Analysis II

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Contents

1 Metric spaces

1.1 Definitions

Definition 1 *Let X be a set. A* **metric** *on X is a mapping*

$$d : X \times X \to \mathbf{R}$$

with the following properties

1. *$d(x, y) \geq 0$ and $d(x, y) = 0 \iff x = y$;*
2. *$d(x, y) = d(y, x) \quad (x, y \in X)$;*
3. *$d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$ (The triangle inequality).*

$d(x, y)$ is the **distence** *between the points x and y.*

A **metric space** is a set X together with a metric.

Examples I. $\mathbf{R}$ is a metric space with the mapping

$$d(x, y) = |x - y|.$$

II. Each subset A of a metric space is itself a metric space (one simply take the restriction of the metric to A. The is called the **induced metric**).

Definition—normed spaces Let V be a real vector space. A **norm** on V is a mapping $\| \ \| : V \to \mathbf{R}$, so that

1. $\|x\| \geq 0$ and $\|x\| = 0 \iff x = 0$;
2. $\|\lambda x\| = |\lambda| \|x\| \quad (\lambda \in \mathbf{R}, x \in V)$;
3. $\|x + y\| \leq \|x\| + \|y\| \quad (x, y \in V)$.

A **normed vector space** is a pair $(V, \| \ \|)$, whereby V is a vector space and $\| \ \|$ is a norm thereon.

We remark that each normed space V is a metric space with the distance function

$$d(x, y) = \|x - y\|.$$

Hence the following examples are further examples of metric spaces.

Examples I. Euclidean spaces. A **euclidean space** is a vector space V with scalar product $(\ |\ )$. (*The* example is $\mathbf{R}^n$ with sclar product

$$(x|y) = \sum_{i=1}^{n} \xi_i \eta_i).$$

The mapping

$$\|x\| \mapsto \sqrt{(x|x)}$$

is then a norm on V. In the above case, this norm is the usual euclidean distance on $\mathbf{R}^n$.
II. Subsets of $\mathbf{R}^n$. The metric of I. induces the structure of a metric space on each subset of $\mathbf{R}^n$.
III. The supremum norm and the ℓ^1-norm on $\mathbf{R}^n$ are defined as follows:

$$\|x\|_\infty = \max_i \{|\xi_i|\};$$

$$\|x\|_1 = \sum_{i=1}^{n} |\xi_i|.$$

IV. The supremum norm $\|\ \|_\infty$ on $C([a,b])$. This is the mapping

$$f \mapsto \sup\{|f(x)| : x \in [a,b]\}.$$

Definitions If x is an element of a metric space and $\epsilon > 0$, then the set

$$U(x,\epsilon) = \{y \in X : d(x,y) < \epsilon\}$$

is the ϵ-**neighbourhood** of x or **the open ball with centre** x **and radius** ϵ. More generally the set N is a **neighbourhood** of x, if $\epsilon > 0$ exists so that $U(x,\epsilon) \subset N$. We denote the family of all neighbourhoods of x with $\mathcal{N}(\S)$.

A subset U of X is **open**, if U is a neighbourhood of each point $x \in U$ (i.e. U is a union of open balls).

A subset A is **closed** if $X \setminus A$ is open. This definition can be reformulated in the following way: $U(x,\epsilon) \cap A \neq \emptyset$ fof each $\epsilon > 0$, then $x \in A$.

One can easily verify that the family τ_d of open subsets has the following properties:

1. $\emptyset$ and X are open;

2. if U and V are open, then so is $U \cap V$;

3. the union of a family of open sets is open.

(These properties are the motivation for the definition of general topological spaces—see the Appendix).

1.2 Convergence

A sequence (x_n) in a metric space X **converges to** $x \in X$, if for each $\epsilon > 0$ there exists $N \in \mathbf{N}$ so that $d(x_n, x) < \epsilon$, whenever $n > N$. A sequence is **Cauchy**, if to each $\epsilon > 0$ there exists $N \in \mathbf{N}$ with $d(x_m, x_n) < \epsilon$ whenever $m, n > N$.

A further characterisation of closed sets is: A is closed if and only if for each sequence in A, which converges in X the limit is in A.
PROOF. Exercise.

■

1.3 Completeness

Just as in the case of $\mathbf{R}$ one shows that each convergent sequence is Cauchy. The converse is not true for general metric spaces. (consider for example the sequence $(\frac{1}{n})$ in $]0,1[$). This leads to the following definition

Definition A metric space is **complete**, if each Cauchy sequence therein converges.

For our purposes, the following characterisation of completenss is sufficient:

Proposition 1 *Each closed subset of* $\mathbf{R}^n$ *is complete.*

1.4 Continuity of functions

Just as in the case of functions on $\mathbf{R}$, we can use the metric to define various continuity properties for functions $f : X \to Y$ (X and Y metric spaces.) f is **continuous at the point** $x \in X$, if $f(x_n) \to f(x)$, whenever $x_n \to x$. (This is equivalent to the condition: for each $\epsilon > 0$ there exists $\delta > 0$, so that $d(f(y), f(x)) < \epsilon$, whenever $d(x, y) < \delta$). f is **continuous on** X, if it is continuous at each point.

f is **uniformly continuous**, if for each $\epsilon > 0$ there exists a δ, so that for each pair $x, y \in X$, $d(f(x)f(y)) < \epsilon$, whenever $d(x, y) < \delta$.

f is **Lipschitz-continuous**, if $K > 0$ exists with the property that

$$d(f(x), f(y)) \leq K d(x, y)$$

for $x, y \in X$. If such a K with $K < 1$ exists then f is a **contraction**.

The following result is the basis for many iteration methods for solving equations:

Proposition 2 *Banach's fixed point theorem Let X be a non-empty complete metric space and $f : X \to X$ a contraction. We choose an initial value x_0 and define a sequence (x_n) recursively as follows:*

$$x_1 = f(x_0), \quad x_2 = f(x_1), \quad \ldots, \quad x_{n+1} = f(x_n).$$

Then (x_n) converges to a fixed point of f i.e. an element x with $f(x) = x$. Furthermore, x is the unique fixed point of f.

PROOF. Exercise.

■

1.5 Compactness

A metric space X is **sequentially compact**, if each sequence (x_n) has a convergent subsequence. It is clear that a closed subspace of a sequentially compact space is also sequentially compact. We can then easily obtain the following version of the theorem of Bolzano-Weierstraß.

Proposition 3 *A subset A of $\mathbf{R}^n$ is sequentially compact if and only if it is closed and bounded.*

PROOF. If A is not closed or not bounded we can find a sequence (x_n) in A for which no subsequence converges in A.

For the converse, it suffices to consider the case $A = \prod_i [a_i, b_i]$. This is proved as in the one-dimensional case. Alternatively, one can use the following fact:

■

Proposition 4 *Satz Let X and Y be sequentially compact metric spaces. $X \times Y$ is sequentially compact.*

PROOF. Let $((x_n, y_n))$ be a sequence in $X \times Y$. (x_n) has a convergent subsequence (x_{n_k}). (y_{n_k}) has a convergent subsequence $(y_{n_{k_r}})$. Then $(x_{n_{k_r}}, y_{n_{k_r}})$ is a convergent subsequence of the original one.

■

Proposition 5 *Let X be sequentially compact and $f : X \to Y$ continuous. Then $f(X)$ is sequentially compact.*

PROOF. Let (y_n) be a sequence in $f(X)$. Each y_n has the form $f(x_n)$, where (x_n) is a sequence in X. (x_n) has a convergent subsequence (x_{n_k}) in X. (y_{n_k}) is then a convergent subsequence of (y_n).

■

Proposition 6 *Let X be a sequentially compact metric space. Then a subset X_0 of X is sequentially compact if and only if it is closed in X.*

PROOF. Exercise.

■

Proposition 7 *Let f be a continuous, real-valued function on a sequentially compact space X (in particular, on a closed, bounded subset of $\mathbf{R}^n$). Then f is bounded and there are points x_0 and x_1, so that*

$$f(x_0) = \sup\{f(x) : x \in X\} \qquad f(x_1) = \inf\{f(x) : x \in X\}.$$

PROOF. As in the one-dimensional case.

■

Proposition 8 *Let f be a continuous function from a sequentially compact metric space X into a metric space Y. Then f is uniformly continuous.*

PROOF. Once again, the proof is as in the case of functions on closed, bounded intervals.

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1.6 Exercises:

Exercise: Let $f : X \to Y$ be continuous. Show that $\Gamma(f)$ and X are homeomorphic.

In the next four exercises, X is a separable metric space.

Exercise: 1) If $(U_i)_{i \in I}$ is a disjoint family of open subsets, then I is countable. 2) The set of isolated points of X is countable. (x is **isolated** in X, if $\{x\}$ is open).

Exercise: Each subspace Y of X is separable.

Exercise: Each open covering of X has a countable subcovering.

Exercise: If $f : X \to \mathbf{R}$ is a genral function, then the set

$$\{y \in X : \lim_{x \to y, x \neq y} f(x) \text{ exists and is distinct from } f(y)\}$$

is countable. (For each couple p, q of rational numbers with $p < q$ the set $\{y \in X : f(y) \leq p < q \leq \lim_{x \to y, x \neq y} f(x)\}$ consists of isolated points).

Exercise: Construct a metric d on $\mathbf{R} \setminus \mathbf{Q}$ so that $(\mathbf{R} \setminus \mathbf{Q}, d)$ is complete and separable.

Exercise: Show that $\| \|_1$ and $\| \|$ are norms on $\mathbf{R}^n$.

Exercise: Show that the mapping

$$\| \|_p : x \mapsto \left(\sum_{i=1}^{n} |\xi_i|^p \right)^{1/p}$$

is a norm on $\mathbf{R}^n$ $(1 < p < \infty)$.

Exercise: Let (X, d) be a metric space, A a non-empty set. Show that the mapping

$$d_A : x \mapsto \inf\{d(x, y) : y \in A\}$$

is Lipschitz-continuous. What is its Lipschitz constant? What is the set of its zeroes?

Exercise: If X is a metric space and $A(\subset X)$ closed. Then

$$A = [d_A < \frac{1}{n}] \quad ([d_A < \frac{1}{n}] := \{x \in X : d_A(x) < \frac{1}{n}\})$$

for each n.

Exercise: Do we have the equality $(X, d) : \overline{[d(x, .) < 1]} = [d(x, .)] \leq 1]$ in a general metric space?

Exercise: Let X be a metrix space, $M := P(X)$. Is $d(A, B) := \sup\{d_A(x) : x \in B\}$ a metrix on M?

Exercise: Let (X, d) be a metric space, $f : \mathbf{R}_+ \to \mathbf{R}_+$ so that

$$f(0) = 0 \quad f(x + y) \leq f(x) + f(y)$$

and f is monotone increasing. Show that: $\overline{d} = f \circ d$ is also a metric on X.

Show that if f is continuous, then $x_n \to x$ for $d \iff x_n \to x$ for $\overline{d}$.

(Examples: $f(x) = \dfrac{x}{1 + x}$ resp. $f(x) = \min(x, 1)$).

Exercise: Let V_1 and V_2 be vector spaces with norms $\| \|_1$ resp. $\| \|_2$ and let $f : V_1 \to V_2$ be linear. Show that f is continuous $\iff$ it is Lipschitz-continuous.

Exercise: Let X, Y, Z be subsets of $\mathbf{R}^n$. Show that $X \sim Y$ and $Y \sim Z$ imply $X \sim Z$. ($\sim$ means "are homeomorphic").

Exercise: Show that the cylinder and the Möbius strip are not homeomorphic. (intuitive proof).

Exercise: Consider the following subset of $\mathbf{R}$ resp. $\mathbf{R}^2$:
$\{\frac{1}{m} + \frac{1}{n} : m, n \in \mathbf{N}\}$; c) $\{(\frac{1}{m}, \frac{1}{n}) : m, n \in \mathbf{N}\}$; d) $\{(t, \cos\frac{1}{t}) : t \in \mathbf{R}_+\}$. Characterise (i) the cluster points (ii) the condensation points (iii) the boundary points.

Exercise: Which of the following relationships are valid?

1. $\overline{A \cup B} = \overline{A} \cup \overline{B}$;
2. $\overline{A \cap B} = \overline{A} \cap \overline{B}$;
3. $(A \cup B)^o = A^o \cup B^o$;
4. $(A \cap B)^o = A^o \cap B^o$.

Exercise: An inverse metric on a set X is a mapping d from $X \times X$ in $\mathbf{R}^+$, so that

1. $d(x, x) = 0$;
2. $d(x, y) = d(y, x)$;
3. $d(x, z) \geq d(x, y) + d(y, z)$.

Give an example of such a mapping.

Exercise: Let $f : \mathbf{R}^n \to \mathbf{R}^n$ be a Lipschitz continuous function with constant < 1. Show that $g = \mathrm{Id} - f$ is injective.

Exercise: Show that g is surjective.

2 Curves and surfaces in $\mathbf{R}^n$, curvilinear integrals.

2.1 Definitions

Definition A **curve** in $\mathbf{R}^n$ is a continuous mapping on an interval I with values in $\mathbf{R}^n$. We employ typically the letter c for curves and write $c = (c_1, \dots, c_n)$, where c_i is the i-th component of c.

Examples I. The circles is the curve

$$c(t) = (r \cos t, r \sin t)$$

with domain of definition $[0, 2\pi]$. $r > 0$ is the **radius**.
II. A **straight line** in $\mathbf{R}^n$ is a curve of the form

$$c(t) = a + tb,$$

whereby $a, b \in \mathbf{R}^n$, $b \neq 0$.
III. A **helix** in $\mathbf{R}^3$ is a curve with parametrisation

$$c(t) = (a \cos t, a \sin t, bt).$$

Definition If $c : I \to \mathbf{R}^n$, we say that c is **differentiable resp. continuously differentiable**, if the same holds for each jedes c_i.

$$c' = (c'_1, \dots, c'_n)$$

is then the **derivative** of c.

We say similarly that c is integrable if this holds for each c_i and then we put

$$\int_a^b c(t)\, dt = \left(\int_a^b c_1(t)\, dt, \dots, \int_a^b c_n(t)\, dt \right).$$

In this connection the following inequality is useful:

$$\left\| \int_a^b c(t)\, dt \right\| \leq \int_a^b \|c(t)\|\, dt.$$

($\| \ \|$ can be any norm on $\mathbf{R}^n$).

PROOF. We remark that the integral $\int_a^b c(t)\,dt$ is the limit of Riemann sums of the form

$$\sum_k c(\xi_k)(t_k - t_{k-1}).$$

As a consequence of the triangle inequality

$$\left\| \sum_k c(\xi_k)(t_k - t_{k-1}) \right\| \leq \sum \|c(\xi_k)\|(t_k - t_{k-1}).$$

The right hand side is a Riemann sum for $\int_a^b \|c(t)\|\,dt$.

■

Example If c and d are differentiable, then so is the scalar function $t \mapsto (c(t)|d(t))$ and

$$\frac{d}{dt}(c(t)|d(t)) = (c'(t)|d(t)) + (c(t)|d'(t)).$$

(This is proved as in the case of the product rule in $\mathbf{R}$).

Arc length Let $c : [a, b] \to \mathbf{R}^n$ be a curve. One defines the **arc length** of c as follows: Consider first the partition $\mathcal{P}$

$$a = t_0 < t_1 < \cdots < t_n = b$$

and form the sum

$$L_{\mathcal{P}} = \sum_{k=1}^{n} \|c(t_i) - c(t_{i-1})\|.$$

The curve is **rectifiable** (with length L), if for each $\epsilon > 0$ there exists $\delta > 0$, so that for each partition with $\max |t_i - t_{i-1}| < \delta$ we have

$$|L_{\mathcal{P}} - L| < \epsilon.$$

Example If the curve c is continuously differentiable, then c is rectifiable and

$$L = \int_a^b \|c'(t)\|\,dt.$$

As a concrete example one calculates immediatley that the length of the circle with radius r is equal to $2\pi r$.

The tangent vector Let $c : I \to \mathbf{R}^n$ be a smooth curve. If $c'(t_0) \neq 0$, then t_0 is a **regular point** on the curve. Otherwise, t_0 is a **singular point**. For example, the origin is a singularit of the curve $c(t) = (t^2, t^3)$ (Neil's Parabola).

The curve is **smooth**, if each $t \in [a, b]$ is a regular point.

If t_0 is a regular point, then

$$\mathbf{T}_c(t_0) = \frac{c'(t)}{\|c''(t_0)\|}$$

is a **unit tangent vector** to the curve at the point t_0. The straight line

$$t \mapsto c(t_0) + tc'(t_0)$$

is then the **tangent** to the curve at the position t_0.

If c and d are curves with $c(t_0) = d(t_1)$ (i.e. the corresponding point is a point of intersection of the curves, then the angle between c and d at this point is the angle θ, whereby

$$\cos\theta = (c'(t_0)|d'(t_1)).$$

Definition—Change of parameter Let $c : [a, b] \to \mathbf{R}^n$ and $d : [a_1, b_1] \to \mathbf{R}^n$ be two curves. A change of parameter between c and d is a continuously differentiable, bijective mapping

$$\phi : [a, b] \to [a_1, b_1]$$

with $\phi'(t) \neq 0$ for t, so that $c = d \circ \phi$. (ϕ^{-1} is then also continuously differentiable).

If $\phi'(t) > 0$ for each t, then ϕ is **direct**, otherwise it is indirect.

Remark I. If ϕ is direct, then

$$\mathbf{T}_c(t) = \mathbf{T}_d(\phi(t)) \quad (t \in [a, b]).$$

(For indirect mappings we have

$$\mathbf{T}_c(t) = -\mathbf{T}_d(\phi(t))).$$

II. Arc length: We have $L(c) = L(d)$.

Arc length parametrisation Let c be a smooth curve. We write

$$s = \phi(t) = \int_a^t \|c'(u)\| \, du.$$

ϕ is a change of parametrisation and we write $c = \gamma \circ \phi^{-1}$, also $\gamma(s) = c(t)$. γ is the **parametrisation by arc length** of c. For curves γ with arc-length parametrisation, we have

$$T_\gamma(s) = \gamma'(s).$$

2.2 Curvilinear integrals

Let $a_1, \ldots, a_n$ be continuous functions on a subset U of $\mathbf{R}^n$, c a continuously differentiable curve in U. We define the **curvilinear integral** $\int_c a_1(x)\, dx_1 + \cdots + a_n(x)\, dx_n$ als the Rieman integral

$$\int_a^b \left[a_1(c(t))c_1'(t) + \cdots + a_n(c(t))c_n'(t) \right] \, dt.$$

The integrand

$$a_1 \, dx_1 + \cdots + a_n \, dx_n$$

above is called a **Pfaffian differential form**. We consider such form as formal expressions, which can be integrated over curves. In physical applications, the differential form is, for example, a force field and the curvilinear integral is the work carried out ba a particle which traverses the curve.

2.3 Planar curves

: Let $c : I = [a, b] \to \mathbf{R}^2$ be a planar.

$$s = \int_a^t \sqrt{(\dot{c}_1)^2 + (dotc_2)^2 dt}$$

is its arc length,

$$L = \int_a^b \sqrt{(\dot{c}_1)^2 + (\dot{c}_2)^2 dt}$$

its total length.

If $(\dot{c}_1)^2 + (\dot{c}_2)^2 = 1$, then the curce has arc length parametrisation. We use the letter s for the independent variable in this case and γ for the parametrisation.

$$\mathbf{T}(s) = (\gamma_1'(s), \gamma_2'(s))$$

is the tangent vector of γ at the point s. Then $||\mathbf{T}|| = 1$. Let $\mathbf{N} = D_{\frac{\pi}{2}}\mathbf{T}$. $\mathbf{N}$ is the **normal vector** of γ and $(\mathbf{T}, \mathbf{N})$ form a orthonormal basis for $\mathbf{R}^2$ (**the moving frame**).

There is a scalar $\kappa(s) \in \mathbf{R}$, so that

$$\frac{d\mathbf{T}}{ds} = \kappa(s)\mathbf{N}(s), \quad \frac{d\mathbf{N}}{ds} = -\kappa(s)\mathbf{T}(s).$$

κ is the **curvature** of γ at s. $\rho = \frac{1}{\kappa}$ is the **radius of curvature**, $C = \gamma(s) + \rho(s)\mathbf{N}(s)$ the **centre of curvature**.

Curvilinear integrals of the second type : For a scalar-valued function f we have

$$\int_C f ds = \int_a^b f(c_1(t), c_2(t))\sqrt{((\dot{c}_1)^2 + (\dot{c}_2)^2)}dt.$$

Remark :
I. If c is a closed curve without self-intersections, then the curvilinear integral

$$\frac{1}{2}\int_c (y\, dx - x\, dy)$$

is the area enclosed by c.
II. If $X = \text{grad} f$ is the gradient field of a scalar Function f (see below), then

$$\int_C \vec{X} d\vec{x} = f(b) - f(a)$$

whereby a, b are the end points of c. In this case, the curvilinear integral is path-independent. This is the case, if $\frac{\partial X_1}{\partial y} = \frac{\partial X_2}{\partial x}$ (assuming that the domain of definition of X has no "holes").

Examples of curvilinear integrals :

Of the first type: $Q = \int_C (c_V dT + \frac{RT}{V} dV)$ whereby

$Q =$ is the heat content of an ideal gas
$V =$ Volume
$c_V =$ specific heat Wärme (for constant V.)
$R =$ the gas constant

Second type: $T = \int_C \frac{ds}{c(x,y)}$ with T the time, taken for a ray of light to traverse the curve C where $c(x, y)$ is the velocity of light at the point (x, y) of an (inhomogeneous) Mediums.

Example Calculate $\int_C X_1\,dx + X_2\,dy$ where

1. $X(x,y) = (y, -\sin x)$, C the curve $c(t) = (t, t^2)$ ($t \in [0,1]$)
2. $X(x,y) = (x+y, x-y)$, C the curve $c(t) = (\cos t, \sin t)$ ($t \in [0, 2\pi]$).

(1) The integral =

$$\int_0^1 (t^2 \cdot 1 + (-\sin t)\cdot 2t)dt = \int_0^1 (t^2 - 2t\sin t)dt$$

(2) The integral =

$$\int_0^{2\pi} ((\cos t + \sin t)(-\sin t) + (\cos t - \sin t)\cos t)dt$$

$$= \int_0^{2\pi} (\cos^2 t - \sin^2 t)dt.$$

2.4 Curves in space

: Let $c(t) = (c_1(t), c_2(t), c_3(t))$ be a curve in $\mathbf{R}^3$. Once again, we define

$$\begin{aligned} s &= \textstyle\int_a^t \sqrt{(\dot c_1)^2 + (\dot c_2)^2 + (\dot c_3)^2}du \\ L &= \textstyle\int_a^b \sqrt{(\dot c_1)^2 + (\dot c_2)^2 + (\dot c_3)^2}du. \end{aligned}$$

c is parametrised by arc length if

$$\frac{ds}{dt} = (\dot c_1)^2 + (\dot c_2)^2 + (\dot c_3)^2 = 1.$$

Then $\mathbf{T}(s) = (\dot c_1, \dot c_2, \dot c_3)$ is a unit vector—the unit tangent vector.

$\mathbf{N}(s) = \dfrac{\mathbf{T}'(s)}{||\mathbf{T}'(s)||}$ is the **normal vector**

$\mathbf{B}(s) = \mathbf{T}(s) \times \mathbf{N}(s)$ is the **binormal**.

Proposition 9 *There is a positive function κ and a real function ω, so that*

$$\begin{aligned} \mathbf{T}' &= \kappa\mathbf{N} \\ \mathbf{N}' &= -\kappa\mathbf{T} + \tau\mathbf{B} \\ \mathbf{B}' &= -\tau\mathbf{N}. \end{aligned}$$

$\kappa(s)$ is the **curvature** of c, $\tau(s)$ the **torsion**, $\rho = \dfrac{1}{\kappa}$ is the **radius of curvature**. (τ is a quantitative measure for the speed with which the curves is moving away from the osculating plane).

Example The helix $c(t) = \dfrac{1}{\sqrt{2}}(\cos t, \sin t, t)$. We have

$$(\dot{c}_1)^2 + (\dot{c}_2)^2 + (\dot{c}_3)^2 = 1$$

i.e. the curve is parametrised by arc length.

$$\begin{aligned}
\mathbf{T}(s) &= \dot{c}(s) = \tfrac{1}{\sqrt{2}}(\sin s, \cos s, 1) \\
\mathbf{T}'(s) &= \tfrac{1}{\sqrt{2}}(\cos s, -\sin s, 0) \\
\mathbf{N}(s) &= (-\cos s, -\sin s, 0) \\
\mathbf{B}(s) &= \mathbf{N}(s) \times \mathbf{T}(s) = \tfrac{1}{\sqrt{2}}(\sin s, -\cos s, 1).
\end{aligned}$$

Since $\mathbf{T}'(s) = \kappa(s)\mathbf{N}(s)$ we have $\kappa(s) = \frac{1}{\sqrt{2}}$. Since $\mathbf{B}'(s) = -\tau(s)\mathbf{N}(s)$ and $\mathbf{N}'(s) = \frac{1}{\sqrt{2}}(\cos s, \sin s, 0)$ we have $\tau(s) = \dfrac{1}{\sqrt{2}}$.

For a general curve $c(t) = (c_1(t), c_2(t), c_3(t))$ gilt

$$\mathbf{T} = \frac{\dot{c}}{||\dot{c}||}, \qquad \mathbf{B} = \frac{\dot{c} \times \ddot{c}}{||\dot{c} \times \ddot{c}|}$$

$$\mathbf{N} = \mathbf{B} \times \mathbf{N}$$

$$\kappa = \frac{||\dot{c} \times \ddot{c}||}{||\dot{c}||^3} \qquad \tau = \frac{[\dot{c}, \ddot{c}, \ddot{c}]}{||\dot{c} \times \ddot{c}^2||}$$

Example For $c(t) = (1 + t^2, t, t^3)$ we have

$$\dot{c} = (2t, 1, 3t^2), \ddot{c} = (2, 0, 6t), \dddot{c} = (0, 0, 6)$$

$$\dot{c} \times \ddot{c} = (-6t, -6t^2, -2), [\dot{c}, \ddot{c}, \dddot{c}] = -12.$$

Hence

$$\begin{aligned}
\kappa &= \tfrac{(36t^2+36t^4+4)^{\frac{1}{2}}}{(4t^2+1+9t^4)^{\frac{3}{2}}} \\
\tau &= \tfrac{-12}{(4t^2+1+9t^4)^{\frac{2}{3}}} \\
\mathbf{T} &= \tfrac{(2t,1,3t^2)}{(4t^2+1+9t^4)^{\frac{1}{2}}} \\
\mathbf{B} &= \tfrac{16t,-6t^2,-2)}{36t^2+36t^4+9t^4)^{\frac{1}{2}}} \\
\mathbf{N} &= \tfrac{(-18t^4+2,-4t-18t^3,6t+12t^3)}{(4t^2+1+9t^4)^{\frac{1}{2}}(36t^2+36t^4+4)^{\frac{1}{2}}}.
\end{aligned}$$

As in $\mathbf{R}^2$ we define vector fields

$$X(x, y, z) = (X_1(x, y, z), X_2(x, y, z), X_3(x, y, z))$$

in $\mathbf{R}^3$ resp. their curvilinear integrals

$$\int_C \vec{X} d\vec{x} = \int_C X_1 dx + X_2 dy + X_3 dz = \int_a^b X_1(c_1(t), c_2(t), c_3(t)) \frac{dc_1}{dt}$$

$$+ \int_a^b X_2(c_1(t), c_2(t), c_3(t)) \left(\frac{dc_2}{dt}\right) dt + \int_a^b X_3(c_1(t), c_2(t), c_3(t)) \left(\frac{dc_3}{dt}\right) dt.$$

For scalar functions f we define

$$\int_C f ds = \int_a^b f(c_1(t), c_2(t), c_3(t)) \sqrt{(\dot{c}_1)^2 + (\dot{c}_2)^2 + (\dot{c}_3)^2} dt.$$

Once again: $\int_C \vec{X} d\vec{x} = f(b) - f(a)$ if $X = \operatorname{grad} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$ is a gradient field. This is the case if

$$\frac{\partial X_i}{\partial x_j} = \frac{\partial X_j}{\partial x_i}$$

for $i, j = 1, 2, 3$.

k-**cubes, chains:** We can generalise the notion of a curve as follows:

Definition A k-**cube** in $\mathbf{R}^n$ is a smooth mapping $c : [0,1]^k \to \mathbf{R}^n$. A **chain** of k-cubes is a formal combination

$$n_1 c_1 + \cdots + n_r c_r$$

of k-cubes, where the coefficients n_i are whole numbers. If c is a k-cube, then its boundary ∂c is the $k-1$-chain

$$\sum_{i=1}^{k-1} (\partial_{\mathrm{o}}^i c - \partial_{\mathrm{u}}^i c),$$

whereby

$$\partial_{\mathrm{o}}^i c : (t_1, \ldots, t_{k-1}) \mapsto c(t_0, \ldots, t_{i-1}, 1, t_{i+1}, \ldots, t_k)$$

resp.

$$\partial_{\mathrm{u}}^i c : (t_1, \ldots, t_{k-1}) \mapsto c(t_0, \ldots, t_{i-1}, 0, t_{i+1}, \ldots, t_k).$$

Example The sphere $x^2 + y^2 + z^2 = R^2$ corresponds to the cube

$$c(s,t) = (R\sin s\cos t, R\sin s\sin t, R\cos s) \quad (0 \leq s \leq \pi, 0 \leq t \leq 2\pi.$$

The sphere has no boundary (more exactly, the boundary is in a certain sense trivial).

The cylinder $x^2 + y^2 = r^2, 0 \leq z \leq h$ corresponds to the cube

$$c(s,t) = (r\cos s, r\sin s, t) \quad 0 \leq s \leq 2\pi, 0 \leq t \leq h.$$

Its boundary consists ot the circles

$$s \mapsto (r\cos s, r\sin s, 0)$$

and

$$s \mapsto (r\cos s, r\sin s, h)$$

(with suitable orientations).

2.5 Exercises

Exercise: Integrate $(x^2 + y)\,dx + (-x + 2z)\,dy + yz\,dz$ along the segment from $(0,3,2)$ to $(-5,2,0)$.

Exercise: Calculate

$$\int_c x\,dy - y\,dx$$

around the circle $c(t) = (r\cos t, r\sin t)$ resp.

$$\int_c \frac{x}{x^2+y^2}\,dy - \frac{y}{x^2+y^2}\,dx.$$

Exercise: Consider the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where $0 < b < a$, with parametrisation

$$(a\cos\theta, b\sin\theta).$$

We put

$$P = (a\cos\theta, b\sin\theta), P' = -P, Q = (a\cos(\theta+\frac{\pi}{2}), b\sin(\theta+\frac{\pi}{2})) = (-a\sin\theta, b\cos\theta), Q' = -Q.$$

QQ' is the conjugate diameter of PP'.

Show

1. The tangent vector to the ellipse at P is parallel to QQ';

2. PP' bisects each secant of the ellipse which is parallel to the tangent vector at P;

3. if the secant GG' is parallel to the tangent vector at P, and M is the centre of the segment GG', then

$$\frac{|GM|^2}{|OQ|^2} = \frac{|PM||P'M|}{|OP|^2};$$

(Show that a ϕ exists so that

$$G = (a\cos(\theta + \phi, b\sin(\theta + \phi)), G'' = (a\cos(\theta - \phi, b\sin(\theta - \phi))$$

4. the area of the parallelogram $PQP'Q'$ is $2ab$;

5. if $F_1 = (-\sqrt{(a^2 - b^2)}, 0)$ resp. $F_2 = (\sqrt{(a^2 - b^2)}, 0)$ are the foci of the ellipse, then $|F_1P| + |F_2P| = 2a$;

6. the bisector of the angle $\angle F_1PF_2$ is perpendicular to the tangent vector at P;

7. if E is the point of intersection of the segments F_1P and QQ', show $|EP| = a$.

Exercise: Let $\mathbf{x}(t) = (x(t), y(t)) = r(t)(\cos\theta(t), \sin\theta(t))$ be a curve in the plane with velocity $\mathbf{v}(t) = \frac{d\mathbf{x}}{dt}$ and acceleration $\mathbf{a}(t) = \frac{d\mathbf{v}}{dt}$. Put

$$\mathbf{u}_r(t) = \frac{\mathbf{x}(t)}{r(t)}, \quad \mathbf{u}_\theta(t) = (-\sin\theta(t), \cos\theta(t)).$$

Show

1. $\frac{d\mathbf{u}_r}{dt} = \frac{d\theta}{dt}\mathbf{u}_\theta$;

2. $\frac{d\mathbf{u}_\theta}{dt} = -\frac{d\theta}{dt}\mathbf{u}_r$;

3. $\mathbf{v} = \frac{dr}{dt}\mathbf{u}_r + r\frac{d\theta}{dt}\mathbf{u}_\theta$;

4. $\mathbf{a} = \left(\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right)\mathbf{u}_r + \frac{1}{f}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)\mathbf{u}_\theta$.

Exercise: The acceleration of a particle is radial if and only if $r^2\frac{d\theta}{dt}$ is constant. (Remark: From the formula

$$A = \frac{1}{2}\int_{t_0}^{t} r(t)^2\theta'(t)\,dt$$

for the area one deduces then that $\frac{dA}{dt}$ is constant).

Exercise: Show that the curve with representation $r(1+\epsilon\cos\theta) = c \quad (|\epsilon| < 1, c > 0)$ in polar coordinates is and ellipse with 0 as focus and the coordinate axes as principal axes. (The corresponding semi-axes are then $\frac{c}{(1-\epsilon^2)}$ resp. $\frac{c}{\sqrt{1-\epsilon^2}}$).

Exercise: A particle moves on the above ellipse with radial acceleration. Show that $\mathbf{a}(t) = -\frac{k^2}{cr^2}\mathbf{u}_r$, where $k = 2\frac{dA}{dt}$.

Exercise: Determine the acceleration (as function of r) of a particle which moves on an ellipse with O as centre and with radial acceleration.

Exercise: Let $\mathbf{x}$ and $\mathbf{y}$ be vector-valued functions. Show that

$$\frac{d}{dt}(\mathbf{x}|\mathbf{y}) = (\frac{d\mathbf{x}}{dt}|\mathbf{y}) + (\mathbf{x}|\frac{d\mathbf{y}}{dt}), \quad \frac{d}{dt}(\mathbf{x}\times\mathbf{y}) = (\frac{d\mathbf{x}}{dt}\times\mathbf{y}) + (\mathbf{x}\times\frac{d\mathbf{y}}{dt}).$$

Show that if $|\mathbf{x}|$ is constant, then $\mathbf{x}' \perp \mathbf{x}$.

