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Eidesstattliche Erklärung

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Abstract This thesis consists essentially of two parts. In the first part, we prove a vector-valued (more precisely UMD-Banach-space-valued) $T(1)$-theorem on spaces of homogeneous type X. The method of proof extends the method of T. Figiel for $X = \mathbb{R}$, whose basic idea is to decompose singular integral operators into a sum of more basic operators (namely paraproducts, shifts and rearrangements) acting on the Haar system. One then applies martingale methods to deduce their L^p-boundedness. Based on dyadic cubes on spaces of homogeneous type constructed by M. Christ, the first step is to define a substitute to Haar functions that are again martingale differences and piecewise constant. Due to the lack of group structure on X, a great deal of our work consists of *defining* suitable shift (resp. rearrangement) operators that can be controlled (by martingale methods) in L^p-norm.

In the second part of the thesis, we move from piecewise constant functions to piecewise linear functions and analyze certain properties of the Franklin orthogonal system. More precisely, we investigate the behavior of sequences of L^∞-norms of (L^2-)projection operators corresponding to increasing sequences of subspaces of L^2 consisting of piecewise linear functions. The *Lebesgue constant* is now defined as the supremum of those L^∞-norms. In the case that the domain for the Franklin functions is the unit interval $[0, 1]$, it was shown by Z. Ciesielski and A. Kamont that the Lebesgue constant for the classical Franklin system is $2 + (2 - \sqrt{3})^2$. We continue this line of research by first giving a short result showing that for equally spaced knots on the unit interval $[0, 1]$, the corresponding sequence of L^∞-operator norms is strictly increasing as the number of points increases. This theorem prepares the main result of the second part of the thesis, which is the proof that the Lebesgue constant of the classical Franklin orthogonal system is different, if one considers them defined on the torus instead of the unit interval $[0, 1]$. In fact, we will determine its exact value to be $2 + \frac{33-18\sqrt{3}}{13}$.

Zusammenfassung Diese Dissertation besteht aus zwei Hauptteilen. Im ersten Teil beweisen wir einen vektorwertigen (genauer, einen UMD-Banachraum-wertigen) $T(1)$-Satz auf Räumen homogenen Typs X. Die Methode des Beweises erweitert die Methode von T. Figiel für $X = \mathbb{R}$, dessen wesentliche Idee die Zerlegung eines singulären Integraloperators in die Summe von grundlegenderen Operatoren (nämlich Paraprodukte, Shift- und Rearrangementoperatoren) ist, die auf dem Haar-System agieren. Dann wendet man Martingalmethoden an, um die L^p-Beschränktheit dieser Operatoren zu zeigen. In unserer Situation für Räume homogenen Typs X ist der erste Schritt, einen geeigneten Ersatz für Haarfunktionen zu finden. Dies wird mit Hilfe eines Systems dyadischer Würfel erreicht, die auf X von M. Christ konstruiert wurden. Die neu definierten "Haar"-Funktionen sind – wie die Haar-Funktionen auf $\mathbb{R}$ – stückweise konstant und bilden eine Martingaldifferenzenfolge. Da auf Räumen homogenen Typs keine Gruppenstruktur vorhanden ist, wird sich anschließend ein Großteil unserer Arbeit damit beschäftigen, geeignete Shift- und Rearrangement-Operatoren zu *definieren*. Diese Operatoren können dann wieder mit Martingalmethoden in der L^p-Norm kontrolliert werden.

Im zweiten Teil der Dissertation gehen wir von stückweise konstanten zu stückweise linearen Funktionen über und analysieren bestimmte Eigenschaften des orthogonalen Franklin-Systems. Genauer untersuchen wir das Verhalten von L^∞-Normen von L^2-Projektionsoperatoren auf Räume stetiger, stückweise linearer Funktionen. Die *Lebesgue-Konstante* einer steigenden Folge solcher Unterräume von L^2 ist dann definiert als das Supremum dieser L^∞-Normen. Im Fall, dass der Definitionsbereich der Franklin-Funktionen das Einheitsintervall $[0, 1]$ ist, wurde von Z. Ciesielski und A. Kamont gezeigt, dass die Lebesgue-Konstante für das klassische Franklin-System den Wert $2 + (2 - \sqrt{3})^2$ hat. Wir setzen Untersuchungen in diese Richtung fort, indem wir zuerst beweisen, dass für gleichmäßig verteilte Punkte auf dem Einheitsintervall $[0, 1]$ die zugehörige Folge von L^∞-Operatornormen streng monoton steigend mit der Anzahl der Zwischenpunkte ist. Das Hauptresultat des zweiten Abschnitts ist allerdings die Bestimmung der Lebesgue-Konstante des klassischen Franklin-Systems auf dem Torus. Diese ist tatsächlich verschieden zu der auf dem Einheitsintervall und hat den Wert $2 + \frac{33-18\sqrt{3}}{13}$.

Preface

In this thesis, we consider two main topics. The first one is a vector-valued $T(1)$ theorem on spaces of homogeneous type X and is based on [MP11]. We thus show the L^p_E-boundedness of certain singular integral operators. Here L^p_E is the Bochner-Lebesgue space of p-integrable E valued function and E is a UMD-Banach space. The abbreviation UMD stands for 'unconditional for martingale differences'. So, the class of UMD-spaces is that subclass of the set of Banach spaces E such that martingale difference sequences converge unconditionally in L^p_E. A space of homogeneous type is a measure space equipped with a metric, such that these two satisfy the doubling condition. The second topic is the determination of the Lebesgue constant of the classical Franklin orthogonal system on the torus. The Lebesgue constant of a certain increasing sequence of subspaces V_n of L^2 is defined as the supremum of L^∞-norms of the L^2-projection operators onto V_n. This result is based on [Pas11b].

We now give an outline of what is happening in every chapter of the underlying thesis. We do this here without any references since these are contained in the corresponding sections.

Chapter 1 is to give some preliminary results, the most of them concerning martingales, that are used then in Chapter 2. The first thing to do in Section 1.1 is to recall the definition of Lebesgue-Bochner spaces and some information about their dual spaces. Then we come to a simple, but essential tool in showing the boundedness of shift and rearrangement operators defined in Chapter 2, namely Kahane's contraction principle. In Section 1.3, we first recall the definitions of conditional expectation and of martingales in order to define UMD-spaces and to state the vector-valued version of Stein's martingale inequality which is due to J. Bourgain. The basic proof-steps in Chapter 2 for showing the L^p_E boundedness of certain operators consist of first using the unconditionality of martingale differences assured by the UMD-property of E. This is done to switch to the Rademacher average. Then on this average either Kahane's contraction principle or Bourgain's version of Stein's inequality applies. If we are done with that, use again the unconditionality of martingale differences to switch back from Rademacher averages. A first application of this technique is to show the L^p_E boundedness of martingale multipliers in Section 1.3.5. Sections 1.3.6 and 1.3.7 are then to show the L^p_E boundedness of paraproduct operators, that, apart from the boundedness of shift and rearrangement operators, is necessary to show our $T(1)$ theorem.

Chapter 2 incorporates the first part of the thesis. For historical comments concerning the problem area for both the first and the second part, we refer to the introduction sections of Chapter 2 and 4 respectively. In Section 2.2 we construct shifts on spaces of homogeneous type X. To do that, we first recall the definition of spaces of homogeneous type and a few of their properties in Section 2.2.1. Then in Section 2.2.2, we collect important properties of dyadic cubes in X. Based on those dyadic cubes, we construct martingale differences on X in Section 2.2.3 that possess analogous properties to the Haar system in $\mathbb{R}$. Next we introduce an isotropic basis in $L^2(X \times X)$ using tensor products $f \otimes g$ of Haar functions and corresponding characteristic functions on cubes where the support of $f \otimes g$ looks like a square (thus the name isotropic). In Section 2.2.5 we decompose the collection of all pairs of dyadic cubes into subcollections that fix their relative distance. These subcollections are then decomposed further in Section 2.2.6 to define injections (shifts). Section 2.2.7 is then to further extract subcollections that satisfy a certain nestedness condition.

