Contractions between Hardy and Bergman spaces

Aleksei Kulikov

NTNU

July 7th 2022

Hardy and Bergman spaces

Definition

For a function f analytic in $\mathbb{D}$ and $0 < p < \infty$ we define its H^p-norm as

$$||f||_{H^p}^p = \lim_{r\to 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta.$$

Definition

For a function f analytic in $\mathbb{D}$ and $0 < p < \infty, \alpha > 1$ we define its A_α^p-norm as

$$||f||_{A_\alpha^p}^p = \int_{\mathbb{D}} (\alpha - 1)|f(z)|^p (1 - |z|^2)^\alpha \frac{dz}{\pi(1 - |z|^2)^2}.$$

Constants are chosen so that for $f(z) \equiv 1$ we have $||f|| = 1$.

Pointwise bounds

As usual for Banach spaces of analytic functions, pointwise evaluations are continuous in A^p_α and H^p

$$|f(z)|^p(1-|z|^2) \leq ||f||^p_{H^p},$$

$$|f(z)|^p(1-|z|^2)^\alpha \leq ||f||^p_{A^p_\alpha}.$$

Moreover, for fixed function f these quantities tend to 0 uniformly as $|z| \to 1$.

Embeddings between Hardy and Bergman spaces

From the pointwise bounds via the Hölder's inequality we get the embeddings between the Bergman spaces

Theorem (Hardy, Littlewood, 1927)

If $0 < p < q < \infty$ and $1 < \alpha < \beta < \infty$ are such that $\frac{p}{\alpha} = \frac{q}{\beta} = r$ then A^p_α is a subset of A^q_β and this embeddings is continuous.

It turns out that the H^r-norm is the limit of A^p_α-norms

$$||f||_{H^r} = \lim_{\alpha \to 1} ||f||_{A^{r\alpha}_\alpha}.$$

Passing to the limit in the above theorem, we get that H^r continuously embeds into $A^{r\alpha}_\alpha$ for all $\alpha > 1$, so it is natural to denote H^r by A^r_1.

It is important to note that this only proves the embedding and not the contractions between our spaces, since Hölder's inequality is never an equality here.

Coefficient estimates

Consider the embedding $H^p \to A^2_{2/p}$, $p < 2$. For a function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ we have

$$||f||^2_{A^2_{2/p}} = \sum_{n=0}^{\infty} \frac{|a_n|^2}{c_{2/p}(n)}, c_{2/p}(n) = \binom{n+2/p-1}{n}.$$

If $||f||_{A^2_{2/p}} \leq ||f||_{H^p}$ for all $f \in H^p$, then for all functions $f(z_1, \dots, z_k) = \sum a_{n_1,\dots,n_k} z_1^{n_1} \dots z_k^{n_k}$ we get

$$\sum \frac{|a_{n_1,\dots,n_k}|^2}{c_{2/p}(n_1) \dots c_{2/p}(n_k)} \leq ||f||^2_{H^p(\mathbb{D}^k)}.$$

Passing to the limit, we get similar inequality for the functions of infinitely many variables. It is crucial that $c_{2/p}(0) = 1$.

Main results

Theorem

Let $G : [0, \infty) \to \mathbb{R}$ be an increasing function. The maximum value of

$$\int\limits_{\mathbb{D}} G(|f(z)|^p(1 - |z|^2)) \frac{dz}{\pi(1 - |z|^2)^2}$$

among the functions with $||f||_{H^p} = 1$ is attained for $f(z) \equiv 1$.

Theorem

Let $G : [0, \infty) \to \mathbb{R}$ be a convex function. The maximum value of

$$\int\limits_{\mathbb{D}} G(|f(z)|^p(1 - |z|^2)^\alpha) \frac{dz}{\pi(1 - |z|^2)^2}$$

among the functions with $||f||_{A^p_\alpha} = 1$ is attained for $f(z) \equiv 1$.