Exercise: Let $\mathbf{x}$ be the position of a particle in space. Show that if the acceleration is radial, then $\mathbf{x}\times\mathbf{v}$ is a constant vector $\mathbf{K}$, where $K = |\mathbf{K}| = 2\frac{dA}{dt}$.

Exercise: Let $\mathbf{a} = -\frac{GM}{r^2}\mathbf{u}_r$ and $\frac{dA}{dt} = \frac{K}{2}$. Then there is a constant vector $\mathbf{e}$, so that $r + (\mathbf{x}|\mathbf{e}) = \frac{K^2}{GM}$. (i.e. the equation of $\mathbf{x}$ in polar coordinates is $r(1+\cos\theta) = \frac{K^2}{GM}$.

Exercise: Let γ be a curve with arc length parametrisation in $\mathbf{R}^3$, so that $\gamma''(s) \neq 0 \quad (s \in [0, L])$. Define

$$\begin{aligned}
\mathbf{T}(s) &= \gamma'(s); \\
\mathbf{N}(s) &= \tfrac{\mathbf{T}'(s)}{\|\mathbf{T}'(s)\|}; \\
\mathbf{B}(s) &= \mathbf{T}(s) \times \mathbf{N}(s).
\end{aligned}$$

Show that there exist scalar functions $\kappa(s)$ and $\tau(s)$, so that the following equations (the Serret-Frenet formulae) hold:

$$\begin{aligned}
\mathbf{T}'(s) &= \kappa(s)\mathbf{N}(s) \\
\mathbf{N}'(s) &= -\kappa(s)\mathbf{T}(s) + \tau(s)\mathbf{B}(s) \\
\mathbf{B}'(s) &= - \tau(s)\mathbf{N}(s).
\end{aligned}$$

(Consider the derivatives of the functions $(\mathbf{T}|\mathbf{T})$, $(\mathbf{T}|\mathbf{N})$ etc.)

Exercise: Calculate $\mathbf{T}$, $\mathbf{N}$, $\mathbf{B}$, τ and κ for the helix.

3 Partial derivatives, differentiability, the Taylor formula, local extrema:

3.1 Definitions

In this chapter we consider the derivative of a function $f : \mathbf{R}^m \to \mathbf{R}^n$. Even the notation provides problems in this case. For example, consider polynomials in two variables

$$p(x, y) = 13 + 5xy + 3xy^3 - x^7 y$$

or in three variables

$$7 + 2z + 3xy + yz + xz - 17x^2 yz.$$

The general polynomial in two variables can be written as

$$a + bx + cy + dx^2 + exy + fy^2 + \dots$$

or

$$a_{00} + a_{10}x + a_{01}y + a_{20}x^2 + a_{21}xy + \dots$$

or

$$\sum_{k,l} a_{kl} x^k y^l.$$

However, such attempts break down for a polynomial of an arbitrary number of variables. We shall employ a notation which was introduced by L. Schwartz to deal with partial differential operators which uses the concept of a multi-index.

Notation We write $x = (x_1, \dots, x_n)$ for a vector in $\mathbf{R}^n$. A **multi-index** is an n-tuple $p = (p_1, \dots, p_n)$, where each $p_i \in \mathbf{N}_0$. x^p denotes the product $x_1^{p_1} \dots x_n^{p_n}$. We also write

$$\begin{aligned} p! &= p_1! \dots p_n! \\ |p| &= p_1 + \dots + p_n \\ \binom{p}{k} &= \frac{p!}{k!(p-k)!} = \binom{p_1}{k_1} \dots \binom{p_n}{k_n} \\ p &\le q \iff p_i \le q_i \text{ for each i.} \end{aligned}$$

The general polynomial can then be written

$$\sum_{p \le k} a_p x^p.$$

As an example of the elegance of this notation, the multinomial theorem can be written as:

Let $x, y \in \mathbf{R}^n$, $p \in (\mathbf{N}_0)^n$ a multi-index. Then

$$(x+y)^p = \sum_{0 \le k \le p} \binom{p}{k} x^k y^{p-k}.$$

Examples of functions of several variables: I. **Linear functionals**:

$$\begin{aligned} m = 2, n = 1 : f(x,y) \quad &= ax + by + c \\ m = 3, n = 1 : f(x,y) \quad &= ax + by + cz + d. \end{aligned}$$

In general: $f(x_1, \ldots, x_m) = a_1 x_1 + a_2 x_2 + \cdots + a_m x_m + d$.
II. **Quadratic functional**:

$$\begin{aligned} m = 2 : f(x,y) \quad &= ax^2 + 2bxy + cy^2 + d \text{ (in particular: } x^2 + y^2, x^2 - y^2, xy, x^2) \\ m = 3 : f(x,y) \quad &= ax^2 + by^2 + cz^2 + dyz + ezx + fxy + d. \end{aligned}$$

In general: $f(x_1, \ldots, x_m) = \sum_{i,j=1}^m a_{ij} x_i x_j + d$.
III. **Polynomials**:

$$\sum_{i_1 + \cdots + i_m = \le k} a_{i_1 \ldots i_m} x_1^{i_1} \ldots x_m^{i_m}.$$

IV. **Function which can be written als combinations of elementary functions of one variable**:

$$f(x,y) = \left(\sqrt{x^2 + y^2}, \frac{1}{\sqrt{x^2 + y^2}}, \sin(x+y), \sin(x^2 + xy + y^2) \right)$$

$$f(x,y,z) = (\ln(x^2 + y^2 + z^2), e^{x+y} \sin(|y+z|)$$

etc.

Definition Let $f : U \to \mathbf{R}$, where U is open in $\mathbf{R}^n$. Then f is partial differential at x_0 with respect to the i-th variable if

$$D_i f(x) = \lim_{t \to 0} \frac{f(x + te_i) - f(x)}{t}$$

exists. (e_i is the i-te basis element $(0, \ldots, 0, 1, 0, \ldots, 0)$, where the 1 is in the i-the position). The limit is then the ***i*-th partial derivative** of f at

x. If each partial derivative $D_i f$ exists at each point in U, then f is **partial differentiable** and the functions

$$x \mapsto D_i f(x)$$

are the **partial derivatives** of f. If these derivatives are continuous, then f is **continuously partially differentiable**.

If $f = (f_1, \ldots, f_m) : U \to \mathbf{R}^m$ is such that each f_i is (continuously) partially differentiable, then we say that f is (continuously) partially differentiable.

Definition Let $f : U \to \mathbf{R}$ be partially differentiable. The vector

$$\operatorname{grad} f(x) = (D_1 f(x), \ldots, D_n f(x))$$

is the **gradient** of f at x.

Directional derivative If $f : U \to \mathbf{R}$ is continuously partially differentiable and v is a vector in $\mathbf{R}^n$, then

$$D_v f(x) = (\operatorname{grad} f(x) | v)$$

is the **directional derivative** of f in the direction v. The notation comes from the fact that $D_v f(x)$ is the derivative of the function

$$t \mapsto f(x + tv)$$

of *one* avriable at 0.

The geometric significance of the gradient vector is described by the following result:

Proposition 10 *Let c be a smooth curve in U and $f : U \to \mathbf{R}$ a continuously partially differentiable function. Then*

$$\frac{d}{dt} f(c(t)) = (c'(t) | \mathit{grad}\, f(c(t)).$$

This follows from the chain rule which will be proved below.

By means of the Cauchy Schwarz inequality one sees that the gradient vector $\operatorname{grad} f(x)$ is the direction of *steepest ascent* of the function f.

Proposition 11 *Let $f : U \to \mathbf{R}$ be continuously partially differentiable where U is a neighbourhood of x. Then for $\xi \in \mathbf{R}^n$ with $\|\xi\| < \epsilon$, where $U(x, \epsilon) \subset U$,*

$$f(x + \xi) = f(x) + (\mathit{grad}\, f(x) | \xi) + \rho(\xi),$$

where $\lim_{\xi \to 0} \dfrac{\phi(\xi)}{\|\xi\|} \to 0$.

The following concept is the direct translation of the definition for the one-dimensional case.

Definition 2 *Let $U \subset \mathbf{R}^n$ be open and $f : U \to \mathbf{R}^m$ a mapping f is* **(totally) differentiable** *at x if there is a linear mapping*

$$A : \mathbf{R}^n \to \mathbf{R}^m$$

so that

$$f(x + \xi) = f(x) + A\xi + ||\xi||\pi(\xi) \text{ mit } \lim_{\xi \to 0} \pi(\xi) = 0.$$

We write $(Df)_x$ for A—the **derivative** *of f at x. If $(Df)_x$ exists for each $x \in U$, then f is* **differentiable** *on U. The mapping $Df : x \mapsto (Df)_x$ (from U to $M_{m,n}$ is then the* **derivative** *of f. If Df is continuous then f is* **continuously differentiable**.

Remark If $U \subset \mathbf{R}^n$ is open and $f : U \to \mathbf{R}^n$, then f is diffentiable if and only if each $f_i : U \to \mathbf{R}$ $(i = 1, \dots, m)$ is differentiable where $f = (f_1, \dots, f_n)$.

In the following result we investigate the relationship between differentiability and the existence of partial derivatives:

Proposition 12 *Let $U \subset \mathbf{R}^n$ be open and $f : U \to \mathbf{R}^m$.*

i) If f is differentiable at $x \in U$, then each f_i is partially differentiable at x and

$$\frac{\partial f_i}{\partial x_j}(x) = a_{ij} \qquad (i = 1, \dots, m; j = 1, \dots, n),$$

where $[a_{ij}]$ is the matrix of $(Df)_x$;

ii) If the f_i are continuously partially differentiable in a neighbourhood of x and

$$a_{ij} := \frac{\partial f_i}{\partial x_j}(x) \text{ for } i = 1, \dots, m \text{ and } j = 1, \dots, n.$$

Then f is differentiable at x and

$$f(x + \xi) = f(x) + A\xi + ||\xi||\pi(\xi),$$

where

$$A = (a_{ij}) \text{ and } \lim_{\xi \to 0} \pi(\xi) = 0.$$

Notation. If $f : U \to \mathbf{R}^m$ is differentiable at x. Then the $(m \times n)$-matrix

$$(Df)(x) = J_f(x) = \left(\frac{\partial f_i}{\partial x_j}(x)\right)$$

is called the **functional matrix** or **Jacobi matrix** of f at x.

PROOF. i) We deduce from the hypothesis that

$$f_i(x+\xi) = f_i(x) + \sum_{j=1}^{n} a_{ij}\xi_j + ||\xi||\pi_i(\xi)$$

with $\lim_{\xi\to 0}\pi_i(\xi) = 0$ for $i = 1, 2, \ldots, m$.

For $j = 1, 2, \ldots, n$ and $h \in \mathbf{R}$ with $x + he_j \in U$ we have:

$$f_i(x + he_j) = f_i(x) + a_{ij}h + |h|\pi_i(he_j)$$

with $\lim_{h\to 0}\pi_i(he_j) = 0$.

This implies

$$\frac{\partial f_i}{\partial x_i}(x) = \lim_{h\to 0}\frac{f_i(x+he_j) - f_i(x)}{h} = a_{ij} + \lim_{h\to 0}\frac{|h|}{h}\pi_i(he_j) = a_{ij}, \text{ q.e.d.}$$

ii) For $i = 1, 2, \ldots, m$ we have

$$f_i(x+\xi) = f_i(x) + \sum_{j=1}^{n}\frac{\partial f_i}{\partial x_j}(x)\xi_j + ||\xi||\pi_i(\xi)$$

with $\lim_{\xi\to 0}\pi_i(\xi) = 0$.

Then

$$f(x+\xi) = f(x) + A\xi + ||\xi||\pi(\xi),$$

where A is the matrix

$$\begin{matrix} \frac{\partial f_1}{\partial x_1}(x) & \ldots & \frac{\partial f_1}{\partial x_n}(x) \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1}(x) & \ldots & \frac{\partial f_m}{\partial x_n}(x) \end{matrix}$$

and $\pi(\xi)$ is

$$\begin{matrix} \pi_1(\xi) \\ \vdots \\ \pi_m(\xi) \end{matrix}$$

with $\lim_{\xi\to 0}\pi(\xi) = 0$, q.e.d.

∎

3.2 Basic results

Proposition 13 *the chain rule Let* $U \subset \mathbf{R}^n$ *and* $V \subset \mathbf{R}^m$ *be open and* $g : U \to \mathbf{R}^m$, $f : V \to \mathbf{R}^k$ *mappings with* $g(U) \subset V$*. If* g *is differnetiable at* $x \in U$ *and* f *at* $y := g(x) \in V$*, then the composition*

$$f \circ g : U \to \mathbf{R}^k$$

is differeniable at x*, and*

$$D(f \circ g)_x = (Df)_y \circ (Dg)_x \text{ resp. } J_{f \circ g}(x) = J_f(g(x))J_g(x).$$

Proof. Let $A := J_g(x)$ and $B := J_f(y)$. Then

$$g(x + \xi) = g(x) + A\xi + ||\xi||\pi(\xi) \text{ with } \lim_{\xi \to 0} \pi(\xi) = 0,$$

$$f(y + \eta) = f(y) + B\eta + ||\eta||\psi(\eta) \text{ mit } \lim_{\eta \to 0} \psi(\eta) = 0.$$

If we choose, in particular,

$$\eta = A\xi + ||\xi||\pi(\xi),$$

then

$$||\eta|| \leq ||A\xi|| + ||\xi|| \, ||\pi(\xi)|| \leq (||A|| + ||\pi(\xi)||)||\xi||.$$

In particular, $\lim_{\xi \to 0} \eta = 0$.

$$\begin{aligned} f(g(x + \xi)) &= f(g(x) + \eta) \\ & f(g(x)) + BA\xi + B||\xi||\pi(\xi) + ||\eta||\psi(\eta). \end{aligned}$$

With

$$\chi(\xi) := B\pi(\xi) + \frac{||\eta||}{||\xi||}\psi(\eta) \text{ for } \xi \neq 0, \chi(0) := 0,$$

we have $||\chi(\xi)|| \leq ||||B\pi(\xi)|| + (||A|| + ||\pi(\xi)||)||\psi(\eta)||$, and so $\lim_{\xi \to 0} \chi(\xi) = 0$, and

$$(f \circ g)(x + \xi) = f(g)(x) + BA\xi + ||\xi||\chi(\xi).$$

Hence $J_{f \circ g}(x) = BA$, q.e.d.

■

Proposition 14 *Let* $V \subset \mathbf{R}^m$, $U \subset \mathbf{R}^n$ *be open and* $f : V \to \mathbf{R}$, $g =: U \to \mathbf{R}^m$ *differentiable mappings with* $g(U) \subset V$*. Then*

$$h := f \circ g : U \to \mathbf{R}$$

is partilly differentiable and for $i = 1, 2, \ldots, n$*:*

$$\frac{\partial h}{\partial x_i}(x_1, \ldots, x_n) = \sum_{j=1}^{m} \frac{\partial f}{\partial y_j}(g_1(x), \ldots, g_m(x)) \frac{\partial g_j}{\partial x_i}(x_1, \ldots, x_n).$$

Proof.

$$J_{f\circ g}(x) = J_h(x) = \left(\frac{\partial h}{\partial x_1}(x), \ldots, \frac{\partial h}{\partial x_n}(x)\right) = \operatorname{grad} h(x).$$

$$J_f(g(x)) = \left(\frac{\partial f}{\partial y_1}(g(x)), \ldots, \frac{\partial f}{\partial y_m}(g(y))\right) = \operatorname{grad} f(g(x)).$$

$J_g(x)$ is

$$\begin{matrix} \frac{\partial g_1}{\partial x_1}(x) & \ldots & \frac{\partial g_1}{\partial x_n}(x) \\ \vdots & & \vdots \\ \frac{\partial g_m}{\partial x_1}(x) & \ldots & \frac{\partial g_m}{\partial x_n}(x) \end{matrix}$$

The result then follows form the fact that $J_{f\circ g}(x) = J_f(g(x))J_g(x)$.

■

Definition 3 *Let $U \subset \mathbf{R}^n$ be open. Then one defines by induction: a function $f : U \to \mathbf{R}$ is* **k-times continuously partially differentiable**, *if f is $(k-1)$-times continuously partially differentiable and each it its $(k-1)$-th partial derivatives*

$$D_{i_{k-1}} \ldots D_{i_2} D_{i_1} f : U \to \mathbf{R} \quad (1 \leq i_\ell \leq n \text{ for } \ell = 1, \ldots, k-1)$$

is continuously partially differentiable.

Proposition 15 *Let $U \subset \mathbf{R}^n$ be open and $f : U \to \mathbf{R}$ twice continuously partially differentiable. Then for each $x \in U$ and for each $i, j = 1, \ldots, n$*

$$D_j D_i f(x) = D_i D_j f(x).$$

Proof. The claim is obviously true for für $i = j$. Suppose thant that $i \neq j$ and $x \in U$ is fixed. We define

$$\pi(s,t) = f(x + se_i + te_j)$$

π is defined in a neighbourhood V of $(0,0) \in \mathbf{R}^2$. It follows directly from the definition that

$$\begin{aligned} D_1\pi(s,t) &= D_i f(x + se_i + te_j), \\ D_2\pi(s,t) &= D_j f(x + se_i + te_j). \end{aligned}$$

Hence

$$\begin{aligned} D_2 D_1 \pi(s,t) &= D_j D_i f(x + s e_i + t e_j), \\ D_1 D_2 \pi(s,t) &= D_i D_j f(x + s e_i + t e_j). \end{aligned}$$

Hence it suffices to show that $D_2 D_1 \pi(0,0) = D_1 D_2 \pi(0,0)$.

1) Choose $t \neq 0$ and put

$$\phi(s) := \pi(s,t) - \pi(s,0).$$

According to the intermediate value theorem of the differential calculus

$$\phi(s) - \phi(0) = s\phi'(s_1),$$

where s_1 lies between 0 and s and

$$\phi'(s_1) = D_1\pi(s_1,t) - D_1\pi(s_1,0) = tD_2D_1(s_1,t_1),$$

where t_1 lies between 0 and t. Hence

$$\pi(s,t) - \pi(s,0) - \pi(0,t) + \pi(0,0) = stD_2D_1\pi(s_1,t_1).$$

2) Similarly

$$\pi(s,t) - \pi(s,0) - \pi(0,t) + \pi(0,0) = stD_1D_2\pi(s_2,t_2)$$

where s_2 (resp. t_2) lies between 0 and s (resp. 0 and t).

According to 1) and 2)

$$D_2D_1\pi(s_1,t_1) = D_1D_2\pi(s_2,t_2)$$

and

$$\lim_{(s,t)\to(0,0)} (s_i,t_i) = (0,0) \text{ for } i = 1,2.$$

Since $D_2D_1\pi$ and $D_1D_2\pi$ are continuous,

$$\begin{aligned} D_2D_1\pi(0,0) = \lim_{(s,t)\to(0,0)} D_2D_1\pi(s_1,t_1) &= \lim_{(s,t)\to(0,0)} D_1D_2\pi(s_2,t_2) \\ &= D_1D_2\pi(0,0), \text{ q.e.d.} \end{aligned}$$

∎

Proposition 16 *If $U \subset \mathbf{R}^n$ is open and $f : U \to \mathbf{R}$ k-times continuously partially differentiable, then*

$$D_{i_k} \dots D_{i_2} D_{i_1} f = D_{i_{\pi(k)}} \dots D_{i_{\pi(2)}} D_{i_{\pi(1)}} f$$

for each permutation π of $1, \dots, k$ and $1 \le i_\ell \le n$ for $\ell = 1, \dots, k$.

The proof is by induction on k. (Use the fact that each permutation is a product of transpositions).

Hence we can define $D^p f$ in a unique fashion for any multi-index p, provided that f is sufficiently smooth.

Proposition 17 *Let $U \subset \mathbf{R}^n$ be open, $f : U \to \mathbf{R}$ k-times continuously partially differentiable, $x \in U, \xi \in \mathbf{R}^n$. Then*

$$t \mapsto g(t) := f(x + t\xi)$$

is defined on an interval $]-\epsilon, \epsilon[\subset \mathbf{R} \quad (\epsilon > 0)$ and is k-times continuously partial differentiable and

$$\frac{d^k g}{dt^k}(t) = \sum_{|\alpha|=k} \frac{k!}{\alpha!} D^\alpha f(x + t\xi) \xi^\alpha.$$

Proof. (1) We claim that

$$\frac{d^k g}{dt^k}(t) = \sum_{i_1, \dots, i_k = 1}^{n} D_{i_k} \dots D_{i_1} f(x + t\xi) \xi_{i_1} \dots \xi_{i_k}.$$

This is proved by induction on k: $k = 1$:

$$\frac{dg}{dt}(t) = (\operatorname{grad} f(x + t\xi) | \xi) = \sum_{i=1}^{n} D_i f(x + t\xi) \xi_i.$$

$k - 1 \to k$:

$$\begin{aligned} \frac{d^k g}{dt^k}(t) &= \tfrac{d}{dt} \textstyle\sum_{i_1, \dots, i_{k-1}=1}^{n} D_{i_{k-1}} \dots D_{i_1} f(x + t\xi) \xi_{i_1} \dots \xi_{i_{k-1}} \\ &= \textstyle\sum_{j=1}^{n} D_j \left(\sum_{i_1, \dots, i_{k-1}=1}^{n} D_{i_{k-1}} \dots D_{i_1} f(x + t\xi) \xi_{i_1} \dots \xi_{i_{k-1}} \right) \xi_j \\ &= \textstyle\sum_{i_1, \dots, i_{k-1}=1}^{n} D_{i_k} \dots D_{i_1} f(x + t\xi) \xi_{i_1} \dots \xi_{i_k}. \end{aligned}$$

(2) If the index 1 occurs in the list $(i_1, \dots, i_k)$ exactly α_1-times, the index 2 exactly α_2-times,..., the index n α_n-times, then

$$D_{i_k} \dots D_{i_1} f(x + t\xi) \xi_{i_1} = D_1^{\alpha_1} \dots D_n^{\alpha_n} f(x + t\xi) \xi_1^{\alpha_1} \dots \xi_n^{\alpha_n}.$$

Since there are $\dfrac{k!}{\alpha_1! \dots \alpha_n!}$ k-tuples $(i_1, \dots, i_k)$ of natural numbers $1 \le i_\ell \le n$, for which ν occurs exactly α_ν-times ($\nu = 1, \dots, n; \alpha_1 + \dots + \alpha_n = k$), we have

$$\sum_{i_1,\dots,i_k=1} D_{i_k} \dots D_{i_1} f(x+t\xi)\xi_{i_1} \dots \xi_{i_k} = \sum_{|\alpha|=k} \frac{k!}{\alpha_1! \dots \alpha_n!} D_1^{\alpha_1} \dots D_n^{\alpha_n} f(x+t\xi)\xi_1^{\alpha_1} \dots \xi_n^{\alpha_n},$$

and so by (1)

$$\frac{d^k g}{dt^k}(t) = \sum_{|\alpha|=k} \frac{k!}{\alpha!} D^\alpha f(x + t\xi)\xi^\alpha, \text{ q.e.d.}$$

■

3.3 Taylor's formula

Proposition 18 *If $U \subset \mathbf{R}^n$ is open, $x \in U$ and $\xi \in \mathbf{R}^n$, so that $x + t\xi \in U$ for each $t \in [0, 1]$. Let $f : U \to \mathbf{R}$ be a $(k + 1)$-times continuously partially differentiable function. Then there is a $\theta \in [0, 1]$, so that*

$$f(x + \xi) = \sum_{|\alpha|\le k} \frac{D^\alpha f(x)}{\alpha!}\xi^\alpha + \sum_{|\alpha|=k+1} \frac{D^\alpha f(x + \theta\xi)}{\alpha!}\xi^\alpha.$$

Proof. The function $g : [0, 1] \to \mathbf{R}$, where $t \mapsto f(x + t\xi)$ is $(k + 1)$-times continuously partially differentiable. According to the Taylor formula for functions of one variable there exists a $\theta \in [0, 1]$, so that

$$g(1) = \sum_{i=0}^{k} \frac{g^{(i)}(0)}{i!} 1^i + \frac{g^{(k+1)}(\theta)}{(k + 1)!} 1^{k+1}.$$

Hence for $i = 0, 1, \dots, k$

$$\frac{g^{(i)}(0)}{i!} = \sum_{|\alpha|=i} \frac{D^\alpha f(x)}{\alpha!}\xi^\alpha$$

and

$$\frac{g^{(k+1)}(\theta)}{(k + 1)!} = \sum_{|\alpha|=k+1} \frac{D^\alpha f(x + \theta\xi)}{\alpha!}\xi^\alpha,$$

which implies the theorem.

■

Proposition 19 *Let $U \subset \mathbf{R}^n$ be open, $x \in U, \xi \in \mathbf{R}^n$ with $x + t\xi \in U$ for each $t \in [0,1]$. Let $f : U \to \mathbf{R}$ be k-times continuously differentiable. Then*

$$f(x+\xi) = \sum_{|\alpha|\leq k} \frac{D^\alpha f(x)}{\alpha!}\xi^\alpha + \pi(\xi)||\xi||^k,$$

where $\lim_{\xi\to 0}\pi(\xi) = 0$.

PROOF. There exists a $\theta \in [0,1]$, so that

$$\begin{aligned} f(x+\xi) &= \textstyle\sum_{|\alpha|\leq k-1} \frac{D^\alpha f(x+\theta\xi)}{\alpha!}\xi^\alpha \\ &= \textstyle\sum_{|\alpha|\leq k} \frac{D^\alpha f(x+\theta\xi)}{\alpha!}\xi^\alpha + \sum_{|\alpha|=k} r_\alpha(\xi)\xi^\alpha \end{aligned}$$

with $r_\alpha(\xi) = \dfrac{D^\alpha f(x+\theta\xi) - D^\alpha f(x)}{\alpha!}$. On account of the continuity of $D^\alpha f$, we have $\lim_{\xi\to 0} r_\alpha(\xi) = 0$.

With

$$\pi(\xi) := \sum_{|\alpha|=k} r_\alpha(\xi)\frac{\xi^\alpha}{||\xi||^k}$$

we deduce the result, since

$$\frac{|\xi^\alpha|}{||\xi||^k} = \frac{|\xi_1^{\alpha_1}\dots\xi_n^{\alpha_n}}{||\xi||^y\alpha_1\dots||\xi||^{\alpha_n}||} \leq 1 \text{ for } |\alpha| = k.$$

■

Example For $f = f(x,y)$ the Taylor series has the form

$$\sum_{m,n} \frac{1}{m!n!}\frac{\partial^{m+n}f}{\partial^m x\partial y^n}\bigg|_{x=0}\bigg| x^m y^n$$

e.g.

$$\sin(xy) = \sum_{m=0}^{\infty}(-1)^{m+1}\frac{x^{2m+1}y^{2m+1}}{(2m+1)!}.$$

Example a) For $f(x,y) = \sin xy + e^{x^2-y^2}$ we have:

$$\begin{aligned} D_1 f(x,y) &= y\cos xy + 2xe^{x^2-y^2} \\ D_2 f(x,y) &= x\cos xy - 2ye^{x^2-y^2} \\ D_{11} f(x,y) &= -y^2\sin xy + 4x^3e^{x^2-y^2} + 2e^{x^2-y^2} \\ D_{22} f(x,y) &= -x^2\sin xy - 4y^2e^{x^2-y^2} - 2e^{x^2-y^2} \\ D_{12} f(x,y) &= \cos xy - xy\sin xy - 5xye^{x^2-y^2} \\ D_{21} f(x,y) &= \cos xy - xy\sin -4xye^{x^2-y^2}. \end{aligned}$$

b) Calculate the Taylor series for e^{x+3y} around $(1,1)$ up to term of order 3.

$$f(1+h,1+k) = e^3 + e^3 h + 2e^3 h + \frac{e^3}{2!}h^2 + 2e^3 hk + \frac{4e^3}{2k^2}k^2$$

$$+\frac{e^3}{3!}h^3 + \frac{2e^3}{2!}h^2 k + \frac{4e^3}{2!}hk^2 + \frac{8e^3}{3!}k^3 + R_4.$$

Example $n = 2, k = 3$

$$f(x_1+\xi_1, x_2+\xi_2) = a + P_1(\xi_1,\xi_2) + P_2(\xi_1,\xi_2) + P_3(\xi_1,\xi_2) + \pi(\xi)||\xi||^3$$

with $\lim_{\xi\to 0}\pi(\xi) = 0$. Here

$$a = f(x_1, x_2)$$

and the P_k are homogeneous polynomials of degree k in ξ_1, ξ_2. Indeed

$$\begin{aligned}
P_1(\xi_1,\xi_2) &= \tfrac{\partial f}{\partial x_1}(x_1,x_2)\xi_1 + \tfrac{\partial f}{\partial x_2}(x_1,x_2)\xi_2), \\
P_2(\xi_1,\xi_2) &= \tfrac{1}{2}\tfrac{\partial^2 f}{\partial x_1^2}(x_1,x_2)\xi_1^2 + \tfrac{\partial^2 f}{\partial x_1 \partial x_2}(x_1,x_2)\xi_1\xi_2 + \tfrac{1}{2}\tfrac{\partial^2 f}{\partial x_2^2}(x_1,x_2)\xi_2^2, \\
P_3(\xi_1,\xi_2) &= \tfrac{1}{6}\tfrac{\partial^3 f}{\partial x_1^3}(x_1,x_2)\xi_1^3 + \tfrac{1}{2}\tfrac{\partial^3 f}{\partial x_1^2 \partial x_2}(x_1,x_2)\xi_1^2\xi_2 + \\
&\quad + \tfrac{1}{2}\tfrac{\partial^3 f}{\partial x_1 \partial x_2^2}(x_1,x_2)\xi_1\xi_2^2 + \tfrac{1}{6}\tfrac{\partial^3 f}{\partial x_2^3}(x_1,x_2)\xi_2^3.
\end{aligned}$$

3.4 Optimisation

Definition 4 *Let $U \subset \mathbf{R}^n$ be open and $f : U \to \mathbf{R}$. A point $x \in U$ is a* **local maximum** *(resp. a* **local minimum***) for f, if there is a neighbourhood $V \subset U$ of x, so that $f(x) \geq f(y)$ (resp. $f(x) \leq f(y)$) for each $y \in V$.*

If we have equality $f(x) = f(y)$ only when $x = y$, then the maximum or minimum is called strict or isolated. A **local extremum** *is a local maximum or minimum.*

Proposition 20 *Let $U \subset \mathbf{R}^n$ be open and $f : U \to \mathbf{R}$ a partially differentiable function. If f has a lokales extremum at the point x, then*

$$grad\, f(x) = 0.$$

PROOF. For $k = 1, \ldots, n$ let

$$g_k(t) := f(x + te_k)$$

g_k is defined on an interval $[-\epsilon, \epsilon[\subset \mathbf{R}$ with $\epsilon > 0$ and is differentiable on this set.

If f has a local extremum at x, so does g_k (at 0) and so $g_k'(0) = 0$. Since

$$g_k'(0) = (\operatorname{grad} f(x)|e_k) = \frac{\partial f}{\partial x_k}(x)$$

we have

$$\operatorname{grad} f(x) = \left(\frac{\partial f}{\partial x_1}(x), \ldots, \frac{\partial f}{\partial x_n}(x)\right) = 0, \text{ q.e.d.}$$

■

Definition 5 *If $U \subset \mathbf{R}^n$ is an open set and $f : U \to \mathbf{R}$ is twice partially differentiable, then the* **Hessean matrix** *of f at $x \in U$ is the $n \times n$ matrix*

$$\text{Hess } (f)(x) := \left[\frac{\partial^2 f}{\partial x_i \partial x_k}\right]$$

This matrix is symmetric since

$$\frac{\partial^2 f}{\partial x_i a x_k}(x) = \frac{\partial^2 f}{\partial x_k \partial x_i}(x).$$

We recall some facts from linear algebra.

Definition 6 *A symmetric matrix $A \in M_n$ is* **positive (negative) definite,** *if all of its eigenvalues are positive (negative).*

The matrix is **indefinite***, if it has at least one positive and one negative eigenvalue. (Recall that all eigenvalues of a real symmetric matrix are real).*

We use the fact that if $A \in M_n$ is symmetric, positive definite with eigenvalues $l_1, \ldots, l_n$ and $l := \min(l_1, \ldots, l_n)$, then for each $x \in \mathbf{R}^n$ we have the inequality $x^T A x \geq l||x||^2$.

Proposition 21 *Let $U \subset \mathbf{R}^n$ be open, $f \in U \to \mathbf{R}$ twice continuously partially differentiable. If $x \in U$ with grad $f(x) = 0$, then we have:*

i) if $(f)(x)$ is positive definite, then f has a relative minimum f at x.

ii) if Hess $(f)(x)$ is negative definite, then f has a relative maximum at x.

iii) if Hess $(f)(x)$ is indefinite, then f has neither a local maximum or minimum at x.

PROOF. The Taylor series of f at x has the form

$$f(x+\xi) = f(x) + \sum_{|\alpha|=2} \frac{D^\alpha f(x)}{\alpha!}\xi^\alpha + \pi(\xi)||\xi||^2,$$

where $\lim_{\xi\to 0}\pi(\xi) = 0$.

Hence

$$Q(\xi) := \sum_{|\alpha|=2} \frac{D^\alpha f(x)}{\alpha!} = \frac{1}{2}\sum_{i,j=1}^{n} D_i D_j f(x)\xi_i\xi_j = \frac{1}{2}\sum_{i,j=1}^{n} \alpha_{ij}\xi_i\xi_j = \frac{1}{2}\xi^T A\xi,$$

mit

$$\xi = \begin{bmatrix}\xi_1\\ \vdots\\ \xi_n\end{bmatrix} \text{ und } (\alpha_{ij}) = A = \text{ Hess } (f)(x).$$

Thus

$$f(x+\xi) - f(x) = Q(\xi) + \pi(\xi)||\xi||^2.$$

i) if A is positive definite and l is the smallest eigenvalue of A, then

$$Q(\xi) \geq \frac{1}{2}l||\xi||^2.$$

There exists $\epsilon > 0$, so that $|\pi(\xi)| \leq \dfrac{l}{4}$ for $||\xi|| < \epsilon$.

$$f(x+\xi) - f(x) = Q(\xi) + \pi(\xi)||\xi||^2 \geq \frac{1}{4}l||\xi||^2 > 0$$

for each ξ with $0 < ||\xi|| < \epsilon$, i.e. x is a relative local minimum for f.

ii) the second case is proved analogously (or apply (ii) to $-f$).

iii) Suppose that A is indefinite. We show that in each neighborurhood of x there are points x' with $f(x') > f(x)$ and points x'' with $f(x'') < f(x)$.