In Chapter 2.3 we use the isotropic basis to decompose integral operators. Then according to the developed decomposition in Chapter 2.2, we split the operator further into a sum of paraproducts, shift and rearrangement operators. The paraproducts are bounded in L^p_E by the results of Section 1.3.7. The shift and rearrangement operators are bounded on collections that satisfy a certain nestedness condition, as proved in Section 2.3.4. Then we utilize the decomposition of Chapter 2.2 to show the boundedness of shift and rearrangements in Section 2.3.5. Finally in Section 2.3.6 we use those results to obtain that integral operators are bounded on L^p_E by a constant that does not depend on the L^2-Norm of the underlying kernel of the integral operator. This finally allows us to treat singular integral operators with that method.

In Chapter 3 we prepare the main theorem of the second part in Chapter 4. Section 3.1 gives a few elementary results about the solutions of the recurrence $f_{k-1} - 4f_k + f_{k+1} = 0$, that will be needed extensively in both Chapters 3 and 4. This is the case, since the inverses of the Gram matrices we consider there can be written in terms of these solutions. In Chapter 3.2 we consider the case for equally spaces knots on the unit interval and the spaces of piecewise linear functions between these knots. In this setting, it is known that the Lebesgue constant is 2. We show that the sequence of L^∞-norms of projection operators is in fact strictly increasing as the number of knots increases.

In Chapter 4, the first step to determine the Lebesgue constant for the classical Franklin system on the torus is to determine the inverse of the Gram matrix for the B-spline basis of degree one for partially equally spaces knots. This is done in Section 4.4.2. It turns out that there is one special parameter choice for which the corresponding sequence of L^∞-norms converges to $2 + \frac{33-18\sqrt{3}}{13}$. This choice will be singled out in Section 4.4.3. Chapter 4.4.4 is then to show that all other choices of parameters lead to L^∞-norms that are strictly less that this value. The main tool for proving this are bounds on the quotient of two consecutive entries of the inverse of the Gram matrix.

Appendix A is to provide some supplementary results to Chapters 2 and 4. In Section A.1, we show that our substitute of the Haar system on spaces of homogeneous type arise if one orthogonalizes characteristic functions. Sections A.2 and A.3 are to apply our a priori main result of Chapter 2 to singular integral operators. Chapter A.4 contains the proof of two lemmas used in Chapter 4 for the bounds of quotients of subsequent entries of the inverse of the Gram matrix.

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Markus Passenbrunner Linz, June 2011

Contents

Chapter 1

Preliminaries

1.1 The Bochner Integral

In the following, let $(X, \mathcal{F}, \mu)$ be a probability space and E a Banach space. For more information concerning definitions and results contained in this section, see [DU77].

1.1.1 Definition and Elementary Properties

Definition 1.1.1. A function $\varphi : X \to E$ is called *simple,* if there exist finitely many $\{e_1, \ldots, e_n\} \subseteq E$ and finitely many $\{A_1, \ldots, A_n\} \subseteq \mathcal{F}$, that are mutually disjoint, such that

$$\varphi = \sum_{j=1}^{n} e_j 1_{A_j},$$

where 1_A is the characteristic function of the set A, that is $1_A(x) = 1$ if $x \in A$ and $1_A(x) = 0$ otherwise. A function $f : X \to E$ is called *μ-measurable* if there exists a sequence of simple functions (f_n) with $\lim_{n\to\infty} \|f_n - f\| = 0$ μ-almost everywhere.

Definition 1.1.2. A μ-measurable function $f : X \to E$ is called *Bochner-integrable*, if there exists a sequence of simple functions (f_n) such that

$$\lim_{n\to\infty} \int_X \|f_n - f\| \; d\mu = 0. \tag{1.1.1}$$

The integral of a simple function $\varphi = \sum_{i=1}^{m} e_i 1_{A_i}$ is defined canonically as

$$\int_X \varphi \; d\mu := \sum_{j=1}^{n} \mu(A_j) e_j.$$

This extends to Bochner-integrable functions $f : X \to E$ by

$$\int_X f d\mu := \lim_{n\to\infty} \int_X f_n d\mu,$$

where (f_n) is a sequence of simple functions such that (1.1.1) holds. This limit exists and is independent of the approximating sequence (f_n). It further can be shown that a measurable function $f : X \to E$ is Bochner-integrable if and only if

$$\int_X \|f\| \, d\mu < \infty.$$

It is also a simple property of the Bochner integral that

$$\left\| \int_X f d\mu \right\| \leq \int_X \|f\| \, d\mu.$$

Let $T : E \to F$ be a continuous linear operator into another Banach space F. Then it holds that if both f and Tf are Bochner-integrable, we have

$$T\left(\int_A f d\mu \right) = \int_A T f d\mu \quad \text{for all } A \in \mathcal{F}.$$

1.1.2 The Lebesgue-Bochner Spaces and their Duals

Definition 1.1.3. (Lebesgue-Bochner spaces). Let $1 \leq p < \infty$. Then $L_E^p(X)$ stands for the space of all (equivalence classes of) E-valued Bochner-integrable functions f defined on X with $\int_X \|f\|^p \, d\mu < \infty$. The norm $\|\cdot\|_{L_E^p(X)}$ is defined as

$$\|f\|_{L_E^p(X)} := \left(\int_X \|f\|^p \, d\mu \right)^{1/p}, \quad f \in L_E^p(X).$$

$L_E^p(X)$ is a Banach space under $\|\cdot\|_{L_E^p(X)}$ and simple functions are dense in $L_E^p(X)$. For the missing exponent $p = \infty$, $L_E^\infty(X)$ may be defined in an analogous fashion.

In order to identify the duals of L_E^p (at least for some Banach spaces E), we have to give a few definitions first.

Definition 1.1.4. An E-valued σ-additive set function $G : \mathcal{F} \to E$ is said to be of *bounded variation*, if the total variation

$$V_G(X) := \sup \left\{ \sum_{k=1}^m \|G(A_k)\| : A_k \in \mathcal{F}, (A_k) \text{ disjoint} \right\}$$

is finite.

Definition 1.1.5. A Banach space E has the *Radon-Nikodým property (RNP) with respect to* $(X, \mathcal{F}, \mu)$, if for each E-valued σ-additive set function G of bounded variation that is absolutely continuous with respect to μ (that is: $\mu(A) = 0 \Rightarrow G(A) = 0$) has an integral representation, i.e. there exists $f \in L_E^1(X)$ such that

$$G(A) = \int_A f d\mu \quad \text{for all } A \in \mathcal{F}.$$

Now, the following theorem concerning the dual of Lebesgue-Bochner spaces holds

Theorem 1.1.6. *For $1 \le p < \infty$, a probability space $(X, \mathcal{F}, \mu)$ and a Banach space E we have that $(L^p_E(X))' = L^{p'}_{E'}(X)$ for the conjugate exponent $p' = p/(p-1)$ if and only if the dual E' of E satisfies RNP with respect to $(X, \mathcal{F}, \mu)$.*

Remark.

i. The dual pairing of $L^p_E(X)$ and $L^{p'}_{E'}(X)$ is given as

$$\int_X \langle f(x), g(x)\rangle \, d\mu(x), \quad f \in L^p_E(X), g \in L^{p'}_{E'}(X), \tag{1.1.2}$$

where $\langle \cdot, \cdot \rangle$ denotes the dual pairing of E and E'. In the future we will also denote the expression in (1.1.2) by $\langle f, g\rangle$.

ii. The set of Banach spaces that possess RNP includes in particular reflexive Banach spaces and separable duals of Banach spaces. For a more complete listing of spaces satisfying RNP, see [DU77].

