Sturm–Liouville M-Functions in Terms of Green's Functions

Fritz Gesztesy (Baylor University, Waco, TX, USA)

Based on joint work with

Roger Nichols (University of Tennessee, Chattanooga, TN, USA)

Complex Analysis, Spectral Theory and Approximation meet in Linz

July 4–8, 2022

Appreciation:

CONGRATULATIONS, PETER!!!!

1. Topics Discussed
2. Basic Facts and Self-Adjoint L^2-Realizations $T_{A,B}$ of S–L Operators
3. M-Functions and Separated Boundary Conditions
4. M-Functions and General (e.g., Coupled) Boundary Conditions
5. Connecting $M(\,\cdot\,)$ and the Spectral Projections $E_{T_{A,B}}(\,\cdot\,)$

Topics Discussed:

- General **three-coefficient** **Sturm–Liouville operators** generated by $\tau = r(x)^{-1}[-(d/dx)p(x)(d/dx) + q(x)]$, $x \in (a,b)$, and their self-adjoint $L^2((a,b);rdx)$-realizations, T.
- The traditional 2×2 matrix-valued M-functions associated with separated boundary conditions (if any) at the endpoints a and b.
- The **connection** of M to the **Green's function** in the separated b.c. case.
- M for general b.c.'s and its **Nevanlinna–Herglotz** property.
- The precise **connection** between the family of spectral projections $E_T(\lambda)$, $\lambda \in \mathbb{R}$, in $L^2((a,b);rdx)$ and the 2×2 matrix-valued spectral measure Ω in the **Nevanlinna–Herglotz** representation of M for general b.c.'s.

Basic Facts and Self-Adjoint L^2-Realizations

Hypothesis 1. (To be assumed throughout this talk.)

Let $-\infty \leqslant a < b \leqslant \infty$. Suppose that p, q, r are Lebesgue measurable on (a, b) with $p^{-1}, q, r \in L^1_{loc}((a, b); dx)$ and real-valued a.e. on (a, b) with $r > 0$ and $p > 0$ a.e. on (a, b).

Introduce the differential expression τ

$$\tau f = \frac{1}{r}\big(-(pf')' + qf\big) \in L^1_{loc}((a, b); r\,dx), \quad f \in \mathfrak{D}((a, b)),$$

where

$$\mathfrak{D}((a, b)) = \big\{f \in AC_{loc}((a, b)) \,\big|\, f^{[1]} \in AC_{loc}((a, b))\big\},$$

and

$$f^{[1]} = pf'$$

is the **first quasi-derivative** of f. The **Wronskian** of f and g is defined as usual by

$$W(f, g)(x) = f(x)g^{[1]}(x) - f^{[1]}(x)g(x), \quad x \in (a, b),\ f, g \in \mathfrak{D}((a, b)).$$

Basic Facts and Self-Adjoint L^2-Realizations (contd.)

Then $W(f,g)$ is locally absolutely continuous on (a,b) and its derivative is

$$W(f,g)'(x) = \big[g(x)(\tau f)(x) - f(x)(\tau g)(x)\big] r(x), \quad x \in (a,b).$$

If $z \in \mathbb{C}$, then the **Wronskian** of two solutions $u_j(z,\cdot\,) \in \mathfrak{D}((a,b))$, $j \in \{1,2\}$, of $(\tau - z)u = 0$ on (a,b) is constant. Moreover, $W(u_1(\lambda,\cdot\,), u_2(\lambda,\cdot\,)) \neq 0$ if and only if $u_1(\lambda,\cdot\,)$ and $u_2(\lambda,\cdot\,)$ are **linearly independent**.

Definition 2.

The differential expression τ is said to be **regular** on (a,b) if $-\infty < a < b < \infty$ (i.e., a and b are finite) and $p^{-1}, q, r, s \in L^1((a,b);dx)$; otherwise, τ is said to be **singular** on (a,b).

If τ is **regular** on (a,b), then for all $f \in \mathfrak{D}((a,b))$, $f, \tau f \in L^2((a,b);rdx)$ (i.e., for all $f \in \operatorname{dom}(T_{max})$) the following limits exist and are finite:

$$f(a) := \lim_{x \downarrow a} f(x), \quad f^{[1]}(a) := \lim_{x \downarrow a} f^{[1]}(x),$$
$$f(b) := \lim_{x \uparrow b} f(x), \quad f^{[1]}(b) := \lim_{x \uparrow b} f^{[1]}(x).$$

Basic Facts and Self-Adjoint L^2-Realizations (contd.)

Maximal, T_{max}, preminimal, $\dot{T}_{min}$, and minimal T_{min}, operators are then defined in a standard manner,

$$T_{max} f = \tau f,$$
$$f \in \mathrm{dom}(T_{max}) = \big\{g \in L^2((a,b); r\,dx) \,\big|\, g \in \mathfrak{D}((a,b)),\ \tau g \in L^2((a,b); r\,dx)\big\}.$$
$$\dot{T}_{min} f = \tau f,$$
$$f \in \mathrm{dom}\big(\dot{T}_{min}\big) = \big\{g \in \mathrm{dom}(T_{max}) \,\big|\, g \text{ has compact support in } (a,b)\big\}.$$
$$T_{min} = \overline{\dot{T}_{min}} = T_{max}{}^*, \quad T_{min}{}^* = T_{max}.$$

(Here $\overline{S}$ denotes the operator closure of S.) The existence of **principal** and **nonprincipal** solutions is closely connected to oscillation theory for $\tau - \lambda$.

Definition 3.

Let $\lambda \in \mathbb{R}$. The differential expression $\tau - \lambda$ is called **oscillatory at a** (resp., b) if some solution of $(\tau - \lambda)u = 0$ has infinitely many zeros accumulating at a (resp., b); otherwise, $\tau - \lambda$ is called **nonoscillatory at a** (resp., b).

Basic Facts and Self-Adjoint L^2-Realizations (contd.)

Theorem 4 (Eckhardt-G-Nichols-Teschl 2013).

Let $\lambda \in \mathbb{R}$ be fixed. If $\tau - \lambda$ is **nonoscillatory** at b, then there exists a real-valued solution $u_b(\lambda, \cdot)$ of $(\tau - \lambda)u = 0$ satisfying the following properties (i)–(ii) in which $\widehat{u}_b(\lambda, \cdot)$ denotes an arbitrary real-valued solution of $(\tau - \lambda)u = 0$ linearly independent of $u_b(\lambda, \cdot)$.

(i) $u_b(\lambda, \cdot)$ and $\widehat{u}_b(\lambda, \cdot)$ satisfy the limiting relation

$$\lim_{x \uparrow b} \frac{u_b(\lambda, x)}{\widehat{u}_b(\lambda, x)} = 0.$$

(ii) $u_b(\lambda, \cdot)$ and $\widehat{u}_b(\lambda, \cdot)$ satisfy

$$\int^b dx\, |p(x)|^{-1} \widehat{u}_b(\lambda, x)^{-2} < \infty \text{ and } \int^b dx\, |p(x)|^{-1} u_b(\lambda, x)^{-2} = \infty.$$

The analogous result holds if $\tau - \lambda$ is **nonoscillatory** at a.

$u_b(\lambda, \cdot)$ is called a **principal** solution (it is unique up to normalization, and the "smallest" solution), $\widehat{u}_b(\lambda, \cdot)$ are called **nonprincipal** solutions of $(\tau - \lambda)u = 0$.

Basic Facts and Self-Adjoint L^2-Realizations (contd.)

Theorem 5 (Eckhardt-G-Nichols-Teschl 2013).

Suppose there exist $\lambda_a, \lambda_b \in \mathbb{R}$ such that $\tau - \lambda_a$ is **nonoscillatory** at a and $\tau - \lambda_b$ is **nonoscillatory** at b. Then $\dot{T}_{min}$ and hence any self-adjoint extension of the minimal operator T_{min} is **lower semibounded**.

In particular, if τ is regular on (a, b), then $\dot{T}_{min}$ and hence every self-adjoint extension of T_{min} is bounded from below.

Definition 6.

The operator $\dot{T}_{min}$ is said to be **bounded from below at a** if there exists $c \in (a, b)$ and $\lambda_a \in \mathbb{R}$ such that

$$\big(u, \dot{T}_{min} u\big)_{L^2((a,b);r\,dx)} \geqslant \lambda_a (u, u)_{L^2((a,b);r\,dx)},$$

$$u \in \operatorname{dom}\big(\dot{T}_{min}\big) \text{ such that } u \equiv 0 \text{ on } (c, b).$$

Analogously one introduces the notion that $\dot{T}_{min}$ is said to be **bounded from below at b.**

Basic Facts and Self-Adjoint L^2-Realizations (contd.)

The celebrated **Weyl alternative** then can be stated as follows:

Theorem 7 (Weyl's Alternative).

