fixed $z \in \Omega$ and small $w \in \mathbb{C}^n$ we have

$$-\log\delta_\Omega(z) \leq \frac{1}{2\pi}\int_0^{2\pi} -\log\delta_\Omega(z + we^{i\theta})\,d\theta.$$

To shorten the notation we will denote

$$\psi(\xi) := -\log\delta_\Omega(z + \xi w) \qquad \xi \in \mathbb{C}, |\xi| \leq 1.$$

Hence we are to show that

$$\psi(0) \leq \frac{1}{2\pi}\int_0^{2\pi} \psi(e^{i\theta})\,d\theta.$$

We will use the fact that every real, continuous function on ∂U can be uniformly approximated by real parts of holomorphic polynomials. (This is a restatement of the fact that real trigonometric polynomials are dense in 2π-periodic, continuous, real functions).

So let p be a holomorphic polynomial in $\mathbb{C}$, such that

$$|\mathrm{Re}\, p(\xi) - \psi(\xi)| < \varepsilon \quad \text{for} \quad \xi \in \partial U.$$

Starting with a smaller ε and adding a constant to p we may assume that $\mathrm{Re}\, p > \psi$ on ∂U.

Take a point $b \in \mathbb{C}^n$ with $|b| \leq 1$ and define

$$\Delta(\xi) := z + \xi w + be^{-p(\xi)} \quad \text{for} \quad \xi \in \overline{U}$$

then Δ is a closed analytic disk. We will show that $\Delta \subset \Omega$. Assume for a moment that this has been shown, then we finish as follows:

$$\Delta(0) = z + be^{-p(0)} \in \Omega.$$

Since b was arbitrary with $|b| \leq 1$ we get that $B(z, |e^{(-p(0)}|) \subset \Omega$, or equivalently $\delta_\Omega(z) \geq |e^{-p(0)}| = e^{-\mathrm{Re}\, p(0)}$. Since $\psi(0) = -\log \delta_\Omega(z)$ we get $\psi(0) \leq \mathrm{Re}\, p(0)$. But then

$$\psi(0) \leq \mathrm{Re}\, p(0) = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{Re}\, p(e^{i\theta})\, d\theta \leq \varepsilon + \int_0^{2\pi} \psi(e^{i\theta})\, d\theta.$$

Take $\varepsilon \to 0$ to get the desired inequality.

It remains to show that $\Delta \subset \Omega$. It will be enough to show that $\partial\Delta \subset \Omega$, so let $\xi \in \partial U$ then

$$|z + \xi w - (z + \xi w + be^{-p(\xi)})| = |b||e^{-p(\xi)}| \leq e^{-\mathrm{Re}\, p(\xi)} < e^{-\psi(\xi)} = \delta_\Omega(z + \xi w).$$

Hence $\Delta(\xi) = z + \xi w + be^{-p(\xi)} \in \Omega$ if $|\xi| = 1$.

2) $\Rightarrow$ 3) Obvious.

3) $\Rightarrow$ 1) Let l be a plush exhaustion function for Ω and let $K \subset\subset \Omega$. Then for some $t \in \mathbb{R}$, $K \subset \{l < t\}$. If $z \in \hat{K}_{P(\Omega)}$ then $l(z) \leq \sup_K l \leq t$ so $z \in \{l < t + \varepsilon\} \subset\subset \Omega$.

2) $\Rightarrow$ 8) We assume now that Ω is C^2 – bounded. We know that δ^*_Ω is C^2 in a neighborhood of $\partial\Omega$ and $(-\log(-\delta^*_\Omega))$ is plush in Ω. It follows that

$$\sum \frac{\partial^2}{\partial z_i \partial \bar{z}_j} (-\log(-\delta^*_\Omega))(z) w_i \overline{w}_j \geq 0$$

for z in an Ω-neighborhood of $\partial\Omega$ and $w \in \mathbb{C}^n$. But

$$\frac{\partial}{\partial \bar{z}_j} \log(-\delta^*_\Omega) = \frac{1}{\delta^*_\Omega} \frac{\partial \delta^*_\Omega}{\partial \bar{z}_j}$$

so

$$\frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log(-\delta^*_\Omega) = -\frac{1}{(\delta^*_\Omega)^2} \frac{\partial \delta^*_\Omega}{\partial z_i} \frac{\partial \delta^*_\Omega}{\partial \bar{z}_i} + \frac{1}{\delta^*_\Omega} \frac{\partial \delta^*_\Omega}{\partial z_i \partial \bar{z}_j}.$$

Hence

$$0 \leq \sum_{i,j} \frac{\partial^2}{\partial z_i \partial \bar{z}_j} (-\log(-\delta^*_\Omega)) w_i \overline{w}_j = \frac{1}{(\delta^*_\Omega)^2} \left| \sum_i \frac{\partial \delta^*_\Omega}{\partial z_i} w_i \right|^2 - \frac{1}{\delta^*_\Omega} \sum_{i,j} \frac{\partial^2 \delta^*_\Omega}{\partial z_i \partial \bar{z}_j} w_i \overline{w}_j.$$

Now take $z \in \Omega$ and w such that $\sum \frac{\partial \delta_\Omega^*}{\partial z_i}(z) w_i = 0$. Since $\delta_\Omega^*(z) < 0$ we get

$$\sum_{i,j} \frac{\partial^2 \delta_\Omega^*}{\partial z_i \partial \bar{z}_j}(z) w_i w_j \geq 0.$$

Passing with $z \to \partial\Omega$ we get that the same is satisfied on the boundary of Ω which implies Levi pseudoconvexity.

(8 $\Rightarrow$ 3) We will begin by showing that $(-\log \delta_\Omega)$ is plush close to $\partial\Omega$.

Let V be a neighbourhood of $\partial\Omega$ in which δ_Ω^* is C^2, then $(-\log \delta_\Omega)$ is C^2 in $V \cap \Omega$. Suppose it is not plush there, then

$$\exists_{z \in \Omega \cap V} \exists_{w \in \mathbb{C}^n} \quad c := \sum_{i,j} \frac{\partial^2 (\log \delta_\Omega)}{\partial z_i \partial \bar{z}_j} w_i \overline{w}_j > 0.$$

Consider $\varphi(\xi) = \log \delta_\Omega(z + \xi w)$, for small $\xi \in \mathbb{C}$. Then expanding φ into the Taylor's formula according to (2.2.4), we get

$$\begin{aligned} \varphi(\xi) &= \varphi(0) + \mathrm{Re}\left(\frac{\partial \varphi}{\partial \xi}(0)\xi + \frac{1}{2}\frac{\partial^2 \varphi}{\partial \xi^2}(0)\xi^2\right) + \frac{\partial^2 \varphi}{\partial \xi \partial \bar{\xi}}(0)|\xi|^2 + o(|\xi|^2) \\ &= \mathrm{Re}(A\xi + B\xi^2) + c|\xi|^2 + o(|\xi|^2). \end{aligned}$$

Hence

$$\delta_\Omega(z + \xi w) = \delta_\Omega(z) e^{\mathrm{Re}(A\xi + B\xi^2)} e^{c|\xi|^2 + o(|\xi|^2)}.$$