1.2 Kahane's Contraction Principle

Theorem 1.2.1 (Kahane, contraction principle)**.** *Let $e_1, \dots, e_m$ be elements in a Banach space E and $r_1, \dots, r_m$ be independent Rademacher functions (for instance the standard choice $r_j(t) = \operatorname{sgn}\sin(2^j \pi t)$). If $a_1, \dots, a_m$ are real numbers with $\sup_{k \le m} |a_k| \le 1$, we have for any $1 \le p < \infty$*

$$\int_0^1 \left\| \sum_{k=1}^m a_k r_k(t) e_k \right\|_E^p dt \le \int_0^1 \left\| \sum_{k=1}^m r_k(t) e_k \right\|_E^p dt.$$

Proof. (Taken from [MP81].) To prove this inequality we first remark that for any function $g : \{-1, 1\}^m \to \mathbb{R}$ we have

$$\int_0^1 g(r_1(t), \dots, r_m(t)) dt = 2^{-m} \sum_{\varepsilon \in \{-1,1\}^m} g(\varepsilon_1, \dots, \varepsilon_m) \tag{1.2.1}$$

by definition of the Rademacher functions and the transformation formula for integrals. If we set $f(a) := \left(\int_0^1 \|\sum_{k=1}^m a_k r_k(t) e_k\|_E^p \, dt \right)^{1/p}$ for $a = (a_1, \dots, a_m)$ we see by the triangle inequality and the homogeneity of norms that f is convex. Since also, the cube $[-1, 1]^m$ in $\mathbb{R}^m$ is convex we see that $a = \sum_j b_j e_j$ with some positive real numbers b_j summing to 1 and extremal points e_j of the cube $[-1, 1]^m$. Applying this representation of a to the convexity of f we obtain

$$f(a) \le \sum_j b_j f(e_j).$$

It is now best seen from (1.2.1) that $f(e_j)$ has the same value for all extremal points e_j and thus we get the assertion $f(a) \le f(1, \dots, 1)$. □

Remark. Let (X, μ) be a probability space and E a Banach space. We will often make use of the contraction principle in the following form: We let $d_k, f_k : X \to \mathbb{R}$ be functions with $|d_k| \leq |f_k|$ pointwise for all k and we want to estimate

$$I := \int_0^1 \left\| \sum_{k=1}^m e_k r_k(t) d_k \right\|_{L^p_E(X)}^p dt.$$

A simple use of Fubini's theorem allows us to write

$$I = \int_X \int_0^1 \left\| \sum_{k=1}^m e_k r_k(t) d_k(x) \right\|_E^p dt d\mu(x)$$

and now we can make use of the contraction principle (applied to $z_k := f_k(x)e_k$) to get

$$I \leq \int_X \int_0^1 \left\| \sum_{k=1}^m e_k r_k(t) f_k(x) \right\|_E^p dt d\mu(x) = \int_0^1 \left\| \sum_{k=1}^m e_k r_k(t) f_k \right\|_{L^p_E(X)}^p dt.$$

1.3 Martingale Preliminaries

1.3.1 Conditional Expectation

For the following, let $(X, \mathcal{F}, \mu)$ a probability space and E a Banach space.

Definition 1.3.1. Let $\mathcal{G} \subset \mathcal{F}$ be a sub-σ-algebra of $\mathcal{F}$ and $f \in L^1_E(X)$. An element $g \in L^1_E(X)$ is called the *conditional expectation* of f relative to $\mathcal{G}$ if g is $\mathcal{G}$-measurable and

$$\int_E g d\mu = \int_E f d\mu \quad \text{for all } E \in \mathcal{G}.$$

If a conditional expectation exists, it is defined uniquely up to equality almost everywhere. In the scalar valued case ($E = \mathbb{R}$), existence follows from the Radon-Nikodým theorem and Jensen's inequality yields that the operator $f \mapsto \mathbb{E}(f|\mathcal{G})$ is a contraction on $L^p(X)$ for $1 \leq p < \infty$. For vector valued functions f (E is an arbitrary Banach space) one defines the conditional expectation canonically for simple functions $f = \sum_{j=1}^n e_j 1_{A_j}$ with $e_j \in E$ and $A_j \in \mathcal{F}$ as

$$\mathbb{E}(f|\mathcal{G}) = \sum_{j=1}^n e_j \mathbb{E}(1_{A_j}|\mathcal{G}),$$

where on the right hand side we have a scalar valued conditional expectation. This operator is again a contraction on $L^p_E(X)$ for $1 \leq p < \infty$ and, since simple functions are dense in $L^p_E(X)$ for $1 \leq p < \infty$, it can be extended uniquely to whole $L^p_E(X)$. For details concerning the construction of the conditional expectation and more facts, we refer to [DU77].

1.3.2 Definition of Martingales

Definition 1.3.2. (Martingale). Let $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots$ be an increasing sequence of $\sigma-$algebras, such that $\mathcal{F}_j \subset \mathcal{F}$ for all indices j. A sequence of functions $(f_j)_{j=1}^{\infty}$ in $L_E^p(X)$ $(1 \leq p < \infty)$ is called a *martingale*, if we have

$$\mathbb{E}(f_{j+l}|\mathcal{F}_j) = f_j \quad \text{for all } l \geq 0 \text{ and } j \geq 1.$$

Definition 1.3.3. (Martingale difference sequence). A sequence of integrable, E-valued random variables $(d_k)_{k\in\mathbb{N}}$ is called a *martingale difference sequence*, if $\mathbb{E}d_1 = 0$ and $\mathbb{E}(d_{k+1}|\sigma(d_1, \dots, d_k)) = 0$ for all $k \in \mathbb{N}$.

If (d_k) is a martingale difference sequence, then the sequence of partial sums $f_n = \sum_{k=1}^{n} d_k$ is a martingale with respect to the sequence of σ-algebras generated by (d_k).

1.3.3 UMD spaces

Remark.(Convergence of Series in Banach spaces). Let $(e_k)_k \in \mathbb{N}$ be a sequence in a Banach space E. The series $\sum_{k=1}^{\infty} e_k$ is said to be *convergent* and equals $e \in E$, if the sequence of partial sums $(\sum_{k=1}^{n} e_k)_{n\in\mathbb{N}}$ converges to e in E. A series $\sum_k e_k$ is *unconditionally convergent*, if $\sum_{k=1}^{\infty} e_{\sigma(k)}$ converges for every permutation σ of $\mathbb{N}$. Well known characterizations of unconditional convergence include (as may be found for instance in [LT77]) A series $\sum_k e_k$ is unconditionally convergent if and only if one (and thus all) of the following statements hold true:

i. The series $\sum_{j=1}^{\infty} e_{k_j}$ converges for every choice of $k_1 < k_2 < k_3 < \cdots$.

ii. The series $\sum_{k=1}^{\infty} \beta_k e_k$ converges for every sequence $(\beta_k)_{k\in\mathbb{N}}$ such that $|\beta_k| \leq 1$ for all $k \in \mathbb{N}$.

iii. The series $\sum_{k=1}^{\infty} \varepsilon_k e_k$ converges for all choices of signs $\varepsilon_k \in \{\pm 1\}$.

Definition 1.3.4. A Banach space E is called a *UMD–space (unconditional for martingale differences)*, if for every $1 < p < \infty$ there exists a constant β_p such that for every E-valued martingale difference sequence $(d_k)_{k\geq 0}$ we have the inequality

$$\left\| \sum_{k=0}^{n} \varepsilon_k d_k \right\|_{L_E^p} \leq \beta_p \left\| \sum_{k=0}^{n} d_k \right\|_{L_E^p} \tag{1.3.1}$$

for all sequences ε of numbers in $\{-1, 1\}$ and all $n \in \mathbb{N}$.

Remark.
1. We remark that if there exists one $1 < p < \infty$ with a constant β_p such that (1.3.1) holds, we have automatically that for all $1 < p < \infty$ there exists a constant β_p with (1.3.1).
2. Hilbert spaces are UMD–spaces, UMD–spaces are reflexive and the UMD–property is a self dual isomorphic invariant (see for instance [Bur01]). The next result is in its vector valued form due to J. Bourgain [Bou86].

Theorem 1.3.5 (Stein, Bourgain)**.** *Let E be a UMD-space and $(X,\mathcal{F},\mu)$ be a probability space. If $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \cdots \subseteq \mathcal{F}_m \subseteq \mathcal{F}$ are σ-algebras in X, we have for any choice of $f_1,\ldots,f_m \in L^p_E(X)$ the following estimate*

$$\int_0^1 \left\| \sum_{k=1}^m r_k(t)\mathbb{E}(f_k|\mathcal{F}_k) \right\|^p_{L^p_E(X)} dt \le \beta_p^p \int_0^1 \left\| \sum_{k=1}^m r_k(t) f_k \right\|^p_{L^p_E(X)} dt,$$

where β_p is the same constant as in Definition 1.3.4 and $r_1,\ldots,r_m$ are independent Rademacher functions.