Assume Hypothesis 1. Then the following alternative holds: Either,

(i) for every $z \in \mathbb{C}$, all solutions ψ of $(\tau - z)\psi = 0$ are in $L^2((a, b); rdx)$ near b (resp., near a),

or,

(ii) for every $z \in \mathbb{C}$, there exists at least one solution ψ of $(\tau - z)\psi = 0$ which is not in $L^2((a, b); rdx)$ near b (resp., near a). In this case, for each $z \in \mathbb{C}\backslash\mathbb{R}$, there exists precisely one solution ψ_b (resp., ψ_a) of $(\tau - z)\psi = 0$ (up to constant multiples) which lies in $L^2((a, b); rdx)$ near b (resp., near a).

Basic Facts and Self-Adjoint L^2-Realizations (contd.)

This yields the **limit circle**/**limit point** classification of τ at an interval endpoint and links self-adjointness of T_{min} (resp., T_{max}) and the **limit point** property of τ at both endpoints as follows.

Definition 8.

Assume Hypothesis 1.

In case (i) in Theorem 7, τ is said to be in the **limit circle** *case* at b (resp., at a). (Frequently, τ is then called **quasi-regular** at b (resp., a).)

In case (ii) in Theorem 7, τ is said to be in the **limit point** *case* at b (resp., at a).

If τ is in the **limit circle** case at a and b then τ is also called **quasi-regular** on (a, b).

Basic Facts and Self-Adjoint L^2-Realizations (contd.)

Theorem 9 (see, e.g., Eckhardt-G-Nichols-Teschl 2013).

If $\dot{T}_{min}$ is bounded from below at a, then there exists $\alpha \in \mathbb{R}$ such that for all $\lambda < \alpha$, $\tau - \lambda$ is **nonoscillatory** at a. An analogous result holds at the endpoint b.

Assuming T_{min} is **lower semibounded** and in the **limit circle** case at a, and given **principal** and **nonprincipal** solutions $u_a(\lambda_0, \cdot\,)$ and $\widehat{u}_a(\lambda_0, \cdot\,)$ of $(\tau - \lambda_0)u = 0$, one introduces generalized boundary values for functions $g \in \mathrm{dom}(T_{max})$ as follows:

$$\widetilde{g}(a) = \lim_{x \downarrow a} \frac{g(x)}{\widehat{u}_a(\lambda_0, x)}, \quad \widetilde{g}\,'(a) = \lim_{x \downarrow a} \frac{g(x) - \widetilde{g}(a)\widehat{u}_a(\lambda_0, x)}{u_a(\lambda_0, x)}, \qquad (*)$$

and similarly at the endpoint b (see, **G-Nichols-Littlejohn 2020**).

Note. When τ is regular at a, then the following boundary values re-emerge,

$$\widetilde{g}(a) = g(a), \quad \widetilde{g}\,'(a) = g^{[1]}(a) = \lim_{x \downarrow a} p(x)g'(x).$$

Hence $\widetilde{g}\,'(a)$ in $(*)$ represents the natural analog of the (quasi) difference quotient at $x = a$.

Basic Facts and Self-Adjoint L^2-Realizations (contd.)

If τ is in the **limit circle** case at a and b, then

$$T_{min} f = \tau f,$$
$$f \in \operatorname{dom}(T_{min}) = \big\{g \in \operatorname{dom}(T_{max}) \,\big|\, \widetilde{g}(a) = \widetilde{g}\,'(a) = 0 = \widetilde{g}(b) = \widetilde{g}\,'(b)\big\},$$

and the Friedrichs (resp., Dirichlet) extension T_F of T_{min} is given by

$$T_F f = \tau f, \quad f \in \operatorname{dom}(T_F) = \big\{g \in \operatorname{dom}(T_{max}) \,\big|\, \widetilde{g}(a) = 0 = \widetilde{g}(b)\big\}.$$

Actually, at this point **ALL** self-adjoint extensions can be described as follows:

Theorem 10.

Assume Hypothesis 1 and that τ is in the **limit circle** case at a and b. In addition, assume that $v_j \in \operatorname{dom}(T_{max})$, $j = 1, 2$, are real-valued solutions v_j, $j = 1, 2$, of $(\tau - \lambda)u = 0$ with $\lambda \in \mathbb{R}$, such that $W(v_1, v_2) = 1$. For $g \in \operatorname{dom}(T_{max})$ we introduce the generalized boundary values

$$\widetilde{g}_1(a) = -W(v_2, g)(a), \quad \widetilde{g}_1(b) = -W(v_2, g)(b),$$
$$\widetilde{g}_2(a) = W(v_1, g)(a), \quad \widetilde{g}_2(b) = W(v_1, g)(b).$$

Then the following items (i)–(iv) hold:

Basic Facts and Self-Adjoint L^2-Realizations (contd.)

Theorem 10 (contd.).

(i) $T_{A,B}$ is a self-adjoint extension of T_{min} if and only if there exist 2×2 matrices A and B (with complex-valued entries) satisfying

$$\operatorname{rank}(A \quad B) = 2, \quad AJA^* = BJB^*, \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

with $T_{A,B}$ given by

$$T_{A,B}f = \tau f, \quad f \in \operatorname{dom}(T_{A,B}) = \left\{ g \in \operatorname{dom}(T_{max}) \,\middle|\, A \begin{pmatrix} \widetilde{g}_1(a) \\ \widetilde{g}_2(a) \end{pmatrix} = B \begin{pmatrix} \widetilde{g}_1(b) \\ \widetilde{g}_2(b) \end{pmatrix} \right\}.$$

(ii) All self-adjoint extensions $T_{\gamma,\delta}$ of T_{min} with separated boundary conditions are of the form

$$\begin{aligned} &T_{\gamma,\delta}f = \tau f, \quad \gamma, \delta \in [0, \pi), \\ &f \in \operatorname{dom}(T_{\gamma,\delta}) = \big\{ g \in \operatorname{dom}(T_{max}) \,\big|\, \sin(\gamma)\widetilde{g}_2(a) + \cos(\gamma)\widetilde{g}_1(a) = 0; \\ &\qquad\qquad \sin(\delta)\widetilde{g}_2(b) + \cos(\delta)\widetilde{g}_1(b) = 0 \big\}. \end{aligned}$$

Basic Facts and Self-Adjoint L^2-Realizations (contd.)

Theorem 10 (contd.).

(*iii*) All self-adjoint extensions $T_{\varphi,R}$ of T_{min} with coupled boundary conditions are of the type

$$T_{\varphi,R} f = \tau f,$$
$$f \in \operatorname{dom}(T_{\varphi,R}) = \left\{ g \in \operatorname{dom}(T_{max}) \,\middle|\, \begin{pmatrix} \widetilde{g}_1(b) \\ \widetilde{g}_2(b) \end{pmatrix} = e^{i\varphi} R \begin{pmatrix} \widetilde{g}_1(a) \\ \widetilde{g}_2(a) \end{pmatrix} \right\},$$

where $\varphi \in [0, 2\pi)$, and R is a real 2×2 matrix with $\det(R) = 1$ (i.e., $R \in SL(2, \mathbb{R})$).

(*iv*) Every self-adjoint extension of T_{min} is either of type (*ii*) (i.e., separated) or of type (*iii*) (i.e., coupled).

Basic Facts and Self-Adjoint L^2-Realizations (contd.)

In the **lower semibounded** case this can be rewritten as follows:

Theorem 11.

Assume Hypothesis 1 and that τ is in the **limit circle** case at a and b. In addition, assume that $T_{min} \geqslant \lambda_0 I$ for some $\lambda_0 \in \mathbb{R}$, and denote by $u_a(\lambda_0, \cdot\,)$ and $\widehat{u}_a(\lambda_0, \cdot\,)$ (resp., $u_b(\lambda_0, \cdot\,)$ and $\widehat{u}_b(\lambda_0, \cdot\,)$) **principal** and **nonprincipal** solutions of $\tau u = \lambda_0 u$ at a (resp., b), satisfying

$$W(\widehat{u}_a(\lambda_0, \cdot\,), u_a(\lambda_0, \cdot\,)) = W(\widehat{u}_b(\lambda_0, \cdot\,), u_b(\lambda_0, \cdot\,)) = 1.$$

Then the following items (i)–(iii) hold:

Basic Facts and Self-Adjoint L^2-Realizations (contd.)

Theorem 11 (contd.).

