ON COMPLEMENTED SUBSPACES OF $H^1(\delta)$ AND ISOMORPHISM BETWEEN H^1-SPACES

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Dipl. Ing. Paul F.X. Müller

angefertigt am Institut für Mathematik der Technisch-Naturwissenschaftlichen Fakultät der Johannes Kepler Universität Linz.

eingereicht bei: o. Univ. Prof. Dr. James Bell Cooper
Univ.Doz. Mag. Dr. Walter Schachermayer

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Abstract of the dissertation

We classify and characterise the subspaces of $H^1(\delta)$ spanned by subsequences of Haarbasis. ℓ^1 and $(\Sigma H_n^1)_{\ell^1}$ and $H^1(\delta)$ are the only isomorphic types which occur in this way.

We also give a necessary and sufficient condition on an increasing sequence of fields (F_n) for $H^1(F_n)$ to be linearly isomorphic to $H^1(\delta)$, thus verifying a conjecture of B. Maurey.

Conditions are isolated on embeddings $i_n : H_n^1 \to H^1(\delta)$ which cause the norm of any projection P from $H^1(\delta)$ onto $i(H_n^1)$ to be big.

Introduction

This thesis consists of 3 Sections. Section one and two contain our main results.

In section three examples, which illustrate our work in the previous sections can be found. No lengthy introductory chapters are included. All proven results are new. In sections one and two, two problems, which were posed some years ago are solved. [10], [15].

We will work in the setting of dyadic H^1 spaces: To the pair (n,i), $n \in \mathbb{N}$, $0 \leqq i \leqq 2^n-1$ we associate the dyadic interval $(2^{-n}i, 2^{-n}(i+1)]$ and the Haarfunction h_{ni} which is 1 on the left half of $(2^{-n}i, 2^{-n}(i+1)]$, -1 on the right half, and zero else where. The σ-algebra generated by the sets $\{(2^{-n}i, 2^{-n}(i+1)] : 0 \leqq i \leqq 2^n-1\}$ is denoted by E_n. Dyadic intervals are nested in the sense that if $I \cap J \neq \emptyset$ then either $I \subset J$ or $J \subset I$.

If $f = \Sigma a_{ni} h_{ni} \in L^1(0,1]$, we write $S(f) = (\sum_{(ni)} a_{ni}^2 h_{ni}^2)^{1/2}$ and $\|f\|_{H^1(\delta)} = \int S(f)$. Then $H^1(\delta) = \{f \in L^1 : \|f\|_{H^1(\delta)} < \infty\}$. The space $H^1(\delta)$ is called the dyadic H^1-space. General results on $H^1(\delta)$ can be found in [10], [5], [6]. H_n^1 denotes the subspace of $H^1(\delta)$ spanned by $\{h_{mj} : m \leqq n, 0 \leqq j \leqq 2^m-1\}$. We can combine these finite dimensional parts of H^1 in an ℓ^1-sum:

$(\Sigma H_n^1)_{\ell^1} = \{(f_n)_{n\in\mathbb{N}} : f_n \in H_n^1 \text{ and } \Sigma\|f_n\| < \infty\}$.
$(\Sigma H_n^1)_{\ell^1}$ is not isomorphic to $H^1(\delta)$. This follows, for example, from the fact that $(\Sigma H_n^1)_{\ell^1}$ does not contain any subspace isomorphic to ℓ^2. Geometric characterisations as given in Theorem 1 indicate that $(\Sigma H_n^1)_{\ell^1}$ is rather "small" compared to $H^1(\delta)$. Given $f \in L^1(0,1]$ and a dyadic interval I we write $f_I = \frac{1}{|I|}\int_I f$ and

$\|f\|_{BMO(\delta)} = \sup \{(\frac{1}{|I|}\int_I |f-f_I|^2)^{1/2} : I \text{ dyadic}\}$

$BMO(\delta) = \{f \in L^1(0,1] : \int f = 0 \text{ and } \|f\|_{BMO(\delta)} < \infty\}$.

Functions in BMO are said to have bounded mean oscillation.

The modern theory of H^1 spaces has its starting point in the following formula, due to C. Fefferman, cf. [6]

$\|f\|_{H^1(\delta)} = \sup\{|\int fg| : \|g\|_{BMO} = 1 \text{ and } g \in L^\infty\}$.

The formula above suggests that $BMO(\delta)$ may be identified with the dual space of $H^1(\delta)$. This is indeed the case, but one has to be careful when defining the pairing [6], [10].

We frequently use the fact that for $f = \Sigma a_{ni}h_{ni}$ we can express the BMO-norm of f by means of the coefficients. In fact: $\|f\|_{BMO} = \sup_{(ni)} (2^n \sum_{(mj)\subset(ni)} 2^{-m}a_{mj}^2)^{1/2}$. This expression was used first by L. Carleson [1] and P. Wojtaszczyk [12] to establish a direct isomorphism between $H^1(\delta)$ and the classical $H^1(D)$ space which are Hardy spaces of holomorphic functions in the disc.

The functions $(h_{ni})_{0\leq i, n\in\mathbb{N}}^{2^n-1}$ form our unconditional basis in $H^1(\delta)$, cf [9, p. 15].
A subsequence of the Haarbasis in $H^1(\delta)$ is specified by a collection of dyadic intervals. Let X denote a subspace of $H^1(\delta)$, spanned by an arbitrary subsequence of Haarbasis (h_{ni}). The fact that the Haarbasis is unconditional in $H^1(\delta)$ implies that X is complemented in $H^1(\delta)$, cf [9, p. 18].

Now we ask:

Question A: Can we classify the isomorphic types of subspaces X which are produced in this way?

Such questions for the classical Banach spaces have a long history and enjoy prominent positions in the theory of Banach spaces. Their solution is a necessary step towards understanding a given Banach space together with its basis. We summarize known results in the following table:

Space	Subspace X	Isomorphic types of X
ℓ^p	infinite dimensional complemented	ℓ^p
$\ell^p \oplus \ell^q$	- " -	ℓ^p, ℓ^q, $\ell^p \oplus \ell^q$
L^p $1<p<\infty$	spanned by subsequences of Haarbasis	ℓ^p, L^p.

For proof and references for the above look into the book of Lindenstrauss, Tzafriri [9].

Theorem 1 in Section 1 answers Question A for $H^1(\delta)$: The isomorphic types of those spaces X, which are spanned in $H^1(\delta)$ by subsequence of the Haarbasis are as follows: ℓ^1, $(\Sigma H_n^1)_{\ell^1}$, $H^1(\delta)$.

Let $\mathcal{B}$ denote the collection of supports of Haar functions used to span X.
The isomorphic type of X can be described in terms of $\mathcal{B}$. The decisive quantities are:

(a) $\sigma = \{t \mid t \in I \in \mathcal{B}, \text{ for infinitely many } I \in \mathcal{B}\}$

(b) $\sup\limits_{J \in \mathcal{B}} \frac{1}{|J|} \sum\limits_{\substack{I \subset J \\ I \in \mathcal{B}}} |I|$

Condition (a) is nothing else than lim sup of $\mathcal{B}$.
Condition (b) says that <u>no</u> interval J in $\mathcal{B}$ contains "many" $I \in \mathcal{B}$, and is sometimes called Carlesons condition on $\mathcal{B}$, cf. [4], [5], [7].
The proofs of Theorem 1, b,c use Pelczynski's decomposition principle, cf [9, p.54].
Many spaces can be shown to be isomorphic by Pelczynski's method, without direct isomorphism between these spaces, having been found, cf. [13], [14].
In general, it can be said that the construction of explicit isomorphisms requires new and different techniques.
A good illustration of the last sentence is given by Carleson's

proof in [1] of Maurey's Theorem [10] on the isomorphism of $H^1(\delta)$ and $H^1(D)$.

In the proof of part c of theorem 1 we require functions which share some properties of the Haar functions. In the fundamental papers of Lindenstrauss-Pelczynski [8] and Enflo-Starbird [2] the use of Leaponof's theorem in such situations (cf. [9, (II), p 159]) is explained. These papers illustrate the power of Leaponof's theorem in situations, where no other means of constructions are available.