Now let $a \in \mathbb{C}^n$ be such that $|a| = \delta_\Omega(z)$, $z + a \in \partial\Omega$ and define

$$\psi(\xi) := z + \xi w + a e^{A\xi + B\xi^2}$$

then $\psi(0) = z + a \in \partial\Omega$. What we will essentially show is that the closed analytic disc ψ has its boundary in Ω and an interior point on $\partial\Omega$ which impossible. We estimate

$$\begin{aligned} \delta_\Omega(\psi(\xi)) &= \delta_\Omega(z + \xi w + a e^{A\xi + B\xi^2}) \\ &\geq \delta_\Omega(z + \xi w) - |a w^{A\xi + B\xi^2}| \\ &= \delta_\Omega(z) e^{\mathrm{Re}(A\xi + B\xi^2)} e^{c|\xi|^2 + o(|\xi|^2)} \\ &- \delta_\Omega(z) e^{\mathrm{Re}(A\xi + B|\xi|^2)} (e^{c|\xi|^2 + o(|\xi|^2)} - 1) \\ &\geq \varepsilon |\xi|^2 \quad \text{for small} \quad \xi \end{aligned}$$

It follows that there exists an $r > 0$ such that for $0 < |\xi| < r$, $\psi(\xi) \in \Omega$ and $\psi(0) \in \partial\Omega$. In particular $\delta^*_\Omega \circ \psi$ has a maximum at 0. Next we consider

$$\begin{aligned}
-\varepsilon|\xi|^2 &\geq \delta^*_\Omega(\psi(\xi)) \\
&= (\delta^*_\Omega \circ \psi)(0) + 2\mathrm{Re}\left(\frac{\partial}{\partial\xi}(\delta^*_\Omega \circ \psi)\Big|_{\xi=0}\xi + \frac{1}{2}\frac{\partial^2}{\partial\xi^2}(\delta^*_\Omega \circ \psi)\Big|_{\xi=0}\xi^2\right) \\
&\quad + \frac{\partial^2}{\partial\xi\partial\bar\xi}(\delta^*_\Omega \circ \psi)\Big|_{\xi=0}|\xi^2| + o(|\xi|^2) \\
&= \mathrm{Re}\left(\frac{\partial^2}{\partial\xi^2}(\delta^*_\Omega \circ \psi)\ |_{\xi=0}\ \xi^2\right) \\
&\quad + \sum \frac{\partial^2\delta^*_\Omega}{\partial z_i\partial\bar z_j}(z+a)\psi_i'(0)\overline{\psi_j'(0)}|\xi|^2 + o(|\xi|^2) \\
&\geq \mathrm{Re}\left(\frac{\partial^2}{\partial\xi^2}(\delta^*_\Omega \circ \psi)\ |_{\xi=0}\ \xi^2\right) + o(|\xi|^2).
\end{aligned}$$

The last inequality holds because $\psi'(0) \in T^{\mathbb{C}}_{z+a}\partial\Omega$ and the Levi form of δ^*_Ω is positive definite on this space. The inequality

$$-\varepsilon|\xi|^2 \geq \mathrm{Re}(\alpha\xi^2) + o(|\xi|^2)$$

cannot hold on any neighbourhood of zero, it is false if $\alpha = 0$ and if $\alpha \neq 0$ than we can always make the right-hand side positive. We have thus proved that $(-\log\delta_\Omega)$ is plush close to $\partial\Omega$.

Now we will show that if Ω has any continuous exhaustion function l which is plush close to $\partial\Omega$ then we also have an exhaustion function which is plush in all of Ω.

Let $C > 1$ be such that l is plush on the set $\{z \in \Omega : l(z) > C\}$. Let $\chi\colon \mathbb{R} \to \mathbb{R}$ be convex, increasing, $\chi = 0$ on $(-\infty, -1]$, $\chi(x) = x$ for $x > 1$. Define

$$\Phi(z) = \chi(l(z) - 2C) + 2C.$$

According to Cor. 5.2.11 Φ is plush when $l > C$ but if $l(z) \leq C + (C-1)$ then $\Phi(z) = 2C$ so Φ is plush on all of Ω. We also have $\Phi = l$ on $\{z : l(z) > 2C+1\}$ so Φ is an exhaustion function.

3) $\Rightarrow$ 4) Let l be a continuous plush exhaustion function for Ω. Let $\Omega^t = \{z \in \Omega : l(z) < t\}$ and $\Omega_\varepsilon = \{z \in \Omega : \delta_\Omega(z) > \varepsilon\}$. On Ω_ε define $\tilde l_\varepsilon = (l|_{\Omega_{3\varepsilon}}) * \varphi_\varepsilon$, where φ_ε are the functions from Lemma 5.4.1.

Then, on $\Omega_{4\varepsilon}$, $\tilde{l}_\varepsilon = l * \varphi_\varepsilon$, hence $\tilde{l}_\varepsilon$ is plush on $\Omega_{4\varepsilon}$ and if we fix a compact $K \subset \Omega$ and take ε small enough for $K \subset \Omega_{4\varepsilon}$ then, by Lemma 5.4.2, we get that $\tilde{l} \searrow l$ on K. Observe also that on $\Omega_\varepsilon \setminus \Omega_{2\varepsilon}$ we have $\tilde{l}_\varepsilon = 0$ so we are able to extend $\tilde{l}_\varepsilon$ smoothly to $\mathbb{C}^n$ and so will consider $\tilde{l}_\varepsilon$ to be C^∞ on $\mathbb{C}^n$.

Now for $j = 1, 2, \dots$ choose ε_j so that $0 < \varepsilon_j < \frac{1}{4}\delta_\Omega(\Omega_{j+2})$ and

$$\Phi_j(z) := \tilde{l}_{\varepsilon_j}(z) + \varepsilon_j |z|^2 \le l(z) + \frac{1}{2} \quad \text{on} \quad \overline{\Omega}^j.$$

It can be achieved since $\tilde{l}_\varepsilon$ converge to l uniformly on $\overline{\Omega}^j$. For every j the function Φ_j is C^∞ on $\mathbb{C}^n$ and strictly plush on Ω^{j+2} ($\overline{\Omega}^{j+2} \subset \Omega_{4\varepsilon}$ so $\tilde{l}_{\varepsilon_j}$ is plush there and the second term makes Φ_j strictly plush).

Now let us describe the strategy. We have functions Φ_j which are strictly plush on larger and larger sets but, of course, need not be plush on all of Ω. We could consider a sum $\sum a_k \Phi_k$, where a_j are chosen inductively to be so large that the addition of $a_j \Phi_j$ to $\sum_{k=1}^{j-1} a_k \Phi_k$ makes the sum strictly plush on Ω^j (second order derivatives of Φ_k are bounded on Ω^j so this can be done). But with large a_k there is no reason for the series to converge. It would be a different story of we new that Φ_j are zero on, say Ω^{j-2}, and strictly plush on $\Omega^{j+1} \setminus \Omega^{j-1}$ because then the series would stabilize and there would be no problems with convergence. This is exactly what we are going to do by replacing Φ_j with new functions Ψ_j.