(i) Introducing $v_j \in \operatorname{dom}(T_{max})$, $j = 1, 2$, via

$$v_1(x) = \begin{cases} \widehat{u}_a(\lambda_0, x), & \text{for } x \text{ near a}, \\ \widehat{u}_b(\lambda_0, x), & \text{for } x \text{ near b}, \end{cases} \quad v_2(x) = \begin{cases} u_a(\lambda_0, x), & \text{for } x \text{ near a}, \\ u_b(\lambda_0, x), & \text{for } x \text{ near b}, \end{cases}$$

one obtains for all $g \in \operatorname{dom}(T_{max})$,

$$\widetilde{g}(a) = -W(v_2, g)(a) = \widetilde{g}_1(a) = -W(u_a(\lambda_0, \cdot\,), g)(a) = \lim_{x \downarrow a} \frac{g(x)}{\widehat{u}_a(\lambda_0, x)},$$

$$\widetilde{g}(b) = -W(v_2, g)(b) = \widetilde{g}_1(b) = -W(u_b(\lambda_0, \cdot\,), g)(b) = \lim_{x \uparrow b} \frac{g(x)}{\widehat{u}_b(\lambda_0, x)},$$

$$\widetilde{g}\,'(a) = W(v_1, g)(a) = \widetilde{g}_2(a) = W(\widehat{u}_a(\lambda_0, \cdot\,), g)(a) = \lim_{x \downarrow a} \frac{g(x) - \widetilde{g}(a)\widehat{u}_a(\lambda_0, x)}{u_a(\lambda_0, x)},$$

$$\widetilde{g}\,'(b) = W(v_1, g)(b) = \widetilde{g}_2(b) = W(\widehat{u}_b(\lambda_0, \cdot\,), g)(b) = \lim_{x \uparrow b} \frac{g(x) - \widetilde{g}(b)\widehat{u}_b(\lambda_0, x)}{u_b(\lambda_0, x)}.$$

In particular, the limits on the right-hand sides above exist.

Basic Facts and Self-Adjoint L^2-Realizations (contd.)

Theorem 11 (contd.).

(ii) All self-adjoint extensions $T_{\gamma,\delta}$ of T_{min} with separated boundary conditions are of the form

$$\begin{aligned} &T_{\gamma,\delta} f = \tau f, \quad \gamma, \delta \in [0,\pi), \\ &f \in \operatorname{dom}(T_{\gamma,\delta}) = \big\{ g \in \operatorname{dom}(T_{max}) \,\big|\, \sin(\gamma)\widetilde{g}'(a) + \cos(\gamma)\widetilde{g}(a) = 0; \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad \sin(\delta)\widetilde{g}'(b) + \cos(\delta)\widetilde{g}(b) = 0 \big\}. \end{aligned}$$

Moreover, $\sigma(T_{\gamma,\delta})$ is simple.

(iii) All self-adjoint extensions $T_{\varphi,R}$ of T_{min} with coupled boundary conditions are of the type

$$\begin{aligned} &T_{\varphi,R} f = \tau f, \\ &f \in \operatorname{dom}(T_{\varphi,R}) = \left\{ g \in \operatorname{dom}(T_{max}) \,\middle|\, \begin{pmatrix} \widetilde{g}(b) \\ \widetilde{g}'(b) \end{pmatrix} = e^{i\varphi} R \begin{pmatrix} \widetilde{g}(a) \\ \widetilde{g}'(a) \end{pmatrix} \right\}, \end{aligned}$$

where $\varphi \in [0, 2\pi)$, and $R \in SL(2,\mathbb{R})$.

Note. For simplicity only, from now on we assume the **lower semibounded** case.

Classical M-Function Theory

Throughout the following we assume Hypothesis 1 and fix $\alpha \in [0, \pi)$.

Associated with the differential expression τ we consider the self-adjoint operator $T_{\gamma,\delta}$ in $L^2((a,b); rdx)$ corresponding to separated boundary conditions (if any) indexed by $\gamma, \delta \in [0, \pi)$, and the usual fundamental system of solutions $\phi_\alpha(z, \cdot\,, x_0)$ and $\theta_\alpha(z, \cdot\,, x_0)$, $z \in \mathbb{C}$, of $\tau u = zu$, with respect to a fixed reference point $x_0 \in (a,b)$, satisfying the initial conditions

$$\begin{aligned} &\phi_\alpha(z, x_0, x_0) = -\theta_\alpha^{[1]}(z, x_0, x_0) = -\sin(\alpha), \\ &\phi_\alpha^{[1]}(z, x_0, x_0) = \theta_\alpha(z, x_0, x_0) = \cos(\alpha), \quad \alpha \in [0,\pi),\ z \in \mathbb{C},\ x_0 \in (a,b). \end{aligned}$$

Again we note that for any fixed $x, x_0 \in (a,b)$, $\phi_\alpha(z, x, x_0)$ and $\theta_\alpha(z, x, x_0)$ are **entire** with respect to z and that

$$W(\theta_\alpha(z, \cdot\,, x_0), \phi_\alpha(z, \cdot\,, x_0))(x) = 1, \quad z \in \mathbb{C},\ x_0 \in (a,b).$$

Classical M-Function Theory (contd.)

Particularly important solutions of $\tau u = zu$ are the **Weyl–Titchmarsh solutions** $\psi_{\alpha,b}(z,\cdot,x_0)$ or $\psi_{\alpha,b,\delta}(z,\cdot,x_0)$ at b (resp., $\psi_{\alpha,a}(z,\cdot,x_0)$ or $\psi_{\alpha,a,\gamma}(z,\cdot,x_0)$ at a) of $\tau u = zu$, uniquely characterized as follows:

(i) If τ is in the **limit point** case at b (resp., a), one introduces $\psi_{\alpha,b}(z,\cdot,x_0)$ (resp., $\psi_{\alpha,a}(z,\cdot,x_0)$) via the requirement

$$\psi_{\alpha,b}(z,\cdot,x_0) \in L^2([x_0,b);rdx), \ \ \big(\text{resp., } \psi_{\alpha,a}(z,\cdot,x_0) \in L^2((a,x_0];rdx)\big),$$

$$\sin(\alpha)\psi^{[1]}_{\alpha,b}(z,x_0,x_0) + \cos(\alpha)\psi_{\alpha,b}(z,x_0,x_0) = 1$$

$$(\text{resp., } \sin(\alpha)\psi^{[1]}_{\alpha,a}(z,x_0,x_0) + \cos(\alpha)\psi_{\alpha,a}(z,x_0,x_0) = 1), \quad z \in \mathbb{C}\backslash\mathbb{R}.$$

The crucial condition is the L^2-property at b (resp., a), which uniquely determines $\psi_{\alpha,b}(z,\cdot,x_0)$ (resp., $\psi_{\alpha,a}(z,\cdot,x_0)$) up to constant (possibly, z-dependent) multiples by the **limit point** case hypothesis of τ at a and b. In particular, for $\alpha,\beta \in [0,\pi)$,

$$\psi_{\alpha,b}(z,\cdot,x_0) = C_b(z,\alpha,\beta,x_0)\psi_{\beta,b}(z,\cdot,x_0)$$

$$(\text{resp., } \psi_{\alpha,a}(z,\cdot,x_0) = C_a(z,\alpha,\beta,x_0)\psi_{\beta,a}(z,\cdot,x_0))$$

for some coefficients $C_b(z,\alpha,\beta,x_0) \in \mathbb{C}$, (resp., $C_a(z,\alpha,\beta,x_0) \in \mathbb{C}$).

Classical M-Function Theory (contd.)

(ii) If τ is in the **limit circle** case at b (resp., a), one introduces $\psi_{\alpha,b,\delta}(z,\,\cdot\,,x_0)$ (resp., $\psi_{\alpha,a,\gamma}(z,\,\cdot\,,x_0)$) by requiring that

$\psi_{\alpha,b,\delta}(z,\,\cdot\,,x_0)$ (resp., $\psi_{\alpha,a,\gamma}(z,\,\cdot\,,x_0)$) satisfies the (**separated**) boundary condition at b (resp., a) of the form, $\sin(\delta)\widetilde{\psi}'_{\alpha,b,\delta}(z,b,x_0)+\cos(\delta)\widetilde{\psi}_{\alpha,b,\delta}(z,b,x_0)=0$ (resp., $\sin(\gamma)\widetilde{\psi}'_{\alpha,b,\gamma}(z,a,x_0)+\cos(\gamma)\widetilde{\psi}_{\alpha,b,\gamma}(z,a,x_0)=0$),

$\sin(\alpha)\psi^{[1]}_{\alpha,b,\delta}(z,x_0,x_0)+\cos(\alpha)\psi_{\alpha,b,\delta}(z,x_0,x_0)=1$

(resp., $\sin(\alpha)\psi^{[1]}_{\alpha,a,\gamma}(z,x_0,x_0)+\cos(\alpha)\psi_{\alpha,a,\gamma}(z,x_0,x_0)=1$), $\quad z\in\mathbb{C}\backslash\mathbb{R}$.

Notational convention. To minimize the case distinctions to be made in the following, we will adopt the notation of case (ii) and should the **limit point** case of τ be present at b or a we simply ignore the extra δ- or γ-dependence.

Classical M-Function Theory (contd.)