Some of the ingredients used in part c of section 1 are again applied in section 2. There we give necessary and sufficient conditions on an increasing sequence (F_n) of fields for $H^1(F_n)$ to be isomorphic to $H^1(\delta)$. The condition used, is related to the condition (a) mentioned above. Useful facts and definitions concerning $H^1(F_n)$ can be found in Garsia's book [6] and Maurey's Acta-paper [10]. Our Theorem 2 is a positive solution to a conjecture of Maurey in this paper - it is proved here for the first time.

Section three illustrate our work in sections one and two. Although this does not appear explicitely, the difficulties here arise partially from the fact that E.M. Stein's Martingale-inequality [11] is valid only for L^p spaces with $p > 1$. In fact, the examples of section three are modelled on counterexamples to E.M. Stein's Martingale inequality. Our proof of Theorem 3b again uses Carleson's conditions and their obvious relation to BMO.

It is an open question whether this proof can be extended to solve one of the following problems:

(1) Is $VMO(T^2)$ isomorphic to a subspace of $VMO(T)$?

(2) Is $H^1(T)$ primary ?

SECTION 1: SUBSEQUENCES OF HAARBASIS IN $H^1(\delta)$

Theorem 1: Let $\mathcal{B}$ be an infinite collection of dyadic intervals. Let X be the closed linear span of $\{h_I \mid I \in \mathcal{B}\}$ in H^1 and $\sigma = \{t \mid t \in I$ for infinitely many $I \in \mathcal{B}\}$.

Then:

(a) If $|\sigma| = 0$ and $\sup_I \left(\frac{1}{|I|} \cdot \sum_{\substack{J\in\mathcal{B}\\ J\subset I}} |J|\right) < \infty$

X is isomorphic to ℓ^1.

(b) If $|\sigma| = 0$ and $\sup_I \left(\frac{1}{|I|} \cdot \sum_{\substack{J\in\mathcal{B}\\ J\subset I}} |I|\right) = \infty$

X is isomorphic to $(\Sigma H^1_n)_{\ell^1}$.

(c) If $|\sigma| > 0$, X is isomorphic to $H^1(\delta)$.

Proof of theorem 1 - part (a):

Suppose

$$\sup_I \frac{1}{|I|} \sum_{\substack{J\in\mathcal{B}\\ J\subset I}} |J| = M < \infty .$$

Then

$$\|\sum_{I\in\mathcal{B}} a_I h_I\|_{H^1} = \sup \{\Sigma a_I b_I |I| : \|\sum_{I\in\mathcal{B}} b_I h_I\|_{BMO} = 1\}$$

$$= \sup \{\Sigma a_I b_I |I| : \sup_I \left(\frac{1}{|I|} \sum_{J\subseteq\mathcal{B}} b_J^2 |J|\right)^{1/2} = 1\}$$

$$\geq \sup \{\Sigma a_I b_I |I| : \sup_I \sup_{J\subset I} |b_I| (\frac{1}{|I|} \sum_{J\subset I} |J|)^{1/2} = 1\}$$

$$\geq \sup \{\Sigma a_I b_I |I| : \sup_I |b_I| . M^{1/2} = 1\}$$

$$= \frac{1}{M^{1/2}} \sum_{I\in\mathcal{B}} |a_I| |I| .$$

Thus $\{h_I \,.\, |I|^{-1}, I \in \mathcal{B}\}$ is equivalent to the unit vector basis in ℓ^1. □

The following Lemmata and Propositions are needed to prove part (b) of the theorem.

Definition 2:

Let $\mathcal{B}$ be a collection of dyadic intervals. Let $I \in \mathcal{B}$.

$G_1(I) = \{J \in \mathcal{B} | J \subset I, J \max\}$

$G_n(I) = \bigcup_{J\in G_{n-1}(I)} G_1(J)$

We enumerate the intervals of $G_n(I)$ in such a way that

$$|I_1| \geq |I_2| \geq |I_3| \ldots ;$$

Let $k(\varepsilon)$ be the smallest integer such that:

$$\sum_{k=1}^{k(\varepsilon)} |I_k| \geq (1-\varepsilon) \sum_{k=1}^{\infty} |I_k| .$$

Then

$$G^{\varepsilon}_{n,n}(I) = \{I_k | k \leq k(\varepsilon)\}$$

$$G^{\varepsilon}_{p,n}(I) = \{K \in G_p(I) | \exists J \in G^{\varepsilon}_{p+1,n}(I) \wedge K \supset J\}$$

$p < n$.

<u>Remark:</u> We will use the fact that $G^{\varepsilon}_{p,n}(I)$ is a <u>finite</u> subset of $G_p(I)$ such that

$$\sum_{J\in G^{\varepsilon}_{p,n}} |J| > (1-\varepsilon) \sum_{J\in G_n(I)} |J|$$

<u>Lemma 1:</u> ([4], Ch. XI, Lemma 3.2)

Let $\mathcal{B}$ be a collection of dyadic intervals. $K \in \mathcal{B}$.
If $n \in \mathbb{N}$, $\gamma < 1$ are given, then:

$$\frac{1}{|K|} \sum_{\substack{J\subset K\\ J\in\mathcal{B}}} |J| > \frac{n}{1-\gamma}$$

implies that there exists $I_o \in \mathcal{B}$, $I_o \subset K$ such that:

$$\frac{1}{|I_o|} \sum_{J\in G_n(I_o)} |J| \geq \gamma$$

<u>Proof:</u> (a) Suppose this is false: Then, for any $I \in \mathcal{B}$ with $I \subset K$,

$$\frac{1}{|I|} \sum_{J\in G_n(I)} |J| < \gamma.$$

But this implies:

$$\frac{1}{|K|} \sum_{\substack{J\subset K\\ J\in\mathcal{B}}} |J| = \frac{1}{|K|} \sum_{r=1}^{n} \sum_{m\in\mathbb{N}} \sum_{J\in G_{n.m+r}(K)} |J| \leq$$

$$\leq \frac{1}{|K|} \sum_{r=1}^{n} \sum_{m\in\mathbb{N}} \gamma^m |K| = \frac{n}{1-\gamma}$$

a contradiction:

<u>Main-Lemma 2:</u>

Let $\mathcal{B}$ be a collection of dyadic intervals. Suppose that there exists $I_o \in \mathcal{B}$, $\varepsilon > 0$ such that

$$\frac{1}{|I_o|} \sum_{J \in G^{\varepsilon}_{n,n}(I_o)} |J| > \gamma_n, \quad 1-4^{-n} < \gamma_n < 1.$$

Then there exists a subspace Y_n which is contained in span $\{h_I | I \in G^{\varepsilon}_{p,n}, 0 \leq p \leq n\}$, 4-complemented in $H^1(\delta)$, and 4-isomorphic to H^1_n.

<u>Proof:</u>

<u>Step 1:</u> $\tilde{h}_{oo} := h_{I_o}$;
<u>Step 2:</u> $E^+_o := E(h_{I_o} = 1)$
$E^-_o := E(h_{I_o} = -1)$

$$\tilde{h}_{10} = \sum_{J \in G^{\varepsilon}_{1,n}(I_o) \cap E^+_o} h_J$$

$$\tilde{h}_{11} = \sum_{J \in G^{\varepsilon}_{1,n}(I_o) \cap E^-_o} h_J$$

We observe that

$$|\text{supp } \tilde{h}_{10}| \geq (\gamma_n - \tfrac{1}{2})|I_o| \text{ and } |\text{supp } \tilde{h}_{11}| \geq (\gamma_n - \tfrac{1}{2})|I_o|$$

<u>Step 3:</u> $j \in \{0,1\}$, $E^+_{1j} = E(\tilde{h}_{1j} = 1)$, $E^-_{1j} = E(\tilde{h}_{1j} = -1)$

$$\tilde{h}_{2,2j} = \sum_{J \in G^{\varepsilon}_{2,n}(I_o) \cap E^+_{1j}} h_J$$

$$\tilde{h}_{2,2j+1} = \sum_{J \in G^{\varepsilon}_{2,n}(I_o) \cap E^-_{1j}} h_J$$

and we observe that:

$$|\operatorname{supp} \tilde{h}_{2,2j}| \geq |\operatorname{sup} \tilde{h}_{1j}| - \frac{1}{4}|I_o| \geq (\gamma_n - \frac{1}{2} - \frac{1}{4})|I_o|$$

$$|\operatorname{supp} \tilde{h}_{2,2j+1}| \geq (\gamma_n - \frac{1}{2} - \frac{1}{4})|I_o|.$$

