Hardy Spaces of Fuchsian Groups for Akhiezer - Levin Points

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Notations

Space L^2 of $\mathbb{T}$

$$\int_{\mathbb{T}} |f(t)|^2 \mu(dt) < \infty$$

μ is the Lebesgue measure

Space $\mathcal{L}^2_{t_0}$

$$\int_{\mathbb{T}} |f(t)|^2 \frac{\mu(dt)}{|t - t_0|^2} < \infty, \quad |t_0| = 1$$

Isometric isomorphism $L^2 \to \mathcal{L}^2_{t_0}$

$$f(t) \to (t - t_0) f(t)$$

Space $\mathcal{L}^2_{t_0}$

$$\int_{\mathbb{T}} |f(t)|^2 \frac{\mu(dt)}{|t - t_0|^2} < \infty, \quad |t_0| = 1$$

Substitute $t = t_0 \frac{x - i}{x + i}$, $x \in \mathbb{R}$

$$F(x) = f(t(x))$$

$$\int_{\mathbb{R}} |F(x)|^2 dx < \infty$$

Basis L^2

Space L^2

$$\int_{\mathbb{T}} |f(t)|^2 \mu(dt) < \infty$$

Basis $\{t^j\}$; Fourier series

$$f(t) = \sum_{j=-\infty}^{\infty} c_j t^j, \quad |t| = 1$$

$$\sum_{j=-\infty}^{\infty} |c_j|^2 < \infty$$

Basis $\mathcal{L}^2_{t_0}$

Space $\mathcal{L}^2_{t_0}$

$$\int_{\mathbb{T}} |f(t)|^2 \frac{\mu(dt)}{|t - t_0|^2} < \infty, \quad |t_0| = 1$$

Substitute $t = t_0 \frac{x - i}{x + i}$, $x \in \mathbb{R}$

$$\int_{\mathbb{R}} |F(x)|^2 dx < \infty, \quad F(x) = f(t(x))$$

Continual Basis $\{e^{i\xi x}\}$; Fourier integral

$$F(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \widetilde{F}(\xi) e^{i\xi x} d\xi, \quad x \in \mathbb{R}$$

$$\int_{\mathbb{R}} |\widetilde{F}(\xi)|^2 d\xi < \infty$$

Hardy Spaces

Space L^2

$$\int_{\mathbb{T}} |f(t)|^2 \mu(dt) < \infty$$

Hardy space $H^2 \subset L^2$ analytic in $\mathbb{D}$

$$f(\zeta) = \sum_{j=0}^{\infty} c_j \zeta^j, \quad |\zeta| \leq 1$$

Hardy space $\mathcal{H}^2_{t_0} \subset \mathcal{L}^2_{t_0}$

$$\mathcal{H}^2_{t_0} = (t - t_0) H^2$$

Corresponds to

$$F(z) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \widetilde{F}(\xi) e^{i\xi z} d\xi, \quad \mathrm{Im}\ z > 0$$

Fuchsian Group

Conformal maps

$$\mathbb{D} \to \mathbb{D}$$

Are of the form

$$\zeta \to \frac{\zeta - \zeta_0}{1 - \zeta\overline{\zeta_0}} c, \quad |\zeta_0| < 1, \quad |c| = 1$$

Equivalently

$$\zeta \to \frac{a\zeta + b}{\overline{b}\zeta + \overline{a}}, \quad |a|^2 - |b|^2 = 1$$

They form a group (composition)

Definition (Fuchsian Group)

Discrete subgroups of this group are called Fuchsian groups.