In either case (i) or (ii), the normalizations employed show that $\psi_{\alpha,{b,\delta \atop a,\gamma}}(z,\,\cdot\,,x_0)$ are of the type

$$\psi_{\alpha,{b,\delta \atop a,\gamma}}(z,x,x_0) = \theta_\alpha(z,x,x_0) + m_{\alpha,{b,\delta \atop a,\gamma}}(z,x_0)\phi_\alpha(z,x,x_0), \quad z \in \mathbb{C}\backslash\mathbb{R},\ x \in (a,b),$$

for some coefficients $m_{\alpha,{b,\delta \atop a,\gamma}}(z,x_0)$, $z \in \mathbb{C}\backslash\mathbb{R}$, the **Weyl–Titchmarsh m-functions** associated with τ, α, γ, δ, and x_0, which contains (half-line) **spectral information**,

$$m_{\alpha,{b,\delta \atop a,\gamma}}(z,x_0) = \cos(\alpha)\psi^{[1]}_{\alpha,{b,\delta \atop a,\gamma}}(z,x_0,x_0) - \sin(\alpha)\psi_{\alpha,{b,\delta \atop a,\gamma}}(z,x_0,x_0), \quad z \in \mathbb{C}\backslash\mathbb{R}.$$

One recalls the fundamental identities

$$\int_{x_0}^{b} r(x)dx\, \psi_{\alpha,b,\delta}(z_1,x,x_0)\psi_{\alpha,b,\delta}(z_2,x,x_0) = \frac{m_{\alpha,b,\delta}(z_1,x_0) - m_{\alpha,b,\delta}(z_2,x_0)}{z_1 - z_2},$$

$$\int_{a}^{x_0} r(x)dx\, \psi_{\alpha,a,\gamma}(z_1,x,x_0)\psi_{\alpha,a,\gamma}(z_2,x,x_0) = -\frac{m_{\alpha,a,\gamma}(z_1,x_0) - m_{\alpha,a,\gamma}(z_2,x_0)}{z_1 - z_2},$$

$$z_1, z_2 \in \mathbb{C}\backslash\mathbb{R},\ z_1 \neq z_2,$$

and concludes

$$\overline{m_{\alpha,{b,\delta \atop a,\gamma}}(z,x_0)} = m_{\alpha,{b,\delta \atop a,\gamma}}(\overline{z},x_0), \quad z \in \mathbb{C}\backslash\mathbb{R}.$$

Classical M-Function Theory (contd.)

Choosing $z_1 = z$, $z_2 = \overline{z}$ one infers

$$\int_{x_0}^{b} r(x)dx \, |\psi_{\alpha,b,\delta}(z,x,x_0)|^2 = \frac{\operatorname{Im}(m_{\alpha,b,\delta}(z,x_0))}{\operatorname{Im}(z)} > 0, \quad z \in \mathbb{C}\backslash\mathbb{R},$$

$$\int_{a}^{x_0} r(x)dx \, |\psi_{\alpha,a,\gamma}(z,x,x_0)|^2 = -\frac{\operatorname{Im}(m_{\alpha,a,\gamma}(z,x_0))}{\operatorname{Im}(z)} > 0, \quad z \in \mathbb{C}\backslash\mathbb{R}.$$

In addition, since $m_{\alpha,{b,\delta \atop a,\gamma}}(\,\cdot\,,x_0)$ are known to be analytic on $\mathbb{C}\backslash\mathbb{R}$, one obtains that $\pm m_{\alpha,{b,\delta \atop a,\gamma}}(\,\cdot\,,x_0)$ are **Nevanlinna–Herglotz** functions.

The Green's function $G_{\gamma,\delta}(z,x,x')$, $z \in \rho(T_{\gamma,\delta})$, $x,x' \in (a,b)$, of $T_{\gamma,\delta}$ then reads

$$G_{\gamma,\delta}(z,x,x') = \frac{1}{W(\psi_{\alpha,b,\delta}(z,\,\cdot\,,x_0),\psi_{\alpha,a,\gamma}(z,\,\cdot\,,x_0))} \times \begin{cases} \psi_{\alpha,a,\gamma}(z,x,x_0)\psi_{\alpha,b,\delta}(z,x',x_0), & a < x \leqslant x' < b, \\ \psi_{\alpha,b,\delta}(z,x,x_0)\psi_{\alpha,a,\gamma}(z,x',x_0), & a < x' \leqslant x < b, \end{cases} \quad z \in \mathbb{C}\backslash\mathbb{R},$$

with

$$W(\psi_{\alpha,b,\delta}(z,\,\cdot\,,x_0),\psi_{\alpha,a,\gamma}(z,\,\cdot\,,x_0)) = m_{\alpha,a,\gamma}(z,x_0) - m_{\alpha,b,\delta}(z,x_0), \quad z \in \mathbb{C}\backslash\mathbb{R}.$$

Classical M-Function Theory (contd.)

Thus (given $z \in \mathbb{C}\backslash\mathbb{R}$, $x \in (a,b)$, $f \in L^2((a,b); rdx)$), for **separated bc's**,

$$((T_{\gamma,\delta} - zI)^{-1} f)(x) = \int_a^b r(x')dx' \, G_{\gamma,\delta}(z,x,x') f(x').$$

For each $x \in \mathbb{R}$, the **diagonal Green's function** of $T_{\gamma,\delta}$, denoted by $g_{\gamma,\delta}(z,x)$, has the **Nevanlinna–Herglotz** property,

$$g_{\gamma,\delta}(\,\cdot\,,x) = G_{\gamma,\delta}(\,\cdot\,,x,x), \quad x \in (a,b), \text{ is a Nevanlinna–Herglotz function.}$$

Given $m_{\alpha,{b,\delta \atop a,\gamma}}(z,x_0)$, introduce the 2×2 matrix-valued **Weyl–Titchmarsh** fct.

$$M_{\alpha,\gamma,\delta}(z,x_0) = \big(M_{\alpha,\gamma,\delta,\ell,\ell'}(z,x_0)\big)_{\ell,\ell'=1,2}$$
$$= \begin{pmatrix} \frac{1}{m_{\alpha,a,\gamma}(z,x_0) - m_{\alpha,b,\delta}(z,x_0)} & \frac{1}{2}\frac{m_{\alpha,a,\gamma}(z,x_0) + m_{\alpha,b,\delta}(z,x_0)}{m_{\alpha,a,\gamma}(z,x_0) - m_{\alpha,b,\delta}(z,x_0)} \\ \frac{1}{2}\frac{m_{\alpha,a,\gamma}(z,x_0) + m_{\alpha,b,\delta}(z,x_0)}{m_{\alpha,a,\gamma}(z,x_0) - m_{\alpha,b,\delta}(z,x_0)} & \frac{m_{\alpha,a,\gamma}(z,x_0) m_{\alpha,b,\delta}(z,x_0)}{m_{\alpha,a,\gamma}(z,x_0) - m_{\alpha,b,\delta}(z,x_0)} \end{pmatrix}, \quad z \in \mathbb{C}\backslash\mathbb{R},$$

and notes that

$$\det{}_{\mathbb{C}^2}(M_{\alpha,\gamma,\delta}(z,x_0)) = -1/4, \quad M_{\alpha,\gamma,\delta}(z,x_0)^* = M_{\alpha,\gamma,\delta}(\overline{z},x_0), \quad z \in \mathbb{C}\backslash\mathbb{R}.$$

By inspection, $M_{\alpha,\gamma,\delta}(z,x_0)$ is a 2×2 matrix-valued **Nevanlinna–Herglotz** fct. since $-m_{\alpha,a,\gamma}(\,\cdot\,,x_0)$ and $m_{\alpha,b,\delta}(\,\cdot\,,x_0)$ are scalar **Nevanlinna–Herglotz** fcts.

Classical M-Function Theory (contd.)

Turning to the connection between $M_{\alpha,\gamma,\delta}(z,x_0)$ and the Green's function $G_{\gamma,\delta}(z,\cdot\,,\cdot\,)$ of $T_{\gamma,\delta}$, still in the **separated b.c. case**, we introduce

$$\big(\partial_1^{[1]} G_{\gamma,\delta}\big)(z,x_0,x') = p(x_1)\partial_{x_1} G_{\gamma,\delta}(z,x_1,x')\big|_{x_1=x_0},$$
$$\big(\partial_2^{[1]} G_{\gamma,\delta}\big)(z,x,x_0) = p(x_2)\partial_{x_2} G_{\gamma,\delta}(z,x,x_2)\big|_{x_2=x_0},$$
$$\big(\partial_1^{[1]}\partial_2^{[1]} G_{\gamma,\delta}\big)(z,x_0,x_0) = p(x_1)\partial_{x_1}\, p(x_2)\partial_{x_2} G_{\gamma,\delta}(z,x_1,x_2)\big|_{x_1=x_0,x_2=x_0}$$
$$= \big(\partial_2^{[1]}\partial_1^{[1]} G_{\gamma,\delta}\big)(z,x_0,x_0), \ \text{ etc.}$$

Then $M_{\alpha,\gamma,\delta}(z,x_0)$ can be rewritten as

$$M_{\alpha,\gamma,\delta,1,1}(z,x_0) = \big(\big[\cos(\alpha)+\sin(\alpha)\partial_1^{[1]}\big]\big[\cos(\alpha)+\sin(\alpha)\partial_2^{[1]}\big]\, G_{\gamma,\delta}\big)(z,x_0,x_0),$$
$$M_{\alpha,\gamma,\delta,1,2}(z,x_0) = M_{\alpha,\gamma,\delta,2,1}(z,x_0)$$
$$= (1/2)\big(\big\{\big[\cos(\alpha)+\sin(\alpha)\partial_1^{[1]}\big]\big[-\sin(\alpha)+\cos(\alpha)\partial_2^{[1]}\big]$$
$$+\big[-\sin(\alpha)+\cos(\alpha)\partial_1^{[1]}\big]\big[\cos(\alpha)+\sin(\alpha)\partial_2^{[1]}\big]\big\}\, G_{\gamma,\delta}\big)(z,x_0\pm 0,x_0\mp 0),$$
$$M_{\alpha,\gamma,\delta,2,2}(z,x_0) = \big(\big[-\sin(\alpha)+\cos(\alpha)\partial_1^{[1]}\big]\big[-\sin(\alpha)+\cos(\alpha)\partial_2^{[1]}\big]\, G_{\gamma,\delta}\big)(z,x_0,x_0),$$
$$z\in\mathbb{C}\backslash\mathbb{R}.$$

Classical M-Function Theory (contd.)