Proof. (see [Mül05] or [FW01]) By (1.2.1), it is equivalent to show that

$$\operatorname*{Ave}_{\varepsilon} \int_X \left\| \sum_{n=1}^m \varepsilon_n \mathbb{E}(f_n|\mathcal{F}_n)(x) \right\|^p_E d\mu(x) \le \operatorname*{Ave}_{\varepsilon} \int_X \left\| \sum_{n=1}^m \varepsilon_n f_n(x) \right\|^p_E d\mu(x).$$

where by $\operatorname{Ave}_\varepsilon v(\varepsilon)$ we mean the average of v for all signs ε:

$$2^{-m} \sum_{\varepsilon\in\{-1,1\}^m} v(\varepsilon).$$

Let $u : E \to \mathbb{R}_0^+$ be the (convex) function that maps $x \in E$ to $\|x\|^p_E$ and for $\varepsilon \in \{-1,1\}^m$ and $x \in X$ define

$$F(\varepsilon,x) = \sum_{n=1}^m \varepsilon_n f_n(x).$$

On the product space $\{-1,1\}^m \times X$ define the increasing σ-algebras

$$\mathcal{G}_n = \sigma(\mathrm{pr}_1,\ldots,\mathrm{pr}_n) \otimes \mathcal{F}_n \quad \text{for } n \le m,$$

where pr_j denotes the projection onto the j-th coordinate. We then get

$$F_n(\varepsilon,x) := \mathbb{E}(F|\mathcal{G}_n)(\varepsilon,x) = \sum_{i\le n} \varepsilon_i \mathbb{E}(f_i|\mathcal{F}_n), \quad \text{for } 1 \le n \le m. \tag{1.3.2}$$

Furthermore, set $F_0 = 0$. Now, define the martingale difference sequence $d_n(f_i)$ for f_i with respect to the sequence of σ-algebras $(\mathcal{F}_j)$ as $d_n(f_i) = \mathbb{E}(f_i|\mathcal{F}_n) - \mathbb{E}(f_i|\mathcal{F}_{n-1})$ for $2 \le n \le m$. By definition of F_n and $d_n(f_i)$ and by (1.3.2) it holds that

$$F_n - F_{n-1} = \varepsilon_n \mathbb{E}(f_n|\mathcal{F}_n) + \sum_{i=1}^{n-1} \varepsilon_i d_n(f_i) \quad \text{for } 1 \le n \le m. \tag{1.3.3}$$

Since $\operatorname{Ave}_\delta \delta_i\delta_n = 0$ for $i \ne n$, we artificially set for $\varepsilon \in \{-1,1\}^m$

$$\sum_{n=1}^m \varepsilon_n \mathbb{E}(f_n|\mathcal{F}_n) = \operatorname*{Ave}_{\delta} \sum_{n=1}^m \left[\varepsilon_n \mathbb{E}(f_n|\mathcal{F}_n) + \sum_{i=1}^{n-1} \delta_i\delta_n\varepsilon_i d_n(f_i) \right],$$

so that by convexity of $\|\cdot\|^p$ we get

$$\left\|\sum_{n=1}^{m} \varepsilon_n \mathbb{E}(f_n|\mathcal{F}_n)\right\|_E^p \leq \operatorname*{Ave}_{\delta} \left\|\sum_{n=1}^{m} \left[\varepsilon_n \mathbb{E}(f_n|\mathcal{F}_n) + \sum_{i=1}^{n-1} \delta_i \delta_n \varepsilon_i d_n(f_i)\right]\right\|_E^p.$$

For fixed $\delta \in \{-1,1\}^m$, averaging over ε or over $(\varepsilon_1\delta_1, \dots, \varepsilon_m\delta_m)$ yields the same result, so if we combine the last display and (1.3.3) we find

$$\operatorname*{Ave}_{\varepsilon} \int_X \left\|\sum_{n=1}^{m} \varepsilon_n \mathbb{E}(f_n|\mathcal{F}_n)(x)\right\|_E^p d\mu(x) \leq \operatorname*{Ave}_{\varepsilon,\delta} \int_X \left\|\sum_{n=1}^{m} \delta_n (F_n(\varepsilon, x) - F_{n-1}(\varepsilon, x))\right\|_E^p d\mu(x).$$

Now, the assertion of the theorem follows from the UMD-property of E applied to the martingale difference sequence $(F_n - F_{n-1})$ and the fact that the conditional expectation operator is a contraction on $L_E^p(X)$. □

1.3.4 The Martingale Convergence Theorem

We begin with the following proposition, taken from [Cha68], which gives information about the convergence of the sequence of conditional expectations of an L_E^p-function.

Proposition 1.3.6. *Let $(X, \mathcal{F}, \mu)$ be a probability space, E a Banach space and $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots$ a strictly increasing sequence of σ-algebras such that $\mathcal{F}$ is generated by the union $\cup_k \mathcal{F}_k$. Further, let $1 \leq p < \infty$ and $f \in L_E^p(X)$. Then we have that*

$$\lim_{k\to\infty} \|\mathbb{E}_k f - f\|_{L_E^p} = 0.$$

Proof. [Cha68] We first assume that f is measurable with respect to $\cup_k \mathcal{F}_k$, which means that $\mathcal{G} := f^{-1}(\mathcal{B}_E) \subset \cup_k \mathcal{F}_k$, where $\mathcal{B}_E$ is the σ-algebra of Borel sets in E. Then there exists an index k_0 such that $\mathcal{G} \subset \mathcal{F}_{k_0}$, since if we assume the contrary, we obtain a subsequence k_j such that

$$\mathcal{G} \cap \mathcal{F}_{k_{j+1}} \supset \mathcal{G} \cap \mathcal{F}_{k_j} \quad \text{with strict inclusion.}$$

Since out of this, we obtain that $\mathcal{G} = \cup_j (\mathcal{G} \cap \mathcal{F}_{k_j})$ is not a σ-algebra [BH77], this is a contradiction since by definition $\mathcal{G}$ is a σ-algebra.

So we conclude for $j \geq k_0$ that $\mathbb{E}_j f = f$ a.s. and thus the conclusion of the proposition for f measurable with respect to $\cup_k \mathcal{F}_k$. If now f is $\mathcal{F}$-measurable, we approximate f by simple functions (that are dense in L_E^p) and then approximate every characteristic function 1_A of a set $A \in \mathcal{F}$ by a characteristic function 1_B of a set $B \in \cup_k \mathcal{F}_k$ such that $\mu(A \Delta B)$ is small. Then we obtain with the first step of the proof the conclusion of the proposition. □

An important result about vector valued martingales is the following theorem, which is one part of the result stated in [Cha68].

Theorem 1.3.7 (Martingale Convergence Theorem)**.** *Let $(X, \mathcal{F}, \mu)$ be a probability space, E a Banach space with the Radon-Nikodym-property with respect to $(X, \mathcal{F}, \mu)$, $\{f_k\}_{k\in\mathbb{N}}$ an E-valued martingale with respect to the filtration $(\mathcal{F}_k)_{k\in\mathbb{N}}$ and $1 \leq p < \infty$. If $\sup_{k\in\mathbb{N}} \|f_k\|_{L_E^p(X)} < \infty$ then there exists $f \in L_E^p(X)$ such that $f_k \to f$ in L_E^p as k tends to ∞.*

As noted in the remark after Theorem 1.1.6, reflexive Banach spaces – and hence UMD-spaces – possess the Radon-Nikodym-property.

1.3.5 Martingale Multipliers

Let $(d_k)_{k\in\mathbb{N}}$ be an $\mathbb{R}$–valued martingale difference sequence in a probability space $(X,\mathcal{F},\mu)$ and E a UMD–space. Kahane's contraction principle and the UMD-property of E show that bounded sequences of scalars acting by multiplication on UMD–valued martingale differences yield L^p–bounded operators. We assume that elements of the form

$$\sum_{k=1}^{n} a_k d_k \quad \text{with } a_k \in E$$

are dense in $L^p_E(X)$ for $1 \leq p < \infty$. Since $(d_k)_{k\in\mathbb{N}}$ are martingale differences, the coefficients in the expansion of $f \in L^p_E(X)$ in its series

$$f = \sum_{k=1}^{\infty} a_k d_k$$

are unique and this series converges unconditionally in $L^p_E(X)$ since E is UMD. Now, we let $f \in L^p(X)$ be a finite linear combination of martingale differences

$$f = \sum_{k=1}^{n} a_k d_k.$$

Then we define the multiplication operator $\mathcal{M}$

$$\mathcal{M}f = \sum_{k=1}^{n} m_k a_k d_k$$

algebraically, where $(m_k)_{k\in\mathbb{N}}$ is a bounded sequence in $\mathbb{R}$. For $\mathcal{M}$, the following theorem holds.