Let v be an eigenvector of A whose eigenvalue $l > 0$. We can suppose that $||v|| = 1$. Put $\xi := tv, t > 0$. Then

$$\xi^T A\xi = t^2 v^T Av = t^2 l = l||\xi||^2,$$

and so

$$f(x+\xi) - f(x) = \frac{1}{2}l||\xi||^2 + \pi(\xi)||\xi||^2 \geq \frac{1}{4}l||\xi||^2 > 0,$$

for t sufficiently small.

We find a point x'' in a similar way, using an eigenvector of a negative eigenvalue.

■

Examples We consider some typical examples of functions $f : \mathbf{R}^2 \to \mathbf{R}$

a) $f(x,y) = a + x^2 + y^2$. f has a relative minimum at $(0,0) \in \mathbf{R}^2$ ein since

$$\frac{1}{2}\text{ Hess }(f) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The graph of f i.e. the set

$$\Gamma_f = \{(x,y,z) \in \mathbf{R}^3 : z = a + x^2 + y^2\}$$

is a paraboloid.

b) $f(x,y) = a - x^2 - y^2$. In this case the origin is a local maximum since

$$\frac{1}{2}\text{ Hess }(f) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

c) $f(x,y) = a + x^2 - y^2$. The gradient of f vanishes at the origin and

$$\frac{1}{2}\text{ Hess }(f) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The Hessean matrix is indefinite and so the origin is neither a local maximum or a minimum.

The graph

$$\Gamma_f = \{(x,y,z) \in \mathbf{R}^3 : z = a + x^2 - y^2\},$$

is a **saddle**.

d) If the Hessean matrix is semi-definite at a point (i.e. it has a zero eigenvalue (equivalently, the matrix is singular), then it cannot be used to determine the nature of a stationary point as the following examples show:

$$\begin{aligned} f_1(x,y) &= x^2 + y^4 \\ f_2(x,y) &= x^2 \\ f_3(x,y) &= x^2 + y^3. \end{aligned}$$

Each of these functions has a vanishing gradient at the origin and

$$\text{Hess }(f_k)(0) = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \qquad (k = 1,2,2).$$

Hence the Hessean matrix is positive semi-definite in each case.

f_1 has a local minimum at 0, f_2 has a local minimum which is not strict (since each point on the y-axis is also a minimum), and for f_3 the origin is neither a local minimum or a maximum.

Vectorfields in $\mathbf{R}^2$ A **vector field** in $\mathbf{R}^2$ is a mapping X from a subset of $\mathbf{R}^2$ into $\mathbf{R}^2$.

Example

$$\begin{aligned} X(x,y) &= (x,y) \\ X(x,y) &= (-y,x) \\ X(x,y) &= (0,y). \end{aligned}$$

Many of the most important vector fields are **gradient fields** i.e. of the form

$$\operatorname{grad} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$$

where f is a mapping of a subset of $\mathbf{R}^2$ into $\mathbf{R}$.

Example For $f(x,y) = \frac{1}{x^2+y^2}$ we have

$$\operatorname{grad} f = \left(\frac{-2x}{(x^2+y^2)^2}, \frac{-2x}{(x^2+y^2)^2}\right).$$

Kurvenintegrale, 1. Art : Falls $X(x,y) = (X_1(x,y), X_2(x,y))$ ein Vektorfeld ist, dann definiert man das Kurvenintegral $\int_C X_1 dx + X_2 dy$ entlang einer Kurve C mit Parametrisierung c als

$$\int_a^b \left(X_1(c_1(t), c_2(t))\frac{dc_1}{dt} + X_2(c_1(t), c_2(t))\frac{dc_2}{dt}\right) dt$$

(andere Schreibweise: $\int_C \vec{X} \cdot d\vec{x}$).

Ein Vektorfeld X bestimmt ein System von Differentialgleichungen

$$\begin{aligned} \frac{dx}{dt} &= X_1(x,y) \\ \frac{dy}{dt} &= X_2(x,y). \end{aligned}$$

Die Lösungen sind Kurven, die zu den Pfeilen des Feldes tangential sind – die sogenannten **Feldlinien**.

Beispiel Für $X(x,y) = (y, -x)$ ist das entsprechende System

$$\begin{aligned} \frac{dx}{dt} &= y \\ \frac{dy}{dt} &= -x \end{aligned}$$

mit Lösung $x(t) = (\cos t, \sin t)$. Für $X(x,y) = (-x-y, x-y)$ ist das entsprechende System,

$$\begin{aligned} \frac{dx}{dt} &= -x-y \\ \frac{dy}{dt} &= x-y \end{aligned}$$

mit Lösung $f(t) = (e^{-t}\cos t, e^{-t}\sin t)$.

3.5 Exercises

Exercise: At which points is the function $f : (x,y) \mapsto y\sqrt{2x^2+y^2}$ partialy differentiable? Calculate the partial derivatives there.

Exercise: Calculate the Jacobi matrix of

$$f : (x,y,z) \mapsto (\cos(xy+z), \sin(x^2y)).$$

Exercise: Let U be an open subset of $\mathbf{R}^n$. For a partially differentiable mapping

$$f : U \to \mathbf{R}^n, x \mapsto (f_1(x), \dots, f_n(x)),$$

let

$$\text{div } f := \sum_{j=1}^{n} \frac{\partial f_j}{\partial x_j} \text{ (\textbf{the divergence} of } f).$$

Prove

a) if $f : U \to \mathbf{R}^n$ and $v : U \to \mathbf{R}$ are partially differentiable, then

$$\text{div } (vf) = (\text{grad}\, v|f) + v \text{ div } f.$$

b) if $v, w : U \to \mathbf{R}$ are partially differentiable then

$$\text{grad}\,(vw) = v\text{grad}\, w + w\text{grad}\, v.$$

Exercise: Let r denote the function $(x_1, \ldots, x_n) \mapsto \sqrt{x_1^2 + \ldots x_n^2}$ from $\mathbf{R}^n$ into $\mathbf{R}$. Show that for $x \in \mathbf{R}^n \setminus \{0\}, \alpha \in \mathbf{R}$:

a) $(\operatorname{grad} r)(x) = \dfrac{x}{r(x)}$.

b) $(\operatorname{grad} r^\alpha)(x) = \alpha[r(x)]^{\alpha-2}(x)$.

Exercise:

$$u(x,y) := \begin{matrix} xy\dfrac{x^2-y^2}{x^2+y^2} & \text{für} & (x,y) \neq (0,0) \\ 0 & \text{für} & (x,y) = (0,0) \end{matrix} .$$

defines a function $u : \mathbf{R}^2 \to \mathbf{R}$.

a) At which points is u continuous?

b) At which points is u partially differentiable?

c) Show that $\dfrac{\partial^2 u}{\partial x \partial y}$ and $\dfrac{\partial^2 u}{\partial y \partial x}$ exist at the origin and are distinct.

Exercise: Let U be an open subset of $\mathbf{R}^n$ and $v : U \to \mathbf{R}$ a twoce partially differentiable function. The **Laplace operator** Δ is defined as follows:

$$\Delta v = \operatorname{div}(\operatorname{grad} v).$$

Show that

a)

$$\Delta v = \sum_{j=1}^{n} \frac{\partial^2 v}{\partial x_j^2}.$$

b) If $v, w : U \to \mathbf{R}$ are twice partially differentiable, then

$$\Delta(vw) = (\Delta v)w + 2(\operatorname{grad} v | \operatorname{grad} w) + v(\Delta w).$$

c) For the function

$$r : \mathbf{R}^n \setminus \{0\} \to \mathbf{R}, x \mapsto ||x||,$$

and $\alpha \in \mathbf{R}$ we have

$$\Delta r^\alpha = \alpha r^{\alpha-2}(\alpha - 2 + n).$$

d) If $u : \mathbf{R} \to \mathbf{R}$ is twice partially differentiable, then

$$\Delta(u \circ r)(x) = u''(r(x)) + \frac{n-1}{r(x)} u'(r(x)).$$

Exercise: (the Laplace operator in polar coordinates) Let p be the mapping

$$(r, \pi) \mapsto (r \cos \pi, r \sin \pi)$$

on $\mathbf{R}_+^* \times \mathbf{R}$ with values in $\mathbf{R}^2$, where $\mathbf{R}_+^* = \{x \in \mathbf{R} : x > 0\}$. If

$$u : \mathbf{R}^2 \to \mathbf{R}$$

is twice partially differentiable show

$$(\Delta u) \circ p = \frac{\partial^2 (u \circ p)}{\partial r^2} + \frac{1}{r} \frac{\partial (u \circ p)}{\partial r} + \frac{1}{r^2} \frac{\partial^2 (u \circ p)}{\partial \pi^2}.$$

Exercise: For natural numbers n, m with $0 < n, m \leq N$ let the function u_{nm} be defined as

$$\begin{aligned} u_{nm} : \mathbf{R}^2 &\to \mathbf{R} \\ (x, y) &\mapsto \sin nx \sin my. \end{aligned}$$

V_N is the linear subspace spanned by the functions $u_{nm}, 0 < n, m \leq N$.
a) Show that each element of V_N is infinitely partial differentiable.
b) Show that the Laplace operator Δ is a linear mapping on V_N.
c) Calculate the eigenvalues and the eigenvectors of $\Delta : V_N \to V_N$.

Exercise: Calculate the Taylor series of the function

$$f : (x, y) \mapsto \frac{x - y}{x + y},$$

on $\mathbf{R}_+^* \times \mathbf{R}_+^*$ at the point (1,1) up to terms of second order.

Exercise: At which points does the function

$$f : (x, y) \mapsto (4x^2 + y^2) \exp(-x^2 - 4y^2)$$

from $\mathbf{R}^2 \to \mathbf{R}$ have a local extremum?

Exercise: Show that $(-1, -1)$ is a local minimum for

$$f(x, y) = 3y^2 - 4xy + 4x^2 + 2y + 4x.$$

Exercise: Calculate the stationary points for the following functions

1. $x^2 - 2xy - y^2 + 4x - 2y$;
2. $x^2 + 4xy + y^2 - 3x + y$;
3. $\sin x \cos y$;
4. $y^2 - \sin^2 x$;
5. $(x^2 - y^2)e^{-(x^2-y^2)}$; $y^2 - x^2y - yz^2 + x^2z^2$.

Exercise: Determine the form of the above stationary points.

Exercise: Let $x_1, \ldots, x_n$ n be distinct points in $\mathbf{R}$ and $y_1, \ldots, y_n$ real numbers. Consider the following function

$$E(a, b) = \sum_{i=1}^{n} \left[y_i - (ax_i + b)^2 \right].$$

For which values of a, b does $E(a, b)$ take on its minimal value?

3.6 Integrals which depend on a parameter, integrals for functions defined on subsets of $\mathbf{R}^n$

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3.7 Definitions

We recall that a function $h : [a, b] \to \mathbf{R}$ is a step function if there is a partition

$$a = x_0 < x_1 < \cdots < x_n = b$$

of the interval $[a, b]$ so that h is constant on the sets $]x_{k-1}, x_k[$ for each k.

It is clear how one defines step functions on $\mathbf{R}^n$. (the sets of constancy are now rectangles). We can thus extend the notion of a Riemann to functions defined on suitable subsets of $\mathbf{R}^n$ in a manner completely analogous to the case of one dimensional functions. However, we shall confine our attention to a more elementary concept which suffices to integrate most functions which arise in applications.

3.8 Integrals which depend on a Parameter

Proposition 22 *Lemma Let $[a, b] \subset \mathbf{R}$ and let Q be a closed rectangle i.e.a product of compact intervals in $\mathbf{R}^n$. Let $f : [a, b] \times Q \to \mathbf{R}$ be continuous and $(y_i)_{i\in\mathbf{N}}$ a sequence in Q with $\lim_{i\to\infty} y_i = c \in Q$. Then the functions $F_i : x \mapsto f(x, y_i)$ converge uniformly to the function $F : x \mapsto f(x, c)$.*

Proof. Let $\epsilon > 0$. Since $[a, b] \times Q$ is compact, the function f is uniformly continuous and so there is a $\delta > 0$, with

$$||(x, y) - (x', y')|| < \delta \Rightarrow |f(x, y) - f(x', y')| < \epsilon.$$

We can then find an N, so that $||c - y_i|| < \delta$ for $i \geq N$. Then forr $i \geq N$

$$|f(x, c) - f(x, y_i)| < \epsilon \text{ for each } x \in [a, b],$$

i.e. $||F - F_i|| = \sup\{|f(x, c) - f(x, y_i)| : x \in [a, b]\} \leq \epsilon$, q.e.d.

■

Proposition 23 *Let $[a, b] \subset \mathbf{R}$, Q be a closed rectangle in $\mathbf{R}^n$ and $f : [a, b] \times Q \to \mathbf{R}$ continuous. Then the function $\pi : Q \to \mathbf{R}$, where $\pi(y) = \int_a^b f(x, y)dx$, is continuous.*

Proof. Let $(y_i)_{i\in\mathbf{N}}$ be a sequence in Q with $\lim_{i\to\infty} y_i = c \in Q$. Then

$$\pi(y_i) = \int_a^b F_i(x)dx \text{ and } \pi(c) = \int_a^b F(x)dx.$$

Since $(F_i)_{i\in\mathbf{N}}$ converges uniformly to F,

$$\lim_{i\to\infty} \int_a^b F_i(x)dx = \int_a^b F(x)dx,$$

and so $lim_{i\to\infty}\pi(y_i) = \pi(c)$, q.e.d.

■

Proposition 24 *Lemma Let $I, I' \subset \mathbf{R}$ be compakt intervals. Let $(y_i)_{i\in\mathbf{N}}$ a sequence in I' with $\lim_{i\to\infty} y_i = c \in I'$ and $y_i \neq c$ for each $i \in \mathbf{N}$. Let $f : I \times I \to \mathbf{R}$ be continuous and continuously partially differentiable with respect to the second variable. Then the functions $F_i : I \to \mathbf{R}$, where*

$$F_i(x) = \frac{f(x, y_i) - f(x, c)}{y_i - c} \quad (i \in \mathbf{N}),$$

converge uniformly to the function $F : I \to \mathbf{R}$, where

$$F(x) = \frac{\partial f}{\partial y}(x, c).$$

Proof. Take $\epsilon > 0$. Since $D_2 f$ by hypothesis is uniformly continuous, there is a $\delta > 0$, so that

$$|y' - y''| < \delta \Rightarrow |D_2 f(x, y') - D_2 f(x, y'')| < \epsilon.$$

By the mean value theorem there is for each $i \in \mathbf{N}$ an η_i between y_i and c with $F_i(x) = D_2 f(x, \eta_i)$. If one chooses N so large that $|y_i - c| < \delta$ for $i \geq N$, then $|\eta_i - c| < \delta$, and so by $(*)$

$$|F_i(x) - F(x)| = |D_2 f(x, \eta_i) - D_2 f(x, c)| < \epsilon \text{ for } i \geq N$$

and for each $x \in I$, q.e.d.

■

Proposition 25 *Let $I, I' \subset \mathbf{R}$ be compact intervals and $f : I \times I' \to \mathbf{R}$ continuous and continuously partially differentiable with respect to the second variable. Then the function $\pi : I' \to \mathbf{R}$, where*

$$\pi(y) := \int_I f(x, y)dx,$$

is continuously differentiable and

$$\frac{d\pi(y)}{dy} = \int_I \frac{\partial f(x, y)}{\partial y} dx.$$

PROOF. Let $c \in I'$ and $(h_\nu)_{\nu \in \mathbf{N}}$ be a real seuqnece with $c + h_\nu \in I'$ and $h_\nu \neq 0$ for each $\nu \in \mathbf{N}$. Then the functions $F_\nu : I \to R$, where

$$F_\nu(x) = \frac{f(x, c + h_\nu) - f(x, c)}{h_\nu},$$

converge uniformy to the function $F : I \to R$, where $F(x) = \dfrac{\partial x}{\partial y}(x, c)$. Hence

$$\lim_{h_\nu \to 0} \frac{\pi(c + h_\nu) - \pi(c)}{h_\nu} = \lim_{h_\nu \to 0} \int_I \frac{f(x, c + h_\nu) - f(x, c)}{h_\nu} dx = \int_I \frac{\partial f}{\partial y}(x, c) dx.$$

Hence $\dfrac{\partial \pi}{\partial y}(y) = |int_I \dfrac{\partial f}{\partial y}(x, y) dx$ exists for each $y \in I'$ and $\dfrac{\partial \pi}{\partial y}$ is continuous since $\dfrac{\partial f}{\partial y}$ is continuous by hypothesis. q.e.d.

■

3.9 The integral of continuous functions with compact support

Now let f be a continuous function of two variables which is defined on th set $[a_1, b_1] \times [a_2, b_2]$. We then define

$$\int f(x, y)\, dx\, dy = \int \left(\int f(x, y)\, dx \right) dy.$$

We can define

$$\int f(x_1, \ldots, x_n)\, dx_1 \ldots dx_n$$

similarly as an iterated integral.

We will se below that this definition is independent of the order in which we integrate. i.e. (for $n = 2$)

$$\int \left(\int f(x, y)\, dx \right) dy = \int \left(\int f(x, y)\, dy \right) dx.$$

We introduce the following notation: $\mathcal{K}(\mathbf{R}^{\int})$ is the set of continuous functions $f : \mathbf{R}^n \to \mathbf{R}$, for which a $K > 0$ exists with $f(x) = 0$ whenever $x \in \mathbf{R}^n, ||x|| > K$. If $f \in \mathcal{K}(\mathbf{R}^{\int})$, then $\int f(x_1, \ldots, x_s) dx_s \in \mathcal{K}(\mathbf{R}^{\int - \infty})$.
PROOF. Firstly it is clear that the integral exists, since the mapping $x_s \mapsto f(x_1, \ldots, x_{s-1}, x_s)$ is in $\mathcal{K}(\mathbf{R})$ for any choice of $(x_1, \ldots, x_{s-1})$ in $\mathcal{K}(\mathbf{R})$. The mapping $(x_1, \ldots, x_{s-1}) \mapsto \int f(x_1, \ldots, x_{s-1}, x_s) dx_s$ has compact support, of course, and is continuous by the above result.

■

Since the function

$$(x_1, \ldots, x_{s-1}) \mapsto \int f(x_1, \ldots, x_{s-1}, x_s) dx_s$$

is in $\mathcal{K}(\mathbf{R}^{\int-\infty})$, we can define the integral

$$\int \left(\int f(x_1, \ldots, x_{s-1}, x_s) dx_s \right) dx_{s-1}$$

and this provides us with a function in $K(\mathbf{R}^{s-2})$. Proceeding in this manner, we can define for each function $f \in \mathcal{K}(\mathbf{R}^{\int})$ the iterated integral

$$I(f) = \int \left(\ldots \left(\left(\int f(x_1, \ldots, x_s) dx_s \right) dx_{s-1} \right) \ldots \right) dx^1.$$

Proposition 26 *The functional I on $\mathcal{K}(\mathbf{R}^{\int})$ has the following properties:*

1. $I(f_1 + f_2) = I(f_1) + I(f_2)$;
2. $I(\alpha f) = \alpha I(f)$ *für* $\alpha \in \mathbf{R}$;
3. $I(f) \geq 0$ *for* $f \geq 0$;
4. $I(E^t f) = I(f)$ *for* $t \in \mathbf{R}^s$, *where* $E^t f$ *is the function* $x \mapsto f(x + t)$;
5. $|I(f)| \leq \prod(b_i - a_i)||f||$, *if Tr* $f \subset \prod[a_i, b_i]$ *(i.e.* $f(x) = 0$, *if* $x \notin \prod[a_i, b_i]$.*)*

PROOF. This follows immediately from thye properties of the one-dimensional integral for piecewise continuous functions.

■

We shall now show that the functional I is essentially (i.e. up to a constant) determined by these properties. This will justify our definition of the integral. We begin by extending our integral from the class $\mathcal{K}(\mathbf{R}^{\int})$ to a more general class of functions with compact support. We do this by means of a versions of Archimedes' exhaustion method which is analogue to the construction of the Riemann integral in one-dimension, with the difference that we approximate functions from above and below continuous ones.

Definition 7 *Let f be a non-negative bounded function with compact support on $\mathbf{R}^n$. We say that f is***integrierbar** *on $\mathbf{R}^n$, if for each $\epsilon > 0$ there are functions $g, h \in \mathcal{K}(\mathbf{R}^{\setminus})$ with $0 \leq g \leq f \leq h$ and $I(g - h) < \epsilon$. The integral $I(f)$ of f is then*

$$I(f) = \sup \subset 0 \leq g \leq g g \in \mathcal{K}(\mathbf{R}^{\setminus})\mathcal{I}(\}) = \inf \subset \{ \leq \langle\langle \in \mathcal{K}(\mathbf{R}^{\int})\mathcal{I}(\langle).$$

For general functions (i.e. not necessarily non-negative) we have:

Definition 8 *A bounded function f is integrable if $f^+ = \max(f,0)$ and $f^- = \max(-f,0)$ are integrable. We then put $I(f) = I(f^+) - I(f^-)$.*

Proposition 27 *If f_1 and f_2 are integrabls, then so are $f_1 + f_2$, αf_1, $f_1 f_2$, $\max(f-1, f_2)$, $\min(f_1, f_2)$ and $|f_1|$.*

Proof. We can confine our attention to non-negative functions. For each $\epsilon > 0$ there exists $g_1, h_1, g_2, h_2 \in \mathcal{K}(\mathbf{R}^\backslash)$ with $0 \le g_1 \le f_1 \le h_1$, $0 \le g_1 \le f_2 \le h_2$, $h_1 \le ||f_1|| + 1$, $h_2 \le ||f_2|| + 1$ and $I(h_i - g_i) < \epsilon$.

Hence: $g_1 + g_2 \le f_1 + f_2 \le h_1 + h_2$ and

$$I((h_1 + h_2) - (g_1 + g_2)) = I(h_1 - g_1) + I(h_2 - g_2) < 2\epsilon.$$

Further $g_1, g_2 \le f_1, f_2 \le h_1, h_2$ and

$$I(h_1, h_2 - g_1, g_2) = I((h_1 - g_1)h_2) + I(g_1(h_2 - g_2)) \le (||f_2|| + 1)I(h_1 - g_1) + (||f_1|| + 1)I(h_2 - g_2),$$

which implies the claim for the product.

Finally

$$|f| = f^+ + f^-, \max(f_1, f_2) = \frac{f_1 + f_2 + |f_1 - f_2|}{2}$$

and

$$\min(f-1, f_2) = \frac{f_1 + f_2 - |f_1 - f_2|}{2}.$$

■

Proposition 28 *If f, f_1, f_2 are integrable, then*

1. $I(f_1 + f_2) = I(f_1) + I(f_2)$
2. $I(\alpha f) = \alpha I(f), \alpha \in \mathbf{R}$
3. $I(f) \ge 0$ *for* $f \ge 0$
4. $I(E^t f) = I(f), t \in \mathbf{R}^s$
5. $|I(f)| \le \prod(b_i - a_i)||f||$, *if* $Tr\, f \subset \prod[a_i, b_i]$.

PROOF. If $f_1 \geq 0,\ f_2 \geq 0$ then 1) is a consequence of the definition. For general $f_1, f_2,\ f_1 + f_2 = (f_1 + f_2)^+_-(f_1 + f_2)^-$ and so $I(f_1 + f_2) = I(f_1 + f_2)^+) - I((f - 1 + f_2)^-)$. On the other hand, $f_1 + f_2 = f_1^+ + f_2^+ - (f_1^- + f_2^-)$ and hence

$$(f_1^+ + f_2^+) + (f_1 + f_2)^- = (f_1 + f_2)^+ + (f_1^- + f_2^-).$$

Since only non-negative functions occur in this formula, we have

$$I(f_1^+ + f_2^+) + I((f - 1 + f_2)^-) = I((f_1 + f_2)^+) + I(f - 1^- + f_2^-).$$

$$\Rightarrow I((f_1+f_2)^+) - I((f_1+f_2)^-) = I(f_1^+) + I(f_2^+) - I(f_1^-) - I(f_2^-) = I(f_1) + I(f_2).$$

The other properties are proved analogously.

■

Proposition 29 *Let $J : \mathcal{K}(\mathbf{R}^\backslash) \to \mathbf{R}$ be a functional with the following properties:*

1. $J(f_1 + f_2) = J(f_1) + J(f_2)$
2. $J(\alpha f) = \alpha J(f), \alpha \in \mathbf{R}$
3. $J(f) \geq 0$ *for* $f \geq 0$
4. $J(E^t f) = J(f), t \in \mathbf{R}^n$
5. $|J(f)| \leq K \prod(b_i - a_i)||f||$ *for each* f *with* $Trf \subset \prod[a_i, b_i]$ *where the constant* K *is independent of* f.

Then there is a $l \geq 0$ *with* $J(f) = lI(f)$ *for each* $f \in \mathcal{K}(\mathbf{R}^\backslash)$.

PROOF. Let J be such a function and $M = \prod[a_i, b_i]$ a compact interval. Then for each $\epsilon > 0$ there are functions $g, h \in \mathcal{K}(\mathbf{R}^\backslash)$ with $g \leq \chi_M \leq h$ and $J(h - g) < \epsilon$. Hence $J(E^t\chi_M) = J(\chi_M)$, since $E^t g \leq E^t\chi_M \leq E^t h$ and $J(E^t h - E^t g) = J(h - g) < \epsilon$.

In particular, if W^1 is the unit cube $0 \leq x^i < 1$, $i = 1, 2, \ldots, s$ and W a cube of the form $a^i \leq x^i < a^i + \frac{1}{k}$, then

$$J(\chi_W) = \frac{1}{k^s} J(\chi_{W^1}),$$

since all of these cubes have the same integral and k^n of them cover W^1. We now partition $\mathbf{R}^n$ in cubes W_i whose sides have length $\frac{1}{k}$ in the natural way. If $f \in K(\mathbf{R}^n)$, then f is uniformly continuous. For sufficiently large k,

$|x-y| < \frac{1}{k}$ implies $|f(x)-f(y)| < \epsilon$. Now let $M = \prod[a_i, b_i]$ be a copact cube with whole number a_i, b_i, which contains $\mathrm{Tr}\, f$. M is covered by finitely many W_i'. Let

$$\lambda_i = \inf_{x \in W_i} f(x) \text{ and } \mu_i = \sup_{x \in W_i} f(x).$$

Then

$$\sum l_i \chi_{W_i} \leq f \leq \sum \mu_i \chi_{W_i}$$

and

$$I\left(\sum \mu_i \chi_{W_i}\right) \leq 2\epsilon \prod (b_i - a_i).$$

Let $J(\chi_{w^1}) = \lambda$. Then $J(\chi_{W_i}) = \frac{\lambda}{k^n}$. It follows from 3) that

$$J\left(\sum \lambda_i \chi_{W_i}\right) \leq J(f) \leq J\left(\sum \mu_i \chi_{W)i}\right)$$

and so

$$\frac{\lambda}{k^n} \sum \lambda_i \leq J(f) \frac{\lambda}{k^n} \sum \mu_i$$

or $\lambda I(\sum \lambda_i \chi_{W_i}) \leq J(f) \leq \lambda I(\sum \mu_i \chi_{W_i})$. On the other hand

$$I\left(\sum l_i \chi_{W_i}\right) \leq I(f) \leq I\left(\sum \mu_i \chi_{W-i}\right)$$

and so

$$|J(f) - \lambda_I(f)| \leq 2\lambda\epsilon \prod (b_i - a_i).$$

Hence $J(f) = \lambda I(f)$ mit $\lambda = J(\chi_{W_1})$.

∎

A simple consequence of this fact ist that the integral in the defininition of $I(f)$ is independent of the order of integration as remarked initially.

Proposition 30 *Let* $J(f) = \int \ldots (\int f(x_1, \ldots, x_s)\, dx_{i_1})\, dx_{i_2} \ldots dx_{i_s}$, *whereby* $\{i_1, i_2, \ldots, i_s\}$ *is a permutation of* $\{1, 2, \ldots, \}$. *Then* $J(f) = I(f)$.

PROOF. $J(f)$ satisfies 1) — 5) and $J(\chi_{W_1}) = 1$.

∎

Hence we can write simply $I(f) = \int_{\mathbf{R}^s} f(x) dx$, where $dx = dx_1 \ldots dx_s$.

We can now define the concepts of area or volume in a precise way:

Definition 9 *If* $A \subset \mathbf{R}^s$ *and its characteristic function* χ_A *is integrable in* $\mathbf{R}^n$, *then* $m(A) = I(\chi_A)$ *is the (n-dimensional)* **measure** *or* **content** *of* A. A *is then said to be* **integrable**. *(For* $n = 2$ *we refer to area, and for* $n = 3$ *volume.)*

Remark In the manner we obtain a class of integrable sets which contains all of those sets likely to arise in paractice.

3.10 The tranformation law

Proposition 31 *Let $g = (g_1, \dots, g_n)$ be a C^1 function on the open subset U of $\mathbf{R}^n$. We suppose that g is injective and $\det J_g(x) \neq 0$ on U. Let f be continuous on $g(U)$. Then*

$$\int_{g(U)} f(x)\, dx = \int_U f \circ g(u) |\det J_g(u)|\, du.$$

PROOF. Sketch of proof We consider the functional

$$f \mapsto \int_U f \circ g(u) |\det J_g(u)|\, du.$$

It can be shown that this satisfies the conditions above which characterise the integral.

■

Example Calculate $\int_V x^2 z\, dxdydz$ where

$$V = \{(x, y, z) : x^2 + y^2 \leq a^2, 0 \leq z \leq h\}.$$

We use cylindrical coordinates

$$\phi : (r, \theta, \zeta) \mapsto (r \cos\theta, r \sin\theta\zeta)$$

$\phi(V') = V$ where

$$V' = \{(r, \theta, \zeta) : 0 \leq r \leq a, \theta \in [0, 2\pi], 0 \leq \zeta \leq h\}.$$

Hence

$$\int_r x^2 z dxdydz = \int_0^h \int_0^{2\pi} \int_0^a (r\cos\theta)^2 r d\zeta d\theta dr = \frac{\pi a^4 h^2}{8}.$$

Example We calculate a formula for the volume enclosed by the surface of revolution $0 \leq x^2 + y^2 \leq (\phi(z))^2$: Then

$$\begin{aligned} V &= \int_a^b \iint_{0 \leq (x^2 y^2) \leq \phi(z)^2} dxdydz \\ &= \int_a^b dz \int_0^{2\pi} d\theta \int_0^{\phi(z)} r\, dr \quad \text{(in cylindrical coordinates)} \\ &= \pi \int_a^b \phi(z)^2 dz. \end{aligned}$$

3.11 Applications of the integral

1. If $\rho(x,y,z)$ is the density (x,y,z) of a body, then

$$\iiint_G \rho(x,y,z)\,dxdydz$$

is the mass of G.

2. Moments: The quantities

$$\begin{aligned} T_x &= \iiint_G x\rho(x,y,z)\,dxdydz, \\ T_y &= \iiint y\rho(x,y,z)\,dxdydz, \\ T_z &= \iiint z\rho(x,y,z)\,dxdydz \end{aligned}$$

are the moments of the body G.

3. The centre of gravity of G is the point $(\overline{x},\overline{y},\overline{z})$, where

$$\overline{x} = \frac{\iiint x\rho(x,y,z)\,dxdydz}{\iiint \rho(x,y,z)\,dxdydz} = \frac{T_x}{V} \text{ etc.}$$

4. The moments of inertia I_x, I_y, I_z with respect to the $x-$ (resp. $y-, z-$) axes are:

$$I_x = \iiint (y^2+z^2)\rho(x,y,z)\,dxdydz, \text{ -- etc.}$$

5. The gravitational potential of a body G with density ρ is

$$f(x,y,z) = \iiint_G \frac{\rho(x,y,z)}{\sqrt{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2}}\,d\xi d\eta d\zeta.$$

(The corresponding force field is grad f).

Exercise 1. Calculate $T_x, T_y, T_z, \overline{x}, \overline{y}, \overline{z}$ for the homogeneous hemisphere $\{(x,y,z) = x^2+y^2+z^2 \le 1, z \ge 0\}$, $T_x = T_y = 0$ (by symmetry).

$$\begin{aligned} T_z = \iiint_V z\,dxdydz &= \int_0^1 z dz \int_0^{\sqrt{1-z^2}} r dr \int_0^{2\pi} d\theta \\ &= 2\pi \int_0^1 \tfrac{1-z^2}{2} z\,dz = \tfrac{\pi}{4}. \end{aligned}$$

Since the total mass is $= \dfrac{2\pi}{3}$ we have

$$\overline{x} = 0, \overline{y} = 0, \overline{z} = \frac{3}{8}.$$

Example Calculate I_x for the ball with unit density. We have

$$\begin{aligned} I_x &= \iiint (y^2+z^2)\,dxdydz = I_y = I_z \text{ by symmetry} \\ &= \tfrac{1}{3}\iiint (x^2+y^2+z^2)\,dxdydz \\ &= \tfrac{2}{3}\int_0^1\!\int_0^{\pi}\!\int_0^{2\pi} r^4 \sin v dr dv du = \tfrac{8\pi}{15}. \end{aligned}$$

3.12 Exercise

Exercise: Show that if A and B are integrable, then so are $A \cup B$, $A \cap B$ and for $B \subset A$ $A \setminus B$. If $A \cap B = \emptyset$, then $m(A \cup B) = m(A) + m(B)$.

Exercise: Show that

$$F(y) := \int_1^2 \ln(xy)dx, \qquad y \in \mathbf{R}_+^*,$$

defines a function $F : \mathbf{R}_+^* \to \mathbf{R}$ which is continuously differentiable and calculate F'.

Exercise: Let $g : \mathbf{R}^2 \to \mathbf{R}$ be the function

$$g(x,y) := \begin{matrix} \frac{2x^3y}{(x^2+y^2)^2} & \text{for} & (x,y) \neq (0,0) \\ 0 & \text{for} & (x,y) = (0,0) \end{matrix}.$$

Show that for

$$f(x) := \int_0^1 g(x,t)dt \text{ and } f^*(x) := \int_0^1 (D_1 g)(x,t)dt.$$

we have: f is differentiable at 0 but $f(0) \neq f^*(0)$.

