Recent developments in the spectral theory of Schrödinger operators, Jacobi and CMV matrices reported in top-ranked mathematical journals and our own new level of understanding of the function theory in infinitely connected domains provide a solid basis for tackling some old and new problems, including: 1) Kotani-Last problem, 2) Killip-Simon problem for general finite-gap sets, 3) Mixed inverse spectral problems for reflectionless Jacobi matrices, 4) Parametric description of spectral surfaces of periodic multi-diagonal operators, 5) Widom condition for resolvent domains of Schrödinger operators. The first two were posed by the leading experts in the field, and we concentrate on their description.
Kotani-Last problem requires proof that the presence of an absolutely continuous component in the spectrum of an ergodic operator implies that it is almost periodic. According to Avila, "this problem has been for a while, and became a central topic of the theory, after recent popularization (by Simon, Jitomirskaya, and Damanik)". He answered negatively to this conjecture. Naturally, it is important and highly challenging not only to prove or disprove the conjecture, but also to explain this interesting phenomenon. There are at least two programs related to the subject: by Kotani, on Grassmann manifold and spectral theory of 1-D Schrödinger operators, and by Remling, on reflectionless Jacobi matrices. Our approach is dual to that of Avila and deals with methods of the inverse spectral theory. Consider a real compact in a generic position in which (1) all reflectionless Jacobi matrices with corresponding spectrum have no singular component, and (2) its complement is a Widom domain. We claim that such operators are ergodic, and there is a kind of tumbler with two positions: Direct Cauchy Theorem holds in the domain, or it fails. In the first case, all reflectionless matrices are almost periodic, and we expect that in the second case all of them are not. We can provide examples of such compact sets thus showing classes of ergodic matrices with purely a.c. spectrum without almost periodicity.
The Killip-Simon theorem is probably one of the main achievements in the spectral theory of Jacobi matrices and orthogonal polynomials in the last decade. In collaboration with Damanik, they generalized this theorem to the periodic case. Although several important partial results were obtained, the problem of an extension of Damanik-Killip-Simon theorem to the general finite gap non-periodic case remains open. The proof of the original theorem is based on Sum Rules, and its generalization to the periodic case on the so-called "Magic Formulas". We suggest a way to find new Sum Rules and a counterpart of the Magic Formula in the non-periodic case.