Theorem 1.3.8. *If E is a UMD–space and $1 < p < \infty$, then $\mathcal{M}$ extends to a bounded linear operator on $L^p_E(X)$, so we have the existence of an $M_p > 0$ s.t. for all $f \in L^p_E(X)$:*

$$\|\mathcal{M}f\|_{L^p_E(X)} \leq M_p \sup_{k\in\mathbb{N}}\{|m_k|\} \, \|f\|_{L^p_E(X)} .$$

Proof. The well known proof uses just the UMD-property of E and Kahane's contraction principle. To wit, take $f = \sum_{k=0}^{n} a_k d_k$ and estimate

$$\|\mathcal{M}f\|^p_{L^p_E(X)} = \left\|\sum_{k=1}^{n} m_k a_k d_k\right\|^p_{L^p_E(X)} \leq \beta_p^p \int_0^1 \left\|\sum_{k=1}^{n} r_k(t) m_k a_k d_k\right\|^p_{L^p_E(X)} dt$$

by the UMD-property (1.3.1). Now, Kahane's contraction principle (see the remark after Theorem 1.2.1) implies further that

$$\|\mathcal{M}f\|^p_{L^p_E(X)} \leq \sup_k |m_k|^p \beta_p^p \int_0^1 \left\|\sum_{k=1}^{n} r_k(t) a_k d_k\right\|^p_{L^p_E(X)} dt.$$

Now, use the UMD-property (1.3.1) again to obtain the conclusion

$$\|\mathcal{M}f\|_{L^p_E(X)} \leq \sup_k |m_k| \beta_p^2 \, \|f\|_{L^p_E(X)} \, . \qquad \square$$

1.3.6 Martingale H^1 and BMO

We let $(X, \mathcal{F}, \mu)$ be a probability space and $\{\mathcal{F}_k\}_{k \in \mathbb{N}_0}$ be a regular (in the sense of [Gar73]) sequence of increasing σ-algebras. That is we assume that there is a constant $c > 0$ such that for any two atoms $A \in \mathcal{F}_{k-1}$ and $B \in \mathcal{F}_k$ with $A \supseteq B$ we have

$$\frac{\mu(A)}{\mu(B)} \leq c.$$

We further assume that $\mathcal{F}$ is generated by the union $\cup_k \mathcal{F}_k$. For $f \in L^1_E(X)$ we introduce the abbreviations

$$\mathbb{E}_k f := \mathbb{E}(f|\mathcal{F}_k) \quad \text{and} \quad \Delta_k := \mathbb{E}_k - \mathbb{E}_{k-1}.$$

Definition 1.3.9. (Hardy space). Let $(X, \mathcal{F}, \mu)$ be a probability space and E a Banach space. Then we define the Hardy space $H^1_E(X, (\mathcal{F}_k))$ as a subset of $L^1_E(X)$ and we say that $h \in H^1_E(X, (\mathcal{F}_k))$ if the martingale maximal function

$$f^* := \sup_k \|\mathbb{E}_k f\|_E \tag{1.3.4}$$

is integrable, i.e.

$$\int_X f^*(x) \; d\mu(x) < \infty$$

and we set

$$\|f\|_{H^1_E(X,(\mathcal{F}_k))} := \|f^*\|_{L^1(X)} \, .$$

Every function $f \in H^1_E(X, (\mathcal{F}_k))$ with average zero may be decomposed in a sum of basic functions, called atoms, where a function $a : X \to E$ is an *atom*, provided it satisfies the following conditions:

1. $\operatorname{supp} a \subseteq Q \quad$ for some atom in a σ-algebra $\mathcal{F}_k$
2. For the same Q we have: $\|a(x)\|_{L^\infty_E(X)} \leq \mu(Q)^{-1}$.
3. $\int_X a(x) \; d\mu(x) = 0$.

The atomic decomposition of a function $f \in H^1_E(X, (\mathcal{F}_k))$ takes the following form:

Proposition 1.3.10. *Let $f \in H^1_E(X, (\mathcal{F}_k))$ with $\int_X f \; d\mu = 0$. Then there exists a sequence of atoms (a_j) and a sequence of scalars λ_j with $\sum |\lambda_j| < \infty$ such that*

$$f = \sum_{j=1}^{\infty} \lambda_j a_j,$$

where convergence is meant in the sense of $L^1_E(X)$.

Definition 1.3.11. (Bounded Mean Oscillation). A function $f : X \to E$ is said to be in $\mathrm{BMO}_E(X, (\mathcal{F}_k))$ if and only if f is in $L^2_E(X)$ and

$$\|f\|_{\mathrm{BMO}_E} := \sup_{k\in\mathbb{N}} \left\|\mathbb{E}_k(\|f - \mathbb{E}_{k-1}f\|^2_E)\right\|^{1/2}_\infty < \infty. \tag{1.3.5}$$

This is a norm, if we factor out the constants.

Remark. No matter what exponent $1 \le p < \infty$ in (1.3.5) is chosen instead of 2, the definition leads to the same space $\mathrm{BMO}(X, (\mathcal{F}_k))$ with equivalent norms (cf. [JN61, Gar73, CW77, Bou86]).

Since we are working on a regular sequence of σ-algebras, the expression

$$\sup_k \left\|\mathbb{E}_k(\|f - \mathbb{E}_k f\|^2_E)\right\|^{1/2}_\infty$$

is an equivalent norm to $\|\cdot\|_{BMO_E}$ and this is the same as

$$\sup_k \sup_{A \text{ atom in } \mathcal{F}_k} \left(\frac{1}{\mu(A)}\int_A \|f - \mathbb{E}_k f\|^2_E \, d\mu\right)^{1/2}.$$

1.3.7 Paraproducts

Let a be a scalar valued function that is $\mathcal{F}_n$-measurable for some n and $f : X \to E$ an integrable function. Then we define a paraproduct operator

$$P(a, f) := \sum_k (\mathbb{E}_k f)(\Delta_{k+1}a),$$

What we want to show now is the boundedness on $L^p_E(X)$ of the operator $P(a, \cdot)$, provided $a \in BMO(X, (\mathcal{F}_k))$. This can be accomplished for UMD-spaces E by an argument in Figiel, Wojtaszczyk [FW01] (see also [Fig89]) which they attributed to J. Bourgain and uses Bourgain's version of Stein's inequality, Theorem 1.3.5. We will elaborate this argument in the sequel and start with the following

Lemma 1.3.12. *If $f : X \to E$ is a bounded measurable function, a a scalar valued function (with mean zero) on X that is $\mathcal{F}_n$-measurable for some n. Then, for $1 < p < \infty$, we have the estimate*

$$\|P(a, f)\|_p \le C_p U \left(\|f\|_\infty \|a\|_p + \|f\|_p \sup_k \|\Delta_{k+1}a\|_\infty\right),$$

where C_p depends only on p and U depends only on (the UMD*-constant of) E.*

The proof will make use of Bourgain's version of Stein's inequality and Kahane's contraction principle

Proof. Note that the argument in the defining sum of $P(a, f)$ is a martingale difference sequence with respect to the $\sigma-$algebras $\mathcal{F}_k$, since

$$\mathbb{E}_k[(\mathbb{E}_k f)(\Delta_{k+1}a)] = (\mathbb{E}_k f)\mathbb{E}_k[\mathbb{E}_{k+1}a - \mathbb{E}_k a] = 0.$$

The UMD property of E now yields

$$\|P(a,f)\|_p^p \leq \beta_p^p \left\| \sum_k \varepsilon_k (\mathbb{E}_k f)(\Delta_{k+1} a) \right\|_p^p, \tag{1.3.6}$$

for any vector ε of signs $\varepsilon_k \in \{-1,1\}$. If we take the average over all signs ε and insert the identity $\mathbb{E}_k = \mathbb{E}_{k+1} - \Delta_{k+1}$, we obtain

$$\begin{aligned} \|P(a,f)\|_p^p &\leq 2^p \beta_p^p \left[\int_0^1 \left\| \sum_k r_k(t) (\mathbb{E}_{k+1} f)\,(\Delta_{k+1} a) \right\|_p^p dt \right. \\ &\qquad \left. + \int_0^1 \left\| \sum_k r_k(t)\,(\Delta_{k+1} f)\,(\Delta_{k+1} a) \right\|_p^p dt \right] \\ &=: 2^p \beta_p^p (I_1 + I_2), \end{aligned}$$

where (r_k) are independent Rademacher functions. We estimate I_1 and I_2 separately and begin with I_1. Using Stein's martingale inequality (Theorem 1.3.5) and the unconditionality of martingale differences, we get

$$\begin{aligned} I_1 &\leq \beta_p^p \int_0^1 \left\| \sum_k r_k(t) f\,(\Delta_{k+1} a) \right\|_p^p dt \leq C \|f\|_\infty^p \int_0^1 \left\| \sum_k r_k(t)\,(\Delta_{k+1} a) \right\|_p^p dt \\ &\leq \beta_p^{2p} \|f\|_\infty^p \|a\|_p^p, \end{aligned}$$