Take $\chi\colon \mathbb{R} \to \mathbb{R}$ being convex, C^∞, $\chi = 0$ on $(-\infty, -1/2]$, $\chi' > 0$ on $(-1/2, +\infty]$, $\chi(0) = 1$ and define

$$\Psi_j = \chi \circ (\Phi_j - j + 1).$$

Of course $\Psi_j \ge 0$ and

$$\frac{\partial^2 \Psi_j}{\partial z_k \partial \bar{z}_l} = \Big(\chi'' \circ (\Phi_j - j + 1)\Big) \frac{\partial \Phi_j}{\partial z_k} \frac{\partial \Phi_j}{\partial \bar{z}_l} + \Big(\chi' \circ (\Phi_j - j + 1)\Big) \frac{\partial^2 \Phi_j}{\partial z_k \partial \bar{z}_l}.$$

Hence

$$\begin{aligned} \sum_{k,l=1}^n \frac{\partial^2 \Psi_j}{\partial z_k \partial \bar{z}_l} w_k \overline{w}_l &= \Big(\chi'' \circ (\Phi_j - j + 1)\Big) \left| \sum_{k=1}^n \frac{\partial \Phi_j}{\partial z_k} w_k \right|^2 \\ &+ \Big(\chi' \circ (\Phi_j - j + 1)\Big) \sum_{k,l=1}^n \frac{\partial^2 \Phi_j}{\partial z_k \partial \bar{z}_l} w_k \overline{w}_l \end{aligned}$$

It follows that Ψ_j is plush on Ω^{j+2} and strictly plush on the subset of Ω^{j+2} where $\Phi_j - j + 1 > -1/2$ (because $\chi' \circ (\Phi_j - j + 1) > 0$ on this set), we also have that $\Psi_j = 0$ on the set where $\Phi_j - j + 1 \leq -1/2$. Now let us get some information on these sets. We have

$$\Phi_j \geq \tilde{l}_{\varepsilon_j} \geq l \quad \text{on} \quad \Omega^{j+2}$$

and $l \geq (j - 3/2)$ on $\Omega \setminus \Omega^{(j-3/2)}$ so

$$\Phi_j - j + 1 \geq -1/2 \quad \text{on} \quad \Omega^{j+2} \setminus \Omega^{(j-3/2)}.$$

Next

$$\Phi_j \leq l + 1/2 \quad \text{on} \quad \overline{\Omega}_j$$

so

$$\Phi_j \leq (j-2) + 1/2 \quad \text{on} \quad \overline{\Omega}^{j-2}$$

hence

$$\Phi_j - j + 1 \leq -1/2 \quad \text{on} \quad \overline{\Omega}^{j-2}.$$

It follows that Ψ_j is ≥ 1 and strictly plush on $\Omega^{j+2} \setminus \Omega^{j-3/2}$ and zero on $\overline{\Omega}^{j-2}$. So now finally we can construct our exhaustion function Ψ as

$$\Psi := \Phi_0 + \sum_{k=1}^{\infty} a_k \Psi_k$$

where a_k are defined as follows: suppose for $k < j$ we have defined $a_k > 0$ so that $\Phi_0 + \sum_{k=1}^{j-1} a_k \Psi_k$ is $\geq l$ and strictly plush on Ω^{j-1}. (This is satisfied for $j = 1$). If we take $a_j \geq j$ then $a_j \Psi_j \geq j > l$ on $\Omega^j \setminus \overline{\Omega}^{j-1}$, so $\Phi_0 + \sum_{k=1}^{j} a_k \Psi_k \geq l$ on Ω^j. Since Ψ_j is strictly plush on a neighborhood of $\overline{\Omega}^{j+1} \setminus \Omega^{j-1}$, using Lemma 5.3.12 we can take a_j so large that the sum $\Phi_0 + \sum_{k=1}^{j} a_k \Psi_k$ becomes strictly plush on $\Omega^j \setminus \Omega^{j-1}$ and it remains strictly plush in Ω^{j-1} because the shorter sum was such and $a_j \Psi_j$ is plush there.

The last thing to observe is that the sum did not change on Ω^{j-2} as Ψ_j is zero there. Hence the series $\Phi_0 + \sum_{k=1}^{\infty} a_k \Psi_k$ converges to a C^∞, strictly plush function Ψ satisfying $\Psi \geq l$. This last inequality guarantees that Ψ is an exhaustion function for Ω.

(4 $\Rightarrow$ 5) The proof of this implication would carry us a little to far from the general subject so we just give an idea. Since we have a C^∞, strictly plush exhaustion function l then it is tempting to define $\Omega_j = \{z \in \Omega : l(z) < t_j\}$, where $t_j \nearrow \infty$ as $j \to \infty$. Since the complex hessian of l is strictly positive definite on Ω it would seem that $\varrho_j = l - t_j$ is a defining function for Ω_j which makes Ω_j strictly pseudoconvex. This would be true if we knew that $\nabla \varrho_j$ does not vanish on the set $\{\varrho_j = 0\}$ but there is no guarantee that this may be achieved. Fortunately, using a lemma of Morse it is possible to find a new exhaustion function $\tilde{l}$ which differs from l by a linear term and has only isolated critical points. Then $\tilde{l}$ is also C^∞ and strictly plush and now we can choose $t_j \nearrow \infty$ so that $\nabla \tilde{l} \neq 0$ on $\{\tilde{l} = t_j\}$ which shows that $\Omega_j = \{\tilde{l} < t_j\}$ are strictly pseudoconvex.

5)&8) $\Rightarrow$ 2) Since Ω_j are strictly pseudoconvex, they are Levi pseudoconvex hence by 8) $\Leftrightarrow$ 2) we see that $(-\log \delta_{\Omega_j})$ is plush. But $\delta_{\Omega_j} \nearrow \delta_\Omega$ so $(-\log \delta_{\Omega_j}) \searrow (-\log \delta_\Omega)$ and, by Proposition 5.3.5, $(-\log \delta_\Omega)$ is plush.

∎

Definition 7.4.9 *A domain is called pseudoconvex if it satisfies any of the equivalent conditions of the previous theorem.*

Corollary 7.4.10 *Pseudoconvexity is a local property i.e., if every $P \in \partial\Omega$ has a neighbourhood V such that $V \cap \Omega$ is pseudoconvex then Ω is pseudoconvex.*

Proof. Let P, V be as above. There is a neighbourhood $V_1 \subset V$ of P such that

$$\delta_{\Omega \cap V} \mid_{V_1 \cap \Omega} = \delta_\Omega \mid_{V_1 \cap \Omega}.$$

(For instance $V_1 = B(P, \delta_V(P)/2)$.) It follows that $-\log \delta_\Omega$ is plush in $V_1 \cap \Omega$. Repeating this for every point of $\partial\Omega$ we see that $-\log \delta_\Omega$ is plush close to the boundary of Ω and, as shown in the proof of the previous theorem $(8) \Rightarrow 3))$, this implies pseudoconvexity of Ω.

∎

Corollary 7.4.11 *Pseudoconvexity is preserved by biholomorphic mappings.*

Proof. It has already been remarked before that biholomorphic mappings preserve plush exhaustion functions.

∎

REMARK. The proof of 8) $\Rightarrow$ 3) in Theorem 7.4.8shows that if Ω is pseudoconvex and $\Delta \subset \overline{\Omega}$ is an analytic disk which has a common point with $\partial\Omega$ then $\Delta \subset \partial\Omega$ (hence the situation shown on Figure 3 is impossible); this is the same with convex domains and open intervals. Hence a domain which has a strictly concave point at the boundary cannot be pseudoconvex (such an analytic disc can be found in the tangent plane).

Figure 3.

7.5 Domains of holomorphy

As mentioned before there exist pairs of domain (Ω_1, Ω_2) in $\mathbb{C}^n$ such that

$$\Omega_1 \subsetneq \Omega_2 \quad \& \quad \forall_{f \in H(\Omega_1)} \exists_{\tilde{f} \in H(\Omega_2)} \left. \tilde{f} \right|_{\Omega_1} = f.$$

As an example one may take $\Omega_1 = B(0,1) \setminus B(0,1/2)$, $\Omega_2 = B(0,1)$ (later we will prove a general result from which it will follow that this is a correct example). It is therefore natural to define a class of 'maximal' domains.

Definition 7.5.1 $\Omega \subset \mathbb{C}^n$ *is a domain of holomorphy if for any two domains* Ω_1, Ω_2 *with* Ω_2 *connected,* $\Omega_2 \not\subset \Omega$, $\Omega_1 \subset \Omega_2 \cap \Omega$ *there is an* $f \in H(\Omega)$ *such that there is no* $\tilde{f} \in H(\Omega_2)$ *with* $f = \tilde{f}$ *on* Ω_1.