At <u>step m</u> we are given $\tilde{h}_{mj}$, $0 \leq j \leq 2^m-1$

$E_{mj}^+ = E(\tilde{h}_{mj} = 1)$, $E_{mj}^- = E(\tilde{h}_{mj} = -1)$

$$\tilde{h}_{m+1,2j} = \sum_{J \in G_{m+1,n}^{\varepsilon}(I_o) \cap E_{mj}^+} h_J$$

$$\tilde{h}_{m+1,2j+1} = \sum_{J \in G_{m+1,n}^{\varepsilon}(I_o) \cap E_{mj}^-} h_J$$

and we get, for $k \in \{2j, 2j+1\}$,

$$|\operatorname{supp} \tilde{h}_{m+1,k}| \geq |\operatorname{supp} \tilde{h}_{m,j}| - \frac{1}{2^{m+1}} |I_o|$$

$$\geq (\gamma_n - \frac{1}{2} - \frac{1}{4} - \dots - \frac{1}{2^{m+1}})|I_o|,$$

As the space Y_n we will take

$$\operatorname{span} \{\tilde{h}_{mj} | 0 \leq m \leq n,\ 0 \leq j \leq 2^m-1\}.$$

We must show that:

(a) Y_n is isomorphic to H_n^1 with constant 4.

(b) Y_n is complemented in $H^1(\delta)$ and the norm of the projection is less than 4.

ad (a): Observe that supp $\tilde{h}_{m+1,2j}$ ∪ supp $\tilde{h}_{n+1,2j+1}$ ⊂ supp $\tilde{h}_{m,j}$, take $(a_{mj})_{m=0,j=0}^{n,2^m-1}$ and estimate:

$$\|\Sigma a_{mj}\tilde{h}_{mj}\|_{H^1} = \int (\Sigma a_{mj}^2 \; S^2(\tilde{h}_{mj}))^{1/2}$$

$$\geq \sum_{i=0}^{2^n-1} \int (\sum_{(mj)\supset(ni)} a_{mj}^2)^{1/2} \chi_{\text{supp } \tilde{h}_{ni}}$$

$$\geq \sum_{i=0}^{2^n-1} (\sum_{(mj)\supset(ni)} a_{mj}^2)^{1/2} (\gamma_n - 1 + 2^n) |I_o| .$$

On the other hand:

$$\|\Sigma a_{mj}\tilde{h}_{mj}\|_{H^1} \leq \sum_{i=0}^{2^n-1} (\sum_{(mj)\supset(ni)} a_{mj}^2)^{1/2} \; 2^{-n} |I_o| .$$

Hence $i_n : H_n^1 \to Y_n$

$$h_{mj} \to \tilde{h}_{mj} \frac{1}{|I_o|} \quad \text{is an isomorphism}$$

with

$$\|i_n\| \cdot \|i_n^{-1}\| \leq \frac{2^{-n}}{(\gamma_n - 1 + 2^{-n})} \leq 4 .$$

<u>ad (b):</u> Y_n is complemented in $H^1(\delta)$ by means of the following projection:

$$P_n : H^1(\delta) \to Y_n$$

$$f \to \Sigma (f, \frac{\tilde{h}_{mj}}{\|\tilde{h}_{mj}\|_2^2}) \; \tilde{h}_{mj}$$

By orthogonality we see that P_n is bounded iff

$$(i_n^{-1} \quad P_n)^* : BMO_n \to BMO$$
$$h_{ni} \to \tilde{h}_{ni}$$

is bounded.

Take any dyadic interval $K \subset [0,1]$.

CASE 1: $I_o \subset K$

$$\frac{1}{|K|} \int_K ((i^{-1}P)^*f - ((i^{-1}P)^*f)_K)^2 = \frac{1}{|K|} \int |(i^{-1}P)^*f|^2 =$$

$$= \frac{|I_o|}{|K|} \int_{[0,1]} |f|^2 \leq \frac{|I_o|}{|K|} \|f\|^2_{BMO}.$$

CASE 2: $I_o \cap K = \emptyset$ or $K \subset K_o \in G_{n+1}(I_o)$

$$\frac{1}{|K|} \int_K (i^{-1}P)^*f - (i^{-1}P)^*f)_K)^2 = 0.$$

CASE 3: $\exists m_o \; \exists K_1 \in G^{\varepsilon}_{m_o,n}(I_o)$ with $K_1 \supset K \wedge \forall K_2 \in G^{\varepsilon}_{m_o+1,n}(I_o)$ $K_2 \cap K \neq \emptyset \Rightarrow K_2 \subset K$. By the construction of $(\tilde{h}_{mj})$ we get the following:

(1) $\tilde{h}_{mj/K} = \text{const}$ for $m_o \geq m-1$

(2) $\int_K \tilde{h}^2_{mj} \leq |K| 2^{m_o-m}$ for $m \geq m_o$

(3) $\int_K \tilde{h}_{mj} = 0$ for $m \geq m_o+1$.

First find j_o so that $K \subset \text{supp}\, \tilde{h}_{m_o,j_o}$, then estimate

$$\frac{1}{|K|}\int_K |(i_n^{-1}P_n)^* f - ((i_n^{-1}P_n)^* f)_K|^2 =$$

$$= \frac{1}{|K|}\int_K \Big(\sum_{(ni)\subset(m_o j_o)} a_{ni}\,\tilde{h}_{ni} - (\Sigma a_{ni}\,\tilde{h}_{ni})_K\Big)^2$$

$$\leqq \frac{4}{|K|}\int_K \Big(\sum_{(ni)\subset(m_o j_o)} a_{ni}^2\,\tilde{h}_{ni}^2\Big)$$

$$= \frac{4}{|K|} \sum_{(ni)\subset(m_o j_o)} a_{ni}^2 \int_K h_{ni}^2$$

$$\leqq \frac{4}{|K|} \sum_{(ni)\subset(m_o j_o)} a_{ni}^2\ |K|\ 2^{m_o - n}$$

$$= 4\ .\ 2^{m_o} \sum_{(ni)\subset(m_o j_o)} a_{ni}^2\ 2^{-n}$$

$$\leqq 4\ \|f\|_{BMO}^2 .$$

<u>Proposition 3:</u>

Let $\mathcal{B}$ be a collection of dyadic intervals so that

(1) $|\sigma| = 0$;

(2) $\sup_{I\in\mathcal{B}} \frac{1}{|I|} \sum_{J\in\mathcal{B}\cap I} |J| = \infty$

Then for any $\varepsilon_n > 0$ and $\gamma_n < 1$, $\mathcal{B}$ can be decomposed into $\mathcal{B}^1$ and $\mathcal{B}^2$ so that for $j \in \{1,2\}$ we get:

(1) $\mathcal{B}^j = \bigcup_{k=1}^{\infty} A_k^j$.

(2) A_k^j is finite and pairwise disjoint.

(3) Any A_k^j contains a dyadic intervall I such that: $G_{p,k}^{\varepsilon_k}(I) \subset A_k^j$ for $p \leq k$ and

$$\frac{1}{|I|} \sum_{J \in G_{k,k}^{\varepsilon_k}(I)} |J| \geq \gamma_k(1-\varepsilon_k).$$

(4) For any A_k^j and any $I \in A_k^j$ we get

$$|I \setminus \bigcup_{l=k+1} \bigcup_{J \in A_l^j} J| \geq \frac{|I|}{2}$$

Proof: Define $\sigma_m = \{I : |I| = 2^{-m},\ I \in \mathcal{B}\}$, $\bar{\sigma}_m := \{\cup I : I \in \sigma_m\}$ $\tau_m = \bigcup_{n=m}^{\infty} \bar{\sigma}_m$. We will use repeatedly the following two observations:

(1) $\sigma = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} \bar{\sigma}_m$ which implies (by the hypothesis on σ) that $|\bigcup_{m=n}^{\infty} \bar{\sigma}_m|$ tends to zero when n goes to infinity.