For I_2 we get, applying Kahane's contraction principle to $\Delta_{k+1} a$ and using again the unconditionality of martingale differences

$$I_2 \leq \sup_k \|\Delta_{k+1} a\|_\infty^p \int_0^1 \left\| \sum_k r_k(t) \Delta_{k+1} f \right\|_p^p dt \leq \sup_k \|\Delta_{k+1} a\|_\infty^p \|f\|_p^p .$$

Hence we get the assertion of the Lemma. □

Corollary 1.3.13. *Let $f : X \to E$ is a bounded measurable function, a a scalar valued function on X that is $\mathcal{F}_n$-measurable for some n. Then we have the estimate*

$$\|P(a,f)\|_{BMO_E(X,(\mathcal{F}_k))} \leq C \|a\|_{BMO(X,(\mathcal{F}_k))} \|f\|_\infty, \tag{1.3.7}$$

for some constant C that depends only on p and the UMD*-constant of E.*

Proof. Let $Q \in \mathcal{F}_k$. First we point out that

$$\Delta_{j+1}(1_Q(a - \mathbb{E}_k a)) = \begin{cases} 0, & \text{if } j \leq k-1, \\ 1_Q(\Delta_{j+1} a), & \text{if } j \geq k. \end{cases} \tag{1.3.8}$$

In order to estimate the BMO_E-norm of $P(a,f)$, we have to look at the expression $1_Q(P(a,f) - \mathbb{E}_k P(a,f))$, which becomes

$$\begin{aligned} 1_Q(P(a,f) - \mathbb{E}_k P(a,f)) &= 1_Q \left(\sum_{j \geq k} (\mathbb{E}_j f)(\Delta_{j+1} a) \right) &\quad (1.3.9) \\ &= P(1_Q(a - \mathbb{E}_k a), 1_Q f), &\quad (1.3.10) \end{aligned}$$

if we use the definition of P and (1.3.8). So we use Lemma 1.3.12 and estimate (again using (1.3.8))

$$\begin{aligned} \|1_Q(P(a,f) - \mathbb{E}_k P(a,f))\|_2 &\leq C(\|1_Q f\|_\infty \|1_Q(a - \mathbb{E}_k a)\|_2 \\ &\quad + \|1_Q f\|_2 \sup_j \|\Delta_{j+1} a\|_\infty). \end{aligned}$$

Since $|\Delta_{j+1}a| = |\mathbb{E}_{j+1}(a - \mathbb{E}_j a)| \leq \mathbb{E}_{j+1}|a - \mathbb{E}_j a| \leq \|a\|_{\mathrm{BMO}}$, we get further, that the right hand side of the last display is less or equal

$$C\mu(Q)^{1/2} \|f\|_\infty \|a\|_{\mathrm{BMO}} .$$

So by definition of the BMO_E-norm, we get the assertion of the corollary. □

Corollary 1.3.14. *Let $f \in L^\infty_E(X)$ be an atom (with respect to some $Q \in \mathcal{F}_k$) and a a scalar valued function that is $\mathcal{F}_n$-measurable for some n. Then we have the estimate*

$$\|P(a,f)\|_{L^1_E} \leq C \|a\|_{BMO(X,(\mathcal{F}_k))} ,$$

where C is a constant that depends only on p and the UMD*-constant of E.*

Proof. It holds also that $\operatorname{supp} P(a,f) \subset Q$, so in particular $\mathbb{E}_k P(a,f) = 0$ and thus we obtain by the properties of an atom f

$$\begin{aligned} \|P(a,f)\|_{L^1_E} &= \|1_Q P(a,f)\|_{L^1_E} \leq \mu(Q)^{1/2} \|1_Q P(a,f)\|_{L^2_E} \\ &= \mu(Q)^{1/2} \|1_Q(P(a,f) - \mathbb{E}_k P(a,f))\|_{L^2_E} . \end{aligned}$$

Now we use Corollary 1.3.13 to estimate this by

$$C\mu(Q) \|a\|_{\mathrm{BMO}} \|f\|_\infty \leq C \|a\|_{\mathrm{BMO}} .$$

Thus we get the assertion of the corollary. □

Now we use Corollaries 1.3.13 and 1.3.14 to conclude that $P(a,\cdot)$ is a bounded operator from $L^\infty_E(X)$ to $BMO_E(X,(\mathcal{F}_k))$ and from $H^1_E(X,(\mathcal{F}_k))$ to $L^1_E(X)$. Thus we are able to to use (real) interpolation (see for instance [BX91])

$$(L^\infty_E(X), H^1_E(X,(\mathcal{F}_k)))_{1-1/p;p} = (BMO_E(X,(\mathcal{F}_k)), L^1_E(X))_{1-1/p;p} = L^p_E(X)$$

to deduce the boundedness of $P(a,\cdot)$ from $L^p_E(X)$ to $L^p_E(X)$ for $1 < p < \infty$. Thus we get that

$$\|P(a,f)\|_{L^p_E(X)} \leq C \|a\|_{BMO(X,(\mathcal{F}_k))} \|f\|_{L^p_E(X)} \tag{1.3.11}$$

for $f \in L^p_E(X)$ and a $\mathcal{F}_n$-measurable for some n. The martingale convergence theorem 1.3.7 now yields that we may define $P(a,f)$ for every $a \in \mathrm{BMO}$ satisfying the same inequality (1.3.11)

Chapter 2

A Decomposition Theorem for Singular Integral Operators on Spaces of Homogeneous Type

Let (X, d, μ) be a space of homogeneous type and E a UMD Banach space. Under the assumption $\mu(\{x\}) = 0$ for all $x \in X$, we prove a decomposition theorem for singular integral operators on (X, d, μ) as a series of simple shifts and rearrangements plus two paraproducts. This gives a $T(1)$ Theorem in this setting.

2.1 Introduction

The $T(1)$-Theorem for scalar valued singular integral operators on $\mathbb{R}^n$ was initially proved by David and Journé ([DJ84]) using Fourier analysis methods. It was later extended to spaces of homogeneous type by Coifman (unpublished, see [Chr90a] and [CJS89]). The structural framework for both proofs is given by Cotlar-Stein theorem on almost orthogonal operators. Consequently, different methods had to be developed to obtain a $T(1)$ theorem for integral operators taking values in general Banach spaces. This was done by T. Figiel ([Fig88] and [Fig90]) who introduced a general method of decomposing integral operators into series of basic building blocks. This decomposition arises canonically by expanding the integral kernel along the isotropic Haar system in $\mathbb{R}^n \times \mathbb{R}^n$. Thus proving boundedness of integral operators is reduced to the following problems:

- Verify a priori norm estimates for the building blocks (this is independent of the underlying integral kernel).
- Verify compensating coefficient estimates arising in the isotropic series expansion of the kernel (the decay of the coefficients depends on the size and smoothness of the kernel under investigation).

The basic building blocks isolated by Figiel are simple rearrangements and shifts plus two paraproducts. These rearrangements and shifts act on the Haar system in $\mathbb{R}$. It is important to note that their definition depends expressly on the group structure of the underlying domain $(\mathbb{R}^n, +)$. Figiel's decomposition was applied later to several singular integral operators beyond the Calderón-Zygmund class. These included applications to Dirichlet kernels of generalized Franklin systems ([KM06]) and interpolatory estimates arising in the theory of compensated compactness ([LMM11]).

In the present paper we extend Figiel's decomposition method to the setting of spaces of homogeneous type. Our extension of this method is based on constructing – without recourse to group structure – a suitable class of rearrangement and shift operators that allow us to decompose singular integral operators on (X, d, μ) into a series of basic building blocks that can be analyzed and estimated by combinatorial means. The central result of this chapter is the convergence of this operator-series (2.3.7).