This definition is somewhat obscure so consult Figure 4. The complication comes from the fact that it may happen that $\Omega_2 \cap \Omega$ is not connected and we do not want to require that an extension from one component of this intersection agrees with f on other components. Think of the complex plane with a ray removed and of extending a holomorphic branch of logarithm from this set. This function can be extended only in the way described in the previous definition.

Figure 4.

We will now present several equivalent formulations of this definition.

Theorem 7.5.2 *Let Ω be a domain in $\mathbb{C}^n$, then the following are equivalent:*

1) $\forall_{K \subset\subset \Omega}\, \delta_\Omega(K) = \delta_\Omega(\hat{K}_\Omega)$ *(where $\hat{K}_\Omega := \hat{K}_{H(\Omega)}$).*

2) Ω *is holomorphically convex (i.e., $H(\Omega)$-convex).*

3) $\exists_{f \in H(\Omega)} \forall_{\Omega_1, \Omega_2}$ *if $\Omega_2 \not\subset \Omega$, Ω_2 connected, $\Omega_1 \subset \Omega \cap \Omega_2$ then there is no $\tilde{f} \in H(\Omega_2)$ with $\tilde{f} = f$ on Ω_1.*

4) Ω *is a domain of holomorphy.*

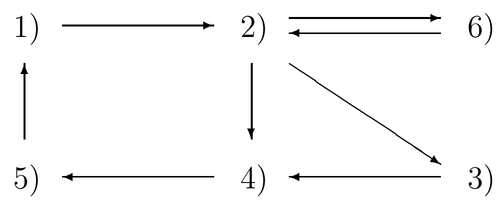
5) $\forall_{z \in \Omega} \forall_{r > \delta_\Omega(z)} \exists_{f \in H(\Omega)}$ *such that f does not extend from $B(z, \delta_\Omega(z))$ to $B(z, r)$.*

6) If $X \subset \Omega$ is infinite with no cluster points in Ω then there is an $f \in H(\Omega)$ which is unbounded on X.

PROOF. Within the proof we will let $\hat{K}$ stand for $\hat{K}_\Omega$. We will introduce an auxiliary condition

a) Assume $K_1 \subset\subset K_2 \subset\subset \ldots \Omega$ be such that $\bigcup_j K_j = \Omega$. For each j let $\xi_j \in \Omega \setminus \hat{K}_j$ and $\vartheta_j \in \mathbb{C}$. Then there exist functions $h_j \in H(\Omega)$ such that $h_j(\xi_j) = \vartheta_j$ and $|h_j| \le 2^{-j}$ on K_j.

Then the proof will run as shown below:



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1) ⇒ 2) It is only necessary to show that $\hat{K}$ is bounded but $\hat{K} \subset \overline{\mathrm{conv}(K)}$, and this larger set is bounded.

2) ⇒ a) Fix j. Since $\xi_j \in \Omega \setminus \hat{K}_j$ there is an $h \in H(\Omega)$ such that $|h(\xi)| > \sup_{K_j} |h|$. Multiplying h by a suitable constant we may assume that $h(\xi) = 1$ and then $c := \sup_{K_j} |h| < 1$. Next observe that $h^m(\xi_j) = 1$ and $\sup_{K_j} |h^m| = c^m$ and take m so large that $c^m \le 2^{-j}/|\vartheta_j|$. It follows that $h_j := \vartheta_j h^m$ satisfies the claim.

2) ⇒ 3) Suppose 3) does not hold then

$$\forall_{f \in H(\Omega)} \exists_{\Omega_1, \Omega_2} \exists_{\tilde{f} \in H(\Omega_2)} \tilde{f}|_{\Omega_1} = f|_{\Omega_1}. \tag{7.5.1}$$

Let $\{w_j\}$ be a dense sequence in Ω and for each j let $B_j := B(w_j, \delta_\Omega(w_j))$. Let $\{K_j\}$ be a sequence of subsets of Ω satisfying the assumptions of a). Since $\hat{K}_j \subset\subset \Omega$ we have $B_j \setminus \hat{K}_j \neq \emptyset$ (consult the definition of B_j) hence we may find $\xi_j \in B_j \setminus \hat{K}_j$. Now take $h_j \in H(\Omega)$, $|h_j| \le 2^{-j}$ on K_j, $h_j(\xi_j) = 1$ and define

$$f = \prod_{j=1}^{\infty} (1 - h_j)^j.$$

Since $\sum_{j=1}^{\infty} j|h_j|$ converges normally on Ω, we have that the product defining f converges normally to a function which is not identically zero hence f is well defined and holomorphic in Ω and $f \not\equiv 0$. Observe that $\frac{\partial^{|\alpha|} f}{\partial z^\alpha}(\xi_j) = 0$ if $|\alpha| < j$.

Apply (7.5.1) to this f and get Ω_1, Ω_2 and $\tilde{f}$. Then observe that the boundary of each component of $\Omega \cap \Omega_2$ has a non-empty intersection with $\partial\Omega$ (Figure 4 can be helpful again). Choose a component V which has non-empty intersection with Ω_1 then $\tilde{f} = f$ on V. We can then find a $z \in V$ such that $\delta_V(z) = \delta_\Omega(z) < \delta_{\Omega_2}(z)$ (just take z close to this part of the boundary of V which lies on $\partial\Omega$).

Let $\varepsilon > 0$ be such that $\delta_\Omega(z) + \varepsilon < \delta_{\Omega_2}(z)$ and let W be the component of $B(z, \delta_\Omega(z) + \varepsilon) \cap \Omega$ which contains $B(z, \delta_\Omega(z))$ then $f = \tilde{f}$ on W. Let $w \in \partial\Omega$ be such that $|z - w| = \delta_\Omega(z)$. Since $\{w_j\}$ is dense in Ω we may find a sequence $w_{j_k} \in W$, $w_{j_k} \to w$. If j_k is large enough then $|w_{j_k} - w| < \varepsilon/2$ so $\delta_\Omega(w_{j_k}) < \varepsilon/2$ and hence $B_{j_k} \subset W$, in particular $\xi_{j_k} \in W$ so

$$\frac{\partial^{|\alpha|} \tilde{f}}{\partial z^\alpha}(\xi_{j_k}) = 0 \qquad \text{for all} \quad |\alpha| < j_k.$$

But as $\xi_{j_k} \to w$ and $\tilde{f}$ is holomorphic in a neighbourhood of w we get that

$$\frac{\partial^{|\alpha|} \tilde{f}}{\partial z^\alpha}(w) = 0 \qquad \text{for all} \quad \alpha$$

which implies that $\tilde{f} \equiv 0$ so $f \equiv 0$ and this is a contradiction.

3) $\Rightarrow$ 4) $\Rightarrow$ 5) Obvious.