Exercise: Let $I \subset \mathbf{R}$ be an open interval, $a \in I$ and $f : I \times I \to \mathbf{R}$ a continuously partially differentiable function. Show that

$$F(y) := \int_a^y f(x,y)dx, \qquad y \in I,$$

is differentiable on I for each $y \in I$ we have

$$F'(y) = f(y,y) + \int_a^y (D_2 f)(x,y)dx.$$

(Hint: show that

$$g(y,z) := \int_a^z f(x,y)dx, \qquad (y,z) \in I \times I,$$

is continuously partially differentiable on $I \times I$ and use the chain rule.

Exercise: Let $a, b, c, d \in \mathbf{R}$, $a < b$ and $\mathcal{R} := \{\{ : [\dashv, \lfloor] \to \mathbf{R} : \{$ is twoce continuously partially differentiable and $f(a) = c,\ f(b) = d\}$. $\ell(f)$ denotes the length of the curve

$$g : [a, b] \to \mathbf{R}^1, t \mapsto (t, f(t)),$$

. Determine the Euler equations for the variational problem

$$\ell(f) = \min \quad (f \in \mathcal{R})$$

and show that the straight line through (a, c) and (b, d) is the unique solution.

Exercise: Calculate

$$\int_R (3x^2y + 2y \sin z) dxdydz$$

where $R = \{(x, y, z) : -1 \leq x \leq 1, 0 \leq y \leq 2, 0 \leq z \leq \frac{\pi}{2}\}$.

Exercise: Let $\operatorname{erf}(x) = \int_0^x e^{-u^2}\, du$. Show thet

1. $\operatorname{erf}^2(x) = \int_R e^{-u^2-v^2} dudv$, where R is a suitable rectangle;
2. $\lim_{x\to\infty} \int_R e^{-u^2-v^2} dudv = \pi$;
3. $\int_0^\infty e^{-u^2}\, du = \frac{\sqrt{\pi}}{2}$.

Exercise: Determine the form of the tranformation rule for the following coordinate transformations: Polar coordinates in $\mathbf{R}^2$, cylindrical coordinates in $\mathbf{R}^3$, spherical coordinates in $\mathbf{R}^3$.

Exercise: Calculate the following integrals:

1.
$$\int_R (x^2 + y^2)\, dx\, dy \text{ where } R = \{(x, y) : 1 \leq x \leq 2, -1 \leq y \leq 1\};$$

2.
$$\int_R x \sin y\, dx\, dy \text{ where } R = \{(x, y) : 0 \leq x \leq 1, x^2 \leq y \leq 2x^2\};$$

3.
$$\int_R (xy+2)\, dx\, dy \text{ where } R = \{(x,y) : 1 \le y \le 2, y^2 \le x \le y^3\};$$

4.
$$\int_R x \cos y \underset{\cdot}{x}\, dy \text{ where } R = \{(x,y) : 0 \le y \le \frac{\pi}{2}, 0 \le x \le \sin y\};$$

5.
$$\int_R xy\, dx\, dy$$

where R is the triangle with vertices $(1,0)$, $(2,2)$ and $(1,2)$.

6.
$$\int_R e^{xy} \underset{\cdot}{x}\, dy \text{ where } R = \{(x,y) : 0 \le x \le 1 + \frac{\log y}{y}, 2 \le y \le 3\};$$

7.
$$\int_R (x^3 + 2xy) \underset{\cdot}{x}\, dy$$

where R is the parallelogram with vertices $(1,3)$, $(3,4)$, $(4,6)$ and $(2,5)$.

Exercise: Calculate
$$\int_0^1 \int_{x^2}^1 x\sqrt{1-y^2}\, dy\, dx.$$

Exercise: Calculate
$$\int_1^2 \int_1^x \frac{x}{\sqrt{x^2+y^2}}\, dy\, dx.$$

Exercise: Calculate
$$\int_1^4 \int_1^{\sqrt{x}} \frac{e^{xy^{-1}}}{y^5}\, dy\, dx.$$

Exercise: Calculate the area between the curves $y = 2x^2$ and $x = 4y^2$.

Exercise: Calculate the area, mass and centre of gravity of the region between the curves $x^2 - y^2 = 1$ and $x = 4$ with density $\rho(x,y) = x$.

Exercise: Calculate the following integrals:

1.

$$\int_R (x+xz-y^2)\,dx\,dy\,dz \text{ where } R = \{(x,y,z) : 0 \le x \le 1, -2 \le y \le 0, 3 \le z \le 5\};$$

2.

$$\int_R (x+z)\,dx\,dy\,dz \text{ where } R = \{(x,y,z) : 0 \le x, 0 \le y, 0 \le z, x+y+2z \le 3\};$$

3.

$$\int_R xyz\,dx\,dy\,dz \text{ where } R = \{(x,y,z) : 0 \le x, 0 \le y, 0 \le z, x^2+y^2+z^2 \le 1\};$$

Exercise: Calculate the volume of the region

$$x_1 \ge 0, x_2 \ge 0, \ldots, x_n \ge 0, x_1 + x_2 + \cdots + x_n \le 1$$

4 The inverse function theorem and the implicit function theorem

4.1 Definitions

Definition 10 *A C^1-mapping $f : U \to \mathbf{R}^m$, where U is open in $\mathbf{R}^n$, is called a C^1-***Isomorphismus** *from U onto $f(U)$, if $f(U)$ is also open in $\mathbf{R}^m$ and a C^1-mapping $g : f(U) \to \mathbf{R}^n$ exists with $g \circ f = Id$ and $f \circ g = Id$ (in other words, f is a bijection and its inverse is also C^1).*

Remark As we already saw in the one-dimensional case, there are bijective C^1-mappings for which the inverse is not differentiable e.g. the mapping $x \mapsto x^3$ on $]-1, 1[$.

4.2 The inverse function theorem

Proposition 32 *If f is a C^1-isomorphism between the open set $U \subset \mathbf{R}^n$ and $g(U) \subset \mathbf{R}^m$, then $m = n$ and $(Df)_x$ is and invertible linear mapping on $\mathbf{R}^n$ for each x.*

Proof. There is an $g : f(U) \to \mathbf{R}^n$ with $g \circ f = \mathrm{Id}$ and $f \circ g = \mathrm{Id}$. Hence by the chain rule we have

$$(Dg)_{f(x)}(Df)_x = \mathrm{Id}$$

and

$$(Df)_x(Dg)_{f(x)} = \mathrm{Id}$$

for each x. Hecne the linear mapping $(Df)_x$ has an inverse and this implies that $m = n$.

∎

Lemma 1 *Let U be an open subset of $\mathbf{R}^n$ and $x, y \in U$ so that the segement from x to y lies in U. Then*

$$|f(y) - f(x)| \leq |y - x| \sup_v ||(Df)_v||,$$

where v runs through this segment.

PROOF.

$$\left|\int_0^1 (Df)_{x+th}(h)dt\right| \leq \int_0^1 |(Df)_{x+th}(h)|dt$$
$$\leq \int_0^1 ||(Df)_{x+th}||\,|h|dt \leq \sup_t ||(Df)_{x+th}||\,|h|.$$

■

Proposition 33 *Let U, x and y be as in the Lemma and f a C^1-function*

$$|f(y) - f(x) - (Df)_{x_0}(y-x)| \leq |y-x| \sup_v ||(Df)_v - (Df)_{x_0}||.$$

PROOF. Put $g(x) = f(x) = (Df)_{x_0}(x)$.

■

Definition 11 *A mapping f is a* **local C^1-isomorphism** *at a point x, if there is an open neighbourhood U of x, so that the restriction of f to U is a C^1-isomorphism from U onto $f(U)$.*

It is then clear that $f \circ g$ is a local C^1-isomorphism at x, when g is a local C^1-isomorphism at x and f a local C^1-isomorphism at $g(x)$.

Proposition 34 *(inverse function theorem) Suppose that U is an open subset of $\mathbf{R}^n$ and $f : U \to \mathbf{R}^n$ is a C^1-mapping. Let $x_0 \in U$ be such that $(Df)_{x_0} : \mathbf{R}^n \to \mathbf{R}^n$ is invertible.*

Then f is a local C^1-isomorphism at x_0.

PROOF. We can assume without loss of generality that $(Df)_{x_0} = \mathrm{Id}$. Further that $x_0 = f(x_0) = 0$. We introduce the function $g(x) = x - f(x)$. Then $g(0) = 0$ and $(Dg)_0 = 0$. Since Id and f are C^1, then $x \mapsto (Dg)_x$ and $x \to ||(Dg)_x||$ are continuous. Since $(Dg)_0 = 0$ there is an $r > 0$, so that for each x with $|x| < 2r$ we have $||(Dg)_x|| < \dfrac{1}{2}$.

From the intermediate value theorem

$$|g(x)| = |g(x) - g(0)| < |x| \sup_{|x|\leq 2r} ||(Dg)_x|| < \frac{|x|}{2}.$$

Hence g mapst the closed ball $\overline{B}(0,r)$ into the closed ball $\overline{B}(0,\dfrac{r}{2})$.

We claim that for each $y \in \overline{B}(0,\dfrac{r}{2})$ there is exactly one $x \in \overline{B}(0,r)$ with $f(x) = y$. This means that f is locally invertible. In order to verify this we consider for each $y \in \overline{B}(0,\frac{r}{2})$ the mapping g_y where

$$g_y(x) = y + x - f(x).$$

If $\|x\| \leq r$, then

$$|g_y(x)| \leq |y| + |x - f(x)| \leq |y| + |g(x) \leq \frac{r}{2} + \frac{r}{2} = r.$$

Hence the mapping g_y maps the closed set $\overline{B}(0, r)$ into itself.

Further

$$|g_y(x_1) - g_y(x_2) = |g(x_1) - g(x_2) = \left| \int_0^1 (dg)_{x_1+t(x_2-x_1)}(x_2 - x_1)dt \right| \leq \frac{x_1 - x_2|}{2}$$

by the intermediate value theorem.

We can thus apply the Banach fixed point theorem to obtaine a unique fixed point x for g_y with $|x| \leq r$. The equality $g_y(x) = x$ means that $f(x) = y$.

Hence we can define a mapping f^{-1} on $\overline{B}(0, \frac{r}{2})$ by putting $f^{-1}(y) = x$. We now show that this mapping f^{-1} is continuous. If we put $x = x - f(x) + f(x) = g(x) + f(x)$, then

$$|x_1 - x_2| \leq |f(x_1) - f(x_2)| + |g(x_1) - g(x_2)| \leq |f(x_1) - f(x_2)| + \frac{1}{2}|x_1 - x_2|$$

and so

$$|x_1 - x_2| = |f^{-1}(f(x_1)) - f^{-1}(f(x_2))| \leq 2|f(x_1) - f(x_2)|,$$

which implies the continuity.

We now show that f^{-1} is differenitable: Let $U = B(0, \frac{r}{2})$ and $y, y_1 \in U$, $x = f^{-1}(y)_1$, $x_1 = f^{-1}(y_1)$. Then $|x| < r$, $|x_1| \leq r$. Hence $||(Dg)_{x_1}|| \leq \frac{1}{2}$, i.e. $||\text{Id} - (Df)_{x_1}|| \leq \frac{1}{2} < 1$. Hence there exists

$$(Df)_{x_1}^{-1} = \sum_{k=0}^{\infty} (\text{Id} - (Df)_{x_1})^k.$$

If f^{-1} is differentiable, then we would have $f \circ f^{-1}(y_1) = y_1$ and $(Df)_{x_1}(Df^{-1})_{y_1} = \text{Id}$, i.e. $(Df^{-1})_{y_1} = (Df)_{x_1}^{-1}$.

Hence we consider the expression

$$|f^{-1}(y) - f^{-1}(y_1) - (Df)_{x_1}^{-1}(y - y_1)| = |x - x_1 - (Df)_{x_1}^{-1}(f(x) - f(x_1))|.$$

Since f is differentiable at the point x_1, we have

$$f(x) = f(x_1) + (Df)_{x_1}(x - x_1) + R(x, x_1) \text{ with } \lim_{x \to x_1} \frac{|R(x, x_1)|}{|x - x_1|} = 0.$$

Hence

$$\begin{aligned}|x - x_1 - (Df)_{x_1}^{-1}(f(x) - f(x_1))| &= |x - x_1 - (Df)_{x_1}^{-1}((Df)_{x_1}(x - x_1) + R(x, x_1))| \\ &\leq |(Df)_{x_1}^{-1}(R(x, x_1)) \leq ||(Df)_{x_1}^{-1}|| \tfrac{|R(x,x_1)|}{|x-x_1|}|x - x_1| \\ &\leq ||(Df)_{x_1}^{-1}|| \tfrac{|R(x,x_1)|}{|x-x_1|} 2|y - y_1| \\ &\Rightarrow \lim_{y\to y_1} \tfrac{|f^{-1}(y)-f^{-1}(y_1)-(Df)_{x_1}^{-1}(y-y_1)|}{|y-y_1|} = 0.\end{aligned}$$

For by the continuity of f^{-1} we have $\lim x = x_1$. Hence f^{-1} is differentiable at y_1 and

$$(Df^{-1})_{y_1} = (Df)_{x_1}^{-1} = (Df)_{f^{-1}(y_1)}^{-1}.$$

Now each of the mappings $y \mapsto f^{-1}(y), x \mapsto (Df)_x$ and $A \in L(\mathbf{R}^n) \mapsto A^{-1} \in L(\mathbf{R}^n)$ are continuous and hence so is the composition $y \mapsto (Df^{-1})_y = (Df)_{f^{-1}(y)}^{-1}$, i.e. f^{-1} is a C^1-isomorphism and f is locally C^1-invertible.

■

Examples $f : \mathbf{R}^n \setminus \{0\} \to \mathbf{R}^n\{0\}$, where $f(x) = \dfrac{x}{|x|^2}$ is a C^1-isomorphism. For f is injective i.e. $f(x_1) = f(x_2)$ implies $x_1 = x_2$. For if $f(x_1) = f(x_2)$, then $|f(x_1)| = |f(x_2)|$.

Hence $\dfrac{|x_1|}{|x_1|^2} = \dfrac{|x_2|}{|x_2|^2}$, i.e. $|x_1| = |x_2|$ and so $x_1 = |x_1|^2 f(x_1) = |x_2|^2 f(x_2) - x_2$.

The mapping is also surjective i.e. its image is $\mathbf{R}^n \setminus \{0\}$, since $f^{-1} = f$ as one can calculate without effort. Since $x \mapsto \dfrac{1}{|x|^2}$ onto $\mathbf{R}^n \setminus \{0\}$is C^1 then f is also C^1.

We can calculate the derivative of this function as follows:

$$(Df)_x(h) = \lim_{t\to 0} \frac{1}{t} \left(\frac{x + th}{|x + th|^2} - \frac{x}{|x|^2} \right) =$$

$$\begin{aligned}&= \lim_{t\to 0} \tfrac{x|x|^2+th|x|^2-x|x|^2-2t(x|h)x-xt^2|h|^2}{t|x+th|^2|x|^2} \\ &= \tfrac{h|x|^2-2(x|h)x}{|x|^4} = \tfrac{1}{|x|^4}(|x|^2 I - 2xx^t)(h).\end{aligned}$$

In the two-dimensional case $n = 2$ for example

$$(f_1(x, y), f_2(x, y)) = \left(\frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2} \right)$$

and

$$(Df)_{(x,y)} = \frac{1}{(x^2 + y^2)^2} \begin{bmatrix} y^2 - x^2, & -2xy \\ -2yx & x^2 - y^2 \end{bmatrix}$$

In this case we thus have

$$\det\,(Df)_{(x,y)} = -1.$$

2) The mapping $g : \mathbf{R}^2 \to \mathbf{R}^2$ where $g(r,\theta) = (r\cos\theta, r\sin\theta)$ has

$$\det\,(Dg)_{(r,\theta)} = \det \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix} = r$$

and so is a local C^1-iomorphism at any point with $r \neq 0$.

If $r = 0$ then g maps any point $(0,\theta)$ onto $(0,0)$ and so is no longer injective in any neighbourhood of the point.

3) A smooth mapping f can be a local C^1-isomorphism at each point with being globally invertible. Such an example is the mapping $z \to e^z$ of the complex plane which has the form

$$(x,y) \mapsto (e^x \cos y, e^x \sin y)$$

as mapping on $\mathbf{R}^2$.

In this case

$$(Df)_{(x,y)} = \begin{bmatrix} e^x\cos y & -e^x\sin y \\ e^x\sin y & e^x\cos y \end{bmatrix}$$

and $\det(Df)_{(x,y)} = e^{2x} \neq 0$. However, $f(x, y+2\pi) = f(x,y)$.

4) The mapping $A \mapsto A^{-1}$ on the set of all invertible $n \times n$-matrices is a C^1-isomorphism. (Note that we can identify M_n, the space of $n \times n$-matrices with the space $\mathbf{R}^{n^2}$ and so talk about differentiability of functions thereon). The function $A \mapsto \det A$ is a polynomial in the coefficients of A and so continuous. Hence the family of invertible matrices, being the pre-image of the singleton 0 under this mapping is open in M_n. The coordinates of A^{-1} have the form $\dfrac{A_{ik}}{\det A}$, where A_{ik} is a suitable subdeterminant and hence are once again polynomials in the coefficients and so continuously differentiable. Hence the mapping $A \mapsto A^{-1}$ is C^1. We calculate its differential as follows: be definition it is the linear mapping $H \mapsto \Phi(H)$ for which

$$\lim_{||H||\to 0} \frac{||(A+H)^{-1} - A^{-1} - \Phi(H)||}{||H||} = 0.$$

Now for small H i.e. $||H|| < \delta$,

$$(A+H)^{-1} - A^{-1} = [A(I + A^{-1}H)]^{-1} - A^{-1} = (I + A^{-1}H)^{-1}A^{-1} - A^{-1}$$

$$= (I - A^{-1}H + (A^{-1}H)^2 \pm \dots)A^{-1} - A^{-1}$$
$$= -A^{-1}HA^{-1} + A^{-1}HA^{-1}HA^{-1} + \dots$$

Hence $\Phi(H) = -A^{-1}HA^{-1}$. The mapping Φ is obviously invertiable. Hence $A \mapsto A^{-1}$ is a local C^1-isomorphismus and so a global C^1-isomorphismus (in fact, it is its own inverse so that the differentiability of the latter is automatic).

As one would expect, the smoothness properties of f^{-1} reflect those of f:

Proposition 35 *Let f be a C^p-mapping from an open subset $U \subset \mathbf{R}^n$ in $\mathbf{R}^n$ with $\det(Df)_0 \neq 0$ for each $x_0 \in U$. Then there is a neghbourhood U of x_0 as in the inverse function theorem so that f^{-1} is a C^p-mapping on $f(U)$.*

PROOF. We use the U of the statement of the inverse function theorem and sho that the partial derivatives

$$\frac{\partial^m (f^{-1})^j}{(\partial y^1)^{i_1} \dots (\partial y^s)^{i_s}}$$

with $m \leq p$ exist and are continuous.

As noted above, the elements of the inverse A^{-1} regarded as functions of the elements of A are C^∞ (in fact they are rational functions). We have

$$(Df^{-1})_y = (Df)^{-1}_{f^{-1}(y)}.$$

Hecne if the partial derivatives of order k exist for$y \mapsto f^{-1}(y)$, then the same is true of $(Df)_{f^{-1}(y)}$ and hence of $(Df^{-1})_y$, i.e. all partial derivatives of order $(k+1)$ for f^{-1} and they are also continuous. We can continue this process up to $k = p - 1$.

■

4.3 Implicit functions

The motivation for the following result is the fact that if a variable y is defined implicitly by an equation of the type $f(x, y) = 0$, then we can solve this to obtain y as a function of x. (Geometrically this means that we can express the level line $f(x, y) = 0$ of the function f as the graph of a function. For example if $x^2 + y^2 - 1 = 0$, then we can solve for y to obtain $y = \sqrt{1 - x^2}$. This simple example already displays the complexity of the situation. The above fact is in general only true locally (the whole circle cannot be obtained as the graph of a function) and in some points (where $x = \pm 1$) is not true at all.

It is the purpose of the implicit function theory to describe exactly in what sense the abvoe vague claim is true.

For the sake of clarity we begin with the two-variable case:

Proposition 36 *(implicit function theorem) Let F be a C^1 real-valued mapping defined on an open set $U \subset \mathbf{R}^2$. Let $(x_0, y_0) \in U$ and suppose that $F_2(x_0, y_0) \neq 0$ then there exists a unique continuously differentiable function f defined on a suitable opn neighbourhood $U(x_0)$ of the point x_0, so that $f(x) = y \iff F(x, f(x)) = 0$ on $U(x_0)$.*

PROOF. We consider the mapping $\phi : U \to \mathbf{R}^2$ whereby

$$\phi \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ F(x,y) \end{bmatrix}.$$

Then

$$\det(D\phi)_{(x,y)} = \det \begin{bmatrix} 1 & 0 \\ f_x(x,y) & F_2(x,y) \end{bmatrix} = F_2(x,y).$$

Since $F_2(x_0, y_0) \neq 0$, ϕ is a local C^1-isomorphism at (x_0, y_0). Hence there exists an open neighbourhood $V(x_0, y_0)$, whose image $\phi(V(x_0, y_0)) = W(x_0, 0)$ is an open neighbourhood of the point

$$\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} x_0 \\ 0 \end{bmatrix}$$

and on which ϕ has a C^1-inverse ψ. This mapping has the form

$$\psi \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ g(x,y) \end{bmatrix}$$

where g is C^1.

Since $\phi\psi(x,y) = (x,y)$ on $W(x_0, 0)$ and $\phi\psi(x,y) = \phi(x, g(x,y)) = (x, F(x, g(x,y)))$ we have $F(x, g(x,y)) = y$ on $W(x_0, 0)$. $W(x_0, 0)$ contains an open set of the form $U(x_0) \times W(0)$ and $f : x \mapsto g(x, 0)$ is continuously differentiable on $U(x_0)$. Further we have $F(x, f(x)) = F(x, g(x, 0) = 0$ which proves the theorem.

■

Examples 1) We consider once again the case $F(x,y) = x^2 + y^2 - 1$. Then $F_2 = 2y \neq 0$ except at the points $(\pm 1, 0)$. As noted above there exists no function f as in the statement near either of these point.

2) Let $F(x,y) = x^2 y^5 - \sin(x - y) + x = 0$ and $(x_0, y_0) = (0, 0)$. Then $F_2 = 5x^2 y^4 + \cos(x - y)$ and so $F_y(0, 0) = 1 \neq 0$. Hence we can find a solution of the equation $f(x, y) = 0$ of the form $y = f(x)$ in a neighbourhood of $x = 0$.

3) $F(x, y) = y^3 - x$. Then $F(0,0) = 0$ and $F_2(x, y) = 3y^2$. Hence $F_y(0,0) = 0$. In this case we can solve the equation $y^3 - x = 0$ in the form of a function but the latter is not differentiable at the crucial point.

4) We consider once again the case of the contour lines of a real-valued, C^1-function f of two variable, defined on the open set U. If for each $(x, y) \in U$ with $f(x, y) = c$ the differential $(Df)_{(x,y)}$ (i.e. the gradient vector), then the set $f(x, y) = c$ discribes a smooth curves (which can be composed of several "pieces"). This follows from the fact that for each $(a, b) \in U$ with $f(a, b) = c$ there is an open neighbourhood in which the set of those (x, y) with $f(x, y) = c$ is a smooth curve through (a, b).

But since $Df = (f_x, f_y) \neq 0$ i.e. either $f_1 \neq 0$ or $f_2 \neq 0$ and so according to the case, the above set has the form of the graph of a function $y = f(x)$ or $x = g(y)$ near (a, b).

We now show how to calculate the derivatives of the function whose existence is ensured by the above result. We have the identity $F(x, f(x)) = 0$. Near the point x_0. Using the chain rule and differentiation we get

$$0 = \frac{d}{dx}F(x, f(x)) = F_x + F_y f'(x).$$

Since $F_y(x_0, y_0) \neq 0$, we have

$$f'(x_0) = -\frac{F_x(x_0, y_0)}{F_y(x_0, y_0)}.$$

(We have used the notation F_x and F_y to denote the partial derivatives of F with respect to x and y).

Example Consider the circle $x^2 + y^2 = 1$. Let (x_0, y_0) be a point with $y_0 > 0$. Then in a neighbourhood of x_0 we have $y = f(x) = \sqrt{1 - x^2}$. Hence we can calculate $f'(x)$ directly. (in fact $f(x) = \frac{-x}{\sqrt{1-x^2}} = -\frac{x}{y}$). Using the methdo sketched above, we get $F_x = 2x$, $F_y = 2y$ and so (once again) $f(x) = -\frac{2x}{2y} = -\frac{x}{y}$.

Example The cases $x^2 + y^2 = 0$ (near $(0,0)$) or $x^2 - y^2$ (at the same point) gives n indication of what can happen to the contour lines at a point where the gradient vanishes.

In our version of the implicit function theorem we assumed that F was C^1 and deduced that f was also C^1. Of course, there is a corresponding result for higher levels of smoothness:

Proposition 37 *If in the hypothese of the implicit function theorem we assume that F is C^p, then f is also p-times continuously differentiable.*

4.4 Calculation of higher derivatives

There are various possibilities. For example, one can differentiate the equation $F_x + F_y f'(x) = 0$ as above to get

$$\begin{aligned} 0 &= \tfrac{d}{dx}(F_x(x, f(x)) + F_y(x, f(x))f'(x)) = F_{xx}(x, f(x)) \\ &+F_{xy}(x, f(x))f'(x) + F_{yx}(x, f(x))f'(x) + F_{yy}(x, f(x))(f'(x))^2 \\ &+F_y(x, f(x))f''(x) \\ &\Rightarrow f''(x) = -\tfrac{F_{xx}+2F_{xy}f'(x)+F_{yy}(F_x)^2}{(F_y)^3}. \end{aligned}$$

If we substitute $f'(x) = -\dfrac{F + x}{F_y}$ then we get the equation

$$f''(x) = \frac{2F_{xy}F_xF_y - F_{xx}F_y^2F_{yy}(F_x)^2}{F_y)^3}.$$

Another possibility is to develop the Taylor series for $F(x_0 + x, y_0 + y) - F(x_0, y_0) = 0$ and then to sets the coefficients $= 0$. We thus get

$$F(x_0 + x + y) - F(x_0, y_0) = F_x x + F_y y + \frac{1}{2}(F_{xx}x^2 + 2F_{xy}xy + F_{yy}y^2) + \dots$$

But

$$y_0 + y = f(x + x_0) = f(x_0) + f'(x_0)x + \frac{f''(x_0)}{2} + \dots$$

and so

$$y = f'(x_0)x + \frac{f''(x_0)}{2}x^2 + \dots$$

If we make the corresponding substitution above we get

$$0 = F_x x+F_y\left(f'x + \frac{f''}{2}x^2 + \dots\right)+\frac{1}{2}F_{xx}x^2F_{xy}x\left(f'x + \frac{f}{2}x^2 + \dots\right)+\frac{1}{2}F_{yy}\left(f'x + \frac{f''}{2}x^2 + \dots\right)^2+$$

The coefficient of x is $F_x + F_y f'$, which gives us our formula for f' once again.

The coefficient of $\dfrac{x^2}{2}$ is

$$F_y f'' + F_{xx} + 2F_{xy}f' + F_{yy}(f')^2,$$

and again we get an expression for f'' $(= -\frac{F_{xx}+2F_{xy}f'+F_{yy}(f')^2}{F_y})$.

Example For $F(x,y) = \log \sqrt{x^2+y^2} - \arctan \frac{y}{x}$ we have

$$F_y = \frac{1}{\sqrt{x^2+y^2}} \frac{y}{\sqrt{x^2+y^2}} - \frac{1}{1+(\frac{y}{x})^2} \frac{1}{x} = \frac{y-x}{x^2+y^2}.$$

We calculate the second derivative of f. Differentiating $F(x,y) = 0$ with respect to x, we get

$$\frac{x+yy'}{x^2+y^2} - \frac{xy'-y}{x^2+y^2} = 0 \text{ oder } x + yy' = xy' - y.$$

$$\Rightarrow y' = \frac{x+y}{x-y}.$$

Differentiating once again we get $1 + (y')^2 + yy'' = xy''$.

$$\Rightarrow y'' = \frac{1+(y')^2}{x-y} = 2\frac{x^2+y^2}{(x-y)^3}.$$

WE now consider a C^1-funktion $F(x_1, x_2, \dots, x_n)$ of n and the question when we can solve for x_n as a function $x_n = f(x_1, \dots, x_{n-1})$ of the remaining variables.

This can be treated analogously: If F is defined on an open set U, $x_0 \in U$ is such that $F(x_0) = 0$ and $F_{x_s}(x_0) \neq 0$, then there exists a C^1-function $f(x_1, \dots, x_{n-1})$, defined on a suitable open neighbourhood $U((x_1^0, \dots, x_{n-1}^0))$ of $(x_1^0, \dots, x_{s-1}^0) \in \mathbf{R}^{n-1}$, so that $f((x_1^0, \dots, x_{n-1}^0)) = x_n^0$ und $F(x_1, \dots, x_{n-1}, f(x_1, \dots, x_{n-1})) = 0$ on this neighbourhood.

This is proved exactly as above. (Consider the function $\phi : U \to \mathbf{R}^n$, where

$$\phi(x_1, x_2 \dots, x_{n-1}, x_n) = (x_1, \dots, x_{n-1}, F(x_1, \dots, x_{n-1}, x_n)).$$

Then $(D\phi)_{x_0} = F_x(x_0) \neq 0$ and so there exists a local inverse ψ. We can then complete the argument as above. If $F \in C^p$, then so does f.

We can regard the above result as a precise formulation of the fact that one can solve one variable of an equation in n-variables as a function of the other variables (under suitable conditions). In general, one would expect to be able to solve for m unknowns in a system of $m+n$ equations. Already in the linear case this is only the case if a suitable determinant is non-vanishing and a similar condition is of course required in the non-linear case.

Proposition 38 *general implicit function theorem Let F be a C^p-mapping on an open subset $U \subset \mathbf{R}^{m+n}$ into $\mathbf{R}^m$. Let $(x_0, y_0) \in U$ with $F(x_0, y_0) = (0, 0, \dots, 0)$. If we assume that the functional determinant*

$$\frac{\partial(F_1, F_2, \dots, F_t}{\partial(y_1, y_2, \dots, y_t)} \neq 0,$$

then there is a C^p-mapping f defined on a suitable open neighbourhood $U(x_0)$ of a point $x_0 \in \mathbf{R}^n$ in $\mathbf{R}^m$, so that $f(x_0) = y_0$ and $F(x, f(x)) = 0$ on $U(x_0)$.

Proof. We consider the mapping $\phi : U \to \mathbf{R}^{m+n}$, whereby

$$\phi \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ F(x,y) \end{bmatrix} \text{ for } x \in \mathbf{R}^n, y \in \mathbf{R}^m.$$

Then

$$\det D\phi_{(x,y)} = \det \begin{bmatrix} 1 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & 0 & \dots & 0 \\ \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} & \dots & \frac{\partial F_1}{\partial x_s} & \frac{\partial F_1}{\partial y_1} & \dots & \frac{\partial F_1}{\partial y_t} \\ \vdots & & & & & & \\ \frac{\partial F_t}{\partial x_1} & \frac{\partial F_1}{\partial x_2} & \dots & \frac{\partial F_t}{\partial x_s} & \frac{\partial F_t}{\partial y_1} & \dots & \frac{\partial F_t}{\partial y_t} \end{bmatrix}.$$

But this is just

$$\det \begin{bmatrix} \frac{\partial F_1}{\partial y_1} & \dots & \frac{\partial F_1}{\partial y_t} \\ \dots & \dots & \dots \\ \dots & \dots & \dots \\ \frac{\partial F_t}{\partial y_1} & \dots & \frac{\partial F_t}{\partial y_t} \end{bmatrix} = \frac{\partial(F_1, \dots, F_t)}{\partial(y_1, \dots, y_t)}.$$

The proof is then completely exactly as above.

∎

Example Consider the system

$$F_1(x,y,z,t) = x^2 + y^2 + z^2 + t^2 - t = 0$$

$$F_2(x,y,z,t) = x^3 + y^3 + z^3 + t^3 - 8 = 0.$$

We consider the question whetehr z and t are expressible as functions of x and y around the point $(0, -1, 2, 1)$. We compute the functional determinant

$$\frac{\partial(F_1, F_2)}{\partial(z,t)} = \det \begin{bmatrix} 2x & 2t \\ 3z^2 & 3t^2 \end{bmatrix} = 6zt(t - z) = -12 \neq 0.$$

Since this is non-zero, we can find a function of the required type i.e. such that $f(x,y) = (z(x,y), t(x,y))$ with $F^i(x,y,z(x,y),t(x,y)) = 0$. We calculate its partial derivatives as follows: consider

$$\frac{\partial}{\partial x} F_i(x,y,z(x,y),t(x,y)) \text{ and } \frac{\partial}{\partial y} F_i(x,y,z(x,y),t(x,y)).$$

In the first case we get

$$2x + 2zz_x + 2tt_x = 0$$

$$3x^2 + 3z^2 z_x + 2t^2 t_x = 0.$$

At $(0,-1,2,1)$ we have

$$2z_x + t_x = 0$$

$$4z_x + t_x = 0.$$

Hence $z_x(0,-1) = 0$ and $t_x(0,-1) = 0$.

For the partial derivatives with respect to y we get

$$2y + 2zz_y + 2tt_y = 0$$

$$3y^2 + 3z^2 z_y + 3t^2 t_y = 0$$

or

$$2z_y + t_y = 1$$

$$4z_y + t_y = -1.$$

Hence $z_y(0,-1) = -1$ and $t_y(0,-1) = 3$.

4.5 Maxima und minima with restraints

In den Anwendungen kommen sehr häufig Probleme der Folgenden Form vor: Man soll Maxima oder Minima für eine Funktion $f(x,y)$ berechnen, wenn x und y nicht unabhängige Veränderliche sind, sondern durch eine Nebenbedingung $g(x,y) = 0$ miteinander verknüpft sind.