(2) If $\sup_{I \in \mathcal{B}} \frac{1}{|I|} \sum_{J \in \mathcal{B} \cap I} |J| = \infty$ then for $\mathcal{B}_k := \mathcal{B} \setminus \bigcup_{m=0}^{k} \sigma_m$ we also get:

$$\sup_{I \in \mathcal{B}_k} \frac{1}{|I|} \sum_{J \in \mathcal{B}_k \cap I} |J| = \infty.$$

We start the iteration with

Step zero: Set $m_0 = 0$, $\sigma_0 = [0,1]$, then find $m_1 \in \mathbb{N}$ such that $|\tau_{m_1}| \leq \frac{1}{2}$ and set $\mathcal{B}_1 = \mathcal{B} \setminus \bigcup_{m=0}^{m_1} \sigma_m$.

Step 1: Find $I \in \mathcal{B}_1$ such that

$$\frac{1}{|I|} \sum_{J \in \mathcal{B}_1 \cap I} |J| \geq \frac{2}{1-\gamma_2} .$$

Then apply Lemma 1 and the Remark after Definition 2. We get $I_2 \in \mathcal{B}_1$ such that

$$\frac{1}{|I_2|} \sum_{J \in G^{\varepsilon_2}_{2,2}(I_2)} |J| \geq \gamma_2(1-\varepsilon_2)$$

Then choose $m_2 \in \mathbb{N}$ so large that $J \in G^{\varepsilon_2}_{2,2}(I_2)$ implies $|J| > 2^{-m_2}$ and such that for $I \in \bigcup_{m=0}^{m_1} \sigma_m$ we get $|\tau_{m_2}| \leq \frac{|I|}{2}$. Set $\mathcal{B}_2 = \mathcal{B}_1 \setminus \bigcup_{m_1}^{m_2} \sigma_m$. Now we continue and arrive at

<u>Step n:</u> First find $I \in \mathcal{B}_n$ such that

$$\frac{1}{|I|} \sum_{J \in \mathcal{B}_n \cap I} |J| \geq \frac{n}{(1-\gamma_n)}.$$

Again apply Lemma 1 and the Remark after Definition 2 to get $I_n \in \mathcal{B}_n$ such that

$$\frac{1}{|I_n|} \sum_{J \in G^{\varepsilon_n}_{n,n}(I_n)} |J| \geq \gamma_n(1-\varepsilon_n).$$

Then choose $m(n+1) \in \mathbb{N}$ such that

(1) $\quad I \in G^{\varepsilon_n}_{n,n}(I_n) \Rightarrow |I| \geq 2^{-m(n+1)}$

(2) $\quad I \in \bigcup_{m=1}^{m(n)} \sigma_m \Rightarrow |\tau_{m(n+1)}| \leq \frac{|I|}{2}$.

Summing up, we have:

$$A_k = \bigcup_{m_{k-1}}^{m_k - 1} \sigma_j; \quad (A_k)_{k\in\mathbb{N}} \text{ is a partition of } \mathcal{B}.$$

$$A_k^1 = A_{2k} \quad \text{and} \quad \mathcal{B}^1 = \bigcup_k A_k^1.$$

$$A_k^2 = A_{2k+1} \quad \text{and} \quad \mathcal{B}^2 = \bigcup_k \mathcal{B}_k^2.$$

So by construction (1), (2), (3) are satisfied and we only have to check (4).

Set $j = 1$. Fix $k \in \mathbb{N}$, and choose $I \in A_k^1$.

$$|I \setminus \bigcup_{l=k+1}^{\infty} \bigcup_{J\in A_l^1} J| \geq |I| - |\tau_{2k+1}| \geq \frac{|I|}{2}.$$

For $j = 2$ we get the same estimate.

Proof of theorem, part (b):

The proof is divided into two steps:

(1) By using mainly property (4) of proposition 3, we show that X is isomorphic to a complemented subspace of $(\Sigma H_n^1)_{\ell 1}$.

(2) By using properties (4) and (3) of proposition 3 and the Mainlemma 2 we show that X contains a complemented subspace isomorphic to $(\Sigma H_n^1)_{\ell 1}$.

Then, using the fact that $(\Sigma H_n^1)_{\ell 1}$ is isomorphic to its

ℓ^1 sum, we apply Pelczynski's decomposition method, and are done.

Choose $\varepsilon_k > 0$, $\gamma_k < 1$, such that:

$$1-4^{-k} < \gamma_k(1-\varepsilon_k).$$

Take the partition of $\mathcal{B}$ as obtained before. Put $X_k = \text{span } \{h_I \mid I \in A_k\}$ and $X = \text{span } \{h_I \mid I \in \mathcal{B}\}$ and let $P_{A_k^j}$ be the natural projection from H^1 onto span $\{h_I \mid I \in A_k^j\}$ Take $f \in X$. We first show that

$$\|f\|_{H^1} .4 \geqq \sum_k \|\mathbb{P}_{A_k^j} f\|_{H^1} + \sum_k \|\mathbb{P}_{A_k^j} f\|_{H^1} \qquad (*)$$

To do so we first observe that

$$2.\|f\|_{H^1} \geqq \|\mathbb{P}_{\mathcal{B}^1} f\|_{H_1} + \|\mathbb{P}_{\mathcal{B}^2} f\|_{H^1}$$

Take $g \in \text{span } \{h_I \mid I \in \mathcal{B}^1\}$. Define $G_k = \bigcup_{J \in A_k^j} J$.

$$\int S(g) = \int (\sum_n (\sum_{I \in A_k^1} a_I^2 h_I^2))^{1/2} \geqq$$

$$\geqq \sum_k \int (\sum_{I \in A_k^1} a_I^2 h^2_{I \setminus \bigcup_{j=k+1} G_j})^{1/2} \geqq$$

$$\geqq \frac{1}{2} \sum_k \int (\sum_{I \in A_k^1} a_I^2 \chi_I)^{1/2} \qquad \text{(by (4) of Prop. 3)}$$

$$= \frac{1}{2} \sum_k \|\mathbb{P}_{A_k^1} f\|_{H^1}$$

For $g \in \text{span}\ \{h_I | I \in \mathcal{B}^2\}$ the same estimate holds. Thus we have verified (*).

To factor the identity on X over $(\Sigma H_n^1)_{\ell^1}$ we first find a sequence n_k such that

$$X_k \subset H_{n_k}^1$$

and define

$$\begin{aligned} i : X &\to (\Sigma H_{n_k}^1)_{\ell^1} \\ f &\to (P_{A_k} f)_{k \in \mathbb{N}} \end{aligned}$$

$$\begin{aligned} P : (\Sigma H_n^1)_{\ell^1} &\to X \\ (f_n) &\to \Sigma P_{A_k} f_k \end{aligned}$$

By the calculation above,we get $P \circ i = id_X$ and $\|i\| . \|P\| \leq C$. On the other hand we factor the identity on $(\Sigma H_n^1)_{\ell^1}$ over X. Using Property (3) of A_k we see,by the Main Lemma 2, that there exist:

$$\begin{aligned} i_n &: H_n^1 \to X_n \\ P_n &: \ X \to i_n(H_n^1) \end{aligned}$$

such that $i_n^{-1} P_n i_n = id_{H_n^1}$ and $\|i_n\| . \|i_n^{-1} P_n\| \leq 4$.

Then define:

$$\begin{aligned} j : (\sum_n H_n^1)_{\ell^1} &\to X \\ (f_n)_{n \in \mathbb{N}} &\to \sum_n i_n f_n \end{aligned}$$

$$\begin{aligned} P : X &\to (\Sigma H_n^1)_{\ell^1} \\ f &\to (i_n^{-1} P_n f)_{n \in \mathbb{N}} \end{aligned}$$

Again we get $P \circ j = id_{(\Sigma H_n^1)_{\ell^1}}$ and $\|P\|.\|j\| \leq C$. The next Proposition is the main step toward proving Theorem 1.c.