Thus, $G_{\gamma,\delta}(z,x_0,x_0)$ and appropriate first quasi-derivatives of $G_{\gamma,\delta}(z,\,\cdot\,,\,\cdot\,)$ at x_0 uniquely determine $M_{\alpha,\gamma,\delta}(z,x_0)$ in a straightforward fashion.

In the particular case $\alpha = 0$, one obtains the remarkably simple formula

$$M_{0,\gamma,\delta}(z,x_0) = \begin{pmatrix} G_{\gamma,\delta}(z,x_0,x_0) & 2^{-1}\Big(\Big[\partial_1^{[1]}+\partial_2^{[1]}\Big]G_{\gamma,\delta}\Big)(z,x_0\pm 0,x_0\mp 0) \\ 2^{-1}\Big(\Big[\partial_1^{[1]}+\partial_2^{[1]}\Big]G_{\gamma,\delta}\Big)(z,x_0\pm 0,x_0\mp 0) & \Big(\partial_1^{[1]}\partial_2^{[1]}G_{\gamma,\delta}\Big)(z,x_0,x_0) \end{pmatrix}, \quad z \in \mathbb{C}\backslash\mathbb{R}.$$

Note. (i) Above, one can of course replace $z \in \mathbb{C}\backslash\mathbb{R}$ by $z \in \rho(T_{\gamma,\delta})$.
(ii) It is possible to take the limit $x_0 \downarrow a$ (resp., $x_0 \uparrow b$) as long as τ is in the **limit circle** case at a (resp., b).

This summarizes the traditional approach to 2×2 **Weyl–Titchmarsh** theory which focuses on **separated boundary conditions at a and b** (if any).

How about **coupled boundary conditions** at a and b? E.g., the **periodic** case?

General M-Function Theory

Let $T_{A,B}$ be a fixed self-adjoint extension of T_{min} with (**separated or coupled**) boundary conditions encoded in the 2×2 matrices $A, B \in \mathbb{C}^{2\times 2}$, and abbreviate the associated Green's function of $T_{A,B}$ by $G_{A,B}(z, \cdot\,, \cdot\,)$, $z \in \mathbb{C}\backslash\sigma(T_{A,B})$.

Inspired by the explicit form of $M_{0,\gamma,\delta}(\,\cdot\,, x_0)$ in terms of the Green's function $G_{\gamma,\delta}(\,\cdot\,, x_0, x_0)$ and some of its first quasi-derivatives, we now introduce the general M-function in exactly the same manner,

$$
\begin{aligned}
&M_{0,A,B}(z,x_0) \\
&= \begin{pmatrix} G_{A,B}(z,x_0,x_0) & 2^{-1}\Big(\big[\partial_1^{[1]}+\partial_2^{[1]}\big]\,G_{A,B}\Big)(z,x_0+0,x_0-0) \\ 2^{-1}\Big(\big[\partial_1^{[1]}+\partial_2^{[1]}\big]\,G_{A,B}\Big)(z,x_0+0,x_0-0) & \Big(\partial_1^{[1]}\partial_2^{[1]}G_{A,B}\Big)(z,x_0,x_0) \end{pmatrix} \\
&= \begin{pmatrix} G_{A,B}(z,x_0,x_0) & 2^{-1}\Big(\big[\partial_1^{[1]}+\partial_2^{[1]}\big]\,G_{A,B}\Big)(z,x_0-0,x_0+0) \\ 2^{-1}\Big(\big[\partial_1^{[1]}+\partial_2^{[1]}\big]\,G_{A,B}\Big)(z,x_0-0,x_0+0) & \Big(\partial_1^{[1]}\partial_2^{[1]}G_{A,B}\Big)(z,x_0,x_0) \end{pmatrix}, \\
&\hspace{8cm} z \in \mathbb{C}_+,\ x_0 \in (a,b).
\end{aligned}
$$

Eventually, we will (indicate how to) prove the **Nevanlinna–Herglotz** property of $M_{0,A,B}(\,\cdot\,, x_0)$.

Note. To simplify matters we restrict ourselves to the simplest case $\alpha = 0$ only.

General M-Function Theory (contd.)

Here we employed the abbreviations

$$\big(\partial_1^{[1]} G_{A,B}\big)(z, x_0 \pm 0, x_0 \mp 0) = \lim_{\varepsilon \downarrow 0} p(x)\partial_x G_{A,B}(z,x,x')\Big|_{\substack{x=x_0\pm\varepsilon \\ x'=x_0\mp\varepsilon}},$$

$$\big(\partial_2^{[1]} G_{A,B}\big)(z, x_0 \pm 0, x_0 \mp 0) = \lim_{\varepsilon \downarrow 0} p(x')\partial_{x'} G_{A,B}(z,x,x')\Big|_{\substack{x=x_0\pm\varepsilon \\ x'=x_0\mp\varepsilon}},$$

$$\begin{aligned}\big(\partial_1^{[1]}\partial_2^{[1]} G_{A,B}\big)(z, x_0, x_0) &= p(x)\partial_x\, p(x')\partial_{x'} G_{A,B}(z,x,x')\Big|_{x=x'=x_0} \\ &= \big(\partial_2^{[1]}\partial_1^{[1]} G_{A,B}\big)(z, x_0, x_0),\end{aligned}$$

and note the explicit formula (for $z \in \rho(T_{A,B})$, $x, x', x_0 \in (a,b)$)

$$\begin{aligned} G_{A,B}(z,x,x') &= G_{A,B}(z,x_0,x_0)\theta_0(z,x,x_0)\theta_0(z,x',x_0) \\ &+ \big[\big(\partial_1^{[1]}\partial_2^{[1]} G_{A,B}\big)(z,x_0,x_0)\big]\phi_0(z,x,x_0)\phi_0(z,x',x_0) \\ &+ \left[\big(\partial_2^{[1]} G_{A,B}\big)(z,x_0\pm 0,x_0\mp 0) + \begin{pmatrix} -1, \\ 0, \end{pmatrix}\right]\theta_0(z,x,x_0)\phi_0(z,x',x_0) \\ &+ \left[\big(\partial_1^{[1]} G_{A,B}\big)(z,x_0\pm 0,x_0\mp 0) + \begin{pmatrix} 1, \\ 0, \end{pmatrix}\right]\phi_0(z,x,x_0)\theta_0(z,x',x_0) \\ &+ \begin{cases} 0, & a<x\leqslant x'<b, \\ [\theta_0(z,x,x_0)\phi_0(z,x',x_0) - \phi_0(z,x,x_0)\theta_0(z,x',x_0)], & a<x'\leqslant x<b. \end{cases}\end{aligned}$$

Note. ϕ_0, θ_0 are **entire !!!!!** All possible spectral information sits in the **rest** !!!!!

General M-Function Theory (contd.)

Theorem 12.