Dual Group Γ^*

Elements $\alpha \in \Gamma^*$

Unimodular multiplicative characters

$$\alpha : \Gamma \to \mathbb{T}$$

$$\alpha(\gamma_1\gamma_2) = \alpha(\gamma_1)\alpha(\gamma_2)$$

Automorphic Hardy Spaces

Recall: space L^2 $$f(\zeta) = \sum_{j=-\infty}^{\infty} c_j \zeta^j, \quad |\zeta| = 1$$

$$\sum_{j=-\infty}^{\infty} |c_j|^2 < \infty$$

Hardy space $H^2 \subset L^2$, analytic in $\mathbb{D}$

$$f(\zeta) = \sum_{j=0}^{\infty} c_j \zeta^j, \quad |\zeta| \leq 1$$

For every $\alpha \in \Gamma^*$ consider subspace $L^2_\alpha \subset L^2$

$$f(\gamma(\zeta)) = \alpha(\gamma) f(\zeta), \quad \gamma \in \Gamma$$

Automorphic Hardy $H^2_\alpha \subset H^2$

Question: H^2_α nontrivial for all α ?

Answered by Widom

Green Function

Green function for the disk $G(\zeta) = -\ln\left|\frac{\zeta - \zeta_0}{1 - \zeta\overline{\zeta_0}}\right|$

Complex Green function $\quad g(\zeta) = \frac{\zeta - \zeta_0}{1 - \zeta\overline{\zeta_0}},$

Let Γ be Fuchsian group of convergent type

Consider Blaschke product over orbit of ζ_0

$$g_{\zeta_0}(\zeta) = \prod_{\gamma\in\Gamma} \frac{\zeta - \gamma(\zeta_0)}{1 - \zeta\overline{\gamma(\zeta_0)}} C_\gamma = \prod_{\gamma\in\Gamma} \frac{\gamma(\zeta) - \zeta_0}{1 - \gamma(\zeta)\overline{\zeta_0}} \widetilde{C}_\gamma$$

Is called automorphic complex Green function

There exists a character α_g of Γ

$$g(\gamma(\zeta)) = \alpha_g(\gamma) g(\zeta)$$

Widom - Pommerenke

Fuchsian group Γ

Complex Green Function

$$g(\zeta) = \prod_{\gamma \in \Gamma} \frac{\gamma(\zeta) - \zeta_0}{1 - \gamma(\zeta)\bar{\zeta}_0} C_\gamma$$

Theorem (Widom)

For every $\alpha \in \Gamma^*$ the space H^2_α contains a non constant function

if and only if

Zeros of $g'(\zeta)$ satisfy Blaschke condition.

Theorem (Pommerenke)

Then $g'(\zeta)$ is of bounded characteristic in $\mathbb{D}$
Moreover, $g' = \frac{B}{O}$, where B Blaschke, O outer, $\|O\|_\infty \leq 1$

Widom Group

Definition

We say Γ is of Widom type if $g'(\zeta)$ is of bounded characteristic

Definition

We say $t_0 \in \mathbb{T}$ is "**good**" if

1. $g(\zeta)$ has unimodular nontangential boundary value $g(t_0)$
2. $g'(\zeta)$ has finite nontangential boundary value $g'(t_0)$

Summary

Γ of Widom type $\Longrightarrow$ almost every $t_0 \in \mathbb{T}$ is "good"

Spaces $\mathcal{H}^2_{t_0,\alpha}$

Space $\mathcal{L}^2_{t_0}$

$$\int_{\mathbb{T}} |f(t)|^2 \frac{\mu(dt)}{|t-t_0|^2} < \infty, \quad |t_0| = 1$$

$\mathcal{H}^2_{t_0} \subset \mathcal{L}^2_{t_0}$, $\qquad \mathcal{H}^2_{t_0} = (t-t_0)H^2$

Theorem (Kh-Yuditskii)

For every $\alpha \in \Gamma^*$ the space $\mathcal{H}^2_{t_0,\alpha}$ contains a non constant function
if and only if

(i) Γ *is of Widom type (g' is of bounded characteristic)*

(ii) t_0 *is good ($|g(t_0)| = 1$, $g'(t_0)$ finite)*

Frostman Theorem

Theorem (Frostman)

Let $g(\zeta)$ be a Blaschke product $\quad g(\zeta) = \prod_k \frac{\zeta - \zeta_k}{1 - \zeta\bar{\zeta}_k} C_k, \quad |\zeta| < 1$