<u>Proposition 4:</u>

Given a sequence $l(n)$, trees (E_{ni}), (F_{ni}) such that $E_{ni} \supset F_{ni}$ and $E_{ni} \in \mathcal{E}_{l(n)}$ and pairwise disjoint collections of dyadic intervals $\mathcal{C}_{ni}$ with finite subsets $\mathcal{K}_{ni}$ such that

1. $\chi_{F_{ni}} \leq |\sum_{I \in \mathcal{C}_{ni}} h_I| \leq \chi_{E_{ni}}$

2. $\|\sum_{I \in \mathcal{C}_{ni}} h_I - \sum_{J \in \mathcal{K}_{ni}} h_J\|_{H^1} \leq 4^{-n}$,

then there exists a subspace Y of span $\{(\sum_{I \in \mathcal{K}_{ni}} h_I) | n \in \mathbb{N}, 0 \leq i \leq 2^n - 1\}$ such that

1. Y is isomorphic to H^1

2. Y is complemented in H^1.

<u>Remark:</u> There are examples for which span $\{(\sum_{I \in \mathcal{K}_{ni}} h_I)\}_{(ni)}$ itself is not complemented. Proposition 4 solves a very special case of the following open <u>problem</u>: Does every subspace $X \subset H^1$ which is isomorphic to H^1 contain a complemented copy of H^1?

<u>Notation:</u> $\mathcal{F}_n$ is the algebra generated by $\{E_{ni} : 0 \leq i \leq 2^n - 1\}$

$m(n) = \sup \{\ln_2 \frac{1}{|I|} : I \in \mathcal{K}_{ni}, 0 \leq i \leq 2^n - 1\}$.

The proof of proposition 4 needs the following:

Lemma 5:

There exists a tree (G_{ni}) and a subsequence $k(n)$ such that:

(a) $G_{ni} \in F_{k(n)}$

(b) $J_{ni} := \{(k(n),j) \mid E_{k(n),j} \subset G_{ni}\}$

$$h_{G_{ni}} := \sum_{(mj)\in J_{ni}} \sum_{I\in K_{mj}} h_I$$

If I is a dyadic interval with $I \subset G_{ni}$, $I \not\subset G_{n+1,2i}$ and $I \not\subset G_{n+1,2i+1}$ we get:

(1) $h_{G_{mj}}/I = \text{const.}$ for $m \leqq n-1$

(2) $|G_{mj} \cap I| \leqq |I| 2^{-m+n}$ for $m \geqq n+2$

(3) $\int_I h_{G_{mj}} = 0$ for $m \geqq n+1$.

Proof of Lemma 5:

Step zero: $G_{oo} := E_{oo}$ and $\tilde{h}_{G_{oo}} := k_{oo}$

Step one: Take a covering of G_{oo} by dyadic intervals of length less than $2^{-m(O)}$, and call it C_{oo}.
On the non atomic measure space (G_{oo}, F) we define a $\mathbb{R}^{|C_{oo}|+1}$ valued vector measure

$$\mu_1 : (G_{oo}, F) \to \mathbb{R}^{|C_{oo}|+1}$$
$$E \to (|E|, (|E \cap I|)_{I\in C_{oo}})$$

We apply Liaponoff's theorem to μ_1 and, for $\varepsilon_1 > 0$, we obtain $k_1 \in \mathbb{N}$ and disjoint sets G_{10}, $G_{11} \in F_{k_1}$ such that

$$G_{10} \cup G_{11} = G_{00}$$

$$\frac{|G_{00}|}{2+\varepsilon_1} \leqq |G_{1j}| \leqq \frac{|G_{00}|}{2-\varepsilon_1} \quad \text{for} \quad j \in \{0,1\}$$

$$|G_{1j} \cap I| \leqq \frac{|I|}{2-\varepsilon_1} \qquad \text{for } I \in C_{00} \text{ and } j \in \{0,1\}.$$

Thus we have verified property (1), (3) for the first step. We continue and arrive at

<u>Step n:</u> Let $(G_{mj})_{m=1}^{n},{}_{j=0}^{2^m-1}$ and $(h_{G_{mj}})_{m=1,j=0}^{n \quad 2^m-1}$ be such that (1), (2), (3) hold. Now take G_{ni}, $0 \leqq i \leqq 2^n-1$, and take a covering C_{ni} of G_{ni} by dyadic intervals of length less than $2^{-m(k(n))}$. Again we have a non atomic $\mathbb{R}^{|C_{ni}|+1}$ valued vector measure

$$\mu_{ni} : (G_{ni}, F) \to \mathbb{R}^{|C_{ni}|+1}$$

$$E \to (|E|, (|E \cap I|)_{I \in C_{ni}})$$

For given $\varepsilon > 0$ we obtain $k_{n+1} \in \mathbb{N}$ and disjoint sets $G_{n+1,2i}$, $G_{n+1,2i+1} \in F_{k_{n+1}}$ such that:

$$G_{n+1,2i} \cup G_{n+1,2i+1} = G_{ni}$$

$$\frac{|G_{ni}|}{2+\varepsilon_n} \leq |G_{n+1,2i+j}| \leq \frac{|G_{ni}|}{2-\varepsilon_n} \qquad \text{for} \quad j \in \{0,1\}$$

$$|G_{n+1,2i+j} \cap I| \leq \frac{|G_{ni} \cap I|}{2-\varepsilon_n} \quad \text{for} \quad I \in C_{ni},\ j \in \{0,1\}$$

Having chosen the (ε_n) such that: $\Pi(1+\varepsilon_n) \leq 2$ we have established the properties (1), (2), (3) up to level (n+1).

<u>Proof of proposition 4:</u>

Take $(G_{mj})_{m=0,j=0}^{\infty\ \ 2^m-1}$, $(h_{G_{mj}})_{m=0,j=0}^{\infty\ \ 2^m-1}$ as obtained in lemma 5.

Let Y be the closed linear span of $\{(h_{G_{mj}}) \mid m\in\mathbb{N}, 0\leq j\leq 2^m-1\}$. By definition Y is a subset of X. We will show that

(a) $i : H^1 \to H^1$

$h_{ni} \to \tilde{h}_{G_{ni}}$ is an embedding.

(b) $P : H^1 \to H^1$

$f \to \sum_{mj} \left(f \left| \frac{h_{G_{mj}}}{\|h_{G_{mj}}\|_2^2}\right.\right) h_{G_{mj}}$ is a bounded idempotent map onto Y.

ad (a):

$$H_{ni} = \{\cup F_{k(n),j} \mid k(n),j \in J_{ni}\}.$$

$$g_{H_{ni}} = \sum_{(m,j)\in J_{ni}} \sum_{I\in C_{(mj)}} h_I$$

By hypothesis (2) $\|g_{H_{ni}} - h_{G_{ni}}\|_{H^1} \leq 4^{-n}$ and by hypothesis (1) there is $C > 0$ such that

$$\frac{1}{C}\|\Sigma a_{ni} g_{H_{ni}}\|_{H_1} \leq \|\Sigma a_{ni} h_{ni}\|_{H^1} \leq C\|\Sigma a_{ni} g_{H_{ni}}\|_{H^1}$$

A standard perturbation result proves (a).

ad (b): Lemma 5 allows us to use the same strategy as in Lemma 2. By orthogonality

$$(i^{-1}P)^*: \text{BMO} \to \text{BMO}$$
$$h_{ni} \to h_{G_{ni}}$$

is bounded iff P is.

To check the continuity of $(i^{-1}P)^*$ we fix a dyadic interval I and find (n,i) such that $I \subset G_{n,i}$ and $I \not\subset G_{n+1,2i}$ and $I \not\subset G_{n+1,2i+1}$. We estimate:

$$\frac{1}{|I|}\int_I (\Sigma a_{mj} h_{G_{mj}} - (\Sigma a_{mj} h_{G_{mj}})_I)^2 dt =$$

$$= \frac{1}{|I|}\int_I ((\sum_{(mj)\subset(ni)} a_{mj} h_{G_{mj}} - (\sum_{(mj)\subset(ni)} a_{mj} h_{G_{mj}})_I)^2$$

$$\leq \frac{4}{|I|}\int_I (\sum_{(mj)\subset(ni)} a_{mj} h_{G_{mj}})^2$$

$$= \frac{4}{|I|}\int_I \sum_{(mj)\subset(ni)} a_{mj}^2 h_{G_{mj}}^2$$

$$= \frac{4}{|I|}\sum_{(mj)\subset(ni)} \int_I a_{mj}^2 h_{G_{mj}}$$

$$\leq \frac{4}{|I|}\sum_{(mj)\subset(ni)} a_{mj}^2 |I| 2^{-m+n}$$

$$= 4(2^n \sum_{(mj)\subset(ni)} a_{mj}^2\ 2^{-m})$$

<u>Proof of Theorem 1, part c:</u>

We will select subsets $\mathcal{K}_{ni}$ and $\mathcal{C}_{ni}$ of $\mathcal{B}$ such that the requirements of Proposition 4 are satisfied. Then we obtain a complemented subspace Y of X which is isomorphic to H^1. By the decomposition method, we are done.