Assume Hypothesis 1, that T_{min} is bounded from below, and let $z \in \mathbb{C}_+$. Then, for each fixed $x_0 \in (a,b)$, $M_{0,A,B}(\,\cdot\,,x_0)$, is a 2×2 **Nevanlinna–Herglotz** matrix with strictly positive imaginary part,

$$\operatorname{Im}(M_{0,A,B}(z,x_0)) > 0, \quad z \in \mathbb{C}_+.$$

Sketch of Proof. (I will have to pull your leg badly, very sorry !!!!) Introduce the graph Hilbert space[1] $H^2_\tau((a,b))$ associated with T_{max} as follows,

$$\begin{aligned} H^2_\tau((a,b)) &= \operatorname{dom}(T_{max}) \\ &= \big\{g \in L^2((a,b);rdx) \,\big|\, g, g^{[1]} \in AC_{loc}((a,b));\, \tau g \in L^2((a,b);rdx)\big\} \end{aligned}$$

with associated graph norm

$$\|f\|^2_{H^2_\tau((a,b))} = \|T_{max} f\|^2_{L^2((a,b);rdx)} + \|f\|^2_{L^2((a,b);rdx)}, \quad f \in \operatorname{dom}(T_{max}),$$

and scalar product

$$(f,g)_{H^2_\tau((a,b))} = (T_{max} f, T_{max} g)_{L^2((a,b);rdx)} + (f,g)_{L^2((a,b);rdx)}, \quad f,g \in \operatorname{dom}(T_{max}).$$

[1] We chose the notation $H^2_\tau((a,b))$ since in the special case $p = r = 1$, $q = 0$, $\tau_0 = -d^2/dx^2$, $H^2_{\tau_0}((a,b))$ coincides with the standard Sobolev space $H^2((a,b)) = W^{2,2}((a,b))$.

General M-Function Theory (contd.)

We also introduce the scale of Hilbert spaces corresponding to the self-adjoint operator $T_{A,B}$, assuming, for simplicity only, that $T_{A,B} \geqslant 0$. Hence, one obtains the chain of strict inclusions,

$$\begin{aligned} &\mathcal{H}_2(T_{A,B}) \subsetneqq H^2_\tau((a,b)) \subsetneqq \mathcal{H}_0(T_{A,B}) = \mathcal{H} = \mathcal{H}^* = \mathcal{H}_0(T_{A,B})^* \\ &\quad \subsetneqq H^2_\tau((a,b))^* \subsetneqq \mathcal{H}_2(T_{A,B})^* = \mathcal{H}_{-2}(T_{A,B}), \end{aligned}$$

with

$$\|\cdot\|_{\mathcal{H}_2(T_{A,B})^*} \leqslant \|\cdot\|_{H^2_\tau((a,b))^*} \leqslant \|\cdot\|_{L^2((a,b);rdx)} \leqslant \|\cdot\|_{H^2_\tau((a,b))} \leqslant \|\cdot\|_{\mathcal{H}_2(T_{A,B})}.$$

At this point we introduce the map

$$\Gamma_{x_0} : \begin{cases} H^2_\tau((a,b)) \to \mathbb{C}^2, \\ u \mapsto \begin{pmatrix} u(x_0) \\ u^{[1]}(x_0) \end{pmatrix}, \end{cases} \qquad x_0 \in (a,b),$$

and record its properties in the following.

General M-Function Theory (contd.)

Lemma 13.

Assume Hypothesis 1 and that $T_{A,B} \geqslant 0$. Then

$$\Gamma_{x_0} \in \mathcal{B}\big(H^2_\tau((a,b)), \mathbb{C}^2\big)$$

and

$$\Gamma^*_{x_0} : \begin{cases} \mathbb{C}^2 \to H^2_\tau((a,b))^* \\ \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \mapsto c_1\delta_{x_0} - c_2 p\delta'_{x_0}, \end{cases} \qquad \Gamma^*_{x_0} \in \mathcal{B}\big(\mathbb{C}^2, H^2_\tau((a,b))^*\big),$$

where

$$\delta_{x_0}(u) = {}_{H^2_\tau((a,b))^*}\langle \delta_{x_0}, u\rangle_{H^2_\tau((a,b))} = u(x_0),$$
$$p\delta'_{x_0}(u) = {}_{H^2_\tau((a,b))^*}\langle p\delta'_{x_0}, u\rangle_{H^2_\tau((a,b))} = -u^{[1]}(x_0), \quad u \in H^2_\tau((a,b)).$$

General M-Function Theory (contd.)

Using mapping properties of resolvents of $\widetilde{T_{A,B}}$ in connection with the chain of Hilbert spaces $\mathcal{H}_s(T_{A,B})$, $s \in \mathbb{R}$, more precisely, using the special cases,

$$\big(\widetilde{T_{A,B}} + \widetilde{I}\big)^{-1} \colon L^2((a,b); rdx) = \mathcal{H}_0(T_{A,B}) \to \mathcal{H}_2(T_{A,B}) \text{ is an isomorphism,}$$
$$\big(\widetilde{T_{A,B}} + \widetilde{I}\big)^{-1} \colon \mathcal{H}_2(T_{A,B})^* = \mathcal{H}_{-2}(T_{A,B}) \to L^2((a,b); rdx) \text{ is an isomorphism,}$$
$$\big(\widetilde{T_{A,B}} + \widetilde{I}\big)^{-2} \colon \mathcal{H}_2(T_{A,B})^* = \mathcal{H}_{-2}(T_{A,B}) \to \mathcal{H}_2(T_{A,B}) \text{ is an isomorphism,}$$

with $\widetilde{I}$ appropriate inclusion maps, one introduces

$$N_{A,B}(z,x_0) = \begin{pmatrix} \operatorname{Im}(G_{A,B}(z,x_0,x_0)) & \partial_2^{[1]}\operatorname{Im}(G_{A,B}(z,x_0,x_0)) \\ \partial_1^{[1]}\operatorname{Im}(G_{A,B}(z,x_0,x_0)) & \partial_1^{[1]}\partial_2^{[1]}\operatorname{Im}(G_{A,B}(z,x_0,x_0)) \end{pmatrix}, \quad z \in \mathbb{C}_+,$$

and computes as follows:

General M-Function Theory (contd.)

$$\begin{aligned}
&(c, N_{A,B}(z,x_0)c)_{\mathbb{C}^2} \\
&= \left(\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \begin{pmatrix} \operatorname{Im}(G_{A,B}(z,x_0,x_0)) & \partial_2^{[1]} \operatorname{Im}(G_{A,B}(z,x_0,x_0)) \\ \partial_1^{[1]} \operatorname{Im}(G_{A,B}(z,x_0,x_0)) & \partial_1^{[1]}\partial_2^{[1]} \operatorname{Im}(G_{A,B}(z,x_0,x_0)) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \right)_{\mathbb{C}^2} \\
&= \Big(c, (2i)^{-1} \Gamma_{x_0} \Big[\big(\widetilde{T_{A,B}} - z\widetilde{I}\big)^{-1} - \big(\widetilde{T_{A,B}} - \overline{z}\widetilde{I}\big)^{-1} \Big] \Gamma_{x_0}^* c \Big)_{\mathbb{C}^2} \\
&= \operatorname{Im}(z) \Big(c, \Gamma_{x_0} \big(\widetilde{T_{A,B}} - z\widetilde{I}\big)^{-1} \big(\widetilde{T_{A,B}} - \overline{z}\widetilde{I}\big)^{-1} \Gamma_{x_0}^* c \Big)_{\mathbb{C}^2} \\
&= \operatorname{Im}(z) \Big(\big(\widetilde{T_{A,B}} - \overline{z}\widetilde{I}\big)^{-1} \Gamma_{x_0}^* c, \big(\widetilde{T_{A,B}} - \overline{z}\widetilde{I}\big)^{-1} \Gamma_{x_0}^* c \Big)_{L^2((a,b);rdx)} \\
&= \operatorname{Im}(z) \big\| \big(\widetilde{T_{A,B}} - \overline{z}\widetilde{I}\big)^{-1} \Gamma_{x_0}^* c \big\|_{L^2((a,b);rdx)}^2 \geqslant 0.
\end{aligned}$$

Thus,

$$\operatorname{Im}(M_{0,A,B}(z,x_0)) = 2^{-1} \big[N_{A,B}(z,x_0) + N_{A,B}(z,x_0)^\top \big] \geqslant 0, \quad z \in \mathbb{C}_+.$$

Finally, proving strict inequality is elementary and hence omitted here. $\square$

Connecting $M_{0,A,B}(\,\cdot\,,x_0)$ and $E_{T_{A,B}}(\,\cdot\,)$

Since $M_{0,A,B}(\,\cdot\,,x_0)$ is a 2×2 matrix-valued **Nevanlinna–Herglotz** function, it permits the representation,

$$M_{0,A,B}(z,x_0) = C_{0,A,B}(x_0) + \int_{\mathbb{R}} d\Omega_{0,A,B}(\lambda,x_0)\bigg(\frac{1}{\lambda - z} - \frac{\lambda}{1+\lambda^2}\bigg), \quad z\in\mathbb{C}\backslash\mathbb{R},$$

where

$$C_{0,A,B}(x_0) = C_{0,A,B}(x_0)^* = \mathrm{Re}(M_{0,A,B}(i,x_0)), \quad \int_{\mathbb{R}} \frac{\|d\Omega_{0,A,B}(\lambda,x_0)\|}{1+\lambda^2} < \infty.$$

The **Stieltjes inversion formula** for the 2×2 nonnegative matrix-valued measure $d\Omega_{0,A,B}(\,\cdot\,,x_0)$ then reads

$$\Omega_{0,A,B}((\lambda_1,\lambda_2],x_0) = \pi^{-1}\lim_{\delta\downarrow 0}\lim_{\varepsilon\downarrow 0}\int_{\lambda_1+\delta}^{\lambda_2+\delta} d\lambda\, \mathrm{Im}(M_{0,A,B}(\lambda+i\varepsilon,x_0)),$$
$$\lambda_1,\lambda_2\in\mathbb{R},\ \lambda_1<\lambda_2,$$

and hence we are now after the connection between $\Omega_{0,A,B}(\,\cdot\,,x_0)$ and $E_{T_{A,B}}(\,\cdot\,)$.