A source of renewed interest in spaces of homogeneous type is the recent development of diffusion wavelets and their multiresolution analysis that was carried out on spaces of homogeneous type by Coifman and Maggioni ([CM06]). We recall further that the vector-valued $T(1)$ theorem on spaces of homogeneous type is an essential first step towards the solution of the open classification problem for the vector valued Banach spaces $H^1_E(X, d, \mu)$. See [Mül95] and [MS91].

2.2 Extracting Rearrangements on Spaces of Homogeneous Type

This section contains an extensive combinatorial analysis of dyadic cubes in spaces of homogeneous type. We recall first basic properties of those cubes and of the martingale differences they generate. Thus we construct orthonormal bases in $L^2(X)$ and $L^2(X \times X)$. Next we introduce a coloring on the collection of all dyadic cubes, so that on each monochromatic subcollection there are well defined rearrangement operators that act like "shifts by q^m units" (Proposition 2.2.11). The complications in the proof of this proposition are due to the fact that we need to have good quantitative control on the numbers of colors involved. This in turn is dictated by the nature of the kernel operators we treat in Section 2.3. Theorem 2.2.17 is the second main result of this section. It provides the combinatorial basis for the norm estimates of the rearrangement operators defined in Section 2.3.3.

2.2.1 Definitions

Definition 2.2.1. Let X be a set. A mapping $d : X \times X \to \mathbb{R}_0^+$ with the properties

1. $d(x, y) = 0 \Leftrightarrow x = y$,
2. $d(x, y) = d(y, x)$,
3. $d(x, y) \leq K(d(x, z) + d(z, y))$ for all $x, y, z \in X$ and some constant $K \geq 1$ that is independent of x, y, z.

is called a *quasimetric* and (X, d) is called a *quasimetric space.*

Given a quasimetric d, we define the ball centered at $x \in X$ with radius $r > 0$ as

$$B(x, r) := \{y \in X : d(x, y) < r\}.$$

Additionally, a set $A \subset X$ is called *open* if and only if for all $x \in A$ there exists $r > 0$ such that $B(x, r) \subseteq A$.

Definition 2.2.2. Let (X, d) be a quasimetric space such that every ball in the quasimetric d is open and μ a Borel measure. If there is an $A > 0$ such that

$$0 < \mu(B(x, 2r)) \leq A\mu(B(x, r)) < \infty, \quad \text{for all } x \in X \text{ and all } r > 0,$$

then (X, d, μ) is called a *space of homogeneous type.* Additionally, if there exist constants b_1, b_2 such that

$$b_1 r \leq \mu(B(x, r)) \leq b_2 r \tag{2.2.1}$$

for all $x \in X$ and all r with $\mu(\{x\}) < r < \mu(X)$, we call the space of homogeneous type (X, d, μ) *normal.*

Remark. We note that if (X, d, μ) is a space of homogeneous type, then for all $\lambda > 0$ there exists A_λ, such that

$$\mu(B(x, \lambda r)) \leq A_\lambda \mu(B(x, r)) \quad \text{for all } x \in X \text{ and all } r > 0.$$

Since for a given quasimetric space (X, d), the balls in X are not necessarily open, we added this condition to the definition. This is the case, if for instance one has a Hölder condition for d: There exists $C < \infty$ and $0 < \beta < 1$ such that for all $x, y, z \in X$ we have

$$|d(x, z) - d(y, z)| \leq Cd(x, y)^\beta \max\{d(x, z), d(y, z)\}^{1-\beta}. \tag{2.2.2}$$

In fact, Macías and Segovia proved in [MS79b] that for every space of homogeneous type there exists an equivalent quasimetric with the desired Hölder property. Here, a quasimetric d' is equivalent to a quasimetric d if there exists a finite constant C such that

$$\frac{1}{C} d(x, y) \leq d'(x, y) \leq Cd(x, y),$$

whenever $x, y \in X$. Furthermore, in [MS79b] it is proved that based on a space of homogeneous type (X, d, μ) one can define a new quasimetric δ as

$$\delta(x, y) = \inf\{\mu(B) : B \text{ is a ball containing } x \text{ and } y\},$$

such that the resulting space of homogeneous type (X, δ, μ) is a normal space and the topologies induced by d and δ coincide.

Standard assumptions on X: In the following, we always assume that the spaces X we work with are spaces of homogeneous type, equipped with a quasimetric d and a Borel probability measure μ. Additionally we impose the restriction that X is normal and that for all $x \in X$ we have $\mu(\{x\}) = 0$, i.e. we have no isolated points.

2.2.2 Dyadic Cubes

In a space of homogeneous type there are analogues for dyadic cubes in $\mathbb{R}^n$ (see [Chr90b] and [Dav91]):

Theorem 2.2.3. *Let (X, d, μ) be a space of homogeneous type. Then there exist a system of open sets*

$$\mathcal{A} := \{Q_\alpha^n \subseteq X \mid n \in \mathbb{Z}, \alpha \in \mathcal{K}_n\},$$

points $z_\alpha^n \in Q_\alpha^n$ and constants $q > 1$, $c_1, c_2, c_3, \eta \in \mathbb{R}^+, N \in \mathbb{N}$ such that we have the following properties

1. *For all $n \in \mathbb{Z}$ we have that $X = \bigcup_{\alpha \in \mathcal{K}_n} Q_\alpha^n$ up to μ-null sets.*
2. *For Q_α^m, Q_β^n with $m \leq n$ and $\alpha \in \mathcal{K}_m$ and $\beta \in \mathcal{K}_n$ we have either $Q_\alpha^m \subseteq Q_\beta^n$ or $Q_\alpha^m \cap Q_\beta^n = \emptyset$. That means that the cubes $\{Q_\alpha^n\}$ are nested.*
3. *For each Q_α^n and every $m \geq n$ there is exactly one $\beta \in \mathcal{K}_m$ such that $Q_\alpha^n \subseteq Q_\beta^m$.*
4. *For all $n \in \mathbb{Z}$ and for all $\alpha \in \mathcal{K}_n$ we have that $B(z_\alpha^n, c_1 q^n) \subseteq Q_\alpha^n \subseteq B(z_\alpha^n, c_2 q^n)$.*
5. *With*
$$\partial_t Q_\alpha^n := \{x \in Q_\alpha^n : d(x, X \backslash Q_\alpha^n) \leq t q^n\},$$
we have
$$\forall n \in \mathbb{Z} \forall \alpha \in \mathcal{K}_n : \mu(\partial_t Q_\alpha^n) < c_3 t^\eta \mu(Q_\alpha^n).$$
6. *For all $n \in \mathbb{Z}$, the set $\mathcal{K}_n$ is countable.*
7. *For all $n \in \mathbb{Z}$ and all $\alpha \in \mathcal{K}_n$ we have $|\{\beta \in \mathcal{K}_{n-1} : Q_\beta^{n-1} \subseteq Q_\alpha^n\}| \leq N$.*
8. *For all $n \in \mathbb{Z}$, $\alpha \in \mathcal{K}_n$ there is a subset E of $\mathcal{K}_{n-1}$ with $|E| \leq N$ such that*
$$Q_\alpha^n = \bigcup_{\beta \in E} Q_\beta^{n-1} \quad \text{up to } \mu\text{-null sets.}$$

Remark. We note that these dyadic cubes were constructed by Christ in [Chr90b] and by David in [Dav91] in a slightly different way. We further remark that in the future use of the dyadic cubes, we neglect μ-null sets in points 1 and 8 of Theorem 2.2.3 and assume equality.

We now collect a few useful definitions, which we will need in the sequel.

Definition 2.2.4. We let

$$\mathcal{A}_n := \{Q_\alpha^n : \alpha \in \mathcal{K}_n\},$$

be the set of dyadic cubes with level $n \in \mathbb{Z}$. Furthermore, let $A \in \mathcal{A}_{n+1}$ and choose $A^*(A) \in \mathcal{A}_n$ arbitrarily (but fixed for all subsequent sections) with $A^*(A) \subseteq A$. Then we set

$$\mathcal{E}(A) := \{B \in \mathcal{A}_n : B \subseteq A \backslash A^*(A)\}.$$

We denote the cardinality $|\mathcal{E}(A)|$ of this set by $N(A)$. Additionally, we define the level of $A \in \mathcal{A}_n$ as

$$\operatorname{lev} A := n.$$

The unique element $A \in \mathcal{A}_{n+1}$ such that for $Q \in \mathcal{A}_n$ we have $Q \subset A$ will be denoted by

$$\operatorname{pre} Q, \tag{2.2.3}$$

which indicates that A is the predecessor of Q. Furthermore, we define a subset of dyadic cubes

$$\mathcal{E}(\mathcal{A}) := \bigcup_{A \in \mathcal{A}} \mathcal{E}(A)$$

Remark. Due to Point 7 of Theorem 2.2.3 we have that the cardinality $N(A)$ of $\mathcal{E}(A)$ is bounded by a uniform constant $N-1$ independent of $A \in \mathcal{A}_{n+1}$.