Let $|t_0| = 1$

1. $g(\zeta)$ *has unimodular nontangential boundary value* $g(t_0)$
2. $g'(\zeta)$ *has finite nontangential boundary value* $g'(t_0)$

If and only if $\quad \sum_k \frac{1 - |\zeta_k|^2}{|t_0 - \zeta_k|^2} < \infty.$ *In this case*

$$g(t_0) = \prod_k \frac{t_0 - \zeta_k}{1 - t_0\bar{\zeta}_k} C_k \quad \text{and} \quad |g'(t_0)| = t_0 \frac{g'(t_0)}{g(t_0)} = \sum_k \frac{1 - |\zeta_k|^2}{|t_0 - \zeta_k|^2}$$

Frostman Theorem

$$g(t_0) = \prod_k \frac{t_0 - \zeta_k}{1 - t_0\overline{\zeta}_k} C_k \quad \text{and} \quad |g'(t_0)| = t_0 \frac{g'(t_0)}{g(t_0)} = \sum_k \frac{1 - |\zeta_k|^2}{|t_0 - \zeta_k|^2}$$

For Green function

$$g(\zeta) = \prod_{\gamma\in\Gamma} \frac{\zeta - \gamma(\zeta_0)}{1 - \zeta\overline{\gamma(\zeta_0)}} C_\gamma = \prod_{\gamma\in\Gamma} \frac{\gamma(\zeta) - \zeta_0}{1 - \gamma(\zeta)\overline{\zeta}_0} \widetilde{C}_\gamma$$

Frostman condition

$$|g'(t_0)| = \sum_{\gamma\in\Gamma} \frac{1 - |\gamma(\zeta_0)|^2}{|t_0 - \gamma(\zeta_0)|^2} = \sum_{\gamma\in\Gamma} \frac{1 - |\zeta_0|^2}{|\gamma(t_0) - \zeta_0|^2} |\gamma'(t_0)| < \infty$$

$$\Longleftrightarrow \sum_{\gamma\in\Gamma} |\gamma'(t_0)| < \infty$$

Theorem Restated

Theorem (Kh-Yuditskii)

For every $\alpha \in \Gamma^*$ the space $\mathcal{H}^2_{t_0,\alpha}$ contains a non constant function
if and only if

(i) Γ *is of Widom type (g' is of bounded characteristic)*

(ii) t_0 *is good ($|g(t_0)| = 1$, $g'(t_0)$ finite)*

Theorem (Kh-Yuditskii)

For every $\alpha \in \Gamma^*$ the space $\mathcal{H}^2_{t_0,\alpha}$ contains a non constant function
if and only if

(i) Γ *is of Widom type (g' is of bounded characteristic)*

(ii) $\sum_{\gamma \in \Gamma} |\gamma'(t_0)| < \infty$

Martin Function

Under assumption $\sum_{\gamma\in\Gamma} |\gamma'(t_0)| < \infty$

Define Martin function

$$\widetilde{M}_{t_0}(\zeta) = \sum_{\gamma\in\Gamma} \frac{1-|\zeta|^2}{|\gamma(t_0)-\zeta|^2} |\gamma'(t_0)|$$

Positive , harmonic , Γ automorphic Automorphic since also

$$\widetilde{M}_{t_0}(\zeta) = \sum_{\gamma\in\Gamma} \frac{1-|\gamma(\zeta)|^2}{|t_0-\gamma(\zeta)|^2}$$

Complex Martin function

$$m_{t_0}(\zeta) = i\sum_{\gamma\in\Gamma} \frac{\gamma(t_0)+\zeta}{\gamma(t_0)-\zeta} |\gamma'(t_0)| = i\sum_{\gamma\in\Gamma} \frac{t_0+\gamma(\zeta)}{t_0-\gamma(\zeta)}$$