Connecting $M_{0,A,B}(\,\cdot\,,x_0)$ and $E_{T_{A,B}}(\,\cdot\,)$ (contd.)

Theorem 14.

Assume Hypothesis 1 and T_{min} is bounded from below. In addition, suppose that $f,g \in C_0^\infty((a,b))$, $F \in C(\mathbb{R})$, $x_0 \in (a,b)$, and $\lambda_1, \lambda_2 \in \mathbb{R}$, $\lambda_1 < \lambda_2$. Then,

$$\begin{aligned}
&(f, F(T_{A,B})E_{T_{A,B}}((\lambda_1,\lambda_2])g)_{L^2((a,b);rdx)} \\
&\quad = \big(\widehat{f}_0(\,\cdot\,,x_0), M_F M_{\chi_{(\lambda_1,\lambda_2]}} \widehat{g}_0(\,\cdot\,,x_0)\big)_{L^2(\mathbb{R};d\Omega_{0,A,B}(\,\cdot\,,x_0))} \\
&\quad = \int_{(\lambda_1,\lambda_2]} \overline{\widehat{f}_0(\lambda,x_0)^\top}\, d\Omega_{0,A,B}(\lambda,x_0)\, \widehat{g}_0(\lambda,x_0) F(\lambda),
\end{aligned}$$

where we introduced the notation (**generalized Fourier coefficients**)

$$\begin{aligned}
&\widehat{h}_{0,1}(\lambda,x_0) = \int_a^b r(x)dx\, \theta_0(\lambda,x,x_0)h(x), \\
&\widehat{h}_{0,2}(\lambda,x_0) = \int_a^b r(x)dx\, \phi_0(\lambda,x,x_0)h(x), \\
&\widehat{h}_0(\lambda,x_0) = \big(\widehat{h}_{0,1}(\lambda,x_0), \widehat{h}_{0,2}(\lambda,x_0)\big)^\top, \quad \lambda \in \mathbb{R},\ h \in C_0^\infty((a,b)).
\end{aligned}$$

Connecting $M_{0,A,B}(\cdot, x_0)$ and $E_{T_{A,B}}(\cdot)$ (contd.)

Here M_G represents the maximally defined operator of multiplication by the $d\Omega^{tr}_{0,A,B}(\cdot, x_0)$-measurable function G in the Hilbert space $L^2(\mathbb{R}; d\Omega_{0,A,B}(\cdot, x_0))$,

$$\big(M_G\widehat{h}\big)(\lambda) = G(\lambda)\widehat{h}(\lambda) = \big(G(\lambda)\widehat{h}_1(\lambda), G(\lambda)\widehat{h}_2(\lambda)\big)^\top \text{ for } d\Omega^{tr}_{0,A,B}(\cdot, x_0)\text{-a.e. } \lambda \in \mathbb{R},$$
$$\widehat{h} \in \operatorname{dom}(M_G) = \big\{\widehat{k} \in L^2(\mathbb{R}; d\Omega_{0,A,B}(\cdot, x_0)) \,\big|\, G\widehat{k} \in L^2(\mathbb{R}; d\Omega_{0,A,B}(\cdot, x_0))\big\},$$

and

$$d\Omega^{tr} = d\Omega_{1,1} + d\Omega_{2,2}$$

denotes the trace measure of a 2×2 matrix-valued nonnegative measure $d\Omega = \big(d\Omega_{\ell,\ell'}\big)_{\ell,\ell'=1,2}$ on $\mathbb{R}$.

The proof of Theorem 14 involves, **Stone's formula** relating the family of spectral projections $E_T(\cdot)$ with nontangential boundary values of the resolvent $(T - zI)^{-1}$, the explicit structure of the Green's function $G_{A,B}(z, \cdot, \cdot)$, $z \in \mathbb{C}\backslash\sigma(T_{A,B})$, the **Stieltjes inversion formula**, and essentially every other trick in the book on (matrix-valued) **Nevanlinna–Herglotz** functions. Here is a sketch of the proof:

Connecting $M_{0,A,B}(\,\cdot\,, x_0)$ and $E_{T_{A,B}}(\,\cdot\,)$ (contd.)

The points of departure are **Stone's formula** and the explicit expression for the Green's function $G_{A,B}(z,x,x')$, $z \in \rho(T_{A,B})$, $x,x' \in (a,b)$, of $T_{A,B}$ in terms of $\phi_0(z,x,x_0)$ and $\theta_0(z,x,x_0)$,

$$\begin{aligned}
&(f, F(T_{A,B})E_{T_{A,B}}((\lambda_1,\lambda_2])g)_{L^2((a,b);rdx)} \\
&\quad = \lim_{\delta\downarrow 0}\lim_{\varepsilon\downarrow 0}\frac{1}{2\pi i}\int_{\lambda_1+\delta}^{\lambda_2+\delta} d\lambda\, F(\lambda)\Big[\big(f,(T_{A,B}-(\lambda+i\varepsilon)I)^{-1}g\big)_{L^2((a,b);rdx)} \\
&\qquad\qquad - \big(f,(T_{A,B}-(\lambda-i\varepsilon)I)^{-1}g\big)_{L^2((a,b);rdx)}\Big] \\
&\quad = \lim_{\delta\downarrow 0}\lim_{\varepsilon\downarrow 0}\frac{1}{2\pi i}\int_{\lambda_1+\delta}^{\lambda_2+\delta} d\lambda\, F(\lambda)\int_a^b r(x)dx\int_a^b r(x')dx'\,\overline{f(x)}g(x') \\
&\qquad \times\Big\{\Big[G_{A,B}(\lambda+i\varepsilon,x_0,x_0)\theta_0(\lambda+i\varepsilon,x,x_0)\theta_0(\lambda+i\varepsilon,x',x_0) \\
&\qquad\quad + \big(\partial_1^{[1]}\partial_2^{[1]}G_{A,B}\big)(\lambda+i\varepsilon,x_0,x_0)\phi_0(\lambda+i\varepsilon,x,x_0)\phi_0(\lambda+i\varepsilon,x',x_0) \\
&\qquad\quad + \big(\partial_2^{[1]}G_{A,B}\big)(\lambda+i\varepsilon,x_0\pm 0,x_0\mp 0)\theta_0(\lambda+i\varepsilon,x,x_0)\phi_0(\lambda+i\varepsilon,x',x_0) \\
&\qquad\quad + \big(\partial_1^{[1]}G_{A,B}\big)(\lambda+i\varepsilon,x_0\pm 0,x_0\mp 0)\phi_0(\lambda+i\varepsilon,x,x_0)\theta_0(\lambda+i\varepsilon,x',x_0)\Big]
\end{aligned}$$

Connecting $M_{0,A,B}(\,\cdot\,, x_0)$ and $E_{T_{A,B}}(\,\cdot\,)$ (contd.)

$$- \Big[\boxed{\text{terms with } \lambda + i\varepsilon \text{ replaced by } \lambda - i\varepsilon}\Big]$$
$$+ \boxed{\text{terms entire in } z \text{ taken at } \lambda + i\varepsilon \text{ minus them taken at } \lambda - i\varepsilon}\Big\}.$$

Freely interchanging the dx and dx' integrals with the limits and the $d\lambda$ integral (since all integration domains are finite and all integrands are continuous) and introducing the notation

$$\Phi_0(z,x,x_0) = \begin{pmatrix} \theta_0(z,x,x_0) \\ \phi_0(z,x,x_0) \end{pmatrix}, \quad x \in (a,b),\ z \in \mathbb{C},$$
$$\widetilde{M}_{0,A,B}(z,x_0) = \begin{pmatrix} G_{A,B}(z,x_0,x_0) & \big(\partial_2^{[1]} G_{A,B}\big)(z,x_0 \pm 0, x_0 \mp 0) \\ \big(\partial_1^{[1]} G_{A,B}\big)(z,x_0 \pm 0, x_0 \mp 0) & \big(\partial_1^{[1]}\partial_2^{[1]} G_{A,B}\big)(z,x_0,x_0) \end{pmatrix}$$
$$z \in \mathbb{C}\backslash\mathbb{R},\ x_0 \in (a,b),$$

then yield

Connecting $M_{0,A,B}(\,\cdot\,, x_0)$ and $E_{T_{A,B}}(\,\cdot\,)$ (contd.)

