Orthogonal rational functions with real poles, root asymptotics, GMP matrices

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Jacobi matrices and orthogonal polynomials

- Let μ be a compactly supported nontrivial probability measure. Define $\{p_n\}_{n=0}^{\infty}$ to be the orthonormal polynomials formed by applying Gram-Schmidt in $L^2(\mu)$ to $\{z^n\}_{n=0}^{\infty}$.
- For $\mathrm{supp}(\mu) \subset \mathbb{R}$, the orthonormal polynomials $\{p_n\}_{n=0}^{\infty}$ satisfy a three term recurrence relation:

$$xp_n(x) = a_{n+1}p_{n+1}(x) + b_{n+1}p_n(x) + a_n p_{n-1}(x), n \geq 1$$
$$xp_0(x) = a_1 p_1(x) + b_1 p_0(x)$$

  for $\{a_n, b_n\}_{n=1}^{\infty} \in (0, \infty) \times \mathbb{R}$ bounded sequences.
- The operator of multiplication by x, $T_{x,\mathrm{d}\mu}$ has a tridagonal matrix representation in the basis $\{p_n\}_{n=0}^{\infty}$:

$$J = \begin{pmatrix} b_1 & a_1 & & \\ a_1 & b_2 & a_2 & \\ & a_2 & \ddots & \ddots \\ & & \ddots & \end{pmatrix}$$

  called a bounded Jacobi matrix.

Universal bounds

- Let $\mathsf{E} = \operatorname{ess\,sup} \mu$ be the essential support of μ and $G_{\mathsf{E}}(\cdot, \infty)$ be the Green function for $\overline{\mathbb{C}} \setminus \mathsf{E}$ at ∞.
- We have the universal inequality

$$\liminf_{n\to\infty} |p_n(z)|^{1/n} \geq e^{G_{\mathsf{E}}(z,\infty)}$$

  for z away from the convex hull of E.
- We have another universal inequality in terms of the coefficients of the Jacobi matrix

$$\limsup_{n\to\infty} \left(\prod_{\ell=1}^{n} a_\ell \right)^{1/n} \leq \operatorname{cap}(\sigma_{\mathrm{ess}}(J))$$

- The latter inequality can be related back to the p_n by the identity $p_n(z) = \frac{1}{\prod_{\ell=1}^{n} a_\ell} z^n + l.o.t.$

Stahl-Totik Regularity

- Equality in

$$\lim_{n\to\infty} |p_n(z)|^{1/n} = e^{G_{\mathsf{E}}(z,\infty)} \tag{1}$$

  is called Stahl-Totik regularity for the measure μ.
- A Jacobi matrix is said to be regular for a set E if $\sigma_{\mathrm{ess}}(J) = \mathsf{E}$ and we have

$$\lim_{n\to\infty} \left(\prod_{\ell=1}^{n} a_\ell \right)^{1/n} = \mathrm{cap}(\sigma_{\mathrm{ess}}(J)). \tag{2}$$

  It was first studied for the case $\mathsf{E} = [-2, 2]$ by Ullman 1972.
- (1) $\iff$ (2).

Orthogonal rational functions

- In our setting, we start with a nontrivial probability measure μ supported on $\overline{\mathbb{R}}$, and a finite sequence $\mathbf{C} = (\mathbf{c}_1, \ldots, \mathbf{c}_{g+1})$ with $\mathbf{c}_k \in \overline{\mathbb{R}} \setminus \mathrm{supp}(\mu)$. We denote $\mathsf{E} = \mathrm{ess\,sup}(\mu)$.
- Our sequence of orthonormal functions come from orthonormalizing the sequence $\{r_n\}_{n=0}^{\infty}$, where $r_0 = 1$ and for $n = j(g+1) + k$, where $1 \leq k \leq g+1$

$$r_n(z) = \begin{cases} \frac{1}{(\mathbf{c}_k - z)^{j+1}} \\ z^{j+1} \end{cases}$$

  Call the sequence $\{\tau_n\}_{n=0}^{\infty}$.
- Orthogonal polynomials are exactly the case $\mathrm{supp}(\mu) \subset \mathbb{R}$, $\mathbf{C} = (\infty)$, and $g = 0$.