5) $\Rightarrow$ 1) Let $K \subset\subset \Omega$ and suppose $\mathcal{D} = D^n(0, \boldsymbol{r})$ is a polydisk such that $K + \mathcal{D} \subset \Omega$ $\;(K + \mathcal{D} := \{z + w : z \in K, w \in \mathcal{D}\})$. If $t < 1$ then $K + t\overline{\mathcal{D}} \subset\subset \Omega$ so if $f \in H(\Omega)$ we have $|f| \le M$ on $K + t\overline{\mathcal{D}}$, for some M. In particular for every $z \in K$, $|f|$ is bounded by M in $z + t\overline{\mathcal{D}} = \overline{D}^n(z, t\boldsymbol{r})$ so, by Cauchy inequalities

$$\left| \frac{\partial^{|\alpha|} f}{\partial z^\alpha}(z) \right| \le M \alpha! t^{-|\alpha|} \boldsymbol{r}^{-\alpha}. \tag{7.5.2}$$

Since derivatives of f are also holomorphic the bounds (7.5.2) extend from K onto $\hat{K}$, this, in turn, implies that for every $z \in \hat{K}$ the power series expansion of f about z converges uniformly in $z + t\mathcal{D}$. Since t can be arbitrarily close to 1 we get that this series converges normally in $z + \mathcal{D}$ if $z \in \hat{K}$. Now let $z_0 \in \hat{K}$ and $r = \delta_\Omega(K)$ then for every polydisk $\mathcal{D} \subset B(0, r)$ the Taylor series of f at z_0 is convergent in $z_0 + \mathcal{D}$ which implies that this series is convergent in $B(z_0, r)$. This means that every $f \in H(\Omega)$ can be extended to $B(z_0, r)$, taking 5) into account we get that $r \leq \delta_\Omega(z_0)$ so $\delta_\Omega(\hat{K}) \geq \delta_\Omega(K)$ and thus they must be equal.

2) ⇒ 6) Suppose X is as in 6) and K_j are as in a). Let $l_1 = 1$, and take $\xi_1 \in X \setminus \hat{K}_1$ ($\hat{K}_1 \subset\subset \Omega$ so such a point exists). Then there is an $l_2 > l_1$ such that $\xi_1 \in K_{l_2}$ and we pick a $\xi_2 \in X \setminus \hat{K}_{l_2}$ and so on. We can now find functions $h_j \in H(\Omega)$ such that $|h_j| \leq 2^{-j}$ on K_{l_j} and

$$|h_j(\xi_j)| \geq j + \sum_{l=1}^{j-1} |h_l(\xi_j)| + 1.$$

Then $\sum h_j$ converges normally in Ω to a holomorphic function f and

$$\begin{aligned} |f(\xi_j)| &= \left| \sum_{l=1}^{\infty} h_l(\xi_j) \right| \\ &\geq |h_j(\xi_j)| - \sum_{l=1}^{j-1} |h_l(\xi_j)| - \sum_{l=j+1}^{\infty} |h_l(\xi_j)| \\ &\geq j + 1 - \sum_{l=j+1}^{\infty} 2^{-l} \\ &\geq j \end{aligned}$$

Hence f is unbounded on X.

6) ⇒ 2) If $K \subset\subset \Omega$ then every $f \in H(\Omega)$ is bounded on K so f is bounded on $\hat{K}$ but, by 6), this implies that every infinite subset of $\hat{K}$ has a cluster point in Ω so $\hat{K} \subset\subset \Omega$.

■

Corollary 7.5.3 *Ω is a domain of holomorphy iff for every $P \in \partial\Omega$ there is an $f \in H(\Omega)$ such that $\overline{\lim}_{\xi \to P, \xi \in \Omega} |f(\xi)| = +\infty$.*

Proof. Take any sequence $z_j \to P$, $z_j \in \Omega$ and apply 6) to $X = \{z_j\}_{j=1}^{\infty}$.

■

Corollary 7.5.4 *A biholomorphic image of a domain of holomorphy is a domain of holomorphy.*

Corollary 7.5.5 *Every convex domain is a domain of holomorphy.*

PROOF. If $K \subset\subset \Omega$ then $\hat{K}_\Omega \subset \overline{\mathrm{conv}(K)} \subset\subset \Omega$.

∎

Corollary 7.5.6 *Every domain of holomorphy is pseudoconvex.*

PROOF. If $f \in H(\Omega)$ then $|f| \in P(\Omega)$ so $\hat{K}_{P(\Omega)} \subset \hat{K}_\Omega$ and thus if $K \subset\subset \Omega$ then $\hat{K}_\Omega \subset\subset \Omega$ so also $\hat{K}_{p(\Omega)} \subset\subset \Omega$ and hence Ω is pseudoconvex.

∎

REMARK. The question whether the reverse implication is true is called the Levi problem. We will show later that it has a positive solution. See Section 8.1 for some consequences this fact has for domains of holomorphy.

REMARK. As we mentioned in the introduction every domain in $\mathbb{C}$ is a domain of holomorphy (exercise) hence pseudoconvex.

8 The $\bar{\partial}$ problem

8.1 The problem and its consequences

Recall that we have introduced an operator $\bar{\partial}$ acting on complex differential forms. So far it was only used to write a condensed form of Cauchy-Riemann equations but now we will show how it can be put to much more use.

Definition 8.1.1 *A differential form ω is said to be $\bar{\partial}$-closed if $\bar{\partial}\omega = 0$ and $\bar{\partial}$-exact if there exists a form α such that $\bar{\partial}\alpha = \omega$.*

Since $\bar{\partial}^2 = 0$, a $\bar{\partial}$-exact form must be $\bar{\partial}$-closed. The question whether every $\bar{\partial}$-closed form is $\bar{\partial}$-exact is called the $\bar{\partial}$-problem. A similar problem can be posed for real forms and the operator d and then Poincaré Lemma says that in a starlike domain (or more generally contractible to a point) every d-closed form is d-exact; in completely arbitrary domains this need not be true.

We will begin by stating some theorems about solutions to the $\bar{\partial}$ problem and listing their consequences.

Theorem 8.1.2 *If ω is a C^1, compactly supported, $\bar{\partial}$-closed, $(0,1)$-form on $\mathbb{C}^n$ $(n \geq 2)$ then there exists a C^1, compactly supported function u such that $\bar{\partial}u = \omega$ and $u \equiv 0$ on the unbounded component of $\mathbb{C}^n \setminus (\operatorname{supp}\omega)$.*

Corollary 8.1.3 (Hartogs extension phenomenon) *Let $\Omega \in \mathbb{C}^n$ $(n \geq 2)$ be a domain and let $K \subset \Omega$ be compact and such that $\Omega \setminus K$ is connected. Suppose $f \in H(\Omega \setminus K)$ then there exists an $F \in H(\Omega)$ such that $F = f$ on $\Omega \setminus K$.*

Proof. Let $\psi \in C_c^\infty(\mathbb{C}^n)$ be such that $\psi = 1$ on K and $\operatorname{supp}\psi \subset \Omega$ then $(1-\psi)f$ can be considered to be in $C^\infty(\Omega)$. Let

$$\omega = \begin{cases} \bar{\partial}((1-\psi)f) & \text{in } \Omega \\ 0 & \text{in } \mathbb{C}^n \setminus \Omega. \end{cases}$$

Since f is holomorphic in $\Omega \setminus K$ and $(1-\psi) = 1$ in a neighbourhood of $\partial\Omega$ we get that $\bar{\partial}((1-\psi)f) = 0$ in a neighbourhood of $\partial\Omega$ so ω is a $(0,1)$-form with $C_c^\infty(\mathbb{C}^n)$ coefficients and of course ω is $\bar\partial$-closed. Let u be the solution to $\bar\partial u = \omega$ given by the previous theorem and define

$$F := (1-\psi)f - u \qquad \text{in} \quad \Omega$$

then

$$\bar\partial F = \omega - \bar\partial u = 0,$$

so F is a holomorphic function in Ω. We know that $\omega = 0$ on the unbounded component of $\mathbb{C}^n \setminus \operatorname{supp}\omega$ and this set has a non-empty intersection with every neighbourhood of $\partial\Omega$, in particular $F = f$ on some open subset of $\Omega \setminus K$ and since $\Omega \setminus K$ is connected $F = f$ on all of $\Omega \setminus K$.