$$\begin{aligned}
&\big(f, F(T_{A,B})E_{T_{A,B}}((\lambda_1,\lambda_2])g\big)_{L^2((a,b);rdx)} \\
&\quad = \int_a^b r(x)dx\,\overline{f(x)}\int_a^b r(x')dx'\,g(x')\lim_{\delta\downarrow 0}\lim_{\varepsilon\downarrow 0}\frac{1}{2\pi i}\int_{\lambda_1+\delta}^{\lambda_2+\delta} d\lambda\, F(\lambda) \\
&\qquad \times \Big[\Phi_0(\lambda+i\varepsilon,x,x_0)^T\widetilde{M}_{0,A,B}(\lambda+i\varepsilon,x_0)\Phi_0(\lambda+i\varepsilon,x',x_0) \\
&\qquad\quad - \Phi_0(\lambda-i\varepsilon,x,x_0)^T\widetilde{M}_{0,A,B}(\lambda-i\varepsilon,x_0)\Phi_0(\lambda-i\varepsilon,x',x_0)\Big].
\end{aligned}$$

With $\boldsymbol{\cdot}$ abbreviating d/dz, one obtains

$$\Phi_0(\lambda\pm i\varepsilon,x,x_0) \underset{\varepsilon\downarrow 0}{=} \Phi_0(\lambda,x,x_0)\pm i\varepsilon\dot{\Phi}_0(\lambda,x,x_0)+O(\varepsilon^2),$$

with $O(\varepsilon^2)$ being uniform with respect to (λ,x) as long as λ and x vary in compact subsets of $\mathbb{R}\times(a,b)$ (recall that f,g have compact support right now!). We also note that for some $C(\lambda_1,\lambda_2,\varepsilon_0,x_0)\in(0,\infty)$,

$$\varepsilon|M_{0,A,B,\ell,\ell'}(\lambda+i\varepsilon,x_0)|\leqslant C(\lambda_1,\lambda_2,\varepsilon_0,x_0),\quad \lambda\in[\lambda_1,\lambda_2],\ 0<\varepsilon\leqslant\varepsilon_0,\ \ell,\ell'=1,2,$$
$$\varepsilon|\mathrm{Re}(M_{0,A,B,\ell,\ell'}(\lambda+i\varepsilon,x_0))| \underset{\varepsilon\downarrow 0}{=} o(1),\quad \lambda\in\mathbb{R},\ \ell,\ell'=1,2,$$

Thus, one arrives at

Connecting $M_{0,A,B}(\,\cdot\,, x_0)$ and $E_{T_{A,B}}(\,\cdot\,)$ (contd.)

$$\begin{aligned}
&\big(f, F(T_{A,B})E_{T_{A,B}}((\lambda_1,\lambda_2])g\big)_{L^2((a,b);rdx)} = \int_a^b r(x)dx\,\overline{f(x)}\int_a^b r(x')dx'\,g(x') \\
&\quad\times \lim_{\delta\downarrow 0}\lim_{\varepsilon\downarrow 0}\frac{1}{\pi}\int_{\lambda_1+\delta}^{\lambda_2+\delta} d\lambda\,F(\lambda)\Phi_0(\lambda,x,x_0)^T \operatorname{Im}(M_{0,A,B}(\lambda+i\varepsilon,x_0))\Phi_0(\lambda,x',x_0) \\
&\quad\times \lim_{\delta\downarrow 0}\frac{1}{\pi}\int_{\lambda_1+\delta}^{\lambda_2+\delta} F(\lambda)\Phi_0(\lambda,x,x_0)^T d\Omega_{0,A,B}(\lambda,x_0)\,\Phi_0(\lambda,x',x_0) \\
&\quad= \int_{(\lambda_1,\lambda_2]} \overline{\widehat{f}_0(\lambda,x_0)^\top}\, d\Omega_{0,A,B}(\lambda,x_0)\,\widehat{g}_0(\lambda,x_0)F(\lambda).
\end{aligned}$$

Here we interchanged the dx, dx' and $d\Omega_{0,A,B}(\,\cdot\,,x_0)$ integrals once more, and employed the **generalized Fourier coefficient**

$$\widehat{h}_0(\lambda,x_0) = \int_a^b r(x)dx\,\Phi_0(\lambda,x,x_0)h(x), \quad \lambda\in\mathbb{R},\ h\in C_0^\infty((a,b)).$$

Connecting $M_{0,A,B}(\,\cdot\,, x_0)$ and $E_{T_{A,B}}(\,\cdot\,)$ (contd.)

One removes the compact support restrictions on f and g in a standard manner: Introduce the unitary map

$$\mathcal{F}_{0,A,B}(x_0)\colon \begin{cases} L^2((a,b); rdx) \to L^2(\mathbb{R}; d\Omega_{0,A,B}(\,\cdot\,, x_0)) \\ h \mapsto \widehat{h}_0(\,\cdot\,, x_0) = \big(\widehat{h}_{0,1}(\,\cdot\,, x_0), \widehat{h}_{0,2}(\,\cdot\,, x_0)\big)^\top, \end{cases}$$

$$\widehat{h}_0(\,\cdot\,, x_0) = \begin{pmatrix} \widehat{h}_{0,1}(\,\cdot\,, x_0) \\ \widehat{h}_{0,2}(\,\cdot\,, x_0) \end{pmatrix} = \operatorname*{s-lim}_{c\downarrow a, d\uparrow b} \begin{pmatrix} \int_c^d r(x)dx\, \theta_0(\,\cdot\,, x, x_0) h(x) \\ \int_c^d r(x)dx\, \phi_0(\,\cdot\,, x, x_0) h(x) \end{pmatrix},$$

where s-lim refers to the $L^2(\mathbb{R}; d\Omega_{0,A,B}(\,\cdot\,, x_0))$-limit. The associated inverse operator is then given by

$$\mathcal{F}_{0,A,B}(x_0)^{-1}\colon \begin{cases} L^2(\mathbb{R}; d\Omega_{0,A,B}(\,\cdot\,, x_0)) \to L^2((a,b); rdx) \\ \widehat{h} \mapsto h_0, \end{cases}$$

$$h_0(\,\cdot\,) = \operatorname*{s-lim}_{\mu_1\downarrow -\infty, \mu_2\uparrow\infty} \int_{\mu_1}^{\mu_2} (\theta_0(\lambda,\,\cdot\,, x_0), \phi_0(\lambda,\,\cdot\,, x_0))\, d\Omega_{0,A,B}(\lambda, x_0)\, \widehat{h}(\lambda),$$

where s-lim now refers to the $L^2((a,b); rdx)$-limit.

Connecting $M_{0,A,B}(\,\cdot\,,x_0)$ and $E_{T_{A,B}}(\,\cdot\,)$ (contd.)

Thus, with $d\Omega^{tr}_{0,A,B}(\,\cdot\,,x_0) = d\Omega_{0,A,B,1,1}(\,\cdot\,,x_0) + d\Omega_{0,A,B,2,2}(\,\cdot\,,x_0)$ representing the **trace measure** of $d\Omega_{0,A,B}(\,\cdot\,,x_0)$, and with M_F denoting the operator of multiplication by the function $F \in C(\mathbb{R})$ in $L^2(\mathbb{R}; d\Omega_{0,A,B}(\,\cdot\,,x_0))$, one obtains the following result.

Theorem 15.

Assume Hypothesis 1 and T_{min} is bounded from below. In addition, let $F \in C(\mathbb{R})$, and $x_0 \in (a,b)$. Then, the **"diagonalization,"**

$$\mathcal{F}_{0,A,B}(x_0) F(T_{A,B}) \mathcal{F}_{0,A,B}(x_0)^{-1} = M_F,$$

holds in $L^2(\mathbb{R}; d\Omega_{0,A,B}(\,\cdot\,,x_0))$. Moreover,

$$\sigma(T_{A,B}) = \operatorname{supp}\big(d\Omega_{0,A,B}(\,\cdot\,,x_0)\big) = \operatorname{supp}\,\big(d\Omega^{tr}_{0,A,B}(\,\cdot\,,x_0)\big).$$

Note. (i) While we focused on the case $\alpha = 0$ for simplicity, the general case $\alpha \in [0,\pi)$ is handled in the same manner.

(ii) Again, boundedness from below of T_{min} is convenient, but not essential. (Krein-type resolvent formulas exist independently of boundedness from below.)

Based on:

• **J. Eckhardt, F.G., R. Nichols, and G. Teschl**, *Weyl–Titchmarsh theory for Sturm–Liouville operators with distributional potentials*, Opuscula Math. **33**, 467–563 (2013).

• **F.G., L. L. Littlejohn, and R. Nichols**, *On self-adjoint boundary conditions for singular Sturm–Liouville operators bounded from below*, J. Diff. Eq. **269**, 6448–6491 (2020).

• **F.G. and R. Nichols**, *Sturm–Liouville M-functions in terms of Green's functions*, preprint, 2022.

• **F.G., R. Nichols, and M. Zinchenko**, *Sturm–Liouville Operators, Their Spectral Theory, and Some Applications*, book in preparation.

Thank you!