Universal inequality for τ_n

We define

$$\mathcal{G}_{\mathsf{E}}(z, \mathbf{C}) = \begin{cases} \frac{1}{g+1} \sum_{k=1}^{g+1} G_{\mathsf{E}}(z, \mathbf{c}_k) & \mathsf{E} \text{ is not polar} \\ +\infty & \mathsf{E} \text{ is polar} \end{cases}$$

Then,

Theorem

For all $z \in \overline{\mathbb{C}} \setminus \overline{\mathbb{R}}$,

$$\liminf_{n \to \infty} |\tau_n(z)|^{1/n} \geq e^{\mathcal{G}_{\mathsf{E}}(z, \mathbf{C})}.$$

Universal inequality on leading coefficients

- Let $\mathcal{L}_n = \operatorname{span}\{r_\ell : 0 \le \ell \le n\}$. By the Gram-Schmidt process, there is a $\kappa_n > 0$ with $\tau_n - \kappa_n r_n \in \mathcal{L}_{n-1}$. We refer to κ_n as a leading coefficient.
- We define

$$\gamma_{\mathsf{E}}^k = \begin{cases} \lim_{z \to \mathbf{c}_k} (G_{\mathsf{E}}(z, \mathbf{c}_k) + \log|z - \mathbf{c}_k|), & \mathbf{c}_k \neq \infty \\ \lim_{z \to \mathbf{c}_k} (G_{\mathsf{E}}(z, \mathbf{c}_k) - \log|z|), & \mathbf{c}_k = \infty \end{cases}$$

and

$$\log \lambda_k = \begin{cases} \gamma_{\mathsf{E}}^k + \sum_{\substack{1 \le \ell \le g+1 \\ \ell \neq k}} G_{\mathsf{E}}(\mathbf{c}_k, \mathbf{c}_\ell) & \mathsf{E} \text{ is not polar} \\ +\infty & \mathsf{E} \text{ is polar} \end{cases}$$

Then:

Theorem

For all $1 \le k \le g+1$, *for the subsequence* $n(j) = j(g+1) + k$,

$$\liminf_{j \to \infty} \kappa_{n(j)}^{1/n(j)} \ge \lambda_k^{1/(g+1)}. \tag{3}$$

Defining C regularity

Theorem

TFAE:

1. *For some* $1 \leq k \leq g+1$, *for the subsequence* $n(j) = j(g+1)+k$,
$$\lim_{j\to\infty} \kappa_{n(j)}^{1/n(j)} = \lambda_k^{1/(g+1)};$$
2. *For all* $1 \leq k \leq g+1$, *for the subsequence* $n(j) = j(g+1)+k$, $\lim_{j\to\infty} \kappa_{n(j)}^{1/n(j)} = \lambda_k^{1/(g+1)}$;
3. $$\lim_{n\to\infty} \left(\prod_{\ell=1}^{g+1} \kappa_{n+\ell} \right)^{1/n} = \left(\prod_{k=1}^{g+1} \lambda_k \right)^{1/(g+1)}$$
4. *For q.e.* $z \in \mathsf{E}$, *we have* $\limsup_{n\to\infty} |\tau_n(z)|^{1/n} \leq 1$;
5. *For some* $z \in \mathbb{C}_+$, $\limsup_{n\to\infty} |\tau_n(z)|^{1/n} \leq e^{\mathcal{G}_{\mathsf{E}}(z,\mathbf{C})}$;
6. *For all* $z \in \mathbb{C}$, $\limsup_{n\to\infty} |\tau_n(z)|^{1/n} \leq e^{\mathcal{G}_{\mathsf{E}}(z,\mathbf{C})}$;
7. *Uniformly on compact subsets of* $\mathbb{C}\setminus\mathbb{R}$, $\lim_{n\to\infty} |\tau_n(z)|^{1/n} = e^{\mathcal{G}_{\mathsf{E}}(z,\mathbf{C})}$.