Im Prinzip könnte man stets eine Veränderliche durch die andere aus $g(x,y) = 0$ ausdrücken und hätte das Problem auf Extrema einer Veränderlichen reduziert. In der Praxis ist das aber sehr oft schwierig. Hier erweist sich eine Methode, die auf Lagrange zurückgeht, als besonders nützlich: Nehmen wir an, daß g auf einer offenen Menge U des $\mathbf{R}^2$ definiert ist und daß $M = \{(x,y) : g(x,y) = 0\}$ eine eindimensionale C^1-Mannigfaltigkeit ist. Das ist definitionsgemäß genau dann der Fall, wenn für jeden Punkt $(a,b) \in M$ das Differential $(Dg)_{(a,b)}$ linear unabhängig, d.h. also von 0 verschieden ist. Wir suchen einen solchen Punkt $(s,b) \in M$, in welchem die Funktion f ein lokales

Extremum annimmt. Dazu betrachten wir eine lokale Parameterdarstellung für M in der Umgebung des Punktes (a, b), etwa $t \mapsto \pi(t) = (\pi_1(t), \pi_2(t))$ mit $\pi(t_0) = (a, b)$.

Dann nimmt die Funktion $t \mapsto f(\pi(t))$ im Punkt t_0 ein lokales Extremum an. Daher gilt $f(\pi(t))' = 0$ im Punkt t_0 oder $(Df)_{(a,b)}(\dot{\pi}(t_0)) = 0$. Weiters ist nach Definition von M die Funktion g auf M identisch 0 und daher gilt $g(\pi(t)) \equiv 0$.

Differenziert man diese Identität im Punkt t_0, so erhält man $(Dg)_{(a,b)}(\dot{\pi}(t_0)) = 0$. Also steht auch der Vektor $(\operatorname{grad} g)(a, b)$ orthogonal auf dem Tangentenvektor $\dot{\pi}(t_0)$. Da im $\mathbf{R}^2$ die Vektoren, die auf einem festen Vektor senkrecht stehen, einen eindimensionalen Teilraum aufspannen, muß gelten $\operatorname{grad} f(a, b) = \lambda \operatorname{grad} g(a, b)$ mit einer Konstanten λ oder anders ausgedrückt, $(Df)_{(a,b)} = \lambda (Dg)_{(a,b)}$. Im Koordinantenschreibweise heißt das:

$$f_x - \lambda g_x = 0 \text{ und } f_y - \lambda g_y = 0.$$

Fassen wir diese Überlegung zusammen, so erhalten wir die **Lagrangesche Regel**: Um die kritischen Punkte der Funktion $f(x, y)$ unter der Nebenbedingung $g(x, y) = 0$ zu bestimmen, betrachte man die neue Funktion $F(x, y) = f(x, y) - \lambda g(x, y)$ mit einem Unbekannten von x und y unabhängigen Multiplikator λ und betrachte die drei Gleichungen

$$\begin{aligned} f_x - \lambda g_x &= 0 \\ f_y - \lambda g_y &= 0 \\ g(x, y) &= 0. \end{aligned}$$

Daraus läßt sich λ wieder eliminieren und die Lösungen $(x, y) = (a, b)$ liefern die gesuchten kritischen Punkte.

Die allgemeine Situation: Sei f eine Funktion von s Veränderlichen und seien $1 \le p \le s$ Bedingungen an die x_i gegeben, etwa $g_1 = g_2 = \cdots = g_p = 0$. Man bestimme lokale Extrema von f unter diesen Nebenbedingungen.

Proposition 39 *Sei $g : U \to \mathbf{R}^p$, (U offen in $\mathbf{R}^n$, $1 \le p \le n$), eine C^1-Abbildung und $M = \{x \in U : g(x) = 0\}$. Für jedes $x \in M$ sei das Differential $(Dg)_x : \mathbf{R}^n \to \mathbf{R}^p$ surjektiv. Sei weiters f eine C^1-Funktion, die auf einer offenen Menge, welche M enthält, definiert ist.*

Dann gilt: Nimmt f in einem Punkt $a \in M$ ein lokales Extremum an, so existieren p reelle Zahlen $l_1, \ldots, l_p$ mit

$$(Df)_a = \sum_{i=1}^{p} l_i (Dg^i)_a.$$

PROOF. Nach Voraussetzung ist M eine k-dimensionale C^1-Mannigfaltigkeit mit $k = s - p$. Wir können daher eine lokale Parameterdarstellung $\pi : U \to \mathbf{R}^n$, $U \subset \mathbf{R}^n$, $U \subset \mathbf{R}^k$ offen, finden. Es existiert $t_0 \in U$ mit $\pi(t_0) = a$ und für jedes $t \in U$ hat das Differential $(D\pi)_t$ den Rang k.

Da $g \equiv 0$ auf m ist, gilt $g\pi \equiv 0$ und daher speziell $(Dg)_a(D\pi)_{t_0} = 0$.

Das bedeutete, daß $(Dg_1)_a, \ldots, (Dg_p)_a$ im Annihilator des $(s-p)$-dimensionalen Teilraumes $\mathrm{Im}\,(D\pi)_{t_0}$ von $\mathbf{R}^n$ liegen. Da dieser Annihilator p-dimensional ist und $(Dg_1)_a, \ldots, (Dg_p)_a$ nach Voraussetzung einen p-dimensionalen Raum aufspannen, bilden diese Funktionale $(Dg^i)_a$ sogar eine Basis des Annihilators $(\mathrm{Im}\,(D\pi)_{t_0})^0$. (Siehe Vorlesung "Lineare Algebra".)

Nach Voraussetzung hat f im Punkt $a \in M$ ein lokales Extremum. Daher hat $f\pi$ in t_0 ein lokales Extremum. Es muß daher gelten $D(f\pi)_{t_0} = (Df)_a(D\pi)_{t_0} = 0$. Das Differential $(Df)_a$ liegt also im Annihilator vom $\mathrm{Im}\,(D\pi)_{t_0}$ und läßt sich daher als Linearkombination der Basiselemente $(Dg_i)_a$ darstellen, $(Df)_a = \sum \lambda_i (Dg_i)_a$.

■

Dieser Satz kann folgendermaßen formuliert werden: Sind in einer differenzierbaren Funktion $y = f(x_1, x_2, \ldots, x_s)$ die s Veränderlichen $x_1, \ldots, x_s$ nicht voneinander unabhängig, sondern durch p voneinenader unabhängige Nebenbedingungen

$$g_1(x_1, \ldots, x_s) = 0, \ldots, g_p(x_1, \ldots, x_s) = 0, \quad 1 \le p < s,$$

miteinander verknüpft, so führt man p zusätzliche Zahlen $\lambda_1, \ldots, \lambda_p$, die sogenannten **Lagrangeschen Multiplikatoren** ein und betrachtet die $s+p$ Gleichungen

$$\frac{\partial F}{\partial x_1} = 0, \ldots, \frac{\partial F}{\partial x_s} = 0, \quad g_1 = 0, \ldots, g_p = 0$$

für die $s+p$-Unbekannten $x_1, \ldots, x_s$, $\lambda_1, \ldots, \lambda_p$, wobei $F = f - (\lambda_1 g_1 + \cdots + \lambda_p g_p)$ gesetzt wurde. Unter den im Satz angegebenen Voraussetzungen muß jedes lokale Extremum eine Lösung dieser Gleichungen darstellen.

Beispiel Calculate the minimum of the function

$$f(x_1, \ldots, x_n) = x_1 + x_2 + \cdots + x_n$$

under the restraints $x_i > 0$ for each i and $g(x_1, \ldots, x_n) = x_1 x_2 \ldots x_n - 1 = 0$.

We use the auxiliary function $F(x_1, \ldots, x_n) = x_1 + \cdots + x_n - \lambda(x_1 \ldots x_{n-1})$. Then

$$\frac{\partial F}{\partial x_i} = 1 - \lambda \frac{x_1 \ldots x_n}{x_i} = 0$$

and $x_1 \dots x_n = 1$.

This implies that $x_i = \lambda$ for each i and $\lambda_n = 1$. The minimum is therefore assumed at the point where $f(1, \dots, 1) = n$.

This can be used to prove the AGM inequality.

Example (Hadamard's inequality Let $A = (a_{ik})$ be a real $n \times n$ matrix. Then

$$|\det A| < \prod_{i=1}^{n} \sqrt{\sum_{k=1}^{n} a_{ik}^2}.$$

We show that $|\det A| < b_1 b_2 \dots b_n$, where $\sum_{k=1}^{n} a_{ik}^2 = b_i^2$.

Let $g_i = \sum a_{ik}^2 - b_i^2$. Then $Dg_i = 2 \sum a_{ik} da_{ik}$. The Dg_i, $i = 1, 2, \dots, n$, span an n-dimensional subspace if and only if $\det A \neq 0$. We can assume without loss of generality that this is the case (since otherwise the Hadamard inequality is trivially valid).

In order to determin a maximum, we caonsider the function

$$\det A + \sum_i \lambda_i \left(\sum_k a_{ik}^2 - b_i^2 \right).$$

We have $\det A = \sum_{k=1}^{n} a_{ik} A_{ik}$, where $A_{ik} = (-1)^{i+k} \det B_{ik}$ with B_{ik} the $(n-1) \times (n-1)$-matrix which one obtains from A by omitting the i-th row and the k-th column.

Hence

$$\frac{\partial \det A}{\partial a_{ik}} = A_{ik}.$$

The Lagrangian equations thus take on the form $A_{ik} + 2\lambda_i a_{ik} = 0, \quad i, k = 1 \dots, n$ and $\sum_k a_{ik}^2 = b_i^2$. Hence $A_{ik} = -2\lambda_i a_{ik}$ and so

$$\det A = \sum_k a_{ik} A_{ik} = -2\lambda_i \sum a_{ik}^2 = -2\lambda_i b_i^2.$$

Hence $(\det A)^n = (-2)^n \lambda_1 \dots \lambda_n b_1^2 \dots b_n^2$.

On the other hand, $\det AA^{-1} = (A_{ik})$ and so

$$(\det A)^n \frac{1}{\det A} = \det(A_{ik}) = \det(-2\lambda_i a_{ik})$$

and hence

$$(\det A)^{n-1} = (-2)^n \lambda_1 \dots \lambda_n \det A.$$

This implies that

$$(\det A)^n = (-2)^n \lambda_1 \dots \lambda_n b_1^2 \dots b_n^2 = (-2)^n \lambda_1 \dots \lambda_n (\det A)^2$$

and so $\det A = b_1 \dots b_n$.

"For the matrix A_0, for which the function $\det A$ is a maximum, we have $\det A_0 = b_1 \dots b_n$ and this implies the required inequality.

4.6 Exercises

Exercise: Let $T : \mathbf{R} \to \mathbf{R}$ be the funciton $x \mapsto \frac{\pi}{2} + x - \arctan x$. Show that T fails to have a fixed point but that $|Tx - Ty| < |x - y|$ for $x, y \in \mathbf{R}$. (Does this contradict Banach's fixed point theorem?)

Exercise: Let $T : \mathbf{R}^n \to \mathbf{R}^n$ be a mapping which maps a closed subset M into a compact set $K \subset M$ and for which $|Tx - Ty| < |x - y|$ for $x, y \in M$. Show that T has a unique fixed point.

Exercise: Let f be a twice continuously partially differentiable function on $\mathbf{R}$. In order to calculate numerically an approximation to a zero x_0 of f one can use the following method of Newton: zu approximieren, kann man nach Newton folgendermaßen vorgehen: One chooses an initial value x_1 and considers point of intersection of the tangent to the curve $y = f(x)$ at the point $(x_1, f(x_1))$ with the x-axis. The corresponding value of x is denoted by x_2. We now iterate i.e. apply the same method but with inital value x_2 and so on. Suppose that f maps the interval $[a, b]$ into itself, $x_1 \in [a, b]$, $f'(x) \neq 0$ and $|\pi'(x)| \leq q < 1$ on $[a, b]$ where $\pi(x) = x - \dfrac{f(x)}{f'(x)}$. Show that the method converges to a solution.

Exercise: Let U be the set of those points $(r, \pi, \theta) \in \mathbf{R}^3$ with $r > 0$, $0 < \theta < 2\pi$, $0 < \pi < \pi$. Show that U is open and that the function f where

$$f(r, \pi, \theta) = (r \sin \pi \cos \theta, r \sin \pi \sin \theta, r \cos \pi)$$

is a diffeomorphism from U onto $f(U)$. Calculate $f(U)$ and the functional determinant.

Exercise: Let $f(x, y, z) = (x + y + z, x^2 + y^2 + z^2, x^3 + y^3 + z^3)$. Determine the set of all points (x, y, z), at which Df is not invertible.

Exercise: Let $\pi(x, y) = (x^2 - y^2, 2xy)$. At which points is π a local diffeomorphism? Is π (globally) invertible? Express π in polar coordinates.

Exercise: Show that

$$f(x,y) = \left(\frac{x^2+y^2-1}{x^2+y^2+2y+1}, \frac{-2x}{x^2+y^2+2y-1}\right)$$

is a diffeomorphism from $U = \{(x,y) : y > 0\}$ onto $x^2+y^2 < 1$.

Exercise: Let $t \mapsto (a_{ik}(t))$ be differentiable from $\mathbf{R}^1$ into the set of invertible $n \times n$ matrices. Calculate $\frac{d}{dt}(A(t))^{-1}$ and more generally $\frac{d}{dt}(A(t))^n$ for $N \in \mathbf{Z}$.

Exercise: Let U be open in $\mathbf{R}^n \times \mathbf{R}^m$ and let $f : U \to \mathbf{R}^p$ be a C^1-mapping. Show that for each $x \in U$

$$(Df)_x = ((D_1 f)_x, (D_2 f)_x),$$

where $(D_1 f)_x$ is the differential of the mapping $x_1 \mapsto f(x_1, x_2)$ when x_2 is held constant $(D_2 f)_x$ is defined correspondingly.

Exercise: Let $F(x,y) = x^3+y^3-6xy = 0$. Sketch this curve and determine those points thereon where y can be expressed locally as a function of x. Calculate $y'(x)$ and $y''(x)$ there.

Exercise: Let $F(x,y) = y^3 + y - x^2 = 0$ and $(x_0, y_0) = (0,0)$. Can y be represented as a function of x in a neighbourhood of x_0? If so, then calculate $y'(x_0)$ and $y''(x_0)$.

Exercise: As above for the set $xy^2 + 3\log x - 4 = 0$ at $(1,2)$.

Exercise: As above for $\sin x + 2\cos y - \frac{1}{2} = 0$ at $(\frac{\pi}{6}, \frac{3\pi}{2})$.

Exercise: At which points (x,y,z) can z be expressed locally as a function of x and y where $f(x,y,x) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0$? Calculate z_x, z_y, z_{xx}, z_{xy} and z_{yy}.

Exercise: Let $F(x,y,z) = 0$ be such that the theorem on implicit functions can be used to solve for z, x and y as $z = z(x,y)$, $x = x(z,y)$ and $y = y(z,x)$. Show that $\frac{\partial z}{\partial x} . \frac{\partial x}{\partial y} . \frac{\partial y}{\partial z} = -1$.

Exercise Suppose that $F_1(x,y,z) = 0$ and $F_2(x,y,z) = 0$ are such that they define two functions $y(x)$ and $z(x)$. Calculate y' und z'.

Exercise: Interpret the implicit function theorem for the case where the $F_i(x,y)$ are linear i.e. of the form $F_i(x,y) = a^i_1 x_1 + \cdots + a^i_{n+m} x_t - b_i$?

Exercise: Let $F(x,y,z) = xz + \sin(xy) + \cos(xz) - 1$. Determine which of the variables can be expressed as a function of the two others near $(0,1,1)$. Calculate the corresponding artial derivatives up to order 2.

Exercise: Let $F_1(x,y,z) = e^x y + \sin xz + \log(1+z) - 2 = 0$ and $F_2(x,y,z) = \sin(x^2 y) + y^2 + z^5 - 4 = 0$. Which of the two variables can be expressed as functions of the third one near the point $(0,2,0)$. Calulate the derivatives of the corresponding function.

Exercise: Do the equations $t^2 + x^3 + y^3 + z^3 = 0$, $t + x^2 + y^2 + z^2 - 1 = 0$, $t + x + y + z = 0$ have a differentiable solution $x(t), y(t), z(t)$ in a neighbourhgood of the point $(t,x,y,z) = (0,-1,1,0)$?

Exercise: Determine the derivatives $y'(x)$ and $y''(x)$, where $y(x)$ is defined by the equation $(x^2 + y^2)^3 - 3(x^2 + y^2) + 1 = 0$. At which points $y(x)$ not defined

Exercise: Suppose that $z = z(x,y)$ is defined implicitly by the equation $x^2 - 2y^2 + 3z^2 - yz + y = 0$. At which points is z? Calculate its differential Dz there.

Exercise: Calculate the minimum of the function $f(x,y) = \dfrac{x^p}{p} + \dfrac{y^q}{q}$, hwere $p > 0$, $q > 0$ and $\dfrac{1}{p} + \dfrac{1}{q} = 1$ under the restrictions $g(x,y) = xy - 1 = 0$, $x > 0$, $y > 0$.

Use this to derive the Hölder inequality

$$\sum_{i=1}^{n} u_i v_i \leq \sqrt[p]{\sum u_i^p} \sqrt[q]{\sum v_i^q}$$

for $u_i \geq 0$, $v_i \geq 0$.

Exercise: Calculate the minimum of the function xy with restraint $x^k + y^k = 1$, $x \geq 0$, $y \geq 0$, $k \in \mathbf{N}$.

Exercise: Use the lagrange method to calculate the distance of the line $ax + by = c$ to the origin.

Exercise: Calculate the distance from the point $(3, 12)$ to the parabola $y^2 = 6x$.

Exercise: Calculate the maximum and minimum of the function $x^2 + y^2$ on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Exercise: Show that for each symmetric $n \times n$ matrix $A = (a_{ik})$ the maximum of $f(x) = (Ax|x)$ on $|x| = 1$ occurs at an eigenvector.

Exercise: Calculate the distance from the origin to the plane $2x_1 + x_2 - x_3 + x_4 = 4$, $x_1 + x_2 - x_3 + x_4 = -6$ in $\mathbf{R}^4$.

Exercise: For $k \geq 2 \in \mathbf{Z}$, calculate the minimum of the function $f(x, y) = x^k + y^k$ on the segment $x + y = a$, $x \geq 0$, $y \geq 0$.

Exercise: Let $f(x, y) = (x - 1)^2 + (y - 2)^2$. Calculate the minimum of f on the set where $g(x, y) = x^2 - 2xy + y^2 = 0$. Do we have a relation of the form $Df = \lambda Dg$ at the minimum?

Exercise: Calculate $\frac{dy}{dx}$, where y is defined implicitly by the equation

1. $x^2y + x^3y^4 = 1$;
2. $\frac{(x + y)}{(\sqrt{x^2 + y^2}} = 2$;
3. $\ln(x^2 + xy) = 1$.

Exercise For

$$x = r\,\cos\theta \text{ resp. } y = r\sin\theta,$$

calculate $\frac{\partial r}{\partial x}$ etc.

Exercise: Minimise

$$f(x, y, z) = x^2 + y^2 + z^2$$

under the restaint

$$(x-1)^2 + (y-2)^2 + (z-3)^2 = 4.$$

Exercise: Determin the extrema of the following f under the corresponding restraints:

1. $f(x, y) = x^3 - xy^2, x^2 + y^2 = 1;$
2. $f(x, y) = x^2 + y^2, x^3 - xy^2 = 1;$
3. $f(x, y) = 2x + 3y, x^2 - 2xy + 2y^2 = 1;$
4. $f(x, y) = x^2 + y^2 + z^2, 3x - y + 2z = 14;$
5. $f(x, y) = x^2 + y^2 + z^2, (x-y)^2 = 1, xyz = 1;$

Exercise: Determine the nearest point on the curve $x^2 - xy + y^2 = 1$ to 0.

Exercise: Determine that pair of points on the curves point(s) on $y^2 = 1 + x^2$ resp. $2y = x$ with minimal distance.

Exercise: Suppose that $(x-z)^2 + (y-z)^2 = 2$. Calculate $\partial^2 y/\partial x\,\partial z$.

Exercise: Let f be a smooth function with $\partial f/\partial z \neq 0$ and let z be determined implicitly by the equation $f(x, y, z) = 0$. Show that

$$\frac{\partial^2 z}{\partial x^2} = \frac{\frac{\partial^2 f}{\partial z^2}\left(\frac{\partial f}{\partial x}\right)^2 + 2\frac{\partial^2 f}{\partial x}\partial y\frac{\partial f}{\partial x}\frac{\partial f}{\partial z} - \left(\frac{\partial f}{\partial z}\right)^2\frac{\partial^2}{\partial x \partial y}}{\left(\frac{\partial f}{\partial z}\right)^3}.$$

Calulate $\partial^2 z/\partial x\partial y$.

5 Vector analysis

5.1 Differential forms

A function from $\mathbf{R}^n$ into $\mathbf{R}^n$ is the mathematical realisation of a vector field. Of course we can generalise the by considering functions on $\mathbf{R}^n$ with values in a (abstract) vektor space V. If V is m-dimensional with basis $(e_1, \dots, e_m)$, then each such field has a representation as an m-tuple $(f_1, \dots, f_m)$ of scalar functions where

$$f = f_1 e_1 + \dots f_m e_m.$$

We will consider the case where V is a space of alternating forms. In this case we use the following bases:

1-forms: the basis is written $(dx_1, \dots, dx_n)$. A typical one-form has thus the form

$$\omega = a_1(x)\, dx_1 + \cdots + a_n(x)\, dx_n.$$

(For $n = 2$ the typical 1-form has the form: $\omega = a\, dx + b\, dy$;

$n = 3$: $\omega = a\, dx + b\, dy + c\, dz$.

Remark: the coefficients a, b etc. are smooth functions.

2-forms: The basis $\{dx_i\, dx_j : 1 \leq i < j \leq n\}$. (Hence we have $\binom{n}{2}$ elements).

(for $n = 2$: $\omega = a\, dx\, dy$.

$n = 3$: $\omega = a\, dy\, dz + b\, dz\, dx + c\, dx\, dy$).

The general situation is as follows: a k-differential form on $U \subset \mathbf{R}^n$ is an expression of the form

$$\omega = \sum_{i_1 < i_2 < \cdots < i_k} a_{i_1 \dots i_k} dx_{i_1} \dots dx_{i_k}.$$

(N.B. in this case the basis has $\binom{n}{k}$ elements).

Operation on differential forms One can

a) add two k-forms on $\mathbf{R}^n$ (one simply sums the corresponding coefficients);

b) multiply a k-form with an l-Form (the result is a $k + l$-form). One multiplies as with numbers, using the rules

$$dx_i dx_j = -dx_j dx_i \text{ and so } dx_i dx_i = 0.$$

c) differentiate a k-form (the result is a $k + 1$-form). To do this, the following rules suffice:

If ω is a function f, then its differential df is defined as

$$df = \frac{\partial f}{\partial x_1}\, dx_1 + \cdots + \frac{\partial f}{\partial x_n}\, dx_n.$$

(We remark that it is convenient to regard functions as 0-forms).

If

$$\omega = \sum_{i_1 < i_2 < \cdots < i_k} a_{i_1 \ldots i_k} dx_{i_1} \ldots dx_{i_k},$$

then

$$d\omega = \sum_{i_1 < i_2 < \cdots < i_k} da_{i_1 \ldots i_k} dx_{i_1} \ldots dx_{i_k}.$$

Examples I. For $\omega = a\, dx + b\, dy$ we have

$$d\omega = \left(\frac{\partial b}{\partial x} - \frac{\partial a}{\partial y} \right) dx\, dy.$$

II. For $\omega = a\, dx + b\, dy + c\, dz$, we have

$$d\omega = \left(\frac{\partial b}{\partial x} - \frac{\partial a}{\partial y} \right) dxdy + \left(\frac{\partial b}{\partial x} - \frac{\partial a}{\partial y} \right) dxdy + \left(\frac{\partial b}{\partial x} - \frac{\partial a}{\partial y} \right) dxdy.$$

III. For

$$\omega = a\, dy\, dz + b\, dz\, dx + c\, dx\, y,$$

we have

$$d\omega = \left(\frac{\partial a}{\partial x} + \frac{\partial b}{\partial y} + \frac{\partial c}{\partial z} \right) dx\, dy\, dz.$$

d) One can complse a k-form

$$\omega = \sum_{i_1 < i_2 < \cdots < i_k} a_{i_1 \ldots i_k} dy_{i_1} \ldots dy_{i_k}$$

on $\mathbf{R}^n$ with a smooth function $\phi = (\phi_1, \ldots, \phi_n)$ from $\mathbf{R}^m$ into $\mathbf{R}^n$, using the rule: one replaces dy_i in ω by

$$d\phi_i = \frac{\partial \phi_i}{\partial x_1}\, dx_1 + \cdots + \frac{\partial \phi_n}{\partial x_m}\, dx_n.$$

Proposition 40 *If ω is a k-form, ω_1 a l-form, then*

1. $d^2\omega = 0$;
2. $d(\omega.\omega_1) = (d\omega).\omega_1 + (-1)^{kl}\omega.(d\omega_1)$;

3. $d(\omega \circ \phi) = (d\omega \circ \phi)$*, where* ϕ *is a suitable smooth function.*

e) Integration of k-forms on k-cubes: Let $c : I^k \to U$, (U open in $\mathbf{R}^n$) be a k-cube. If ω is a k-Form on U, then $\omega \circ c$ is a k-form on $\mathbf{R}^k$ and thus has the form

$$a(x_1, \ldots, a_k)\, dx_1 \ldots dx_k,$$

where a is a smooth function. We define

$$\int_c \omega = \int_{I^k} a(x_1, \ldots, x_k)\, dx_1 \ldots dx_k.$$

5.2 Stokes' theorem

Proposition 41 *Stokes' Integral formula If* c *is a* k*-cube in* U*, and* ω *is a* $(k-1)$*-form on* U*, then*

$$\int_{\partial c} \omega = \int_c d\,\omega.$$

We bring the important special cases of this theorem:

Proposition 42 *Green's theorem Let* $\omega = P\,dx + Q\,dy$ *be a* 1*-form on* $U \subset \mathbf{R}^2$*,* c *a* 2*-cube in* U*. Then*

$$\int_{\partial c} \omega = \int_c \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right] dxdy.$$

Example—path independence Let ω be a 1-form on U with $d\omega = 0$. (This holds in particular, if ω has the formn df, where f is a smooth function). Then $\int_c \omega = 0$, if c is a closed curve which is the boundary of a 2-cube in U.

Surfaces A **parametrisied surface** in $\mathbf{R}^3$ is a continuously differntiable function ϕ from an open subset U of $\mathbf{R}^2$ into $\mathbf{R}^3$.

Example I. A mapping of the form

$$\phi(u, v) = x_0 + u.x_1 + v.x_2,$$

where $x_0 \in \mathbf{R}^3$ and x_1, x_2 are linearly independent in $\mathbf{R}^2$, is a parametrised plane. II. The mapping

$$\phi : (u, v) \mapsto (\sin u \cos v, \sin u \sin v, \cos u)$$

is a parametrisation of the unit sphere.

A parametrised surface ϕ is **regular** or **smooth**, if the Jacobi matrix of ϕ has rank 2 at each point. Then the two vectors $\phi_1 = D_1\phi$ and $\phi_2 = D_2\phi$ are linearly independent. We define the normal $\mathbf{N}(u, v)$ to the surface at the point with parameter (u, v) to be the vector

$$\mathbf{N}(u, v) = \frac{\phi_1(u, v) \times \phi_2(u, v)}{\|\phi_1(u, v) \times \phi_2(u, v)\|}.$$

Example We have the following formula for $\phi_1 \times \phi_2$:

$$\left(\frac{\partial(\phi^2, \phi^3)}{\partial(u, v)}, \frac{\partial(\phi^3, \phi^1)}{\partial(u, v)}, \frac{\partial(\phi^1, \phi^2)}{\partial(u, v)}\right),$$

where $\phi = (\phi^1, \phi^2, \phi^3)$ i.e. ϕ^1 etc. are the coordinates of ϕ (not the partial derivatives ϕ_1 etc.).

The plane

$$\{x : (\mathbf{N}|x) = (\mathbf{N}|p)\}$$

is then the **tangent plane** of S at p.

If S is defined implicitly by the equation

$$S = \{(x, y, z) : f(x, y, z) = 0\}$$

then

$$\mathbf{N} = \frac{\operatorname{grad} f}{||\operatorname{grad} f||}.$$

If ϕ is a parametrised surface, then we define the surface area to be

$$\int_A dF = \int_U \left|\frac{\partial\phi}{\partial s} \times \frac{\partial\phi}{\partial t}\right| dsdt.$$

The 2-form

$$dF = \mathbf{N}_1 dydz + \mathbf{N}_2 dzdx + \mathbf{N}_3 dxdy$$

is called the surface element of the surface. (Remark: since we can regard a 2-cube c as a parametrised surface, these definition apply also in this case).

Definition 12 *If f is a scalar field i.e. a smooth mappin from U into $\mathbf{R}$, we write ∇f for the gradient field of F. Then*

$$\begin{aligned} \nabla(f+g) &= \nabla f + \nabla g; \\ \nabla(f.g) &= f.\nabla g + \nabla f.g; \\ \nabla(\frac{f}{g}) &= \tfrac{(g.\nabla f - f.\nabla g)}{g^2}. \end{aligned}$$

The rotation of a vector field Let $f = (f_1, f_2, f_3)$ be a vector field in $\mathbf{R}^3$. We consider $d\omega$, where ω is the 1-form $f_1\,dx + f_2\,dy + f_3\,dz$ $d\omega$ is also a vecto field as 2-form. We denote this field by $\mathrm{curl} f$ (or $\mathrm{rot}\, f$). As one can easily verify

$$\mathrm{curl} f = (D_2 f_3 - D_3 f_2, D_3 f_1 - D_1 f_3, D_1 f_2 - D_2 f_1).$$

Sometimes the notation $\nabla \times f$ for $\mathrm{curl} f$ is used.

One can regard the rotation $\mathrm{curl} f$ as the formal determinant

$$\det \begin{bmatrix} e_1 & e_2 & e_3 \\ D_1 & D_2 & D_3 \\ f_1 & f_2 & f_3 \end{bmatrix}$$

where e_1, e_2, e_3 is the canonical basis of $\mathbf{R}^3$.

We then have

$$\begin{aligned} \mathrm{curl}(f+g) &= \mathrm{curl} f + \mathrm{curl} g; \\ \mathrm{curl}(\phi.f) &= \phi.\mathrm{curl} f + \nabla\phi \times f. \end{aligned}$$

(ϕ a scalar field).

The field f is **rotationsfrei**, if $\mathrm{curl} f = 0$. A gradient field $\mathrm{grad}\,\phi$ is always (since $d^2 = 0$).

The divergence If f is a vector field, then $\mathrm{div}\, f$ is the scalar field

$$D_1 f_1 + \dots D_n f_n.$$

This is sometimes written as $(\nabla | f)$.

Then

$$\begin{aligned} \mathrm{div}\,(f+g) &= \mathrm{div}\, f + \mathrm{div}\, g; \\ \mathrm{div}\,(\phi.f) &= \phi.\mathrm{div}\, f + (\nabla\phi | f). \end{aligned}$$

Further

$$\mathrm{div}\,(\mathrm{curl} f) = 0$$

for each vector field f.

The Laplace operator This is the operator

$$\Delta : f \mapsto \mathrm{div}\,(\nabla\phi),$$

which is defined for scalar fields ϕ. (Sometimes written ∇^2).

Surface integrals If S is a surface with parametrisation $\phi : U \to \mathbf{R}^3$, we define the surface area A of S as follows:

$$A = \iint_U \|\phi_1(u,v) \times \phi_2(u,v)\| \, dudv.$$

More generally, we define for a scalar field f on the subset V of $\mathbf{R}^3$, with $\phi(U) \subset V$

$$\int_S f.dS = \iint_U f(\phi(u,v))\|\phi_1(u,v) \times \phi_2(u,v)\| \, dudv.$$

Remark If we use the formula

$$(x \times y | x \times y) = \|x\|^2\|y\|^2 - (x|y)^2$$

we obtain the equality

$$A = \iint_U \sqrt{EG - F^2} \, dudv,$$

where $E = (\phi_1|\phi_1)$, $F = (\phi_1|\phi_2)$, $G = (\phi_2|\phi_2)$.