■

A few words of comment are in order. One might be surprised why take ω of the form $\bar\partial g = \omega$ and then seek solution to $\bar\partial u = \omega$, when $u = g$ seems a handy answer. The point is that u has an additional property of being zero in a neighbourhood of $\partial\Omega$, thus it does not change g on the set where g is already holomorphic, it changes it however and makes holomorphic on the other part of Ω.

Corollary 8.1.4 *If $\Omega \subset \mathbb{C}^n \quad (n \geq 2)$ is open and $f \in H(\Omega)$ then f has no isolated zeroes.*

Proof. Suppose z_0 is the only zero of f in $B(z_0, \varepsilon)$ then $1/f$ is holomorphic in $B(z_0, \varepsilon) \setminus \{z_0\}$ so, by Hartogs extension phenomenon, can be extended to $B(z_0, \varepsilon)$ which is impossible since $\lim_{z\to z_0} |1/f| = +\infty$.

■

The above proof can be used to show much more, namely that no component of the zero set of f may be compact, in other words every such component must extend to the boundary of Ω. This is again completely different from the situation we have in $\mathbb{C}$.

Theorem 8.1.5 *If $\Omega \subset \mathbb{C}^n$ is pseudoconvex and ω is a C^∞, $\bar\partial$-closed, $(p, q+1)$-form on Ω then there exists a C^∞, (p, q)-form α such that $\bar\partial\alpha = \omega$. If $q = 0$ then every other form $\tilde\alpha$ satisfying $\bar\partial\tilde\alpha = \omega$ also has to be of class C^∞.*

Corollary 8.1.6 *Let $\Omega \subset \mathbb{C}^n$ be pseudoconvex and $\mathcal{D} = \Omega \cap \{z_n = 0\}$. Then $\mathcal{D}$ can be considered as an open subset of $\mathbb{C}^{n-1}$ and if $f \in H(\mathcal{D})$ then there is an $F \in H(\Omega)$ such that $F|_{\mathcal{D}} = f$.*

Proof. Consult Figure 5 for illustration.

Figure 5.

Let π be the projection onto $\mathbb{C}^{n-1}$ defined by $\pi(z', z_n) = z'$. Let $\mathcal{B} = \pi^{-1}(\mathbb{C}^{n-1} \setminus \mathcal{D}) \cap \Omega$ and let $\psi \in C^\infty(\Omega)$ be equal to 1 on a Ω-neighbourhood $\mathcal{E}$ of $\mathcal{D}$ and to 0 on $\Omega \setminus \mathcal{B}$. We will try to find F in the form

$$F = (f \circ \pi)\psi + z_n v$$

where v remains to be defined. The fact that $f \circ \pi$ is not defined outside $\mathcal{B}$ does not matter since ψ was taken such that $(f \circ \pi)\psi$ can be extended to Ω by putting 0 on $\Omega \setminus \mathcal{B}$.

Applying $\bar{\partial}$ to the above equality we get

$$\bar{\partial}F = (f \circ \pi)\bar{\partial}\psi + z_n\bar{\partial}v$$

hence if F is to be holomorphic v must satisfy

$$z_n\bar{\partial}v = -(f \circ \pi)\bar{\partial}\psi$$

or equivalently

$$\bar{\partial}v = -\frac{(f \circ \pi)\bar{\partial}\psi}{z_n}. \tag{8.1.1}$$

Observe that the right-hand side of (8.1.1) is a well defined, C^∞, $(0,1)$-form in Ω since $\bar{\partial}\psi$ vanishes on $\mathcal{E}$ so division by z_n does not introduce problems. We also have

$$\bar\partial\Big(f\circ\pi\frac{\bar\partial\psi}{z_n}\Big)=(f\circ\pi)\,\bar\partial\Big(\frac{\bar\partial\psi}{z_n}\Big)$$

and $\bar\partial\psi/z_n = 0$ on $\mathcal{E}$ and $\bar\partial(\bar\partial\psi/z_n) = \bar\partial(\bar\partial\psi)/z_n = 0$ off $\mathcal{E}$. (We have repeatedly used the fact that $\bar\partial$ vanishes on holomorphic functions.) It follows that (8.1.1) can be solved so F can be found in the desired form and then obviously $F|_{\mathcal{D}} = f$.

■

Once again we draw attention to the technique that was used. It was no problem to extend f to $\mathcal{B}$, $f\circ\pi$ does just that, then by taking $(f\circ\pi)\psi$ we got a function which was fine on $\mathcal{E}$ but was not holomorphic, so we used the solution to the $\bar\partial$-problem to find an extra term that would make $(f\circ\pi)\psi$ holomorphic without changing it on $\mathcal{D}$, this makes the proof similar in principle to the proof of Cor. 8.1.3.

Exercise. Modify the arguments of the above proof to show that if $\Omega\subset\mathbb{C}^n$ is such that for every C^∞, $\bar\partial$-closed, $(0,q)$-form ω in Ω there exists a $(0,q+1)$-form α with $\bar\partial\alpha=\omega$ then Ω has the extension property of Cor. 8.1.6 for all $\bar\partial$-closed, C^∞, $(0,q)$-forms on $\mathcal{D}$.

Corollary 8.1.7 (Solution of the Levi problem) *Every pseudoconvex set is a domain of holomorphy.*

Proof. We proceed by induction. If $n=1$ then every open set is a domain of holomorphy so there is nothing to prove. Assume the corollary is true for $n-1$. Fix $z_0\in\Omega$. We will want to satisfy condition 5) of Theorem 7.5.2.

Take $P\in\partial\Omega\cap B(z_0,\delta_\Omega(z_0))$, by changing coordinates we may assume that $z_0=0$ and $P\in\mathcal{D}=\Omega\cap\{z_n=0\}$. Since $\delta_{\mathcal{D}}=\delta_\Omega$ on $\mathcal{D}$ we see that $(-\log\delta_{\mathcal{D}})$ is plush so $\mathcal{D}$, or rather each of its components, is pseudoconvex and hence, by the inductive hypothesis, is a domain of holomorphy.

It follows that an $f\in H(\mathcal{D})$ can be found which cannot be extended from $B_{\mathbb{C}^{n-1}}(0,\delta_\Omega(0))$ onto any larger ball. Now use the previous corollary to produce an extension F of f onto Ω. If F could be extended from $B_{\mathbb{C}^n}(0,\delta_\Omega(0))$ onto a larger ball then so could be f hence F is the function we wanted to find.

■

Exercise. Modify the arguments of the above proof to show that if $\Omega \subset \mathbb{C}^n$ is such that for every C^∞, $\bar\partial$-closed, $(0,q)$-form ω in Ω there exists a $(0,q+1)$-form α with $\bar\partial\alpha = \omega$ then Ω is a domain of holomorphy. (Use the previous exercise.)

This exercise, together with Theorem 8.1.5, shows that the fact that $\bar\partial$-problem is solvable in Ω is equivalent to pseudoconvexity of Ω (and to Ω being a domain of holomorphy).

Corollary 8.1.8 *If Ω_j are domains of holomorphy with $\Omega_j \subset \Omega_{j+1}$ then their union $\Omega = \bigcup_j \Omega_j$ is also a domain of holomorphy.*

Proof. Since Ω_j are pseudoconvex we have $(-\log \delta_{\Omega_j})$ plush and

$$(-\log \delta_{\Omega_j}) \searrow (-\log \delta_\Omega)$$

so Ω is pseudoconvex hence a domain of holomorphy. (Being a domain of holomorphy is a local property.)