Martin Function

$$m_{t_0}(\zeta) = i \sum_{\gamma \in \Gamma} \frac{\gamma(t_0) + \zeta}{\gamma(t_0) - \zeta} |\gamma'(t_0)|$$

Theorem (Kh-Yuditskii)

For every $\alpha \in \Gamma^*$ the space $\mathcal{H}^2_{t_0,\alpha}$ contains a non constant function
if and only if

(i) $\sum_{\gamma \in \Gamma} |\gamma'(t_0)| < \infty$

(ii) *Zeros of $m'_{t_0}(\zeta)$ satisfy Blaschke condition*

$\Longrightarrow$

$m'_{t_0}(\zeta)$ is of bounded characteristic

Moreover, $m'_{t_0} = \frac{B}{O}$, B Blaschke, O outer, $\|O\| \leq 1$

Outline of the proof

Lemma

Let

$$\sum_{\gamma\in\Gamma} |\gamma'(t_0)| < \infty$$

Then

$$g' \text{ bounded characteristic} \iff m'_{t_0} \text{ bounded characteristic}$$

$\Longleftarrow$ straightforward

$\Longrightarrow$

g' bounded characteristic $\Longrightarrow$ zeros g' Blaschke $\Longrightarrow$ zeros m' Blaschke $\Longrightarrow$ m' bounded characteristic

Having this

$$|\Phi|^2 = \left|\frac{g'}{m'}\right| \quad \text{and} \quad \mathcal{H}^2_{t_0,\alpha} = \Phi H^2_{\alpha'}$$

Finitely connected approximation

To prove

$$\text{zeros } m' \text{ Blaschke} \Longrightarrow m' \text{ bounded characteristic}$$

Approximate $\Omega = \mathbb{D}/\Gamma$ by finitely connected domains

Pommerenke (for Green function)

$$G(\lambda) > \epsilon$$

Finitely connected

However

$$M(\lambda) > \epsilon$$

Still infinitely connected

Kh-Yuditskii (for Martin function)

$$\Gamma = \bigcup \Gamma_n$$

Γ_n finitely generated

Danjoy domain

Compact $E \subset \mathbb{R}$

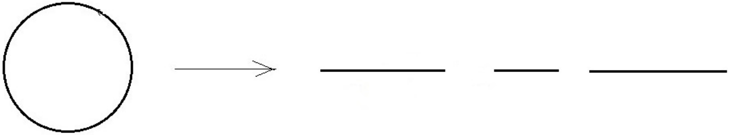
Domain

$$\Omega = \overline{\mathbb{C}} \setminus E$$

Uniformization

Universal cover

$$\Lambda(\zeta): \qquad \mathbb{D} \quad \to \quad \Omega = \overline{\mathbb{C}} \setminus E$$



Exists Fuchsian group $\Gamma \sim \pi_1(\Omega)$

$$\Lambda(\gamma(\zeta)) = \Lambda(\zeta), \quad \gamma \in \Gamma$$

and

$$\Lambda(\zeta_1) = \Lambda(\zeta_2) \quad \Longleftrightarrow \quad \zeta_1, \zeta_2 \in \text{same orbit}$$

$$\Omega \simeq \mathbb{D}/\Gamma$$

Green Function

Regular compact $E \subset \mathbb{R}$

Domain $\Omega = \overline{\mathbb{C}} \setminus E, \quad z_0 \in \Omega$

Definition (Green function)

1. $G(\lambda)$ positive, harmonic in $\Omega \setminus \{z_0\}$
2. Continuous up to $\partial\Omega$ and $G(\lambda) = 0$ on $\partial\Omega$

Universal cover $\Lambda : \mathbb{D} \to \Omega$. Fuchsian group $\Gamma \sim \pi_1(\Omega)$