Example Let $\phi(u,v) = (u, v, 1-u^2-v^2)$ $(u^2+v^2 \leq 1)$ be the hemisphere in $\mathbf{R}^3$. Then the surface integral

$$\begin{aligned}\int_S d|S| &= \textstyle\iint_{u^2+v^2\leq 1} ||(1,0,-u(1-u^2-v^2)^{\frac{1}{2}}) \times (0,1,-v(1-u^2-v^2)^{\frac{1}{2}})||dudv \\ &= \textstyle\iint_{u^2+v^2\leq 1} \frac{1+u^2+v^2+1-u^2-v^2}{(1-u^2-v^2)} dudv \\ &= \textstyle\iint_{u^2+v^2\leq 1} (1-u^2-v^2)^{-1} dudv\end{aligned}$$

Surface integrals of the first type Let S be as above, $\mathbf{F}$ a vector field on S. Then we define the surface integral $\int_S \mathbf{F} d\vec{S}$ or $\int(\mathbf{F}|\mathbf{N})dS$ as

$$\iint_U (\mathbf{F} \circ \phi | \mathbf{N}(u,v)) dudv.$$

This is precisely the integral $\int_S \omega$, where $\omega = F_1 dx_2 dx_3 + F_2 dx_3 dx_1 + F_3 dx_1 dx_2$ i.e.

$$\int_S (F_1 dx_2 dx_3 + F_2 dx_3 dx_1 + F_3 dx_1 dx_2).$$

Example We calculate

$$\int_S xzdydz + yzdzdx + x^2dxdy = \int_S \vec{F}d\vec{S}$$

where $f(x,y,z) = (xz, yz, z^2)$ and S has the parametrisation

$$(\sin u \cos v, \sin u \sin v, \cos u) \quad (u,v) \in \left[0, \frac{\pi}{2}\right] \times [0, 2\pi]$$

.

$$\begin{aligned} D_1\phi &= (\cos u \cos v, \cos u \sin v, -\sin u) \\ D_2\phi &= (-\sin u \sin v, \sin u \cos v, 0). \end{aligned}$$

$$\begin{aligned} D_1\phi \times D_2\phi &= (-\sin^2 u \cos v, \sin^2 u \sin v, \cos u \sin u) \\ ||D_1\phi \times D_2\phi|| &= \sin^4 u \cos^2 v + \sin^4 u \sin^2 v + \cos^2 u \sin^2 u \\ &= \sin^4 u + \cos^2 u \sin^2 u = \sin^2 u. \end{aligned}$$

The integral

$$\begin{aligned} &= \int_0^{\frac{\pi}{2}} \int_0^{2\pi} ((\cos u \sin u \cos v, \cos u \sin u \sin v, \sin^2 v \cos^2 v)(-\sin u \cos v \sin u \sin v, \cos u) dudv \\ &= \int_0^{\frac{\pi}{2}} \int_0^{2\pi} (\cos u(-\sin^2 u) \cos^2 v + \cos u \sin^2 u \sin^2 v + \cos u \sin^2 u \cos^2 v) dudv \\ &= \int_0^{\frac{\pi}{2}} \cos u \sin^2 u du \int_0^{2\pi} -\sin^2 v dv. \end{aligned}$$

For a mas which is distributed over a surface F, we have

$$\begin{aligned} T_x &= \iint_F x\rho(x,y,z)d|S| \\ T_y &= \iint_F y\rho(x,y,z)d|S| \\ T_z &= \iint_F z(x,y,z)d|S| \end{aligned}$$

resp. a mass which is distributed over a curve C:

$$\begin{aligned} T_x &= \int_C x\rho(x,y,z)ds \\ T_y &= \int_C y\rho(x,y,z)ds \\ T_z &= \int_C z\rho(x,y,z)ds. \end{aligned}$$

Example Calculate the surface area of the hemisphere

$$z = f(x,y) = \sqrt{(R^2 - x^2 - y^2)}.$$

Since
$$\frac{\partial f}{\partial x} = -\frac{x}{\sqrt{(R^2 - x^2 - y^2)}} \quad \frac{\partial f}{\partial y} = -\frac{y}{\sqrt{R^2 - x^2 - y^2}}$$

we have

The area

$$= R \int_{0 \le x^2+y^2 \le 1} \int \frac{dxdy}{\sqrt{(R^2-x^2-y^2)}}$$
$$= R \int_0^{2\pi} \int_0^R \frac{rdr}{\sqrt{R^2-r^2}} = R \int_0^R \frac{rdr}{\sqrt{(R^2-r^2)}} = 2\pi R^2.$$

Proposition 43 *Stokes' theorem Let c be a 2-cube in $\mathbf{R}^3$, f a vector field on an open subset U, which contains the image S of c. Then*

$$\int_S (curl f|\mathbf{N})ds = \int_{\partial c} f_1 dx_1 + f_2 dx_2 + f_3 dx_3.$$

(Green's theorem is the special case where the cube c lies in $\mathbf{R}^2$).

Proposition 44 *Gauß' integral formula Let c be a 3-cube $U \subset \mathbf{R}^3$, $\mathbf{f} = (f_1, f_2, f_3)$ a vector field on U. Then*

$$\int_c div\,\mathbf{f} dxdydz = \int_{\partial c} (\mathbf{f}|\mathbf{N})dF.$$

Proposition 45 *Gauß' integral theorem Let V be a region in $\mathbf{R}^3$ with boundary surface S. If F is a vector field on V, then*

$$\iiint_V \text{div } F dxdydz = \iint_S \vec{F} d\vec{S}.$$

5.3 Exercises

Exercise: Calculate $\int_c y\, ds$, where c is the curve $y = \sqrt{x}$ from $x = 2$ to $x = 6$.

Exercise: Calculate the centre of gravity of the curve

$$\mathbf{x}(t) = (t, t^2, 1) \quad (1 \le t \le 3)$$

with density $\rho(x, y, z) = \dfrac{yz}{x}$.

Exercise: Integrate $xy^2dx+ydx$ $(xy^2dx+zxdy+xydz)$ along the following curves from $(0,0)$ to $(1,1)$:

1. the straight line from $(0,0)$ to $(1,1)$;
2. the curve $y=x^2$;
3. the curve $x=y^2$.

Exercise: Calculate $\int_c (x^2-2xy+y^2)\,ds$, where c is the curve $(2\cos t, 2\sin t)$ $(0 \leq t \leq 1)$.

Exercise: Calculate $\int_T (y+z)dydz+(x+z)dzdy+(x-y)dxdy$, where T is the triangle with vertices $(1,2,-1),(3,1,0),(0,1,1)$.

Exercise: Consider the following parametrisation of the n-sphere:

$$\begin{aligned}
x_1 &= r\cos\theta\cos\phi_1\dots\cos\phi_{n-2}\\
x_2 &= r\sin\theta\cos\phi_1\dots\cos\phi_{n-2}\\
x_3 &= r\sin\phi_1\cos\phi_2\dots\cos\phi_{n-2}\\
&\vdots\\
x_{n-1} &= r\sin\phi_{n-3}\dots\cos\phi_{n-2}\\
x_n &= r\sin\phi_{n-2}.
\end{aligned}$$

(This for $0\leq r\leq 1, 0\leq\theta\leq 2\pi, -\frac{\pi}{2}\leq\phi_i\leq\frac{\pi}{2}$). Calculate the Jacobi determinant of this transformation and hence the integral

$$\int_B dx_1\dots dx_n$$

where B is the unit ball in $\mathbf{R}^n$.

Exercise: Calculate

$$\int_S xdydz - zdzdx - dxdy,$$

resp.

$$\int_S y^2dydz + 2dzdx - dxdy,$$

where

1. S is the upper hemisphere;
2. S is the lower hemisphere;
3. S is the cone $x = \cos\theta, y = r\sin\theta, z = 3 - 3r$, $0 \le r \le 1$, $0 \le \theta \le 2\pi$;
4. S is the paraboloid $x = r\cos\theta, y = \sin\theta, z = -3 + 3r$, $1 \le r \le 1, 0 \le \theta \le 2\pi$.

Exercise: Let S be the surface $\{(x, y, z) : z = f(x, y), ((x, y) \in r\}$. Show

$$\int_S dS = \int_R \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2} dxdy.$$

Exercise: Let S be the surface $x = r\cos\theta, y = r\sin\theta, z = f(r, \theta) \quad (r, \theta) \in R$. Show

$$\int_S dS = \int_R \sqrt{1 + \left(\frac{\partial f}{\partial r}\right)^2 + r^{-2}\left(\frac{\partial f}{\partial \theta}\right)^2} rdrd\theta.$$

Exercise: Calculate the surface area of the torus

$$x = (a + b\cos\theta)\sin\phi, y = (a + b\cos\theta)\cos\phi, z = b\sin\theta,$$

where $0 < b < a$.

Exercise: Calculate $\omega \circ c$, where

$$\omega = 4dx - dy + 3dz$$

and

$$c(t) = (2, 5, -3) + t(-3, 2, 7).$$

Exercise: Calculate the 1-form $d\left(\frac{1}{r}\right)$.

Exercise: Calculate $\omega \circ \phi$, where

$$\omega = 2dydz - dzdx + 3dxdy$$

and ϕ is the natural affine mapping from

$$U = [(0, 0), (1, 0), (0, 1)] \text{ to } [(0, 1, -2), (3, 1, 0), (-2, 2, 1)].$$

Exercise: Let

$$\begin{aligned} x &= u^2 + v^2 + w^2 \\ y &= uv + uw + vw \\ z &= uvw. \end{aligned}$$

Calculate dx, dy, dz and use Cramer's rule to calculate du, dv, dw and hence $\dfrac{\partial u}{\partial x}$ usw.

Exercise Prove the following identitities:

$$\int_{\partial c} uv\, dx + uv\, dy = \int_c \left\{ v \left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} \right) + u \left(\frac{\partial v}{\partial x} - \frac{\partial v}{\partial y} \right) \right\} dxdy;$$

$$\frac{1}{2} \int_{\partial c} \left(v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x} \right) dx + \left(u\frac{\partial v}{\partial y} - v\frac{\partial u}{\partial y} \right) dy = \int_c \left(u\frac{\partial^2 v}{\partial x \partial y} - v\frac{\partial^2 u}{\partial x \partial y} \right) dx\, dy.$$

Exercise: Show that

$$\int_{\partial c} v\frac{\partial u}{\partial n}\, ds = \int_c (v\nabla^2 u + (\nabla u | \nabla v))\, dx\, dy;$$

$$\int_{\partial c} \left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n} \right) ds = \int_c (v\nabla^2 u - u\nabla^2 v)\, dx\, dy.$$

Exercise: Let a closed curve be the boundary ∂c of a 2-cube. Show that the area enclosed by the curve is $A = \frac{1}{2}\int_{\partial c}(x\, dy - y\, dx) = \int_{\partial c} x\, dy$.

Exercise: Calculate the area of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

resp. the astroid

$$x^{2/3} + y^{2/3} = a^{2/3}$$

by means of curvilinear integrals.

Exercise: Prove the formula

$$\int_c (\text{grad}\, f | \text{grad}\, g)\, dx\, dy + \int_c f \Delta g\, dx\, dy = \int_{\partial c} f(gradg|\mathbf{N})\, ds;$$

resp.

$$\int_c (f\Delta g - g\Delta g) dx\, dy = \int_{\partial c} (f(\text{grad}\, g|\mathbf{N}) - g(\text{grad}\, g|\mathbf{N}))\, ds.$$

Exercise: Calculate the centre of gravity of the semicircle $x^2+y^2 \leq r^2 \quad x \geq 0$.

Exercise: Calculate the volum of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1.$$

Exercise: Calculate

$$\int_A \frac{dxdydz}{(x+y+z+1)^3}$$

over the region A enclosed by the coordinate planes and the plane $x+y+z = 1$.

Exercise: Calculate the area of the ball with radious R.

Exercise: Calculate a formula for the area of a cube of the form

$$(s,t) \mapsto (s,t,f(s,t)).$$

Exercise: Calculate the surface of the torus generated by rotating the circle $(x-a)^2 + z^2 = r^2$ around the z-axis.

6 Appendix—Topological spaces

The modern theory of topology draws its roots from two main sources. One is the theory of convergence and the related concepts of approximation which play such a central role in modern mathematics and its applications. Since the problems dealt with are of such complexity, the earlier ideal of obtaining exact and explicit solutions in closed forms usually must be replaced by methods which provide successive approximations to the solution. A familiar example of this is the Newton method for obtaining approximate roots of an equation of the form $f(x) = 0$ where f is a suitable function. The second source is that branch of geometry which is often referred to as "rubber sheet geometry" i.e. the study of those properties of a geometrical object which remain unchanged under continuous deformations (in contrast, say, to Euclidean geometry which is concerned with those properties which remain unchanged under congruence). It is remarkable that the same abstract theory—topology—provides the framework for both topics. It is also rather fortunate, since an introduction to the more informal and geometrical aspects of topology is rather more digestible than a dry axiomatic approach which has no intuitive material to draw on. For this reason, we begin with a brief survey of topological notions for subsets of $\mathbf{R}^n$. The usual (euclidean) metric plays an important role here and this leads naturally to the introduction of the general notion of a metric in the second chapter.

With this more intuitive material available, abstract topological spaces are introduced in the next chapter and the various topological concepts are clarified in this context. We also provide a small collection of pathological spaces. The next chapter deals with the various methods of constructing topological spaces. In particular, the construction of the quotient space provides an opportunity for giving a brief survey of classical results on two-dimensional spaces. the next three chapters are devoted to special topological properties which eliminate much of the pathology that can occur in the most general situations. We then treat two variations of the notion of a topological space—uniformities and compactologies. In fact, the concept of a topological space is, in a certain sense, unsatisfactory and owes its pre-eminence more to a historical accident. For many applications, it can be profitably be replaced by one of the above ones. this will be particularly useful in the next section where we consider structures on spaces of mappings between topological spaces. We conclude the first chapter with a discussion of three of the most important special spaces—the Cantor set, the irrationals and the Hilbert cube.

In the second chapter, we turn to the analytical aspects of topology. This contains a brief introduction to basic functional analysis and can be regarded as a start on the theory of infinite dimensional topology.

In the third and final chapter, we return to the more geometrical aspects of topology by considering algebraic topology. Our aim has been to provide enough machinery to give rigorous proofs of the famous results (such as the Brouwer fixed point theorem, the theorem of Borsuk-Ulam etc.) which have been covered in an intuitive manner in the first section.

As we have seen, in many of the definitions of the last section, the open sets play a more fundamental role than the metric. If we recall the basic stability properties of the family of open sets in a metric space, we are led naturally to the following definition:

Definition: Let X be a set. A **topology** on X is a family τ of subsets which satisfies the following conditions: a) $\emptyset$ and X are in τ; b) the union of a subfamily of τ is a set of τ; c) the intersection of a finite subfamily of τ is in τ.
A **topological space** is a set$X$, together with a topology thereon. We will refer then to the topological space (X, τ) or simply X if it is clear from the context which topology we are dealing with. We now reformulate some of the definitions of the last chapter in this context.

If (X, τ) is a topological space, we refer to the sets in τ as the **open sets** (or, more precisely, the τ-open sets). A subset A of X is **closed** if its complement is open.

If τ_1 and τ_2 are topologies on X, we say that τ_1 is **finer** than τ_2 (alternatively, that τ_2 is **coarser** than τ_2), if $\tau_1 \supset \tau_2$. Two topological spaces (X_1, τ_1) and (X_2, τ_2) are **homeomorphic** if there is a bijection f from X_1 onto X_2 so that a subset U of X_1 is open if and only if $f(U)$ is open in X_2. Such an f is called a **homeomorphism**.

A mapping $f : X_1 \to X_2$ is **continuous** if for each open subset U of Y the pre-image $f^{-1}(U)$ is open in X_1. Hence a bijection is a homeomorphism if and only if both it and its inverse are continuous.

We note without proof that most of the simple facts about continuous functions on subsets of the line translate without difficulty to the more general situation. These the sum and product of two continuous, real-valued functions are continuous as is the composition of two continuous functions. If a sequence of continuous, real-valued functions on a space X converges uniformly, then the limit function is also continuous.

If A is a subset of a topological space, we define its **interior** A^o to be the union of all open subsets which are contained in A. Clear A^o is open and in fact is the largest open set which is contained in A. Furthermore, A is open if and only if $A = A^o$.

Similarly, we define the **closure** $\overline{A}$ of A to be the intersection of all closed subsets C which contain A. It is the smallest closed set containing A and the latter is closed if and only if $A = \overline{A}$.

Examples: If X is a set, then in τ_D, every subset is closed and hence equal to its closure. On the contrary, the closure of every subset of (X, τ_I) (with the exception of the empty set) is equal to X. In $(\mathbf{N}, \tau_{cf}$, every finite subset is closed and so equal to its closure, while the closure of any infinite subset is the whole of $\mathbf{N}$.

We remark here that one can use the concepts of closure and closedness to characterise the continuity of a function. Thus for a function $f : X \to Y$ between topological spaces, the following conditions are equivalent to the fact that f is continuous: a) if $C \subset Y$ is closed, so is $f^{-1}(C)$; b) for each $A \subset X$, $f(\overline{A}) \subset \overline{f(A)}$.

We list some simple properties of interiors and closures: If A and B are subsets of a topological space X, then

a) $x \in A^o$ if and only if there is an open subset U with $x \in U$ and $U \subset A$;
b) $x \in \overline{A}$ if and only if every open set U containing x meets A.
c) $\overline{A} = X \backslash (X \backslash A)^0$;
d) $(A^o)^o = A^o$, $\overline{\overline{A}} = \overline{A}$;
e) if $A \subset B$, then $A^0 \subset B^0$ and $\overline{A} \subset \overline{B}$;
f) $\overline{A \cup B} = \overline{A} \cup \overline{B}$ and $(A \cap B)^o = A^o \cap B^0$.

Note that it is *not* true that $(A \cup B)^o = A^o \cup B^o$ or that $\overline{A \cap B} = \overline{A} \cap \overline{B}$ as the following examples show. Firstly, we take $A = [0, 1]$, $B = [1, 2]$. Then the equality $(A \cup B)^o = A^o \cup B^o$ is violated. On the other hand, if $A =]0, 1[$ and $B =]1, 2[$, then $\overline{A \cap B} \neq \overline{A} \cap \overline{B}$.

Suppose now that A is a subset of a topological space X. Then $x \in X$ is called **a limit point** of A if it lies in the closure of A i.e. if every open set U containing x meets A; **a cluster point** of A if every open set U containing x meets A in a point other than x (i.e. if x is a limit point of $A \backslash \{x\}$); **an isolated point** of A if it is in A but is not a cluster point thereof i.e. there is an open set U so that $U \cap A = \{x\}$.

The set of all cluster points of A is called the **derived set** of A and denoted by A^d. Then $\overline{A} = A \cup A^d$. A is said to be **dense in itself** if $A = A^d$.

A subset A of a topological space X is **dense** in X if $\overline{A} = X$ (i.e. if every non- empty subset of X meets A). X is **separable** if it has a countable dense subset. The classical example of a dense subset is the rational numbers as a subset of the real line. This shows that the latter space is separable.

If A is a subset of a topological space, then we define its **boundary** ∂A to be the set $\overline{A} \cap \overline{X \backslash A}$. Thus x is in the boundary of A if and only if each open subset U which contains x meets both A and its complement. The reader will easily check that for domains (i.e. open sets) in $\mathbf{R}^n$, this coincides with the intuitive notion of a boundary.

Note that x is in the boundary of A if and only if it is a limit point of both A and its complement.

The boundary of a set A is closed and we have the inclusion $\partial(\partial A) \subset \partial A$. However, we do not always have equality as the example A the subset of $[0,1]$ consisting of the rational elements of the latter shows. (Here the boundary of A is the interval $[0,1]$ and the boundary of the latter is the two-point set $\{0,1\}$).

The reader can verify the following simple facts:

$$\begin{aligned}
A^o &= A \backslash \partial A \\
\overline{A} &= A \cup \partial A \\
\partial(A \cup B) &\subset \partial A \cup \partial B \\
\partial(A \cap B) &\subset \partial A \cup \partial B \\
\partial(X \backslash A) &= \partial A \\
\partial U &= \overline{U} \backslash U \text{if } U \text{is open} \\
\partial A &= \emptyset \text{if and only if } A\ oxisbothopenandclosed.
\end{aligned}$$

A subset U of a topological space X is **regularly open** if it is equal to the interior of its closure. Dually, it is **regularly closed** if it is the closure of its interior. The subset $]-1,1[\backslash\{0\}$ of the real line is an example of an open set which is not regularly open. Note that if A is an arbitrary subset of a topological space, then the interior of its closure is regularly open. Also if U and V are regularly open, then so is their intersection (but not necessarily their union as the above example shows). Corresponding results hold for regularly closed sets and are obtained by complementation.

In specifying a topology resp. in interpreting various topological concepts, it is often sufficient to consider only open sets of a special type. For example, for a function f between metric spaces, it suffices for f to be continuous that the pre-images of open *balls* under f be open as the reader can easily verify. The appropriate concept in the case of a general topological space is that of a **basis** i.e. a subfamily $\mathcal{B}$ of a topology τ so that each open set U is the union of sets in $\mathcal{B}$. More generally $\mathcal{B}$ is a **subbasis** if the family $\tilde{\mathcal{B}}$ consisting of the sets which are intersections of finite collections from $\mathcal{B}$ forms a basis

For example, the family $\{x\}_{x \in X}$ of singletons in X forms a basis for the discrete topology. The set of pairs $\{x,y\}$ $(x, y \in X, x \neq y)$ forms a subbasis (if the cardinality of X is at least three), but not a basis. In the real line, the family of open intervals is a basis for the usual topology, whereas the family of open intervals of length 1 is merely a subbasis. As indicated by the introductory remarks above, the family of open balls in a metric spaces forms a basis for the metric topology.

It is often more convenient to specify a topology by describing a basis rather than *all* open sets and we shall take advantage of this in the following. Of course, a given family can be a basis for at most *one* topology so there is no ambiguity in specifying a topology in this way.

We recall here that a topological space is defined to be separable if it has a dense countable subset. This condition is satisfied by most of the spaces which arise in applications. However, as we shall see, it has certain disadvantages and it is often useful to replace it by the following stronger condition:

Definition: A space X is said to be **countably generated** (or to satisfy the first axiom of countability) if it has a countable basis. Any such space is separable since a sequence with one element from each set of a countable basis is easily seen to be dense. Conversely, if (X, d) is a separable *metric* space, then it has a countable basis. In fact, if (x_n) is a dense sequence, then the countable family $\{U(x_n, 2^{-n})$ forms a basis.

As remarked above, our list of metric spaces supplies us with a variety of topological spaces, including most of those which are useful in analysis or geometry. Here we shall concentrate on more pathological spaces whose topologies cannot be defined by a metric. We begin with a series of topologies which are defined on *any* set X and are independent of any structure which the latter may have.

I. The **cofinal topology** τ_{cf} on X has as open sets the empty set and those subsets of X whose complements are finite. Similarly, the **co-countable topology** τ_{cc} consists of the empty set, together with those subsets whose complements are countable.

II. If X is a set and x_0 is a distinguished point in X, then the **particular point topology** has as open sets all subsets of X which contain x_0 (plus the empty set of course). The special case where the cardinality of X is 2 is called the **Sierpinski space**. It can be represented schematically as in figure 1.

We now turn to a series of topologies which are related to specific structures on the underlying spaces:

III. **The Niemitsky half-plane:** The underlying space is the upper half-plane

$$H_+ = \{(\xi_1, \xi_2) \in \mathbf{R}^2 : \xi_2 \geq 0\}.$$

It is generated by the set of all open balls in the open half-plane (i.e. those x

in H_+ with $\xi_2 > 0$) plus sets of the form $U \cup \{x\}$ where U is an open ball in the upper half-plane which touches the x-axis at the point x (cf. figure 2).
IV. **Topologies defined by orders** The topology of $\mathbf{R}$ is intimately related to its order structure and it can be conveniently generalised as follows. Suppose that a set X has a partial ordering $<$. Then the open intervals i.e. the sets of the form

$$]x, y[= \{z \in X : x < z < y\}$$

form a basis for the **order topology** on X.

Particularly interesting among the ordered topologies are those on the so-called **ordinal spaces**. Suppose that Γ is a limit ordinal. Then we can regard the intervals $[0, \Gamma]$ and $[0, \Gamma[$ as topological spaces in the above way. The most interesting case is where Γ is the smallest uncountable ordinal. This is a useful source of counterexamples.

A further example of a topological space which arises from an order structure is:
V. **The Sorgenfrey line:** This is the real line, provided with the topology τ_{sorg} generated by the family

$$\{[x, y[: x, y \in \mathbf{R}, x < y\}$$

of half-open intervals as basis. This topology is finer than the natural one. Closely related are the topologies τ_R and τ_L on $\mathbf{R}$ (R and L for right and left) which are generated by the families

$$\{]x, \infty[: x \in \mathbf{R}\}$$

resp.

$$\{]-\infty, y[: y \in \mathbf{R}\}.$$

For applications of topology to analysis, two concepts are of crucial importance – convergence (of sequences) and continuity (of functions). That of convergence allows one to provide a framework for the rigorous treatment of questions of approximation. There is a close connection between them. Thus for metric spaces we can define continuity in terms of convergence as follows: a function between metric spaces is continuous if and only if it maps convergent sequences onto convergent sequences, more precisely, if whenever $x_n \to x$ in X, then $f(x_n) \to f(x)$ in Y. On the other hand, we can characterise convergence in terms of continuity as follows. Let (x_n) be a sequence in a metric space X, x a point of X. Then $x_n \to x$ if and only if the following function is continuous. We denote by Y the subspace of $\mathbf{R}$ consisting of the origin and all points of the form $\frac{1}{n}$ where n runs through the set $\mathbf{N}_0$. Then the sequence converges to x if and only if the function f from Y into X which maps $\frac{1}{n}$ onto x_n and 0 onto x is continuous.

The definition of convergence for sequences in metric spaces can be given in the following version which carries over to the case of general topological spaces:

Definition: A sequence (x_n) in a topological space (X, τ) converges to a point x if and only if for each open set U containing x there is an $N \in \mathbf{N}$, so that $x_n \in U$ for $n \geq N$.

However, the following examples show that some rather peculiar things can happen with respect to convergence in general topological spaces and this will lead us to generalise the notion of convergence shortly:

Examples: This last example shows that two distinct topologies on a set can induce the same notion of convergence of sequences. As mentioned above, this fact makes it necessary to consider convergence for a more general class of objects than sequences. One possibility is the use of **filters**:

Definition: A **filter** on a set X is a non-empty collection $\mathcal{F}$ of subsets of X so that a) each $A \in \mathcal{F}$ is non-empty; b) if A and B belong to $\mathcal{F}$, then so does their intersection $A \cap B$; c) if $A \in \mathcal{F}$, then every superset of A is also in $\mathcal{F}$.

The most important example of a filter is the **neighbourhood filter** $\mathcal{N}(\S)$ of a point in a topological space i.e. the set of all subsets A of X which contain an open U with $x \in U$ (such sets A are called **neighbourhoods** of x).

Filters are often conveniently specified by the use of so-called **filter bases** which we define as follows: A collection $\mathcal{F}$ of non-empty subsets of a set X is called a **filter basis** if it satisfies condition a) above and, in addition, the condition b′) if A and B belong to $\mathcal{F}$, then there is a $C \in \mathcal{F}$ with $C \subset A \cap B$. Then the collection

$$\tilde{\mathcal{F}} = \{B \subset X : oxthereisan\, A \in \mathcal{F} \wr \S \sqsupseteq \rangle \sqcup \langle\, \mathcal{A} \subset \mathcal{B}\}$$

is a filter on X. It is called the **filter** generated by $\mathcal{F}$.

Further examples of filters are:

I. If A is a non-empty subset of a set X, then $\mathcal{F}(\mathcal{A})$, the family of subsets of X which contain A, is a filter. It is generated by the filter basis $\{A\}$.

II. The following filter on $\mathbf{N}$ is of fundamental importance. It is called the **Fréchet filter**. The family of sets

$$\mathbf{N}_m = \{n \in \mathbf{N} : n \geq m\}$$

forms a filter basis in $\mathbf{N}$. The filter that it generates is the Fréchet filter. In other words, the Fréchet filter consists of those subsets which contain *almost all* positive integers.
III. If $\mathcal{F}$ is a filter on a set X and f is a mapping from X into Y, then, as the reader will have no trouble in verifying, the family

$$f(\mathcal{F}) = \{\mathcal{A} \subset \mathcal{Y} : \{^{-\infty}(\mathcal{A}) \in \mathcal{F}\}$$

is a filter on Y. It is called the **image** of $\mathcal{F}$ under f. Note that this is not, in general, the same thing as the family of images of the sets of $\mathcal{F}$. However, the latter family *is* a filter basis which generates the above filter.

One example of this construction is as follows: suppose that we have a sequence (x_n) in a set X. Then we can regard this as a mapping from $\mathbf{N}$ into X and so we can define the **Fréchet filter** of the sequence to be the image of the corresponding filter in $\mathbf{N}$ which we defined in II.

We are now ready to introduce the notion of convergence for filters. A filter $\mathcal{F}$ on a topological space X converges to a point x there if it contains the neighbourhood filter of x. In terms of a filter basis $\mathcal{G}$ which generates $\mathcal{F}$ we can restate this as follows: for each $U \in \mathcal{N}(\S)$, there exists an $A \in \mathcal{G}$ so that A is contained in U.

If we apply this to the Fréchet filter of a sequence (x_n), we see that the latter converges to x in the sense defined above if and only if its Fréchet filter converges.

All topological notions can be expressed in terms of filters. As an example, we consider the closure of a set. In contrast to the case of a metric space, it is not true in general that a subset is closed if it contains the limits of all sequences therein as we have seen above. However, we do have a corresponding and valid version of this result if we replace sequences by filters. Suppose that A is a non-empty subset of X and $\mathcal{F}$ is a filter on A. Then A, while it need not be a filter on X, *is* a filter basis and so *generates* a filter on X. (This filter is just the image of $\mathcal{F}$ under the embedding of A in X in the sense of III above).

Proposition 46 *Let A be a subset of a topological space X. Then a point x of X lies in the closure $\overline{A}$ of A if and only if there is a filter on A which is such that the filter it generates in X converges to x. Hence A is closed if and only if every filter on A which generates a convergent filter in X is such that the limit is in A.*

PROOF. We show that if x is in the closure of A, then there is a filter in A which generates one on X which converges to x. But it is clear that $\{A \cap N : N \in \mathcal{N}(\S)\}$ has the required property.

■

If $\mathcal{F}$ is a filter, a **cluster point** for $\mathcal{F}$ is a point x so that each neighbourhood of x meets each $A \in \mathcal{F}$ (i.e. the set of cluster points is just the intersection $\bigcap_{A\in\mathcal{F}} \overline{A}$). Cluster points can be characterised as follows:

Proposition 47 *x is a cluster point of the filter $\mathcal{F}$ if and only if there is a finer filter $\mathcal{G}$ which converges to x. ($\mathcal{G}$ is finer than $\mathcal{F}$ means simply that $\mathcal{F} \subset \mathcal{G}$).*

Proof. It is clear that if $\mathcal{G}$ converges to x, then x is a cluster point of $\mathcal{G}$ and so of $\mathcal{F}$ (a limit of a convergent filter is clearly a cluster point). On the other hand, if x is a cluster point, then each neighbourhood of x meets each $A \in \mathcal{F}$. Hence

$$\{C \cap N : C \in \mathcal{F}, \mathcal{N} \in \mathcal{N}(\S)\}$$

is a filter base and so generates a filter $\mathcal{F}$ which is finer than both $\mathcal{F}$ and $\mathcal{N}(\S)$.

■

Recall that a mapping f between topological spaces was defined to be continuous if inverse images of open sets are open. This is equivalent to each of the following conditions (where f maps X into Y and $\mathcal{B}_\infty$ resp. $\mathcal{B}_\in$ are bases for the topologies of X and Y respectively).

for each $U \in \mathcal{B}_\in$, $f^{-1}(U)$ is open in X; for each $x \in X$ and $U \in \mathcal{B}_\in$ containing $f(x)$, there is a $V \in \mathcal{B}_\infty$ containing x with $f(V)$ contained in U.

We now display a natural characterisation of continuity which involves convergence of filters:

Proposition 48 *Let $f : X \to Y$ be a function between topological spaces. Then a) f is continuous at x_0 if whenever $\mathcal{F} \to \S_\prime$ in X, then $f(\mathcal{F}) \to \{(\S_\prime)$ in Y; b) f is continuous on X if whenever a filter $\mathcal{F}$ converges to a point x in X, the image $f(\mathcal{F})$ converges to $f(x)$.*

Another possibility for generalising the concept of convergence is the use of nets. A **net** in a set X is a family which is indexed by a directed set A i.e. it is a mapping from A into X (written $(x_\alpha)_{\alpha\in A}$). Such a net converges to a point x if for every open set U containing x there is a $\beta \in A$ so that $x \in U$ for $\alpha \geq \beta$. Then the following characterisations of closedness and continuity hold: x lies in the closure of A if and only if it is the limit of a convergent net in A; $f : X \to Y$ is continuous if and only if $x_\alpha \to x$ in X implies $f(x_\alpha) \to f(x)$ in Y. This result and similar ones can be proved by using the following correspondence between nets and filters. If (x_α) is a net, then just as in the case of sequences, the set $\{F_\beta : \beta \in A\}$ where $F_\beta = \{x_\alpha : \alpha \geq \beta\}$ is a filter basis and we see that the corresponding filter converges to x if and only if the original net does. On the other hand, if $\mathcal{F}$ is a filter, then we can construct a net as follows: we define the index set A to be the set

$$\{(x, B) : B \in \mathcal{F} \wr\S\dashv\backslash\lceil\ \S \in \mathcal{B}\}.$$

This is directed under the ordering defined by specifying that $(x, B) \leq (x_1, B_1)$ if and only if $B \supset B_1$. Then we can define a net (x_α) where $x_\alpha = x$ whenever $\alpha = (x, B)$. This net converges to a point x_0 if and only if the original filter does.

6.1 Construction of topological spaces

In order to enrich our collection of examples of topological spaces, we now describe some simple ways of constructing new spaces from old ones. Firstly we note that any subset of a topological space has itself a natural topology. As open sets in A we take the intersections of A with open subsets of X. More formally, the family

$$\tau_A = \{U \cap A : U \in \tau\}$$

is a topology on A – called the **topology induced** on A by τ. A with this topology is called a **topological subspace** of X. We make the following simple remarks about induced topologies:
I. If $\mathcal{B}$ is a basis (resp. a subbasis) for τ, then

$$\mathcal{B}_{\mathcal{A}} = \{\mathcal{U} \cap \mathcal{A} : \mathcal{U} \in \mathcal{B}\}$$

is a basis (subbasis) for τ_A.
II. On a subset A of a metric space (X, d), the topology induced by the metric topology coincides with that defined by the restriction of the metric to A. This follows easily from the fact that if $x \in A$, then the ϵ-ball in A with x as centre (defined by the restriction of the metric to A), is just the intersection of the ϵ-ball in X with A.
III. If we regard $\mathbf{R}^m$ as a subspace of $\mathbf{R}^p$ in the natural way (for $m \leq p$), then the usual topology on the latter induces the usual topology on the former.
IV. If $f : X \to X_1$ is a mapping between topological spaces which takes its values in a subset A of X_1, then f is continuous (from X into X_1) if and only if it is continuous from X into A (the latter with the induced topology).
V. A subset C of the subspace A of X is closed for the induced topology if and only if it has the form $C_1 \cap A$ for a closed subset C_1 of X.
VI. If we regard the x-axis as a subset of the Niemitsky half-plane, then the induced topology is the discrete topology. This provides an example of a separable space with a subspace which is *not* separable.

VII. Suppose that f is a mapping from the space X into Y and that A is a subset of X. Then the restriction of f to A is continuous for the induced topology. If $\mathcal{F}$ is a filter in A which converges to a point x in A, then the filter generated by $\mathcal{F}$ in X converges to x there (and conversely).

A topological property is called **hereditary** if each subspace of a space with the property also enjoys the property. For example, we have seen that the property of being separable is not hereditary, whereas that of being countably generated is.