<u>Step zero:</u> Here we construct the trees (E_{ni}), (F_{ni}): An $\mathbb{R}^2$ valued non atomic vector measure is given:

$$\mu : ([0,1],\mathcal{B}) \to \mathbb{R}^2$$
$$E \to (|E|,|E \cap \sigma|)$$

Repeated applications of Liaponoff's theorem give a sequence $k(n)$, a constant $c > 0$ and trees (E_{ni}) and $(E_{ni} \cap \sigma)$ such that

$$E_{ni} \in \mathcal{E}_{k(n)}$$

$$\frac{1}{c}\, 2^{-n} \leqq |E_{ni}| \leqq c\, 2^{-n}$$

$$\frac{1}{c}\, 2^{-n}|\sigma| \leqq |E_{ni} \cap \sigma| \leqq c\, 2^{-n}|\sigma|.$$

<u>Step one:</u> Put $\mathcal{C}_{oo} = \{I \in \mathcal{B} \mid I \subset E_{oo} \wedge I \text{ maximal}\}$. Then:

$$\chi_\sigma \leqq \Big|\sum_{I \in \mathcal{C}_{oo}} h_I\Big| \leqq \chi_{E_{oo}}.$$

Put $\mathcal{B}_1 = \mathcal{B} \setminus \mathcal{C}_{oo}$.

Step two: For $j \in \{0,1\}$ we put

$C_{1j} = \{I \in B_1 | I \subset E_{1j}, \text{ I maximal}\}$. Then

$$\chi_{E_{1j} \cap \sigma} \leq | \sum_{I \in C_{1j}} h_I | \leq \chi_{E_{1j}} .$$

Take

$$B_2 = B_1 \setminus \{C_{10} \cup C_{11}\}.$$

We continue and arrive at

Step n: For $j \in \{0,\ldots,2^n-1\}$ we put

$C_{nj} = \{I \in B_{n-1} | I \subset E_{nj}, \text{ I maximal}\}$. Then

$$\chi_{E_{nj} \cap \sigma} \leq | \sum_{I \in C_{nj}} h_I | \leq \chi_{E_{nj}} .$$

Take

$$B_n = B_{n-1} \setminus \{ \bigcup_{j=1}^{2^n-1} C_{nj} \}.$$

Finally, we remark that by choosing any sufficiently large subset K_{ni} of C_{ni}, we can fulfill all the conditions, necessary to apply Proposition 4.

SECTION 2: GENERAL MARTINGALE H^1:

Consider an increasing sequence (F_n) of finite fields on a probability space (Ω, F, P) such that F is the σ-algebra generated by $\bigcup_n \cup \{A | A \in F_n\}$. Given a P integrable function f we set:

$$S(f)(t) = (\sum_n (E(f|F_n) - E(f|F_{n-1}))^2)^{1/2}(t)$$

$$f^*(t) = \sup_n E(f|F_n)(t)$$

$$H^1((F_n)) = \{f \in L^1(\Omega, F, P) : \|S(f)\|_{L^1} < \infty\}$$

$$BMO((F_n)) = \{f \in L^1(\Omega, F, P) : \sup_n \|E((f-f_{n-1})^2 | F_n)\|_\infty^{1/2} < \infty\}$$

We use the following

<u>Theorem (DAVIS):</u> $\frac{1}{c}\|f^*\|_{L^1} \leq \|S(f)\|_{L^1} \leq c\|f^*\|_{L^1}$
for some constant c.

<u>Theorem (MAUREY):</u> $H^1((F_n))$ is isomorphic to a complemented subspace of $H^1(\delta)$.

<u>Remark:</u> This theorem holds without any further condition on (F_n).

In this section we will give a nessesary and sufficient condition on (F_n) such that $H^1((F_n))$ is isomorphic to $H^1(\delta)$. We thus prove a conjecture of B. Maurey.

<u>Definition:</u> $A_k^\varepsilon = \cup\{B | B \text{ is Atom in } F_k \wedge P(B) < \varepsilon\}$

$$A^\infty = \bigcap_{\varepsilon>0} \bigcup_{k\in\mathbb{N}} A_k^\varepsilon.$$

<u>Theorem 2:</u> $H^1((F_n))$ is linearly isomorph to $H^1(\delta)$ if and only if $P(A^\infty) > 0$.

Proof:

We show first that $P(A^\infty) > 0$ is a sufficient condition. By the theorem of Maurey it is enough to find a complemented subspace in $H^1((F_n))$ which is isomorphic to $H^1(\delta)$.

Step 1: Observe that $(\Omega \cap A^\infty, F, P)$ is a non atomic measure space. Fix a sequence (ε_j) such that $\Pi(1+\varepsilon_j) \leq 1,5$. For $\varepsilon_1 > 0$ find $k_o \in \mathbb{N}$ and $A_{oo} \in F_{k_o}$ such that $A_{oo} \supset A^\infty$ and $P(A_{oo} \setminus A^\infty) < \varepsilon_1$. Set $C_{oo} = \{B \mid B \text{ atom in } F_{k_o} \text{ and } B \subset A_{oo}\}$.

Step 2: Apply Liaponoff's theorem to the measure

$$\mu : (A^\infty, F, P) \to \mathbb{R}^n$$
$$E \to (P(E), P(B \cap E)_{B \in C_{oo}})$$

and obtain, for $\varepsilon_2 > 0$ and disjoint sets A_{10}, A_{11}, a natural number $k_1 > k_o$ such that

(1) $A_{10}, A_{11} \in F_{k_1}$ and $A_{10} \cup A_{11} \subset A_{oo}$

(2) $P(A_{1j} \cap B) \overset{\varepsilon_2}{\sim} \dfrac{P(A^\infty \cap B)}{2}$, $j \in \{0,1\}$, $B \in C_{oo}$

(3) $P(A_{1j}) \overset{\varepsilon_2}{\sim} \dfrac{P(A^\infty)}{2}$, $j \in \{0,1\}$

We define the "Haar function":

$$h_{A_{oo}} = \sum_{B \in C_{oo}} \frac{P(B \cap A^\infty)}{2P(B \cap A_{10})} \chi_{A_{10} \cap B} - \frac{P(B \cap A^\infty)}{2P(B \cap A_{11})} \chi_{A_{11} \cap B}$$

We continue and arrive at

<u>Step n:</u> We are given A_{ni} in F_{k_n} and $C_{ni} = \{B | B \subset A_{ni} \wedge B \text{ Atom in } F_{k_n}\}$. Define a vector measure

$$\mu : (A^\infty \cap A_{ni}, F) \to \mathbb{R}^{|C_{ni}|+1}$$
$$E \to (P(E), P(B \cap E)_{B \in C_{ni}})$$

As an application of Liaponoff's theorem we get, for a given ε_n disjoint sets $A_{n+1,2i}$, $A_{n+1,2i+1}$ and a natural number k_{n+1} such that

(1) $A_{n+1,2i}, A_{n+1,2i+1} \in F_{k_{n+1}}$ and $A_{n+1,2i} \cup A_{n+1,2i+1} \subset A_{ni}$

(2) $P(A_{n+1,2i+j}) \overset{\varepsilon_n}{\sim} \frac{1}{2} P(A_{ni} \cap A^\infty)$, $j \in \{0,1\}$

(3) $P(A_{n+1,2i+j} \cap B) \overset{\varepsilon_n}{\sim} \frac{1}{2} P(B \cap A^\infty)$, $B \in C_{ni}$, $j \in \{0,1$

We use these sets to define

$$h_{A_{ni}} = \sum_{B \in C_{ni}} \frac{P(B \cap A^\infty)}{2P(A_{n+1,2i} \cap B)} \chi_{A_{n+1,2i} \cap B} -$$
$$- \frac{P(B \cap A^\infty)}{2P(A_{n+1,2i+1} \cap B)} \chi_{A_{n+1,2i+1} \cap B}$$