Let $\widetilde{G}(\zeta) = G(\Lambda(\zeta)), \quad |\zeta| < 1$

Let $\Lambda(\zeta_0) = z_0$

Let $$g(\zeta) = \prod_{\gamma \in \Gamma} \frac{\zeta - \gamma(\zeta_0)}{1 - \zeta\overline{\gamma(\zeta_0)}} C_\gamma = \prod_{\gamma \in \Gamma} \frac{\gamma(\zeta) - \zeta_0}{1 - \gamma(\zeta)\overline{\zeta}_0} \widetilde{C}_\gamma$$

Then $\widetilde{G}(\zeta) = -\ln|g(\zeta)|$

Martin function

Regular compact $E \subset \mathbb{R}$; Domain $\Omega = \overline{\mathbb{C}} \setminus E$, $x_0 \in E$

Definition (Martin function)

1. $M(\lambda)$ positive, harmonic in Ω
2. Continuous up to $\partial\Omega$ except at x_0
3. $M(\lambda) = 0$ on $\partial\Omega \setminus \{x_0\}$

May not be unique

Universal cover $\Lambda : \mathbb{D} \to \Omega$; Fuchsian group $\Gamma \sim \pi_1(\Omega)$

Let $\widetilde{M}(\zeta) = M(\Lambda(\zeta))$

Martin function

Regular compact $E \subset \mathbb{R}$; Domain $\Omega = \overline{\mathbb{C}} \setminus E$, $x_0 \in E$

Definition (Martin function)

1. $M(\lambda)$ positive, harmonic in Ω
2. Continuous up to $\partial\Omega$ except at x_0
3. $M(\lambda) = 0$ on $\partial\Omega \setminus \{x_0\}$

Universal cover $\Lambda : \mathbb{D} \to \Omega$;

Let $\widetilde{M}(\zeta) = M(\Lambda(\zeta))$, $x_0 = \Lambda(t_0)$

Integral representation $(|\zeta| < 1)$

$$\widetilde{M}(\zeta) = M(\Lambda(\zeta)) = \int\limits_{\mathbb{T}} \frac{1 - |\zeta|^2}{|t - \zeta|^2} \sigma(dt)$$

$$\sigma(\{t_0\}) = \lim_{\zeta \to t_0} \widetilde{M}(\zeta) \frac{|t_0 - \zeta|^2}{1 - |\zeta|^2}$$

Martin Function

$$\sigma(\{t_0\}) = \lim_{\zeta \to t_0} \widetilde{M}(\zeta) \frac{|t_0 - \zeta|^2}{1 - |\zeta|^2}$$

Theorem (Volberg-Yuditskii)

$\sigma(\{t_0\}) > 0 \iff \sigma$ *pure point*

$\sigma(\{t_0\}) = 0 \iff \sigma$ *is continuous singular*

Theorem

$$\sigma(\{t_0\}) > 0 \iff \sum_{\gamma \in \Gamma} |\gamma'(t_0)| < \infty$$

$$\sigma(\{t_0\}) = 0 \iff \sum_{\gamma \in \Gamma} |\gamma'(t_0)| = \infty$$

Martin Function

Integral representation $(\mathrm{Im}\lambda > 0)$

$$M(\lambda) = \frac{1}{\pi} \int\limits_{\mathbb{R}} \frac{\mathrm{Im}\lambda}{|x-\lambda|^2} M(x) dx + A \, \frac{\mathrm{Im}\lambda}{|x_0-\lambda|^2}, \quad A \geq 0. \tag{1}$$

Definition

We say $x_0 \in E$ is an Akhiezer-Levin point
If exists Martin function of x_0 with $A > 0$.

Integral representation $(|\zeta| < 1)$

$$\widetilde{M}(\zeta) = M(\Lambda(\zeta)) = \int\limits_{\mathbb{T}} \frac{1-|\zeta|^2}{|t-\zeta|^2} \sigma(dt) \tag{2}$$

Let $\quad \Lambda(t_0) = x_0$

Theorem

$$A > 0 \Longrightarrow \sigma(t_0) > 0 \Longleftrightarrow \sum_{\gamma \in \Gamma} |\gamma'(t_0)| < \infty.$$