∎

Corollary 8.1.9 *If $\Omega \subset \mathbb{C}^n$ and every $P \in \partial\Omega$ has a neighbourhood V such that $V \cap \Omega$ is a domain of holomorphy then Ω is a domain of holomorphy.*

Proof. Use Cor. 7.4.10.

∎

The above corollary says that in order to prove that Ω is a domain of holomorphy it is enough that for each $P \in \partial\Omega$ we construct a holomorphic function in $V \cap \Omega$ which cannot be extended 'across' P.

Corollary 8.1.10 *Let $\Omega \subset \mathbb{C}^n$ be pseudoconvex and $k \le n$ be fixed. Let $\Omega_k = \Omega \cap \{z_1 = \ldots = z_k = 0\}$ and $H_k(\Omega) = \{f \in H(\Omega) : f|_{\Omega_k} = 0\}$, then*

$$\forall_{f\in H_k(\Omega)} \forall_{j=1,\ldots,k} \exists_{f_j \in H(\Omega)}\ f(z) = \sum_{j=1}^k z_j f_j(z).$$

Proof. We use induction on k with n arbitrary. If $k = 1$ then $f(0, z_2 \dots, z_n) = 0$ so the power series expansion of f about any such point will have coefficient a_α vanishing whenever $\alpha_1 = 0$, in particular all terms of the series have a common factor z_1 and so $f_1 = f/z_1$ is locally a sum of a power series, hence is holomorphic in Ω and $f = z_1 f_1$.

Now consider $k > 1$ and assume the claim has been proved for $k - 1$. Define $\widetilde{\Omega} = \Omega \cap \{z_k = 0\}$ then as in the previous proof, $\widetilde{\Omega}$ can be considered to be pseudoconvex in $\mathbb{C}^{n-1}$. Let $\tilde{f} = f|_{\widetilde{\Omega}}$ then $\tilde{f} \in H_{k-1}(\widetilde{\Omega})$ so, by the inductive hypothesis, we can find $\tilde{f}_j$, $j = 1, \dots, k-1$ with

$$\tilde{f} = \sum_{j=1}^{k-1} z_j \tilde{f}_j.$$

Using Cor. 8.1.6 again, we extend $\tilde{f}_j$ to $f_j \in H(\Omega)$. Then

$$f - \sum_{j=1}^{k-1} z_j f_j = 0 \quad \text{on} \quad \Omega \cap \{z_k = 0\}$$

and arguing like in the case of $k = 1$ we may take

$$f_k = \Big(f - \sum_{j=1}^{k-1} z_j f_j\Big)/z_k$$

■

8.2 Solution for compactly supported forms

Lemma 8.2.1 *Let $\mathcal{D} \subset \mathbb{C}$ be a C^1-bounded domain and g a C^1-function in a neighbourhood of $\overline{\mathcal{D}}$ then for every $z \in \mathcal{D}$*

$$g(z) = \frac{1}{2\pi i} \int_{\partial \mathcal{D}} \frac{g(\xi)}{\xi - z}\, d\xi - \frac{1}{\pi} \int_{\mathcal{D}} \frac{\partial g}{\partial \bar{\xi}}(\xi) \frac{1}{\xi - z}\, d\nu(\xi). \tag{8.2.1}$$

Proof. According to Lemma 3.2.1

$$\frac{1}{2\pi i} \int_{\partial \mathcal{D}_\varepsilon} \frac{g(\xi)}{\xi - z}\, d\xi = \frac{1}{\pi} \int_{\mathcal{D}_\varepsilon} \frac{\partial}{\partial \bar{\xi}} \left(\frac{g(\xi)}{\xi - z} \right) d\nu(\xi)$$

where $\mathcal{D}_\varepsilon = \mathcal{D} \setminus D(z, \varepsilon)$, for small ε. Since $\partial\mathcal{D}_\varepsilon = \partial\mathcal{D} \cup \partial D(z, \varepsilon)$ then, remembering that the orientation on $\partial D(z, \varepsilon)$ was reversed, we get

$$\frac{1}{2\pi i}\int_{\partial D(z,\varepsilon)} \frac{g(\xi)}{\xi - z}\, d\xi = \frac{1}{2\pi i}\int_{\partial\mathcal{D}} \frac{g(\xi)}{\xi - z}\, d\xi - \frac{1}{\pi}\int_{\mathcal{D}_\varepsilon} \frac{\partial}{\partial\bar{\xi}}\left(\frac{g(\xi)}{\xi - z}\right) d\nu(\xi)$$

We also have

$$\frac{1}{2\pi i}\int_{\partial D(z,\varepsilon)} \frac{g(\xi)}{z - \xi}\, d\xi = \frac{1}{2\pi}\int_0^{2\pi} g(z + \varepsilon e^{i\theta})\, d\theta \stackrel{\varepsilon\to 0}{\longrightarrow} g(z)$$

and

$$\begin{aligned}&\frac{1}{\pi}\int_{\mathcal{D}_\varepsilon} \frac{\partial}{\partial\bar{\xi}}\left(\frac{g(\xi)}{\xi - z}\right) d\nu(\xi)\\ &= \frac{1}{2\pi}\int_{\mathcal{D}_\varepsilon} \frac{\partial g}{\partial\bar{\xi}}(\xi)\frac{1}{\xi - z}\, d\nu(\xi) \stackrel{\varepsilon\to 0}{\longrightarrow} \frac{1}{\pi}\int_{\mathcal{D}} \frac{\partial g}{\partial\bar{\xi}}(\xi)\frac{1}{\xi - z}\, d\nu(\xi).\end{aligned}$$

■

Corollary 8.2.2 *If $g \in C^1_c(\mathbb{C})$ then for every $z \in \mathbb{C}$*

$$g(z) = -\frac{1}{\pi}\int_{\mathbb{C}} \frac{\partial g}{\partial\bar{\xi}}(\xi)\frac{1}{\xi - z}\, d\nu(\xi).$$

Proof. Take $\mathcal{D}$ such that $(\operatorname{supp} g) \subset \mathcal{D}$ and $z \in \mathcal{D}$. We apply Lemma 8.2.1 and observe that since $g = 0$ on $\partial\mathcal{D}$, the first integral on the right-hand side of (8.2.1) vanishes.

■

Proof of Theorem 8.1.2. Let $\omega = \sum_{j=1}^n f_j d\bar{z}_j$. We will show that for each j

$$u_j(z_1, \ldots, z_n) := \frac{1}{\pi}\int_{\mathbb{C}} \frac{f_j(z_1, \ldots, z_{j-1}, \xi, z_{j+1}, \ldots, z_n)}{\xi - z_n}\, d\nu(\xi)$$

is a C^k-function on $\mathbb{C}^n$, vanishing on the unbounded component of $\mathbb{C}^n \setminus (\operatorname{supp}\omega)$ and satisfying $\bar{\partial}u_j = \omega$.