Möbius invariance

The measure $dm(z) = \frac{dz}{\pi(1-|z|^2)^2}$ is invariant with respect to the linear fractional transformations $h(z) = c\frac{z-w}{1-z\bar{w}}, w \in \mathbb{D}, |c| = 1$.
With slight adjustments, the same is true for the Bergman spaces as well:
For a function $f \in A^p_\alpha$ the function

$$g(z) = f\left(c\frac{z-w}{1-z\bar{w}}\right)\frac{(1-|w|^2)^{\alpha/p}}{(1-z\bar{w})^{2\alpha/p}}$$

has the same A^p_α-norm and the same distribution of the function $|f(z)|^p(1-|z|^2)^\alpha$ with respect to the measure m.

True main result

Let f be an analytic function in $\mathbb{D}$ such that the function $u(z) = |f(z)|^p(1-|z|^2)^\alpha$ is bounded and $u(z) \to 0, |z| \to 1$.
Put $\mu(t) = m(A_t), A_t = \{z : u(z) > t\}$ and $t_0 = max_{z\in\mathbb{D}} u(z)$.

Theorem

The function $g(t) = t^{1/\alpha}(\mu(t)+1)$ is decreasing on $(0, t_0)$.

If $f(z) \equiv 1$ then $g(t) \equiv 1$ for $0 < t < 1$.

Proof of the monotonicity theorem

The proof consists of four steps. We begin with computing the derivative of $\mu(t)$:

$$-\mu'(t) = \int_{u=t} |\nabla u|^{-1} \frac{|dz|}{\pi(1-|z|^2)^2}.$$

This is based on the fact that ∇u is orthogonal to the curve $\partial A_t = \{z : u(z) = t\}$ and it's pointing inside of A_t.

Next step is a Cauchy–Bunyakovsky–Schwarz inequality

$$\left(\int_{\partial A_t} \frac{|dz|}{\sqrt{\pi}(1-|z|^2)}\right)^2 \leq \left(\int_{\partial A_t} |\nabla u|^{-1} \frac{|dz|}{\pi(1-|z|^2)^2}\right) \left(\int_{\partial A_t} |\nabla u||dz|\right).$$

Left-hand side is a square of the hyperbolic length, while the first term in the right-hand side is $-\mu'(t)$, so we have to understand the second one.

Let ν be an outward normal to ∂A_t. ∇u is parallel to it but it is pointing in the opposite direction. Thus, $|\nabla u| = -\nabla u \cdot \nu$.
For $z \in \partial A_t$ we have $u(z) = t$, therefore

$$\frac{|\nabla u|}{t} = \frac{|\nabla u|}{u} = -\frac{\nabla u \cdot \nu}{u} = -(\nabla \log u) \cdot \nu.$$

From this we get by Green's theorem

$$\int_{\partial A_t} |\nabla u||dz| = -t \int_{\partial A_t} (\nabla \log u) \cdot \nu |dz| = -t \int_{A_t} \Delta \log u(z) dxdy.$$

$\Delta \log u(z) = p\Delta \log |f(z)| + \alpha \Delta \log(1 - |z|^2) = -4\alpha \frac{1}{(1-|z|^2)^2}$.
Plugging this in we get

$$\int_{\partial A_t} |\nabla u||dz| = 4\pi\alpha t\, m(A_t).$$

Combining everything, we arrive at

$$-\mu'(t) \geq \frac{\ell(\partial A_t)^2}{4\pi\alpha t\, m(A_t)}.$$

Our last ingredient is an isoperimetric inequality

$$\ell(\partial A_t)^2 \geq 4\pi m(A_t) + 4\pi\, m(A_t)^2.$$

Using it, we get

$$-\mu'(t) \geq \frac{1 + m(A_t)}{\alpha t} = \frac{1 + \mu(t)}{\alpha t}.$$

Recall our goal function $g(t) = t^{1/\alpha}(\mu(t) + 1)$. We have

$$g'(t) = t^{1/\alpha}\left(\frac{\mu(t) + 1}{\alpha t} + \mu'(t)\right) \leq 0.$$

Thank you for your attention!