6.2 Products:

We now show how to regard Cartesian products of topological spaces as topological spaces. For the sake of simplicity, we begin with finite products. Later, we shall show how to deal with infinite products. Suppose that X is the product $\prod_{k=1}^n X_k$ where each X_k is provided with a topology τ_k. Then the sets of the form

$$U_1 \times U_2 \times \cdots \times U_n$$

where U_i is open in X_i form a basis for a topology on X. We call it the **product topology**. Its characteristic property is summarised in the next result:

Proposition 49 *The projection mappings $\pi_k : X \to X_k$ are all continuous and if f maps a topological space Y into X, then f is continuous if and only if for each k, $\pi_k \circ f$ is continuous from Y into X_k.*

A less formal way of stating this result is as follows: a continuous function from Y into X is just a system $(f_1, \dots, f_n)$ of continuous functions where f_k takes its values in X_k. (f_k is the mapping $\pi_k \circ f$).

A similar criterium for convergence is valid and can be derived immediately from the above result by using the trick mentioned on p. ????

Proposition 50 *If (x_n) is a sequence in the product space, then it converges to x there if and only if $\pi_k(x_n) \to \pi_k(x)$ for each k.*

In other words, convergence is determined by that of components.

Another simple fact about products which is useful to remember is that if A_k is a subspace of X_k, then the product $\prod A_k$ is a *topological* subspace of the product of the X_k (i.e. the product topology on the latter induces the product topology on the former).

We have already met a number of examples of products. For example, the n-torus tT^n is just the product of n copies of the circle.

6.3 Quotient spaces:

This method of constructing topological spaces can be illustrated by the following simple example: if we take a copy of the unit interval and bend it round so that we can join the end-points, then we obtain a copy of the unit circle. We can say that S^1 is the space obtained from a closed interval by identifying the endpoints. The general construction is as follows: X is a topological space and f is a surjection from X onto a *set* Y. Then we can define a topology τ_1 on Y as follows. A subset of the latter is defined to be open if its inverse image under f is open in X.

In our applications, the function f will arise in the following way. We have an equivalence relationship on X and Y is the set of equivalence classes, f being the natural surjection which maps an element to the equivalence class to which it belongs.

The decisive property of the above topology is as follows: a mapping g from Y into a topological space Z is continuous if and only if the composition $g \circ f$ is continuous.

Examples: We begin with three spaces which are useful as counterexamples and which can be most easily defined as quotient spaces: I. The line with two origins (figure 1). This is the quotient of the union of two copies X_1 and X_2 of the real line whereby we identify the points in pairs, with the exception of the origin. More formally, consider the following two subsets of the plane:

$$X_1 = \{(\xi_1, 0) : \xi_1 \in \mathbf{R}\} \quad X_2 = \{(\xi_1, 1) : \xi_1 \in \mathbf{R}\}$$

and put $X = X_1 \cup X_2$ and $Y = X |_\sim$ where

$$x \sim y \; oxifandonlyif \; x = y \; oxor \; x = (\xi_1, \eta), \; y = (\xi_2, \eta) \; oxwith \; \eta \neq 0.$$

II. The interval with three endpoints (figure 2). This is $X |_\sim$ where

$$X = \{(\xi_1, 0) : \xi_1 \in [0, 1]\} \cup \{(\xi_2, 1) : \xi_1 \in [0, 1]\}$$

and

$$x \sim y \; oxifandonlyif \; x = y \; oxor \; x = (\xi_1, 0) \; oxand \; y = (\xi_1, 1) \; oxwith \; \xi_1 > 0.$$

III. The pinched plane. This is an example of a quotient of a metrisable space which is not metric. It is typical in the sense that many desirable properties of topological spaces can be lost in the passage to a quotient space. The pinched plane is obtained from the plane by shrinking a line down to a point.

More formally, it is $\mathbf{R}^2 |_\sim$ where $x \sim y$ if and only if $x = y$ or x and y lie on the real axis. (The quotient space is not metrisable since the special point in the pinched plane fails to have a countable neighbourhood basis).

The above examples illustrate two special types of quotient space which arise frequently, especially in geometric topology, namely spaces obtained by shrinking subsets to points resp. by pasting spaces together along suitable subsets. Because of the frequency of their occurrence, it is worth describing such constructions in their natural generality.

I. Spaces obtained by shrinking subsets to a point. Here X_0 is a subset of a topological space X and the equivalence relation is defined as follows:

$$x \sim y \ oxifandonlyif \ x = y \ oxor \ x \ oxand \ y \ oxarein \ X_0.$$

Of course, example III above is a special case of this construction.

II. Spaces obtained by pasting. Here X and Y are topological spaces which we suppose to be disjoint (as sets) and X_0 resp. Y_0 are homeomorphic subsets, h_0 being a suitable homeomorphism from X_0 onto Y_0. Then we can paste X onto Y along this "common" subset as follows: we consider the quotient space of the union $X \cup Y$ under the following equivalence relationship:

$$x \sim y \ oxifandonlyif \ x = y \ oxor \ x \in X_0, y \in Y_0 \ oxand \ y = h(x).$$

For example we obtain the **Bretzel** by taking two tori, cutting a hole in each of them and pasting the spaces together along the edges of the hole (see figure 3). Perhaps the simplest examples of this construction are those which involve joining two spaces at a point. Here we have two topological spaces X and Y, with distinguished points x_0 and y_0 resp. The space $X \vee Y$ is then the quotient of the disjoint union $X \amalg Y$ obtained by identifying x_0 and y_0. Thus $\mathbf{S}^1 \vee \mathbf{S}^1$ is the figure of eight (see figure 4). (Strictly speaking, this wedge product depends on the particular choices of the distinguished points and this should have been incorporated in the notation. However, in the cases we are interested in, the choice of points is irrelevant so that this pedantry is unnecessary). We remark that the wedge product of X and Y can also be identified with the subset $X \times \{y_0\} \cup \{x_0\} \times Y$ of the product space $X \times Y$. For example, when both spaces are the circle, this means that we are identifying the figure of eight with the subset of the torus indicated in the diagram 5.

Example: The cone over a space. If X is a topological space, the cone over X is the quotient of the product $X \times I$ of X with the unit interval which is obtained by identifying the points of the form $(x, 1)$ i.e. we pinch the top of the cylinder over X together. Thus the cone over $\mathbf{S}^1$ is just the classical

form of a conical dunce's cap (figures 6,7). The following more exotic cones are often used as counterexamples.
I. The cone over $\mathbf{Z}$.
II. The cone over the space $\{\frac{1}{n} : n \in \mathbf{N}\} \cup \{0\}$.

We can picture both of these spaces as subsets of the plane (see figures 8 and 9) but the reader is warned that the quotient topologies described above do not coincide with the topologies induced from the natural one of the plane.

Example: The suspension of a space. This is defined as the quotient of the product of X with the unit interval under the equivalence relationship

$$(x, s) \sim (y, t) \quad \text{if and only if} \quad x = y \quad \text{and} \quad s = t \quad \text{or} \quad s = 1 \quad \text{and} \quad t = 1 \quad \text{or} \quad s = 0q \quad \text{and}$$

In other words, we pinch together both the top and the bottom of the cylinder as in figure (10). For example, the suspension of $\mathbf{S}^1$ is $\mathbf{S}^2$. More generally, the suspension of $\mathbf{S}^n$ is $\mathbf{S}^{n+1}$ for each n.

6.4 Quotient topologies on subspaces

Suppose that $f : X \to Y$ is a surjection and that Y has the corresponding quotient topology. Then this need not apply to subspaces of Y. More precisely, if A is a subspace of Y and if B is a subset of X with $f(B) = A$, then the topology induced on A from Y need not coincide with the quotient of the subspace topology on B. An example where this fails is the following: we take for X the interval I, for Y the circle and for B the interval $]0, 1]$. It is easy to see, however, that the quotient topology is always finer than the subspace topology.

We remark here that if $f : X \to Y$ is a quotient mapping, then it is not necessarily open or closed. On the other hand, any open or closed continuous surjection is automatically a quotient mapping. Just when a quotient mapping is open or closed can be simply characterised as follows:

Proposition 51 *A surjective mapping $f : X \to Y$ from the topological space X to the set Y is open (resp. closed) when Y is provided with the quotient topology if and only the following condition holds: for each open subset U, $f^{-1}(f(U)$ is open resp. for each closed subset C of X, $f^{-1}(f(C))$ is closed.*

The set $f^{-1}(f(A))$ which occurs in the above formulation is called the **saturation** of A. It has the following more intuitive description. $f^{-1}(f(A))$

is the union of all those equivalence classes which have non-empty intersection with A.

However, the reader can verify that this *is* the case if any of the following conditions are met: a) A is open or closed; b) f is open or closed.

As a further example of pathology which can occur in the case of quotient spaces, we note that if X and Y are product spaces, say $X = \prod X_\lambda$, $Y = \prod Y_\lambda$, and if f_λ is a quotient mapping from X_λ onto Y_λ for each λ, then this need not be true of the product mapping from X onto Y. However, if each f_λ is open, then it *is* a quotient mapping (in fact an open mapping).

We now consider two very general constructions which contain all of the previous methods as special cases:

Initial and final topologies: Let X be a set, $\{(X_\alpha, \tau_\alpha)\}_{\alpha \in A}$ be a family of topological spaces and, for each $\alpha \in A$, let f_α be a mapping from X into X_α. Then $\mathcal{B}$, the set of those subsets of the form $f_\alpha^{-1}(V)$ for some $\alpha \in A$ and some open set V in X_α, is a subbasis for a topology τ on X. The latter is called the **initial topology on** X **induced** by the f_α. It can be characterised as the coarsest topology on X for which each f_α is continuous.

Dual to the above is the construction of final topologies. Here X and the X_α are as above but now the f_α are mappings from X_α into X. Then the set of those subsets U of X which have the property that $f_\alpha^{-1}(U)$ is in τ_α for each α forms a topology on X – the **final topology** induced by the f_α. It is the finest topology so that each f_α is continuous.

An important aspect of the above topologies is the following description of continuous mappings: if X is a topological space with the initial topology induced by the mappings $f_\alpha : X \to X_\alpha$, then a mapping f from a further space Y into X is continuous if and only if $f_\alpha \circ f$ from Y into X_α is continuous for each α; if X has the final topology induced by the mappings f_α from X_α into X, then a mapping f from X into a further space Y is continuous if and only if $f \circ f_\alpha$ is continuous for each α.

These characterisations follow immediately from the descriptions of the open subsets of X.

Examples: Examples of initial topologies are subspaces and products. More precisely, if A is a subset of a topological space X, then the topology induced on A from X is precisely the initial topology corresponding to the inclusion mapping from A into X. Similarly, if X is the product $\prod_{k=1}^n X_k$ of a finite number of topological space, then the product topology on X is just the initial topology induced by the mappings $\pi_1, \dots, \pi_n$.

Examples of final topologies are quotient spaces. A further example is that of disjoint unions. Let (X_α, τ_α) be a family of topological spaces and let X be the (set-theoretical) disjoint union of the X_α. Then we can regard X as a topological space, by providing it with the final topology induced by the natural embeddings of the X_α in X. In other words, a subset U of X is open if and only its intersection with each X_α is open for τ_α. Thus a mapping f from X into a second topological space is continuous if and only if its restrictions to the X_α are continuous. X with this topology is called the **topological disjoint union** of the X_α. We note in passing that each X_α is then a clopen subset of X and that a subset C is closed if and only if $C \cap X_\alpha$ is closed in X_α for each α.

6.5 Infinite products:

The above description of finite products as initial structures should make it clear how we shall define infinite products. Suppose that we have an arbitrary family $(X_\alpha, \tau_a)_{\alpha \in A}$ of topological spaces and that X is their set-theoretical product. Then for each α there is a natural projection π_α from X into X_α. Then we regard X as a topological space with the corresponding initial topology. Suppose that U_β is open in X_β. Then the pre-image of this set under π_β is the product $U_\beta \times \prod_{\alpha \neq \beta} X_\alpha$. Hence the sets of this form are a subbasis for the product topology. From this it follows that a basis consists of all sets of the form $\prod_{\alpha \in A} U_\alpha$ where each U_α is open in X_α and all but finitely many of the U_α are equal to the whole space X_α. Using this description of the open sets it is not difficult to prove that the product topology has the natural properties, of which we mention explicitly two:

a) a function f from a space Y into X is continuous if and only if $\pi_\alpha \circ f$ is continuous into X_α for each α; b) a filter $\mathcal{F}$ in X converges to x if and only if $\pi_\alpha(\mathcal{F})$ converges to $\pi_\alpha(x)$ for each α.

6.6 Projective limits:

One of the most ubiquitous constructions in mathematics is that of projective limits. It is based on the following set-theoretical construction. Suppose that we have a family $(X_\alpha)_{\alpha \in A}$ of sets which are indexed by a directed set A. In addition we have, for each pair α, β in A with $\alpha \leq \beta$, a mapping $\pi_{\beta\alpha}$ from X_β into X_α so that the following compatibility conditions are satisfied: a) $\pi_{\alpha\alpha} = \ oxid$; b) $\pi_{\beta\alpha} \circ \pi_{\gamma\beta} = \pi_{\gamma\alpha}$ $(\alpha \leq \beta \leq \gamma)$. Such a family is called a **projective spectrum**. From now on we shall assume that the directed set is $\mathbf{N}$ in order to simplify the notation but the reader will have no difficulty in seeing that most of our considerations are equally valid in the general situation. In the case of spectra indexed by $\mathbf{N}$, it suffices to know the linking mappings $\pi_{n+1,n}$ for each n since condition b) implies that

$$\pi_{mn} = \pi_{n+1,n} \circ \pi_{n+2,n+1} \circ \cdots \circ \pi_{m,m-1}.$$

Suppose then that we have a projective spectrum

$$\{\pi_{mn} : X_m \to X_n : n \leq m\}.$$

Its **projective limit** is then defined to be the family of **threads** in the Cartesian product $\prod X_n$ i.e. the set of those sequences (x_n) for which $\pi_{n+1,n}(x_n + 1) = x_n$ for each n.

We begin with an example to display the motivation for this construction. We consider the space of continuous functions on the real line which we display as a projective limit in the following way. For each $n \in \mathbf{N}$, we let X_n denote the space of continuous real-valued functions on the interval $[-n, n]$. and define $\pi_{n+1,n}$ to be the natural restriction mapping. Then the (X_n) form a projective spectrum and we identify its limit as follows: a typical element of the latter is a thread (x_n) of continuous function where x_n is defined on $[-n, n]$. Further these functions are compatible in the sense that if $n < m$, then x_n is just the restriction of x_m. Clearly, this implies that the functions can be combined to define a continuous function on the whole line. Conversely, the restrictions of a continuous function on the line to the appropriate intervals defines a thread. Thus we can identify the projective limit with the space of continuous functions on the line.

This example can be generalised to arbitrary topological spaces, in which case we are forced to use directed sets of arbitrary cardinality. We shall return to this topic later.

We now turn to the situation where each of the X_n is a topological space. We then provide the projective limit of the spectrum, which we denote by $\overline{lim}\ X_n$, with a topology as follows. By definition, the limit is a subspace of the Cartesian product and so we simply regard it as a topological space with the induced topology.

We remark that there is a natural mapping from the projective limit X into X_n which we denote by π_n. It is simply the restriction of the projection from the product into the component X_n. Of course, this mapping is continuous and the reader will recognise from the above description of the topology that it is precisely the initial topology induced on X by the family of mappings (π_n).

Examples of projective limits: I. Intersections: Suppose that (X_n) is a decreasing sequence of subsets of a topological space X. Then the intersection $\bigcap X_n$

can be identified in a natural way with the projective limit of the spectrum

$$\{i_{mn} : X_m \to X_n, m \leq n\}$$

(where i_{mn} is the natural inclusion).
II. The following type of space, which is most naturally defined as a projective limit, is of importance in descriptive topology. These are spaces which have a representation as the projective limit of a spectrum $\{\pi_{mn} : X_m \to X_n, n \leq m\}$ where each X_n is countable and discrete (i.e. is homeomorphic to $\mathbf{N}$). Such spaces are then complete metric spaces. In fact, they are homeomorphic to closed subspaces of $\mathbf{N}^{\mathbf{N}}$ by the very definition of the projective limit.

Projective limits can degenerate into triviality as the following examples show.

Examples: I. We consider the sequence $([n, \infty[)$ of subsets of the real line. If we regard them as a projective spectrum as in I above, then their projective limit is equal to their intersection, which is, of course, the empty set.,
II. A rather more interesting example is the following. Let S be an uncountable set and consider the system $\{X_A : A \in \mathcal{F}(\mathcal{S})\}$ which is indexed by the family $\mathcal{F}(\mathcal{S})$ of finite subsets of S (of course, this is an uncountable indexing set — it is ordered by inclusion). X_A denotes the family of all injective mappings from A into $\mathbf{N}$. This forms a projective system (with the restriction mapping from X_B into X_A where $A \subset B$ as linking mapping). Since the only possible members of the projective limit are injective mappings from S into $\mathbf{N}$ and there are *no* such mappings, the projective limit is empty. This example is interesting because, in contrast to example I above, the linking mappings are all surjective but the mappings from the projective limit to the components are far from being surjective. We shall see below that this can only happen for indexing sets which are more complicated than $\mathbf{N}$.

It is important in some applications to know *a priori* that a given projective limit is non-empty. We cite here a result that ensures that this will be the case under certain special conditions. Its proof can be regarded as an abstract version of that of the classical theorem of Mittag-Leffler on the existence of meromorphic functions with pre-ascribed principal parts at its poles and for this reason the result is often referred to as the abstract Mittag-Leffler theorem.

Proposition 52 *Let (X_n) be a projective system of complete metric spaces, whereby the linking mapping $\pi_{n+1,n}$ are supposed to be Lipschitz continuous. If for each n $\pi_{n+1,n}$ is dense in X_n, then the image of the projective limit X in X_n under π_n is also dense.*

PROOF.

■

This is a case of a theorem which has no useful generalisation to the case where the index set is uncountable.

For the sake of completeness, we include the much more shallow result referred to above. (It is of purely set-theoretical nature):

Proposition 53 *If* $\{\pi_{mn} : X_m \to X_n, n \leq m\}$ *is a projective system of sets indexed by* $\mathbf{N}$ *whereby the linking mappings* π_{mn} *are all surjective, then the corresponding mappings from the projective limit to each* X_n *are also surjective.*

We leave the simple proof to the reader.

Dual to the construction of projective limits is that of inductive limits. In this case, we have a sequence (X_n) of topological spaces and, for each n, a mapping i_n from X_n into X_{n+1}. We can then define, for $m \leq n$ a mapping i_{mn} from X_m into X_n. The inductive limit X of this spectrum is defined to be the quotient of the disjoint union X_n under the following equivalence relation: $x \sim y$ if and only if there are m, n, p with p larger than both m and n so that x is in X_m, y is in X_n and $i_{mp}(x) = i_{np}(y)$.

The main property of the space constructed in this manner is that a continuous mapping from X into a topological space Y is defined by a sequence (f_n) of continuous mappings, where f_n maps X_n into Y and $f_n \circ i_{mn} = f_m$ whenever $n < m$. bs

Examples: We begin with the following remarks about this construction. In most of the application, the mappings i_n are homeomorphisms from X_n into X_{n+1}. Thus we have the case of unions.

We close this chapter by bringing applications of the quotient structure to two branches of geometrical topology — graphs and surfaces.

6.7 Graphs:

A graph is a topological space G which is a quotient of a disjoint union X of a finite number of copies of the unit interval under an equivalence relationship of the following simple type. Suppose that the intervals $I_1, \dots, I_n$ have endpoints (P_i, Q_i) and further that we have a partition $S_1, \dots, S_r$ of the finite set $\{P_1, \dots, P_n, Q_1, \dots, Q_n\}$. Then points x and y from X are defined to be equivalent if they are equal or if they are endpoints which belong to the same element S_k of the partition.

Examples: see figure 11.

The equivalence class of an endpoint is called a **vertex** of the graph and the copies of the interval are called its **edges**. The **order** of a vertex is defined to be the number of edges which have this vertex as an endpoint i.e. it is the cardinality of the set S_k of the partition to which the endpoint belongs. The vertex is **odd** or **even** according as its order is odd or even. A basic fact about graphs is that the number of odd endpoints is always even.

PROOF. Let n_i denote the number of vertices of order i. Then the number of vertices is clearly

$$N = n_1 + n_2 \ + \dots$$

while the number of odd vertices is

$$N_{\text{odd}} = n_1 + n_3 + \dots$$

Now the number of original endpoints in X (before identification) is

$$N_{\text{tot}} = n_1 + 2n_2 + 3n_3 + \dots$$

Of course, the latter number is even. Now the difference $N_{\text{tot}} - N_{\text{odd}}$ is

$$2n_2 + 2n_3 + 4n_4 + \dots$$

which is also even. Hence N_{odd} is even.

■

A **path** in a graph is a finite sequence $(v_0, \dots, v_n)$ of vertices which are such that for each i there is an edge from v_i to v_{i+1}. v_0 is called the **initial point** of the path, v_n the **endpoint**. If these coincide, then the path is **closed**. A **cycle** is a path with $v_0 = v_n$. A **circuit** is a trail with $v_0 = v_n$. A **Hamiltonian path** is one which contains all vertices. A **Hamiltonian cycle** is a cycle which contains all vertices. An **Eulerian circuit** is a circuit which contains all edges and a **Eulerian trail** is one which contains all edges.

Then we have the following result:

Proposition 54 *A non-trivial connected graph has a Eulerian circuit if and only if each vertex has even degree. A connected graph has an Eulerian trail from a vertex v to a vertex w (whereby these vertices are distinct) if and only if v and w are the only vertices with odd degree.*

Example: The question whether the graph (figure ????) has an Eulerian path is the question whether there is a closed walk in the above configuration (figure ???) which involves crossing each bridge exactly once (the Koenigsberg bridge problem).

6.8 Surfaces as quotient spaces:

In the first section we defined the torus as the product of two circles. It can also be described as a quotient of a square in the following simple manner. Consider the following equivalence relationship on $\mathbf{I}^2$.

$$
\begin{aligned}
(s,t) \sim (s',t') \text{if and only if} \quad & s = s' \text{and} t = t' \text{or} \\
& s = 0, s' = 1 \text{and} t = t' \text{or} \\
& s = 1, s' = 0 \text{and} t = t' \text{or} \\
& s = s', t = 0 \text{and} t' = 1 \text{or} \\
& s = s', t = 1 \text{and} t' = 0.
\end{aligned}
$$

Instead of this hopelessly unwieldy description, the above relationship can be displayed graphically as in figure (13). Here the fact that opposite sides are labelled with the same letter means that they are to be identified, while the arrows indicate that points are to be identified with their mirror images in the lines parallel to the appropriate sides which bisect the square. In the following examples, we shall use this more informal method of describing such equivalence relationships.

The cylinder: This is the quotient of $\mathbf{I}^2$ displayed in figure 14.

The Möbius band: Figure 15. Here the fact that the arrows point in different directions indicates that the points on the appropriate sides are identified with their mirror images in the centre.

The Klein bottle: Figure 16

Note that in the above representation of the Klein bottle as a subset of three-dimensional we have cheated in the sense that the bottle is self-intersecting. This is unavoidable since there is no subset of $\mathbf{R}^3$ which is homeomorphic to the Klein bottle. One has to go into the fourth dimension to obtain a representation of it.

The projective plane: Figure 17

As in the case of the Klein bottle, the projective plane cannot be realised in space without introducing a self-intersection. We can visualise it as a "sphere with cross-cap".

We now introduce a convenient algebraic notation for describing such representations of surfaces as quotients of squares. Consider the concrete case of the torus with the above representation (figure 13).

We denote this by the symbol $ab^{-1}a^{-1}b$ which is obtained as follows. We start at an arbitrary side (in this case the top a) and traverse the circumference of the square (say in the clockwise direction, although this does not matter), writing down successively the symbols for the sides. If the arrow on a side points in the direction in which we are travelling, we write down the letter of the alphabet with which it is labelled. If the arrow points backwards, we add the index -1. Thus the Möbius band has symbol ??? the Klein bottle ????? and the projective plane ?????

Exactly the same method can be used if we replace the square by *any* regular polygon. This allows us to construct more intricate surfaces. For example, the **handle** is the quotient of the regular pentagon indicated in figure 18. It has symbol $acb^{-1}a^{-1}b$. (Here the letter c occurs only once to indicate that this side of the pentagon is not identified with any other one. It then forms part of the boundary of the resulting surface. By the way, we are now using the word "boundary" in its everyday sense, not in the technical sense introduced in I.3)

With this convention any string of symbols of the form

$$a_1^{\epsilon_1} \dots a_n^{\epsilon_n}$$

where the indices ϵ_i are either 1 (in which case we do not reproduce it) or -1 represents a surface, provided that each letter appears at most twice. The surface is uniquely determined by this string of symbols (although a given surface can have several such representations, depending, for example, on the edge at which one starts).

Using the symbolic representation, we can distinguish between certain types of surfaces. A surface is **closed** or **without boundary** if each letter appears exactly twice. Otherwise, we have a surface **with boundary**. For example, the torus, the Klein bottle and the projective plane are closed surfaces, whereas the Möbius band and the cylinder are with boundary.

We also distinguish between **two-sided** (or oriented) surfaces and **one-sided** (or non-oriented) surfaces. Examples of the former are the torus and the cylinder and of the latter the Möbius band, the Klein bottle and the projective plane. The nomenclature one- or two-sided is self-explanatory (try to find a second side on a Möbius band!). The term oriented or non-oriented comes from the fact that on a surface like the Möbius it is possible to change the orientation of a coordinate frame by traversing the surface. This is impossible to do on an oriented surface such as a cylinder or the sphere.

One can determine whether a surface is orientable or not from its algebraic symbol as follows. The surface is non-orientable if one symbol occurs twice with the same index. Otherwise it is orientable.

Representations of surfaces as quotients of polygons can be used to decide the results of simple experiments involving cutting up surfaces which often crop as party games. Consider, for example, the well-known parlour trick of cutting a Möbius band along a central line. This can be carried out without scissors and paper as in figure 19. Hence the result is a cylinder. In fact, if actually carried out with a real Möbius band, what one obtains is a Möbius band with two twists. This is homeomorphic to the cylinder (but is not embedded in space in the same way).

As a further example, consider figure 20 which shows that the projective plane is obtained by "closing" the Möbius band with a disc. In other words, if we cut a disc out of the projective plane, we get a Möbius band.

We can construct new surfaces by means of the exotically named construction of the **smash product**. This is defined as follows. We cut discs out of each of the surfaces X and Y and paste them together along the edges of the wholes (which are homeomorphic to $\mathbf{S}^1$). (see figure 21) The resulting manifold is denoted by $X\sharp Y$. As in the case of the wedge product, for those surfaces which we consider the resulting surface is independent of exactly where and how we cut out these discs.

For example, the smash product of two copies of the sphere is again the sphere. More interestingly, the smash product of two tori is the **Bretzel** (figure 23)

We can determine the algebraic symbol of a smash product from the symbols of the components. We do this for the Bretzel as the smash product of two tori but the method is completely general.

The smash product of two projective planes is the Klein bottle. This is rather difficult to visualise directly, but we can also see it formal manipulations with the polygon representations as follows. We begin with two projective planes as in figure 24.

The spaces which we have been discussing are examples of what are called two-dimensional manifolds (or just 2-manifolds). The fact that they are two-dimensional can be expressed mathematically by noting that each point on them has a neighbourhood which is homeomorphic either to a) the open disc in $\mathbf{R}^2$; or b) half of this disc (together with the bounding diameter. (figure 25) The handle, for example, has points of both types (figure 26). Points of the second type are just those on the boundary as described above i.e. they lie on edges of the polygon which are not identified with a second edge.

The surfaces which we obtain by our constructions have two topological properties which will be discussed in detail in later chapters. They are compact and connected. For our present purposes, the first conditions means that they are homeomorphic to closed, bounded sets in some euclidean space. The second means that each pair of points can be joined by a continuous curve *which lies on the surface* (figure 27). The first condition excludes surfaces such as that in figure 28, the second surfaces which are composed of several pieces.

Using the concept of smash products, we can describe *all* compact, connected surfaces as follows: a) each oriented compact, connected surface is homeomorphic the sphere or a smash product of tori; b) each non-oriented compact, connected surface is a smash product of a finite number of tori with a copy of the projective plane or of the Klein bottle. (This is a deep result of topology which we shall not prove here).

For this reason it is convenient to introduce the following notations: nT denotes a space which is homeomorphic to the smash product of n tori; (nT, P) is a smash product of n tori with a copy of the projective plane; (nT, K) is the smash product of n tori with a copy of the Klein bottle.

6.9 The Euler characteristic:

The Euler characteristic of a surface is an example of a so-called **topological invariant** and is calculated as follows. Consider a closed surface, such as a sphere. On the sphere we draw a network as in figure 29 and calculate the number $V-E+F$ where V denotes the number of vertices, E the number of edges and F the number of faces. It turns out that for a given surface (up to homeomorphism) this number is invariant i.e. independent of the network. It is called the Euler characteristic of the surface and denoted by the symbol $\chi(X)$. Such topological invariants will be discussed in some detail in Chapter III. Here we note that it can be calculated in terms of the surface's polygonal representation as the number $m-n+1$ where the surface is represented by a $2n$-gon (the number of faces must be even since we are considering *closed* surfaces) and m is the number of *distinct* points represented by the vertices of the polygon.

Examples: The method of representing the smash product of two surfaces can be used to show that we have the following formula for the Euler characteristic of $X\sharp Y$:

$$\chi(X\sharp Y) = \chi(X) + \chi(Y) - 2.$$

Using this formula, we can calculate very simply the characteristics of the surfaces listed above. We have the following result: $\chi(n, T) = 2-2n$; $\chi(nT, K) = -2n$; $\chi(nT, P) = 1-2n$. From these we can deduce the following criterium for the equivalence of compact, connected surfaces. Two surfaces X and Y are homeomorphic if and only if a) they have the same orientability properties. and b) they have the same Euler characteristic.

6.10 Connectedness:

In the introductory chapter on geometrical topology, we noted that the topological difference between the unit interval I and the circle S^1 could be pinpointed by using the fact that the removal of a point from I splits the space into two parts (provided that we do not remove an endpoint). We shall study the corresponding topological notion in this chapter.

Definition: A topological space (X, τ) is **connected** if it has no representation $X = A \cup B$ where A and B are open and disjoint. An equivalent condition is that the only subsets of X with are clopen are X and $\emptyset$. More generally, a subset A of X is connected (in X) if it is connected in the induced topology. This means that if U and V are disjoint open subsets of X whose union contains A, then either $A \subset U$ or $A \subset V$.

For example, a space with the indiscrete topology is connected (since X and are the only *open* sets). On the contrary, no space with the discrete topology is connected (with the trivial exceptions of the empty set or a one-point set). An infinite set with the co-finite topology is connected. The set of irrationals (with the natural topology) or the real line with the Sorgenfrey topology are not connected. For example, if α is an irrational number then

$$]-\infty, \alpha[\cap \mathbf{Q} =]-\infty, \alpha] \cap \mathbf{Q}$$

is clopen in $\mathbf{Q}$. On the other hand, $\mathbf{R}$ with the natural topology, is connected. In fact, the connected subsets of the real line are just the intervals as we shall see shortly. We begin with a simple Lemma:

Proposition 55 *Suppose that A is a connected subset of a topological space X. Then the closure $\overline{A}$ of A is also connected. (More generally, any set which lies between A and $\overline{A}$ is connected).*

PROOF. We suppose that $\overline{A}$ is contained in the union $U \cup V$ of two disjoint, open subsets of X. Then since A is connected, we have either $A \subset U$ or $A \subset V$. In the first case, A is a subset of the closed set $X \setminus V$ and hence so is $\overline{A}$. i.e. $\overline{A} \subset U$.

■

Proposition 56 *A subset of the real line is connected if and only if it is an interval.*

Proof. We show firstly that intervals are connected, beginning with the line itself. Let U be a clopen subset of $\mathbf{R}$, which we suppose to be neither the whole space or the empty set. Then it is the disjoint union of at most countably many disjoint intervals. (This is a standard result from an elementary Analysis course). At least one of the endpoints of these intervals is finite (otherwise U is $\mathbf{R}$). Such an endpoint is clearly in the closure of U but not in U which contradicts the fact that U is closed.

■

Once we know that the line is connected it follows that any open interval has the same property since it is homeomorphic to $\mathbf{R}$ and connectedness is clearly a topological property. We can then deduce the connectedness of any interval, since each interval is the closure (in itself) of its interior (in $\mathbf{R}$).

We now show that the only connected subsets of the line are intervals. Suppose that A is connected and put

$$\beta = \sup\{x : x \in A\}, \quad \alpha = \inf\{x : x \in A\}$$

(of course, these can be infinite). Now use the simple remark that if a and b are in A, then the interval $[a, b]$ is a subset of A (for if $\xi \in]a, b[$ were not in A, then the disjoint sets $U =]-\infty, \xi[$ and $V =]\xi, \infty[$ cover A). In order to be concrete, we suppose that α and β are not in A and that both are finite. Then we claim that A is the open interval $]\alpha, \beta[$. For if $x \in]\alpha, \beta[$, there are a, b in A with $a \leq x \leq b$ and so $x \in A$. Hence $]\alpha, \beta[\subset A$ and so is equal to A. (For the closed interval contains A and neither endpoint is in A). The remaining cases are dealt with in a similar fashion.

In order to extend our list of examples of connected spaces, we establish some simple stability properties. Firstly, it is clear that the union of connected sets need not be connected. However, if we can pin them down with a fixed set as in the next proposition, then we do obtain connectedness:

Proposition 57 *Let A be a non-empty connected subset of X and $\mathcal{A}$ a family of connected subsets, each of which intersects A. Then $A \cup (\bigcup \mathcal{A})$ is connected.*

Proof. Suppose that U and V are disjoint open sets whose union covers the relevant set. Then it also covers A and so we can assume, without loss of generality, that A is contained in U and disjoint from V. Now each $B \in \mathcal{A}$ must, for the same reason, also lie either in U or in V. But the fact that B intersects A rules out the second possibility. Hence all of the relevant sets are contained in U and therefore so is their union.

■

In most applications, A is a singleton. For example, we can deduce immediately from this result that $\mathbf{R}^n$ is connected as the union of the family of all lines which pass through the origin. More generally we can prove that products of connected spaces are connected, but in this case we have to use the general form of the result. We prove this for the product $X \times Y$ of two spaces but the result holds for arbitrary products. Without loss of generality we can assume that X and Y are non-empty (the case where one of them is empty is trivially true). Suppose then that b is an element of Y. The subset $A = X \times \{b\}$ of the product is homeomorphic to X and so is connected. Now $X \times Y$ is the union of A and the sets of the form $\{x\} \times Y$ $(x \in X)$ and the conditions of the above result are fulfilled.