Stahl-Totik regularity and C regularity

Theorem

Let $\mathbf{C}_1, \mathbf{C}_2$ be two finite sequences of elements from $\overline{\mathbb{R}} \setminus \operatorname{supp}\mu$, not necessarily of the same length. Then μ is $\mathbf{C}_1$-regular if and only if it is $\mathbf{C}_2$-regular.

Since Stahl-Totik regularity is the case $\mathbf{C} = (\infty)$, this immediately yields:

Corollary

Let $\operatorname{supp}\mu \subset \mathbb{R}$. Let $\mathbf{C}$ be a finite sequence of elements from $\overline{\mathbb{R}} \setminus \operatorname{supp}\mu$. Then μ is $\mathbf{C}$-regular if and only if it is Stahl–Totik regular.

Conformal invariance

Conformal invariance follows from the previous Corollary.

Theorem

Let $f \in PSL(2,\mathbb{R}) \rtimes \{\mathrm{id}, z \mapsto -z\}$. If μ is a Stahl–Totik regular measure on $\mathbb{R}$ and $\infty \notin \operatorname{supp}(f_*\mu)$, then the pushforward measure $f_*\mu$ is also Stahl–Totik regular.

Weak convergence of zero counting measure

Define

$$\nu_n = \frac{1}{n} \sum_{w:\tau_n(w)=0} \delta_w$$

and for nonpolar E,

$$\rho_{\mathsf{E},\mathbf{C}} = \frac{1}{g+1} \sum_{j=1}^{g+1} \omega_{\mathsf{E}}(\mathrm{x}, \mathbf{c}_j).$$

Theorem

Let μ be a probability measure on $\overline{\mathbb{R}}$. Assume that E is not a polar set.

- (a) *If μ is* **C** *regular, then* $\text{w-lim}_{n\to\infty} \nu_n = \rho_{\mathsf{E},\mathbf{C}}$.
- (b) *If* $\text{w-lim}_{n\to\infty} \nu_n = \rho_{\mathsf{E},\mathbf{C}}$, *then μ is* **C** *regular or there exists a polar set $X \subset \mathsf{E}$ such that $\mu(\overline{\mathbb{R}} \setminus X) = 0$.*

GMP matrices

- For a sequence $\mathbf{C} = (\mathbf{c}_1, \ldots, \mathbf{c}_{g+1})$ with $\mathbf{c}_k = \infty$, we call the matrix representation for $T_{x,d\mu}$ with respect to the basis of ORF a GMP matrix. They were introduced in a Yuditskii '18.
- GMP matrices are tridiagonal block matrices

$$A = \begin{bmatrix} B_0 & A_0 & & & \\ A_0^* & B_1 & A_1 & & \\ & A_1^* & B_2 & A_2 & \\ & & A_2^* & \ddots & \ddots \\ & & & \ddots & \end{bmatrix}$$

where B_0 is a $k \times k$ matrix, A_0 is a $k \times (g+1)$ matrix, and A_j, B_j for $j \geq 1$ are $(g+1) \times (g+1)$ matrices.
- GMP matrices have the property that resolvents at the $\mathbf{c}_\ell$, $\ell \neq k$ also have the above form.
- For the sequence $\mathbf{C} = (\infty)$, the matrix representation for $T_{x,d\mu}$ is a Jacobi matrix.

Regular GMP matrices

- A relation between the κ_n and the nonzero entry on the outermost diagonal of the associated GMP matrix allows us to find a notion of regularity of a measure purely in terms of the coefficients of the GMP matrix.

Theorem

Fix a probability measure μ with $\operatorname{supp}\mu \subset \mathbb{R}$ and a sequence $\mathbf{C} = (\mathbf{c}_1, \dots, \mathbf{c}_{g+1})$ with $\mathbf{c}_k = \infty$. Then

$$\limsup_{j\to\infty} \left(\prod_{\ell=1}^{j} \beta_\ell \right)^{1/j} \le \lambda_k^{-1}. \tag{4}$$

Moreover, the measure μ is Stahl–Totik regular if and only if

$$\lim_{j\to\infty} \left(\prod_{\ell=1}^{j} \beta_\ell \right)^{1/j} = \lambda_k^{-1}. \tag{5}$$