To prove Theorem 2, we must first verify that for any sequence (a_{mj}) we have the estimates

$$\|\Sigma a_{mj} h_{mj}\|_{H^1(\delta)} < C_1 \|\Sigma a_{mj} h_{A_{mj}}\|_{H^1(F_n)}$$

$$\leq C_2 \|\Sigma a_{mj} h_{mj}\|_{H^1(\delta)}.$$

Define $K_j = \{m \in \mathbb{N} \mid k_j \leq m < k_{j+1}\}$, $j \in \mathbb{N}$

By construction we get:

$$E(h_{A_{ni}} \mid F_j) = h_{A_{ni}} \qquad \text{for } j \in K_{n+1} \cup K_{n+2} \cup \ldots$$

$$E(h_{A_{ni}} \mid F_j) = 0 \qquad \text{for } j \in \{k_n\} \cup K_{n-1} \cup K_{n-2} \ldots$$

$$|h_{A_{ni}}|(t) \leq 2\chi_{A_{ni}}(t) \qquad \text{for } t \in \Omega.$$

Fix $f = \Sigma a_{mj} h_{A_{mj}}$, a finite linear combination.
Fix $j \in K_n$ then

$$E(f \mid F_j)(t) = E(\sum_i a_{ni} h_{A_{ni}} \mid F_j)(t) + \sum_{m<n} a_{mj} h_{A_{mj}}(t)$$

and so

$$\int \sup_{j \in \mathbb{N}} |E(f \mid F_j)| \leq$$

$$\leq \int \sup_{n \in \mathbb{N}} \sup_{j \in K_n} |E(\Sigma a_{ni} h_{A_{ni}} \mid F_j)| + \int \sup_n \Big| \sum_{\substack{m<N \\ 0 \leq j \leq 2^m - 1}} a_{m_j} h_{A_{mj}} \Big|$$

$$\leq \int \sup_n \sum_{i=0}^{2^n-1} |a_{ni}| |\chi_{A_{ni}}|(t)\, dP(t) + C \|\Sigma a_{mj} h_{mj}\|_{H^1(\delta)}$$

$$\leq \int \Big(\sum_{n \in \mathbb{N}} \sum_{0=i}^{2^n-1} a_{ni}^2 |\chi_{A_{ni}}| \Big)^{1/2} dP(t) + C \|\Sigma a_{mj} h_{mj}\|_{H^1(\delta)}$$

$$\leq \qquad C. \|\Sigma a_{mj} h_{mj}\|_{H^1(\delta)}.$$

The proof of the reverse estimate is similar. Hence the closed linear hull of $\{h_{A_{ni}} : n \in \mathbb{N}, 0 \leq i \leq 2^n-1\}$ in $H^1((F_n))$ is isomorphic to $H^1(\delta)$ and the mapping

$$i : H^1(\delta) \to H^1((F_n))$$
$$h_{ni} \to h_{A_{ni}}$$

is an embedding.

We must now ensure that $i(H^1(\delta))$ is complemented in $H^1((F_n))$.

Define the projection

$$P : H^1((F_n)) \to H^1((F_n))$$
$$f \to \Sigma\left(f, \frac{h_{A_{ni}}}{\|h_{A_{ni}}\|_2^2}\right) h_{A_{ni}}$$

P is bounded iff

$$(i^{-1}P)^* : BMO(\delta) \to BMO((F_n))$$
$$h_{ni} \to h_{A_{ni}}$$

is bounded.

A glance at the construction shows that if $i \in K_n$ and B is an atom in F_i, then

$$h_{A_{mj}/B} = \text{const.} \qquad \text{for } m < n$$

$$\int_B |h_{A_{mj}}|^2 \leq c\, P(B)\, 2^{-m+n} \qquad \text{for } m \geq n$$

$$\int_B h_{A_{mj}} = 0 \qquad \text{for } m > n.$$

Take $f = \Sigma a_{mj} h_{A_{mj}}$. Fix $I \in F_j$ and $J(\supset I) \in F_{j-1}$. Take the largest m_1 such that $h_{A_{m_1 i}}$ is F_{j-1} measurable, and find i_1 such that $J \in \operatorname{supp} h_{A_{m_1 i_1}}$. Therefore $j-1 \in K_{m_1}$ and $j \in K_{m_1} \cup \{k_{m_1+1}\}$.
Hence we estimate as in section 1

$$E(f-f_{j-1}|F_j)/I =$$

$$= \frac{1}{P(I)} \int_I \Big(\sum_{m>m_1} a_{mi}\, h_{A_{mi}} - \big(\sum_{m>m_i} a_{mi}\, h_{A_{mi}} \big)_J \Big)^2$$

$$\leq \frac{c}{P(I)} \sum_{m>m_1} a_{mi}^2 \int_I h_{A_{mi}}^2 + \frac{c}{P(J)} \sum_{m>m_1} a_{mi}^2 \int_J h_{mi}^2$$

$$\leq 2.c \sum_{\substack{m>m_i \\ (m_i)\subset(m_1 i_1)}} a_{mi}\, 2^{m_1-m} = \|\Sigma a_{mj}\, h_{mj}\|_{BMO(\delta)}.$$

To show that $P(A^{\infty}) > 0$ is a necessary condition, we simply conclude that $P(A^{\infty}) = 0$ implies that ℓ^2 does not embed in $H^1((F_n))$.

Let e_i be equivalent to the unit vector basis of ℓ^2 in $H^1((F_n))$. e_i tends to zero in the $\sigma(H^1, BMO)$ topology. By taking a subsequence if necessary, we may suppose that for any sequence (λ_n)

$$\|\Sigma\lambda_i e_i\|_{H^1(F_n)} \geq \frac{1}{4} \int (\Sigma\lambda_i^2 |e_i|^2(t))^{1/2}\, dP(t)$$

We claim that for some $\delta > 0$ the numbers $P(E(e_n > \delta))$,

$P(E(e_i < -\delta))$ tend to zero if i tends to infinity.
Indeed, choose $\mathcal{B}_i \in \cup F_n$ such that $A, B \in \mathcal{B}_i \Rightarrow A \cap B = \emptyset$ and $c_B \in \mathbb{R}$, $B \in \mathcal{B}_i$ such that

$$e_i \, \chi_{E\{e_i > \delta\}} = \sum_{B \in \mathcal{B}} c_B \, \chi_B .$$

By the hypothesis on e_i, $\sup_{B \in \mathcal{B}_i} P(B)$ tends to zero as i tends to infinity. Therefore by hypothesis on A^∞ which says that the <u>union</u> of small atoms is small, the claim is verified. So we can suppose (by taking a subsequence) that:

$$P(E(|e_j|^2 > \delta) \setminus \bigcup_{i=j+1}^{\infty} E(|e_i|^2 > \delta)) > \tfrac{1}{2} P(E(|e_j|^2 > \delta)) .$$

We put everything together and estimate as in proof of theorem 1, part b.

$$(\Sigma \lambda_i^2)^{1/2} \geq C_1 \int (\Sigma \lambda_i^2 |e_1|^2)^{1/2}$$

$$\geq C_2 \int (\Sigma \lambda_i^2 |e_i|^2 \chi_{E(|e_i| > \delta)})^{1/2} - C_2 \, (\Sigma \lambda_i^2)^{1/2}$$

$$\geq C_3 \, \Sigma |\lambda_i| - C_4 \, \delta (\Sigma \lambda_i^2)^{1/2}$$

a contradiction.

Section 3: Examples of badly complemented H_n^1 spaces in $H^1(\delta)$.

In this section we construct isometric copies of H_n^1 in H^1. We isolate properties of embeddings $i_n : H_n^1 \to H^1$ which couse the norm of projections onto $i_n(H_n^1)$ to be large. (C.f. Theorem 3, part a).
These properties in extreme constrast to those which cause a copy of H_n^1 to be "nicely" complemented (cf. Theorem 3, part b). Hence Theorem 3 sheds some light on the ideas behind the proofs in the previous sections.

Construction

Fix $n_o \in \mathbb{N}$. E_{ni} denotes a tree in $[0,1]$ such that $|E_{ni}| = 2^{-n}$. C_{ni} denotes a collection of dyadic intervals such that a) $I, J \in C_{ni}$ implies $I \cap J = \emptyset$.

b) $\bigcup_{I \in C_{ni}} I = E_{ni}$.