We remark that it follows from a result which we shall prove below (Proposition ???) that the converse is valid i.e. a product can only be connected if each component is connected. (However, in this case we must assume that all of the spaces are non-empty).

A further consequence of the above result is that if X has the property that for any pair x, y of points therein, there exists a connected subset which contains x and y, then X is connected., For if we fix x and choose for each y in X a connected set C_y which contains both x and y, then $X = \bigcup_{y \in X} C_y$ is connected. This implies, for example, that convex subsets of $\mathbf{R}^n$ are connected. For any two points in such a set lie on the segment joining them and this is connected (being homeomorphic to an interval).

Another consequence of the Proposition is the following: suppose that we have a sequence (A_n) of connected subsets so that for each n, $A_n \cap A_{n+1}$ is non-empty. Then their union is connected.

If x is a point in a topological space, then we define the **component** $C(x)$ of x to be the union of those connected subsets which contain x. Note that this is connected and so is the *largest* connected set containing x. It is closed since its closure is connected and so cannot be larger.

A further simple stability result is the

Proposition 58 *Suppose that Y is the continuous image of a connected space. Then Y is also connected.*

Proof. The hypothesis means that there is a connected space X and a continuous $f : X \to Y$ with $f(X) = Y$. Suppose then that $Y = U \cup V$ where U and V are open and disjoint. Then $X = f^{-1}(U) \cup f^{-1}(V)$ and these pre-images are clearly open and disjoint. Hence one of the latter sets, say the first one, is the whole of X. Then $U = Y$ since f is surjective.

■

If we apply this result to a real-valued, continuous function on a connected space X, we see that its image $f(X)$ must also be connected and so is an interval. This can be restated as the following abstract version of the intermediate-value theorem from elementary calculus: if x and y are in X and c is a real number which lies between the values of f at x and y, then there is a z in X for which $f(z) = c$.

We remark further that it follows from the above proposition that a quotient of a connected space is itself connected. Thus the torus, Klein bottle and the Möbius band (more generally, all of the surfaces which we described as quotients of polygons) are connected.

We now consider some variants of the concept of connectedness:

Definition: A space X is **locally connected** if each point has a neighbourhood basis consisting of open, connected sets. The properties of being connected or locally connected are not comparable, as the following examples show:

Examples: Firstly it is easy to find a space which is locally connected but not connected. For example, a disjoint union of two copies of the interval is locally connected, but not connected. The following is an example of a space which is connected, but not locally connected. it is called the topologist's **sine curve**. One takes the subset of the plane which consists of the closure of the graph of the function $x \mapsto \sin \frac{1}{x}$ (defined on the interval $]0,1[$). (in other words, it is the union of this graph with the interval $[-1,1]$ in the y-axis). This space is connected (since the above graph is homeomorphic to the real line. Hence so is its closure). However, a glance at figure ??? shows that it is not locally connected. We remark that if X is locally connected, then $C(x)$ is open for each x (and hence clopen). The converse is also true i.e. if $C(x)$ is always open, then X is locally connected.

Definition: X is **arcwise connected** if any two points can be connected by a continuous curve i.e. for each x,y in X, there is a continuous function $c : [0,1] \to X$ with $c(0) = x, c(1) = y$. It follows from the criterium for connectedness in ??? that this implies connectedness, but the converse is not true. For example, the topologist's sine curve is connected but not arcwise connected since it is impossible to joint any point on the graph of $\sin \frac{1}{x}$ to the point $(0,1)$ say.

There is also a local version of the above property, called **local arcwise connectedness**. This means that each point has an open neighbourhood basis consisting of arcwise connected sets. The following is an example of a space which is arcwise connected, but not locally arcwise connected.

Example: Consider the subset of the plane indicated in figure ??? i.e. the union of the collection of segments from $(0,1)$ to the points $\{0, 1, \frac{1}{2}, \frac{1}{3}, \dots\}$ on the x-axis. Then this is arcwise connected, but not locally arcwise connected).

Typical examples of locally arcwise connected spaces are open subsets of $\mathbf{R}^n$. For such spaces, the difference between connectedness and arcwise connectedness vanishes:

Proposition 59 *If X is locally arcwise connected, then it is connected if and only if it is arcwise connected.*

PROOF. We need only show that if X is connected and locally arcwise connected, then it is arcwise connected. In order to do this, we fix a point x in X and define U to be the set of points which can be joined to x by an arc in X, resp. V to be the complement of U. We wish to show that $X = U$. Since X is connected and U is non-empty (x is a member), it suffices to show that U and V are open. But this follows from the fact that X is locally arcwise connected.

■

We remark that this result implies that for open subsets of $\mathbf{R}^n$, the concepts of connectedness and arcwise connectedness coincide.

6.11 Separation properties

As we saw above, in a general topological space a sequence can have more than one limit. We now discuss a number of conditions (which are known as **separation** properties) which ensure that such pathological behaviour cannot occur. Topologies are very general structures and this generality is paid for by the lack of depth of the results which can be proved and by the number of pathologies which can arise. The conditions which we shall introduce in this chapter can be seen as a means of reducing the possible types of space which we consider in order to arrive at a more coherent and richer theory.

We start with three simple separation axioms. In fact, for our purposes only the third will be of any importance but we bring the first two for the sake of completeness.

Definition: A topological space (X, τ) is

1. "" T_0 if whenever x and y are distinct points, then there is an $N \in \mathcal{N}(\S)$ with $y \notin N$ or an $N \in \mathcal{N}(\dagger)$ with $x \notin N$; T_1 if whenever x and y are distinct points, there is an $N \in \mathcal{N}(\S)$ with $y \notin N$; T_2 if whenever x and y are distinct points there is an $N_1 \in \mathcal{N}(\S)$ and an $N_2 \in \mathcal{N}(\S)$ with $N_1 \cap N_2 = \emptyset$.

(See figure 1). Of course, these three conditions are in increasing order of restrictiveness.

The following examples show that they are in fact distinct. An indiscrete space (with more than two elements) is not T_0. The Sierpinski space is T_0 but not T_1. If X is an infinite set, then (X, τ_{cf}) is T_1 but not T_2. Finally, any metric space is T_2.

Another trivial but useful fact is that if X is a topological space with one of the above T-properties, then any finer topology on X also possesses it. They are also preserved by subspaces and products. For example to see that the product of a family of T_2 spaces is again T_2, we proceed as follows. if $x = (x_\alpha)$ and $y = (y_\alpha)$ are distinct elements of the product $\prod X_\alpha$, then there is a $\beta \in A$ with $x_\beta \neq y_\beta$. Hence there are disjoint $N_1 \in \mathcal{N}(\S_\beta)$ and $N_2 \in \mathcal{N}(\dagger_\beta)$. Then $\tilde{N}_1 = (\prod_{\alpha \neq \beta} X_\alpha) \times N_1$ resp. $\tilde{N}_2 = (\prod_{\alpha \neq \beta} X_\alpha) \times N_2$ resp. are the required neighbourhoods.

The T_1-property can be usefully characterised in the following ways:

Proposition 60 *Let X be a topological space. Then the following are equivalent: a) X is T_1; b) for each $x \in X$, the singleton $\{x\}$ is closed; c) τ is finer than τ_{cf}; d) for each $x \in X$, $\{x\} = \bigcap_{N \in \mathcal{N}(\S)} N$.*

These are simple reformulations of the definition. It follows immediately from b) or c) above that there is precisely one T_1 topology on a finite set – the discrete topology.

If x is an element of the T_1-space X and $y_1, \dots, y_n$ are points of X which are distinct from x, then it is clear that there is a neighbourhood N of x which does not contain any of the y's. From this it follows that if x is a cluster point of a subset M of a T_1-space, then every neighbourhood of x contains infinitely many points of M. For suppose if possible that U is a neighbourhood of x which contains only finitely many points of M – say $\{y_1, \dots, y_n\}$. Then there is a neighbourhood V of x which fails to contain any of the y's. $U \cap V$ is a neighbourhood of x and so contains a point in M which is distinct from x. This is a contradiction.

We now turn to the T_2-spaces and show that these are precisely those ones in which convergent filters have unique limits. Almost all of the interesting topological spaces have this property.

Proposition 61 *The following conditions on a topological space X are equivalent:*
a) X is T_2;
b) for each x in X, $\{x\} = \bigcap_{N \in \mathcal{N}(\S)} \overline{N}$;
c) if a filter $\mathcal{F}$ on X converges simultaneously to two points x and y, then x and y coincide;
d) the diagonal set $\Delta = \{(x, x) : x \in X\}$ is closed in the product $X \times X$.

PROOF. Once again, these are all simple manipulations of the definition. We prove the equivalence of a) and c). a) implies c): Suppose that $\mathcal{F} \to \S$ and $\mathcal{F} \to \dagger$. If these two points were distinct, there would be disjoint neighbourhoods N_1 of x and N_2 of y. But this contradicts the fact that both of these sets are in $\mathcal{F}$. c) implies a): Suppose that X is not T_2. Then there are distinct points x and y so that $N_1 \cap N_2 \neq \emptyset$ for each pair N_1, N_2 where the first set is a neighbourhood of x and the second one a neighbourhood of y. Then the union of the families $\mathcal{N}(\S)$ and $\mathcal{N}(\dagger)$ forms a filter basis and the filter which it generates clearly converges to both x and y.

■

Condition d) above implies the following simple fact which is the basis of many uniqueness proofs: if two continuous functions agree on a dense set, then they agree *everywhere* (i.e. are identical functions). We remark that it is rather easy to forget that the range space must satisfy the T_2-condition for it to hold):

Proposition 62 *Let f and g be continuous mappings from a space X into a T_2-space Y. Then the set $\{x \in X : f(x) = g(x)\}$ where f and g coincide is closed. In particular, if f and g coincide on a dense subset of X, then they are equal.*

PROOF. The above set is the pre-image of the diagonal set in $Y \times Y$ (which we know to be closed) under the continuous mapping $x \mapsto (f(x), g(x))$ from X into $Y \times Y$ and hence is closed.

■

A typical example of an application of this theorem is the following: suppose that we have a projective spectrum $\{\pi_{m,n} : X_m \to X_n, n \leq m\}$ of topological spaces. Then, by the very definition, the projective limit is homeomorphic to a subspace of the product $\prod X_n$. Then we claim that in the case where the X_n are T_2-spaces, then it is actually homeomorphic to a *closed* subspace of the product. This is because the image of the projective limit under the homeomorphism is the set $\bigcap_{n \leq m} V_{mn}$ where

$$V_{mn} = \{(x_n) \in \prod X_n : \pi_{mn}(x_m) = x_n\}$$

and this is the coincidence set of two continuous functions with values in the T_2-space X_n.

6.12 Compactness

This is one of *the* most fundamental notions of analysis. The original example of its use is in the proof of the fact that a continuous function on a bounded, closed interval is bounded and attains its supremum and infimum. The abstract definition may at first glance hardly seem connected with this proof but the relationship will become clear in the course of the chapter.

Definition: If X is a set and A is a subset thereof, then a **covering** of A (in X) is a family $\mathcal{U}$ of subsets of X whose union contains A. A **subcovering** of $\mathcal{U}$ is a family $\mathcal{V} \subset \mathcal{U}$ which also covers A. If, in addition, X has a topology, then the covering is said to be **open** if each set therein is open.

A subset A of a topological space X is **quasi-compact** if each open covering of A has a finite sub-cover. In particular, X is quasicompact if it is quasi-compact as a subset of itself. We remark that quasi-compactness is an intrinsic property of a space i.e. if A is a subspace of X, then A is quasi-compact in X if and only if the space (A, τ_A) is quasi-compact.

X is **compact** if it is quasi-compact and T_2.

We note some simple properties of these notions. The definition can be reformulated as follows: X is quasi-compact if and only if each family $\mathcal{C}$ of closed subsets of X with the finite intersection property has non-empty intersection. (The finite intersection property means that each finite subfamily of $\mathcal{C}$ has non-empty intersection). This follows immediately from the original definition by taking complements.

The union of finitely many quasi-compact sets is quasi-compact. Hence finite sets, for example, are quasi-compact. It follows that in a T_2-space, finite unions of compact sets are compact. This is *not* true in the absence of the separation property as the example of the interval with three endpoints shows (it is not T_2 and therefore not compact but it *is* the union of two copies of the unit interval).

Examples of compact sets are closed, bounded subsets of $\mathbf{R}^n$ (this is essentially the Heine-Borel theorem of elementary Analysis). We shall prove below a more general result from which this follows. A space with the indiscrete topology is always quasi-compact but it is not compact if it has more than one point. A discrete space is compact if and only if it is finite.

In order to produce less trivial examples, we shall require some theory.

Proposition 63 *Lemma Let A be a compact subset of a T_2-space X, x a* *point of X which does not lie in A. Then we can separate x from A in the sense that there are disjoint open sets U and V with $x \in U, A \subset V$.*

PROOF. Since X is T_2, we can find, for every $y \in A$, disjoint open sets U_y and V_y with $x \in U_y$ and $y \in V_y$. Then $\{V_y : y \in A\}$ is an open cover of A and so we can find a finite subcover $\{V_{y_1}, \dots, V_{y_n}\}$.

Then

$$U = \bigcap_{n=1}^{n} U_{y_i} \quad oxand \quad V = \bigcup_{i=1}^{n} V_{y_i}$$

are the required sets.

Proposition 64 *Let A and B be disjoint, compact subsets of a T_2-space X. Then there are disjoint open sets U and V in X with $A \subset B$ and $B \subset V$.*

PROOF. We repeat the method of the proof of the Lemma. For each $x \in A$, we find open disjoint sets U_x and V_x with $x \in U_x$ and $B \subset V_x$. The proof then proceeds in the same way, using the cover $\{U_x : x \in A\}$ of A.

■

Proposition 65 *Let A be a subset of a topological space X. Then a) if X is T_2 and A is compact, A is closed; b) if X is quasi-compact and A is closed, A is quasi-compact.*

PROOF. a) follows immediately from the Lemma above since if x is not in A, we can find an open neighbourhood of x which is disjoint from A. b) If $\mathcal{U}$ is an open cover of A in X, then $\mathcal{U} \cup \{\mathcal{X} \backslash \mathcal{A}\}$ is an open covering of X and so has a finite sub-covering. This clearly provides a finite subcovering of $\mathcal{U}$ for A.

■

Of course, these results imply that quasi-compact spaces are normal and that compact spaces are T_4. They also imply that a subset of a compact space is closed if and only if it is compact (this is important since it provides an *intrinsic* characterisation of closedness in this situation).

We now investigate the behaviour of quasi-compactness under continuous mappings.

Proposition 66 *If $f : X \to Y$ is a continuous, surjective mapping and X is quasi-compact, then so is Y.*

PROOF. If $\mathcal{U}$ is an open covering of Y, then $f^{-1}(\mathcal{U}) = \{\{^{-\infty}(\mathcal{U}) : \mathcal{U} \in \mathcal{U}\}$ is an open covering of X and so has a finite subcovering. The images of these form the required subcovering of $\mathcal{U}$.

■

This simple result has a number of interesting corollaries. Firstly note that it implies that quotients of quasi-compact spaces are quasi-compact. Thus the Klein bottle and other surfaces, which we described as quotients of polygons, are quasi-compact (and hence compact).

A further consequence of this result is that a continuous mapping from a compact space X into a T_2-space Y is closed. For if A is a closed subset of X, then it is compact and hence so is its image. This implies that $f(A)$ is closed. The following special case of this result is important enough to be quoted as a Proposition:

Proposition 67 *If $f : X \to Y$ is a continuous bijection from a compact space onto a T_2-space, then f is a homeomorphism.*

In other words, it is impossible to weak a compact topology on a set without losing the T_2-property.

The connection between our abstract definition and more classical concepts of compactness is provided by a rather deep characterisation of compactness in metric spaces which we now consider.

6.13 Characterisations of compactness in a metric spaces:

Definition A metric space $(X, d))$ is **precompact** (or totally bounded) if for every positive ϵ there is a finite subset $\{x_1, \dots, x_n\}$ so that $X \subset \bigcup_{i=1}^n U(x_i, \epsilon)$.

It is clear that any compact metric space is precompact (consider the open covering $\{U(x, \epsilon) : x \in X\}$). The converse is not true since, for example, the open interval $]0, 1[$ is precompact but not compact.

We note that precompactness is not a topological concept. The following *are*:

Definition: A topological space (X, τ) is **sequentially compact** if every sequence in X has a convergent subsequence. It is **countably compact** if every sequence has a cluster point i.e. a point x so that every neighbourhood of x contains infinitely many elements of the sequence (more precisely, for each open neighbourhood U of x and each $N \in \mathbf{N}$, there is an $n > N$ with $x_n \in U$).

We note that a compact topological space is countably compact and that a sequentially compact space is also countably compact. In general, there is no other relationship between these notions. (In order to prove that a compact space is countably compact, consider the closed sets $B_m = \overline{\{x_m, x_{m+1}, \dots\}}$. This family has the finite intersection property and so its intersection is non-empty. But this intersection is precisely the set of cluster points of the sequence).

Our main result shows that the situation is quite different for metric spaces:

Proposition 68 *Let (X, d) be a metric space. Then the following conditions are equivalent: a) (X, τ_d) is compact; b) (X, d) is precompact and complete; c) (X, τ_d) is sequentially compact; d) (X, τ_d) is countably compact.*

The proof will be divided up into a series of Lemmata, several of which are of interest in their own right. At this point we note that the above result contains the fact that a subset of $\mathbf{R}^n$ is compact if and only if it is closed and bounded. For we already know that completeness is equivalent to the fact that it is closed and the reader can verify for himself that for subsets of $\mathbf{R}^n$ total boundedness is equivalent to boundedness. (Needless to say, this fact is not valid for general metric spaces. The typical example is provided by any infinite dimensional Banach space).

Proposition 69 *Lemma If X is a sequentially compact metric space, then it is separable.*

PROOF. For each positive ϵ we find a maximal set A_ϵ so that for each pair x, y of distinct points in $A_\epsilon, d(x, y) \geq \epsilon$. (??????) Now each A_ϵ is finite (otherwise we could extract a sequence (x_n) of distinct elements from A_ϵ — this has the property that $d(x_m, x_n) \geq \epsilon$ if $m \neq n$. Of course, any subsequence has the same property and so cannot be Cauchy). We claim that the union B of the $A_{\frac{1}{n}}$ (which is of course countable) is dense in X. For if x were a point of X which does not lie in the closure of B, there is an $n \in \mathbf{N}$ so that $U(x, \frac{1}{n})$ is disjoint from B and so from $A_{\frac{1}{n}}$. This contradicts the maximality of the latter.

■

Proposition 70 *Lemma If X is sequentially compact, then it is compact.*

Proof. We know that X is separable. Hence it has a countable basis $\mathcal{B} = \{\mathcal{U}_\backslash : \backslash \in \mathbf{N}\}$. Now suppose that $\mathcal{U}$ is an open covering. Then we can reduce it to a countable cover as follows: for each $x \in X$, there is a $U \in \mathcal{U}$ with $x \in U$. Since $\mathcal{B}$ is a basis, there is an n_x so that $x \in U_{n_x}$ and the latter is a subset of U. The family $\mathcal{V}$ of all U_{n_x} which arise in this way is a basis and, of course, countable. Hence it suffices to show that every countable open cover $\mathcal{V} = (\mathcal{V}_\backslash)$ has a finite subcover. Suppose that this is not the case. Then for each positive integer n there is an x_n which is not in the union of the first n V_n's. Consider the sequence (x_n). By the hypothesis, this has a convergent subsequence (x_{n_k}). Let x be the limit of this sequence. There is an $N \in \mathbf{N}$ so that $x \in V_N$. Then almost all of the x_{n_k} are in V_N and this clearly contradicts the construction of the sequence (x_n).

■

Proposition 71 *Lemma: If X is compact, then it is sequentially compact.*

Proof. Notice first that if (x_n) is a sequence in X, then it has a limit point as we remarked above. We can now find a subsequence of (x_n) which converges to x as follows. We choose n_1 so that x_{n_1} is in $U(x, 1)$. We then choose n_2 larger than n_1 so that x_{n_2} is in $U(x, \frac{1}{2})$. Continuing in this manner, we can construct a subsequence (x_{n_k}) with $x_{n_k} \in U(x, \frac{1}{k})$ and this subsequence converges to x.

■

Proposition 72 *Lemma If X is a compact metric space, then it is precompact and complete.*

Proof. We already know that it is precompact. Suppose now that (x_n) is a Cauchy sequence. Then there is a subsequence (x_{n_k}) which converges, say to x, since X is sequentially compact. We now show that $x_n \to x$. For ϵ positive there is an integer N so that $d(x_m, x_n) \leq \epsilon$ if $m, n \geq N$. Hence $d(x_m, x_{n_k}) \leq \epsilon$ if $m \geq N, n_k \geq N$. If we now let n_k tend to infinity in this inequality, we have that $d(x_m, x) \leq \epsilon$ if $m \geq N$ which shows that the original sequence converges to x.

■

Proposition 73 *If the metric space X is precompact and complete, then it is compact.*

Proof. First note that if X is precompact, then every sequence (x_n) has a Cauchy subsequence. For we can cover X by finitely many balls of radius at most 1. Then there must be one of these balls, say U_1, which contains infinitely many terms of the sequence. Hence we can find a subsequence in U_1 which, for reasons which will soon be apparent, we denote by (x_n^1). Proceeding in exactly the same way we find a subsequence (x_n^2) of this sequence which lies in a ball U_2 of radius at most $\frac{1}{2}$ which is contained in U_1. In this manner, we can construct a decreasing sequence (U_k) of balls, where U_k has radius at most 2^{-k}, and, for each k, a subsequence (x_n^k) of the original sequence which lies in U_k. Further (x_n^k) is a subsequence of (x_n^{k-1}). We can display this sequence of sequences as a square array and form the diagonal sequence (x_n^n) which consists of the circled terms. This has the property that it is a subsequence of each of the subsequences constructed above, up to the first k terms. In particular, this sequence lies in each U_k (again up to a finite number of terms). This clearly implies that it is Cauchy as claimed. Now this Cauchy sequence converges since X is complete and we have thus shown that X is sequentially compact and hence compact by the previous Lemma.

∎

The proof of this Lemma completes that of the main result as the reader can verify.

As a Corollary to the final Lemma, we have the following:

Proposition 74 *Let X be a metric space. Then it is precompact if and only if its completion is compact.*

Proof. If the latter is compact, then it is precompact and hence so is X (as a subspace of a precompact space). On the other hand, the precompactness of X implies that of the completion as can easily be verified and so the latter is compact by the last Lemma.

∎

As a final contribution to this circle of ideas we mention the following property of compact spaces which is exactly the one required to make rigorous the compactness arguments employed in the first Chapter.

Proposition 75 *Let $\mathcal{U}$ be an open covering of a compact metric space X. Then there is a positive η so that each subset of X of diameter less than η is contained in a set of $\mathcal{U}$.*

Proof. It suffices to prove the result for a finite covering $U_1, \dots, U_k$ (since $\mathcal{U}$ possesses a finite subcovering). If the claim were false, we could find for each n a set A_n of diameter at most $\frac{1}{n}$ which is not contained in any U_k. Choose $x_n \in A_n$. By going over to a subsequence, we can assume that (x_n) converges, say to x. Then x belongs to one of the U's, say U_i. Choose a positive ϵ so that $U(x, \epsilon)$ is a subset of U_i and n so large that $d(x, x_n) \leq \frac{\epsilon}{3}$ resp. $oxdiam\ A_n < \frac{\epsilon}{3}$. Then it is clear that A_n is contained in U_i which is a contradiction.

■

The positive number η whose existence is ensured by the above Lemma is called **a Lebesgue number** for the covering $\mathcal{U}$.

We indicate briefly how this result can be used to stop the holes in the proofs given in Chapter 1.

We conclude this section with the remark that the above lemma can be restated as follows:

Proposition 76 *Let $C_1, \dots, C_n$ be closed, non-empty subsets of a compact metric space and suppose that their intersection is empty. Then there exists a positive ϵ so that any subset of X which meets each C_i has diameter at least ϵ.*

As a Corollary, we have that if $C_1, \dots, C_n$ is a closed covering of a compact metric space, then there is a positive ϵ which is such that if any set A of diameter ϵ meets the sets $C_{i_1}, \dots, C_{i_r}$, the intersection of the latter is non-empty.

6.14 Tychonov's theorem:

We now turn to one of the most important results on compact space, the theorem mentioned in the paragraph title which states that products of compact spaces are compact. In order to prove this, we introduce the concept of an ultrafilter:

Definition: An **ultrafilter** on a set X is a filter $\mathcal{F}$ which is maximal in the sense that if $\mathcal{G}$ is a second filter which is finer than $\mathcal{F}$, then $\mathcal{F}$ and $\mathcal{G}$ coincide.

It follows immediately from Zorn's Lemma, applied to the family of all filters on a set which are finer than a given one (ordered by inclusion), that every filter can be refined to an ultrafilter.

Ultrafilters can be characterised as follows:

Proposition 77 *A filter $\mathcal{F}$ on a set X is an ultrafilter if and only if the following condition is satisfied: for each subset A of X, either A or its complement is in $\mathcal{F}$.*

PROOF. Suppose firstly that the filter is an ultrafilter and that there is a subset A for which the above condition fails i.e. neither A nor $X \setminus A$ are in $\mathcal{F}$. Then we can define a filter $\mathcal{G}$ which is finer than $\mathcal{F}$ as follows:

$$\mathcal{G} = \{\mathcal{B} \subset \mathcal{X} : \mathcal{A} \cup \mathcal{B} \in \mathcal{F}\}.$$

This is *strictly* finer since $X \setminus A \in \mathcal{G}$. This contradiction shows that ultrafilters have the above property.

Suppose on the other hand, that $\mathcal{F}$ is not an ultrafilter. Let $\mathcal{G}$ be a filter which is strictly finer than $\mathcal{F}$. Then there is a subset A which is in $\mathcal{G}$ but not in $\mathcal{F}$. This A fails the above condition. For if its complement were in $\mathcal{F}$ it would also be in $\mathcal{G}$. Then both A and its complement would belong to the same filter.

■

If a is a point in a set X, then the filter generated by the one point set $\{a\}$ is easily seen to be an ultrafilter,. Such filter are called **fixed ultrafilters.** Of course, an ultrafilter is fixed if and only if the intersection of its elements is non-empty. Such ultrafilters are not very interesting. Filters whose intersections are empty are called **free**. Free ultrafilters are "constructed" by applying the above existence statement to free filters, the typical example being an ultrafilter which is finer than the Fréchet filter on the integers. The relevance of ultrafilters for the proof of Tychonov's theorem is based on the following facts:

I. if $\mathcal{F}$ is an ultrafilter on a topological space and x is a cluster point of $\mathcal{F}$, then $\mathcal{F} \to \S$. For we know that the fact that x is a cluster pint of $\mathcal{F}$ means that a finer filter converges to x. But, apart from $\mathcal{F}$ itself, there *is no* finer filter than $\mathcal{F}$.

II. A topological space X is quasi-compact if and only if every ultrafilter on X converges. This is essentially a restatement of the definition of quasi-compactness (in terms of the finite intersection property). For example, we shall show here that if X is quasi-compact then every ultrafilter $\mathcal{F}$ converges. Note that the family $\{\overline{A} : A \in \mathcal{F}\}$ has the finite intersection property and so its intersection is non-empty. Hence $\mathcal{F}$ has a cluster point to which it must converge by I.

III. The same proof provides the following characterisation: a space X is quasi-compact if and only if every filter on X has a cluster point.

We are now in a position to state and prove Tychonov's theorem. Due to its importance in analysis, it is perhaps worth mentioning that its proof uses the Axiom of Choice in an essential way (in the form of the existence of ultrafilters). In fact, it is known that the result stated here implies this Axiom and so its use is unavoidable.

Proposition 78 *Let* $(X_\alpha)_{\alpha \in A}$ *be a family of quasi-compact spaces. Then their Cartesian product is quasi-compact. (Of course, the corresponding result for compact spaces holds also).*

PROOF. Let $\mathcal{F}$ be an ultrafilter on the product. Then for each α the image filter $\pi_\alpha(\mathcal{F})$ has a cluster point x_α. The reader will have no difficulty in verifying that $x = (x_\alpha)$ is then a cluster point for $\mathcal{F}$.

■

6.15 Projective limits of compacta:

We mentioned previously that general projective limits of sets can be trivial. However, as we shall now see, in the case of compact components such pathologies cannot occur. A typical example is the fact that the intersection of a decreasing sequence of non-empty, compact subsets of a given space is non-empty. This follows from the characterisation of compactness using the finite intersection property. The latter is a special case of the following more general result which we state for a general projective limit indexed by an directed set A which need not be the integers:

Proposition 79 *Let*

$$\{\pi_{\alpha\beta} : K_\beta \to K_a, \alpha \leq \beta, \alpha, \beta \in A\}$$

be a projective spectrum of compact sets. Then the projective limit K *is compact and we have the formula*

$$\pi_\alpha(K) = \bigcap_{\beta \geq \alpha} \pi_{\beta\alpha}(K_\beta)$$

for each $\alpha \in A$*. In particular, if the* K_α *are non-empty, then so is* K *and if each of the* $\pi_{\beta\alpha}$ *is surjective, then so are the* π_α*.*

PROOF.

■

We remark that the above theorem is also true under the weaker assumption that for each $\alpha \leq \beta$ and each $x_\alpha \in K_\alpha$, the pre-image $\pi_{\beta\alpha}^{-1}(x_\alpha)$ of x_α in K_β is compact.

We can also give a version of this theorem which relates to suitable families of mappings between the components of projective spectra: Suppose that we have two such spectra:

$$\{\pi_{\beta\alpha} : X_\beta \to X_\alpha, \alpha \leq \beta, \alpha, \beta \in A\}$$

and

$$\{\pi'_{\beta\alpha} : Y_\beta \to Y_\alpha, \alpha \leq \beta, \alpha, \beta \in A\}.$$

Suppose further that we have a collection (f_α) of mappings, where f_α is continuous from X_α into Y_α and where the f_α are compatible with the linking mappings in the sense that $f_\alpha \circ \pi_{\beta\alpha} = \pi'_{\beta\alpha} \circ f_\beta$ whenever $\alpha \leq \beta$ (see the commutative diagram ?????). Then we can define in a natural way a mapping f from the projective limit X of the first spectrum into Y, the limit of the second one, by defining the image of a thread (x_α) in X to be the thread $(f_\alpha(x_\alpha))$. (That this *is* a thread follows from the compatibility condition).

In general, we cannot deduce interesting properties of f from those of f_α for the simple reason that, as we have seen, one or both of the limits can trivialise. However, in the presence of compactness, we have, for example, the following result:

Proposition 80 *If the spaces X_α and Y_α are all compact and the f_α are surjective, then so is f.*

6.16 Locally compact spaces:

Despite the importance of the concept of compactness in analysis, the basic space for the latter (namely, the real line) does not enjoy this property. However, it does satisfy the condition that every point has a compact neighbourhood and this suffices to allow one to use compactness arguments for many purposes. This is formalised in the following

Definition: A T_2-space is **locally compact** if each point has a compact neighbourhood.

This is equivalent to either of the following

1. every point has a closed, compact neighbourhood (for X then is automatically T_2);

2. X is T_2 and every neighbourhood of a point x of X contains a compact neighbourhood of x.

Of course $\mathbf{R}^n$, with its usual topology, is locally compact, as is any discrete space. $\mathbf{Q}$ is clearly not locally compact. The interval with two endpoints is a space which is not locally compact despite the fact that every point has a compact neighbourhood (this shows that the closedness condition in (1) above is essential).

Subsets of locally compact spaces need not be locally compact (as the rationals as a subset of the reals shows). However, we de have the following result:

Proposition 81 *Let U be an open (resp. C a closed subset) of the locally compact space X. Then U and C are themselves locally compact.*

PROOF. We prove this for U and leave the (easier) case of closed subsets to the reader. If x is a point of U, then U is a neighbourhood of x in X and so there is a closed neighbourhood V of x contained in U. There is also, by definition, a compact neighbourhood W of x in X. Then $W \cap V$ is a compact neighbourhood of x in U.

■

As a Corollary, we note the fact the every subset of the form $C \cap U$ (i.e. the intersection of a closed and an open set) is locally compact, being an open subset of the locally compact space C. A subset of the above form is called **locally closed**. In fact, if a subset of a locally compact space is itself locally compact, it must be locally closed (Exercise).

It is clear that a disjoint union of locally compact spaces is locally compact. On the other hand, a product $\prod X_\alpha$ of non-trivial locally compact spaces can only be locally compact if all of the constituent spaces are locally compact and all but a finite number are compact.

Another stability property is that if a continuous function maps a locally compact space onto a T_2-space Y and the mapping is open, then Y is locally compact.

A locally compact space X is said to be **σ-compact** if it can be expressed as a union of countably many compact subsets. We can then find a sequence (K_n) of compacta in X whose union is X and which is such that each K_n is contained in the interior of its successor K_{n+1}. In this case, any compact subset of X is contained in some K_n.
PROOF.

■

We remark that a closed subspace of a σ-compact locally compact space is of the same type. However, this is not true of open subsets since any locally compact set is an open subset of a *compact* one, as we shall now show.

6.17 The Alexandrov compactification:

If X is a locally compact space, there is a standard method of embedding it into a compact space by adding one point. As a model for this construction, recall the usual method of "compactifying" the real line by adding a point at infinity. The resulting space is then homeomorphic to S^1. The abstract construction is as follows. Let (X, τ) be a topological space (at this point we shall not assume that it is locally compact). We introduce a new set X_∞ which is X together with a point which we shall denote by ∞ for obvious reasons. On X_∞ we define a topology τ_∞ as follows: $U \in \tau_\infty$ if and only if $U \subset X$ and $U \in \tau$ or $\infty \in U$ and $X \backslash U$ is compact in X. It is clear that (X_∞, τ_∞) is a topological space which contains X as a subspace. It is quasi-compact and is T_1 provided that X is. The important point to note is that it is T_2 (and so compact) precisely when X is locally compact. This construction shows that each locally compact space is homeomorphic to an open subspace of a compact space. In fact, the latter condition is a characterisation of local compactness since we saw above that it is necessary. A simple consequence is that each locally compact space is $T_{3\frac{1}{2}}$ (since each compact space is T_4 and the $T_{3\frac{1}{2}}$-property is hereditary).