By a change of coordinates in the integral we get

$$u_j(z_1, \ldots, z_n) = \frac{1}{\pi}\int_{\mathbb{C}} f_j(z_1, \ldots, z_{j-1}, z_j + \xi, z_{j+1}, \ldots, z_n)/\xi\, d\nu(\xi)$$

this allows us to claim that u_j is of the same class of smoothness as f and we also get

$$\frac{\partial}{\partial \bar{z}_l} u_j(z_1, \dots, z_n) = \frac{1}{\pi} \int_{\mathbb{C}} \Big(\frac{\partial}{\partial \bar{z}_l} f_j(z_1, \dots, z_{j-1}, z_j + \xi, z_{j+1}, \dots, z_n) \Big) / \xi \, d\nu(\xi).$$

But we have assumed that ω is $\bar{\partial}$-closed so

$$0 = \bar{\partial}\omega = \sum_{l,j} \frac{\partial f_j}{\partial \bar{z}_l} d\bar{z}_l \wedge d\bar{z}_j = \sum_{l<j} \left(\frac{\partial f_j}{\partial \bar{z}_l} - \frac{\partial f_l}{\partial \bar{z}_j} \right) d\bar{z}_l \wedge d\bar{z}_j$$

hence

$$\frac{\partial f_j}{\partial \bar{z}_l} = \frac{\partial f_l}{\partial \bar{z}_j}.$$

Applying above equality we get

$$\begin{aligned} \frac{\partial}{\partial z_l} u_j(z_1, \dots, z_n) &= \frac{1}{\pi} \int_{\mathbb{C}} \Big(\frac{\partial}{\partial \bar{z}_j} f_l(z_1, \dots, , z_{j-1}, z_j + \xi, z_{j+1}, \dots, z_n) \Big) / \xi \, d\nu \\ &= \frac{1}{\pi} \int_{\mathbb{C}} \frac{\partial}{\partial \bar{\xi}} \Big(f_l(z_1, \dots, z_{j-1}, \xi, z_{j+1}, \dots, z_n) \Big) / (\xi - z_j) \, d\nu(\xi) \\ &= f_l(z_1, \dots, z_n). \end{aligned}$$

(the last equality comes from Cor. 8.2.2). So indeed

$$\bar{\partial} u_j = \sum_l \frac{\partial u_j}{\partial \bar{z}_l} d\bar{z}_l = \sum_l f_l d\bar{z}_l = \omega.$$

It remains verify the claim about the support of u_j. Let V be the unbounded component of $\mathbb{C}^n \setminus (\operatorname{supp} \omega)$ then $\bar{\partial} u_j = 0$ on V so u_j is holomorphic on V and it is enough to show that it vanishes on some open subset of V. Let $l \neq j$ then there is an $R > 0$ such that

$$W = \{ z \in \mathbb{C}^n : |z_l| > R \} \subset V.$$

For z with $|z_l| > R$, $f_j(z) = 0$ so consulting the definition of u_j we see that $u_j(z) = 0$ when $z \in W$, hence $u_j = 0$ on V. This ends the proof but let us observe one more thing. Namely since $\bar{\partial}(u_j - u_l) = 0$ on $\mathbb{C}^n$ we have $(u_j - u_l) \in H(\mathbb{C}^n)$ but $(u_j - u_l)$ vanishes on V, hence on all of $\mathbb{C}^n$. We have thus proved that $u_j = u_l$ so the definition of the solution does not depend on the choice of j.

■

8.3 Elements of Hörmander's solution for smooth forms

Definition 8.3.1 *Let X, Y be a Hilbert space and $E \subset X$ a dense subspace. An operator $T\colon E \to Y$ is called closed if its graph*

$$\{(x, Tx) : x \in E\}$$

is closed in $X \times Y$.

It is standard to write $T : X \to Y$ even though T is only defined on a dense subspace of X.

Definition 8.3.2 *If $\Omega \subset \mathbb{C}^n$ is open and $\varphi : \Omega \to \mathbb{R}$ a continuous function then we define $L^2_{p,q}(\Omega, \varphi)$ as the space of all (p, q)-forms $\omega = \sum f_{\alpha,\beta} dz^\alpha \wedge d\bar z^\beta$ satisfying*

$$\int_\Omega |f_{\alpha,\beta}|^2 e^{-\varphi}\, d\nu < \infty.$$

$L^2_{p,q}(\Omega, \varphi)$ is equipped with scalar product

$$\langle \omega_1 | \omega_2 \rangle_\varphi = \sum_{\alpha,\beta} \int_\Omega f_{\alpha,\beta} \bar g_{\alpha,\beta} e^{-\varphi}\, d\nu.$$

It is a trivial exercise to show that $L^2_{p,q}(\Omega, \varphi)$ is a Hilbert space.

Given spaces $L^2_{p,q}(\Omega, \varphi_1)$, $L^2_{p,q+1}(\Omega, \varphi_2)$, $\varphi_2 \geq \varphi_1$, one observes that there is a way to extend the definition of the operator $\bar\partial$ to obtain an operator $\bar\partial_{p,q}$ which is closed. This operator is not defined on all of $L^2_{p,q}(\Omega, \varphi_1)$ but only on a dense subspace and is not continuous but this is normal when dealing with differential operators, it suffices to know that it is closed. The definition of this extension involves differentiating functions in the sense of distributions.

Having defined $\bar\partial_{p,q} : L^2(\Omega, \varphi_1) \to L^2_{p,q+1}(\Omega, \varphi_2)$ one can address the new $\bar\partial$ problem. To do this one has to know which are the $\bar\partial$-closed forms that we seek solution for. For this purpose we consider

$$\bar\partial_{p,q+1} : L^2_{p,q+1}(\Omega, \varphi_2) \to L^2_{p,q+2}(\Omega, \varphi_3).$$

This is again a closed operator so its kernel is a closed subspace of $L^2_{p,q+1}(\Omega, \varphi_2)$. With luck this kernel will be contained in the range of $\bar\partial_{p,q}$ which means exactly that there is a solution to $\bar\partial$-problem for these forms. At this stage we must remind that we are operating on forms which may have discontinuous coefficients so even if we take a smooth $\bar{\partial}$-closed form ω in $L^2_{p,q+1}$ then the solution we get need not be smooth and we will have $\bar{\partial}_{p,q}\alpha = \omega$ but $\bar{\partial}\alpha$ will not make sense. There is also another problem, it is always possible to find a smooth, $\bar{\partial}$-closed, $(p, q+1)$-form ω which lies outside of $L^2_{p,q+1}(\Omega, \varphi_2)$ so we cannot even start looking for a solution.

The second problem is not to difficult to overcome, we have to fix $\varphi_1, \varphi_2, \varphi_3$ after the form ω was chosen. We must only know that this is always possible. The non-smoothness problem is more serious and requires additional considerations involving Sobolev spaces.

The Sobolev space $W^s(\Omega, loc)$ is defined as consisting of all functions f satisfying the condition that all derivatives of f of orders up to s are locally square integrable. Again derivatives are taken in the weak sense.

Hörmander proved that his solutions have an additional property, namely if $\varphi_1, \varphi_2, \varphi_2$ are chosen and $\omega \in L^2_{p,q+1}(\Omega, \varphi_2)$ is $\bar{\partial}$-closed and in addition the coefficients of ω are in $W^s(\Omega, loc)$ then Hörmander's solution also has coefficients in $W^s(\Omega, loc)$. Now the proof of Theorem 8.1.5 can be concluded because if ω is a $\mathbb{C}^\infty$, $\bar{\partial}$-closed, $(p, q+1)$-form then it has coefficients in $W^s(\Omega, loc)$ for every s, it follows that Hörmander's solution also has coefficients in $W^s(\Omega, loc)$, for every s and this, by a theorem of Sobolev implies that the coefficients are of class C^∞.

Finally we show that for $q = 0$ all solutions have to be of class C^∞. Let us assume for simplicity that $p = 0$, the general case is done the same way, looking separately at coefficients. So we have one C^∞ solution u, and suppose v is another solution. Then $\bar{\partial}(u - v) = 0$ so this difference is a holomorphic function, hence v is of class C^∞. This argument looks very simple but there is one thing that has to be mentioned. A priori v is only an $L^2(\Omega, loc)$ function so $\bar{\partial}(u - v)$ is taken it the *weak* sense and one has to use a fact which says that a function (or in fact even distribution) which is $\bar{\partial}$-closed is a true holomorphic function.

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