Define

$$\tilde{h}_{ni} = \sum_{I \in C_{ni}} h_I$$

$$Y_{n_o} = \text{span}\,\{\tilde{h}_{ni} : n < n_o,\ 0 \leq j \leq 2^n - 1\}.$$

It is easily seen that $H_{n_o}^1 \to H^1$ is an isometry
$h_{ni} \to \tilde{h}_{ni}$ onto Y_{n_o}.

Theorem 3

Fix $n_o \in \mathbb{N}$.

a) If for any (m,j), (n,i), $I \in C_{mj}$, $J \in C_{ni}$, $I \subset J$ implies $m < n$, then we get, for any projection P_{n_o} from $H^1(\delta)$ onto Y_{n_o}, $\|P_{n_o}\| \geq \frac{1}{10} \sqrt{n_o}$.

b) If $E_{n+1,2i} = E(\sum_{C_{ni}} h_I = 1)$ and

$$E_{n+1,2i+1} = E(\sum_{C_{ni}} h_I = -1)$$

then there exists a projection P_{n_o} from $H^1(\delta)$ onto Y_{n_o} such that $\|P_{n_o}\| \leq 4$.

Remark:

We prove theorem 3, part a, without using Bourgain's result on projections onto the image of order-inverting embeddings in H^1. The concrete (and specialized) situation above allowes a different (and simple) proof which "lives" entirely in BMO.

Condition "b" connects the tree E_{ni} strongly with C_{ni} and is, in fact, the exact opposite of property "a".

Proof:

Let P_{n_o} be a projection from $H^1_{n_o}$ onto Y_{n_o}. Arguing as in [16], p. 49 there exists a linear map $\xi_{n_o} : BMO_{n_o} \to BMO$ such that:

$$\|\xi_{n_o}\| \cdot \|(\xi_{n_o}^{-1}/\xi_{n_o}(BMO_{n_o})\| \leq \sqrt{2}\,\|P_{n_o}\| \qquad (*)$$

$$\xi_{n_o} h_{ni} \in \text{span}\,\{h_I : I \in C_{ni}\},\ n \leq n_o. \qquad (**)$$

Let Q_{ni} denote $\{I \in C_{ni} : |\xi_{n_o} h_{ni}| > \delta\}$.
For $\xi_{n_o} h_{ni} = \sum_{I \in C_{ni}} h_I \alpha_I$ we get, by the special form of C_{ni},

$$Q_{ni} = \{I \in C_{ni} : |\alpha_I| > \delta\}.$$

Now define: $\mathcal{B} := \bigcup_{n=0}^{n_o} \bigcup_{i=0}^{2^n-1} Q_{ni}$, $h_n := \sum_{i=0}^{2^n-1} h_{ni}$,
$R_n := \bigcup_{i=0}^{2^n-1} \bigcup_{I \in Q_{ni}} I$, $S_n := [0,1] \setminus R_n$, $M := \|\xi_{n_o}^{-1}/\xi_{n_o}(BMO_{n_o})\|$
and for δ take $\frac{1}{2M}$.

<u>CLAIM 1:</u>

$$\sup_{I \in \mathcal{B}} \frac{1}{|I|} \sum_{J \subset \mathcal{B} \cap I} |J| \geq \left(\frac{1}{M} - \delta\right)^2 n_o.$$

<u>Proof of Claim 1:</u> Choose $\alpha_n \in \mathbb{R}$

$$\frac{1}{M}(\Sigma \alpha_n^2)^{1/2} \leq \left\|\sum_{n=0}^{n_o} \alpha_n \xi_{n_o} h_n\right\|$$

$$\leq \left\|\left(\sum_{n=0}^{n_o} \alpha_n \xi_{n_o} h_n\right)\chi_{R_n}\right\| + \left\|\left(\sum_{n=0}^{h} \alpha_n \xi_{n_o} h_n\right)\chi_{S_n}\right\|$$

$$\leq \sup|\alpha_n| \cdot \sup\left(\frac{1}{|I|}\sum_{J \in \mathcal{B}} |J|\right)^{1/2} + \delta\left(\sum_{n=0}^{n_o} \alpha_n^2\right)^{1/2}$$

Thus we obtain

$$\left(\frac{1}{M} - \delta\right) \frac{\left(\sum_{n=0}^{n_o} \alpha_n^2\right)^{1/2}}{\sup|\alpha_n|} \leq \sup_{I \in \mathcal{B}} \left(\frac{1}{|I|} \sum_{J \in \mathcal{B}} |J|\right)^{1/2}$$

This last estimate proves claim 1.

CLAIM 2:

There exists a sequence $j(n)$, $0 \leq j(n) \leq 2^n-1$, $n \leq n_o$ such that

$$E_{1,j(1)} \supset \ldots \supset E_{n,j(n)} \supset \ldots \supset E_{n_o,j(n_o)}$$

and such that for

$$A = \bigcup_{n=1}^{n_o} Q_{n,j(n)}$$

we get

$$\sup_{I\in A} \frac{1}{|I|} \sum_{J\in A\cap I} |J| \geq \left(\frac{1}{M} - \delta\right)^2 n_o.$$

Proof of Claim 2: By the hypothesis on (C_{ni}) we may assume that there exist $j(n_o)$ and $I_o \in C_{n_o,j(n_o)}$ such that

$$\frac{1}{|I_o|} \sum_{J\subset B\cap I_o} |J| \geq \frac{1}{2}\left(\frac{1}{M} - \delta\right)^2 n_o$$

$(n_o, j(n_o))$ defines uniquely a sequence of nested dyadic intervals $(E_{n,j(n)})$ which contain $E_{n_o,j(n_o)}$.
Again, by the hypothesis on C_{ni},

$$\frac{1}{|I_o|} \sum_{J\subset B\cap I_o} |J| = \frac{1}{|I_o|} \sum_{n=1}^{n_o} \sum_{J\subset I_o\cap Q_{n,j(n)}} |J|$$

and this proves Claim 2.

Now we come back to the proof of theorem 3.b. Take $I_o \subset E_{n_o,j(n_o)}$ and $(n,j(n))$, $n \leq n_o$ as obtained in Claim 2.
We will now estimate $\|\Sigma\xi h_{n,j(n)}\|_{BMO}$ from below.

$$\|\Sigma\xi h_{n,j(n)}\|^2_{BMO} \geq \frac{1}{|I_o|} \int_{I_o} \Big(\sum_n \xi h_{n,j(n)} - \big(\sum_n \xi h_{n,j(n)}\big)_{I_o}\Big)^2 =$$

$$= \frac{1}{|I_o|} \int_{I_o} (\sum_n \xi h_{n,j(n)})^2 = \frac{1}{|I_o|} \int_{I_o} \sum_n (\xi h_{n,j(n)})^2 =$$

$$= \frac{1}{|I_o|} \int_{I_o} \sum_{\substack{I \subset C_{n,j(n)} \\ m=1}}^{n_o} h_I^2 \, \alpha_I^2 \geq$$

$$\geq \frac{1}{|I_o|} \sum_n \sum_{\substack{I \in Q_{n,j(n)} \\ I \subset I_o}} |I| \, |\alpha_I|^2 \geq$$

$$\geq \delta^2 \frac{1}{|I_o|} \sum_{\substack{I \in Q_{n,j(n)} \\ I \subset I_o}} |I| \geq \delta^2 (\tfrac{1}{M} - \delta)^2 n_o$$

□

On the other hand we get:

$$\| \sum_{n=1}^{n_o} h_{n,j(n)} \|_{BMO} \leq 4.$$

Hence

$$\|P_{n_o}\| \geq \|\xi\| \, \|\xi\|^{-1} \geq \frac{1}{\delta} \, 4 \, \frac{\|\sum_n \xi h_{n,j(n)}\|}{\|\sum h_{n,j(n)}\|} \geq \frac{1}{\delta} \, \frac{\delta^2}{4} n_o^{1/2}.$$

Using the fact that i_{n_o} is an isometry and (*) we obtain the estimate $\delta \geq \frac{1}{3}$ and consequently $\|P_{n_o}\| > c \, n_o^{1/2}$.

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