Finite gap sets and the isospectral torus

- We specialize to finite gap sets,

$$\mathsf{E} = [\mathbf{b}_0, \mathbf{a}_0] \setminus \bigcup_{k=1}^{g} (\mathbf{a}_k, \mathbf{b}_k),$$

  and denote by $\mathcal{T}_{\mathsf{E}}^{+}$ the set of almost periodic half-line Jacobi matrices with $\sigma_{\mathrm{ess}}(J) = \sigma_{\mathrm{ac}}(J) = \mathsf{E}$. This set is called the isospectral torus.
- We consider the metric on bounded Jacobi matrices given by

$$d(J, \tilde{J}) = \sum_{k=1}^{\infty} e^{-k} (|a_k - \tilde{a}_k| + |b_k - \tilde{b}_k|). \tag{6}$$

  as well as the distance to $\mathcal{T}_{\mathsf{E}}^{+}$,

$$d(J, \mathcal{T}_{\mathsf{E}}^{+}) = \inf_{\tilde{J} \in \mathcal{T}_{\mathsf{E}}^{+}} d(J, \tilde{J}) = \min_{\tilde{J} \in \mathcal{T}_{\mathsf{E}}^{+}} d(J, \tilde{J}).$$

The Nevai and Cesàro–Nevai conditions and Simon's Conjecture

- Denote by S_+ the right shift operator on $\ell^2(\mathbb{N})$, $S_+ e_n = e_{n+1}$. The condition
$$d((S_+^*)^m J S_+^m, \mathcal{T}_{\mathsf{E}}^+) \to 0$$
as $m \to \infty$ is called the Nevai condition.
- Remling 2011, the Nevai condition implies regularity. The converse is false. However, Simon 2009 conjectured

Theorem

If $\mathsf{E} \subset \mathbb{R}$ *is a compact finite gap set and* J *is a regular Jacobi matrix with* $\sigma_{\mathrm{ess}}(J) = \mathsf{E}$*, then*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{m=1}^{N} d((S_+^*)^m J S_+^m, \mathcal{T}_{\mathsf{E}}^+) = 0. \tag{7}$$

where (7) *is the Cesàro–Nevai condition.*

Special cases

- Proved by Simon 2009 in the special case when E is the spectrum of a periodic Jacobi matrix with all gaps open.
- The method of proof relied on the periodic discriminant and techniques from Damanik–Killip–Simon 2010.
- Proved by Kruger 2010 for the case $\inf_n a_n > 0$ using completely different methods.

The Ahlfor's function and the Yuditskii discriminant

- Our paper proves the general case extending Simon's methods and uses techniques of Yuditskii 2018; in particular GMP matrices and the Ahlfor's function.
- The Ahlfor's function Ψ for $\overline{\mathbb{C}} \setminus \mathsf{E}$ is the analytic function $\Psi : \overline{\mathbb{C}} \setminus \mathsf{E} \to \mathbb{D}$ with $\Psi(\infty) = 0$ that maximizes $\mathrm{Re}(\Psi'(\infty))$. It has one zero $\mathbf{c}_k \in (\mathbf{a}_k, \mathbf{b}_k)$ for each $1 \leq k \leq g$; with ∞, these are the only zeros.
- Our discriminant is

$$\Delta_{\mathsf{E}}(z) = \Psi(z) + \frac{1}{\Psi}.$$

It is a rational function with poles at the $\mathbf{C}_{\mathsf{E}} = (\mathbf{c}_1, \dots, \mathbf{c}_g, \infty)$:

$$\Delta_{\mathsf{E}}(z) = \lambda_{g+1} z + d + \sum_{k=1}^{g} \frac{\lambda_k}{\mathbf{c}_k - z}.$$

Proving Simon's Conjecture

- We show: regularity of $J \implies$ regularity of A and its resolvents $\implies$ the block Jacobi matrix $\mathcal{J} = \Delta_{\mathsf{E}}(A)$ is regular in the sense of Damanik-Pushnitski-Simon $\implies$ $\mathcal{J}$ satisfies a Cesàro-Nevai condition.
- By modifying arguments of Yuditskii 2018, this implies J satisfies the Cesàro-Nevai condition.

Thank you!