2.2.3 Martingale Differences

Let (X, d, μ) be a space of homogeneous type with $\mu(X) = 1$. Then we have $X = Q_1^0$, $\mathcal{K}_0 = \{1\}, \mathcal{A}_0 = \{X\}$ and $\mathcal{K}_n = \emptyset$ for all $n \in \mathbb{N}$. We use then dyadic cubes to build an orthonormal basis in $L^2(X, d, \mu)$ consisting of martingale differences. Fix $n \in -\mathbb{N}$, $A \in \mathcal{A}_{n+1}$ and enumerate the elements in $\mathcal{E}(A)$ in the way that $\mathcal{E}(A) = \{Q_1, \dots, Q_{N(A)}\}$. Additionally we set $Q_{N(A)+1} := A^*(A)$. We define the following functions, supported on A.

Definition 2.2.5. We define for $1 \leq k \leq N(A)$ and $x \in X$ (see Figure 2.1)

$$d_{Q_k}(x) := c_{Q_k} \begin{cases} 0, & \text{if } x \in \bigcup_{j=1}^{k-1} Q_j \cup (X \backslash A) \\ \sum_{j=k+1}^{N(A)+1} \mu(Q_j), & \text{if } x \in Q_k \\ -\mu(Q_k), & \text{if } x \in \bigcup_{j=k+1}^{N(A)+1} Q_j \end{cases},$$

where we choose c_{Q_k} such that

$$\|d_{Q_k}\|_2 = 1. \tag{2.2.4}$$

Remark. The functions defined in Definition 2.2.5 are obviously a martingale difference sequence. We record here also that these martingale differences are just the result of the Gram Schmidt orthogonalization process applied to the indicator functions

$$1_A, 1_{Q_1}, \dots, 1_{Q_{N(A)}}. \tag{2.2.5}$$

For a proof of this assertion, see Appendix A.1.

Now we enumerate all the functions d_Q, $Q \in \mathcal{E}(A)$ where $A \in \mathcal{A}_{n+1}, n \in -\mathbb{N}$ in a canonical way, we set

$$d_0 := 1_X$$

and get the constructed functions that are a basis in the constant functions on $\{Q_1, \dots, Q_{N(X)}, A^*(X)\}$, where $Q_i \in \mathcal{E}(X)$ for $1 \leq i \leq N(X)$ and set

$$d_1 = d_{Q_1}, \dots, d_{N(X)} = d_{Q_{N(X)}}.$$

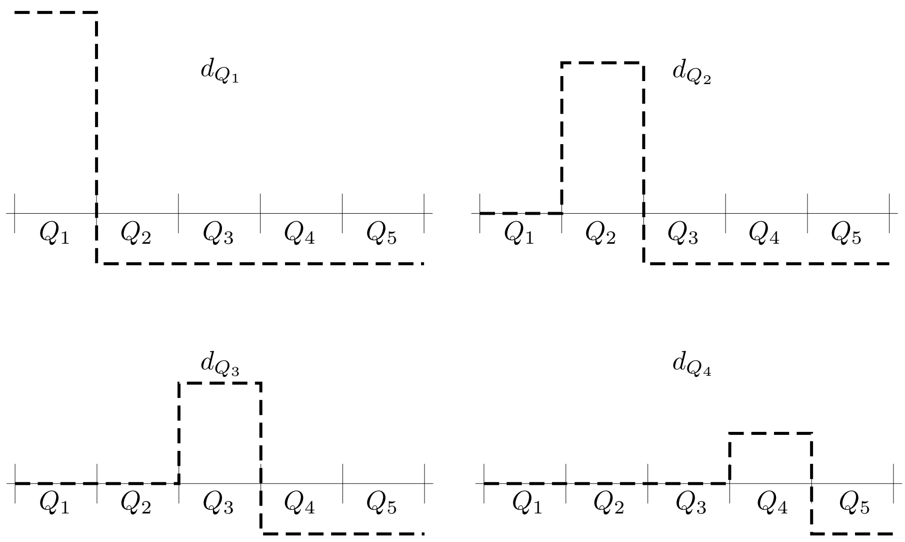

Figure 2.1: Schematic plots of functions in Definition 2.2.5, where N is set to 5.

We continue with this procedure on every Q_i, so we get an enumeration of all functions d_Q, $Q \in \mathcal{E}(A)$ for $A \in \mathcal{A}_{n+1}$, $n \in -\mathbb{N}$ such that the order is preserved in the following way

$$k \leq j \Rightarrow \operatorname{lev} R \geq \operatorname{lev} Q \quad \text{for } d_k = d_R \text{ and } d_j = d_Q.$$

We refer to the functions d_Q as Haar functions. According to this enumeration we define σ-algebras:

$$\mathcal{F}_i := \sigma(d_0, \ldots, d_i) \quad \text{for } i \in \mathbb{N}_0.$$

With respect to this filtration, the collection $\{d_k\}_{k\in\mathbb{N}}$ is a martingale difference sequence, since we have $\mathbb{E}(d_k|\mathcal{F}_{k-1}) = 0$ for every $k \in \mathbb{N}$. Another important sequence of σ-algebras that we need later is a suitable subsequence of the σ-algebras just created. We set

$$\mathcal{F}_k^{\text{lev}} := \sigma(\mathcal{A}_{-k}) \quad \text{for } k \in \mathbb{N}_0, \tag{2.2.6}$$

where the superscript lev should indicate that $\mathcal{F}_k^{\text{lev}}$ is the σ-algebra generated by all dyadic cubes of level $-k$.

As in the case $X = \mathbb{R}$ with the standard Haar functions, we have that the L^∞ norm of an L^2 normalized Haar function d_Q is (approximately) $\mu(Q)^{-1/2}$, which is a simple consequence of Theorem 2.2.3 and the normality of X.

Lemma 2.2.6. *There exists a constant $c < \infty$ depending only on X such that*

$$c^{-1}\mu(Q)^{-1/2} \leq \|d_Q\|_\infty \leq c\mu(Q)^{-1/2} \quad \textit{for all } Q \in \mathcal{E}(\mathcal{A}).$$

Proof. Let $Q \in \mathcal{A}_n$ for $n \in -\mathbb{N}, A = \operatorname{pre} Q$. Then, due to the fact that d_Q is L^2 normalized we get

$$\int_X d_Q^2 d\mu = 1.$$

But $d_Q = \sum_{j=1}^{N(A)+1} a_j 1_{Q_j}$ where $\{Q_1, \dots, Q_{N(A)+1}\} = \mathcal{E}(A) \cup \{A^*(A)\}$ and a_j are suitable numbers. Thus

$$1 = \sum_{j=1}^{N(A)+1} a_j^2 \mu(Q_j).$$

Since by Theorem 2.2.3, $B(z_{Q_j}, c_1 q^n) \subset Q_j \subset B(z_{Q_j}, c_2 q^n)$ for some point $z_{Q_j} \in X$, it follows from the normality of X that

$$b_1 c_1 q^n \sum_{j=1}^{N(A)+1} a_j^2 \leq 1 \leq b_2 c_2 q^n \sum_{j=1}^{N(A)+1} a_j^2.$$

With the fact that $\max_j |a_j| = \|d_Q\|_\infty$, we conclude

$$\frac{1}{\sqrt{b_2 c_2 (N(A)+1) q^n}} \leq \|d_Q\|_\infty \leq \frac{1}{\sqrt{b_1 c_1 q^n}},$$

Since $N(A)+1$ is uniformly bounded by a finite constant N, it follows that the left and right expressions in the previous display are comparable to $\mu(Q)^{-1/2}$ and so the conclusion of the lemma follows. □

Another simple consequence of Theorem 2.2.3 is

Lemma 2.2.7. *$\cup_{l=1}^\infty \mathcal{F}_l$ generates the Borel σ-algebra on X.*

Proof. We show that every open set in X is a countable union of dyadic cubes. This yields the conclusion of the lemma, since every dyadic cube is a Borel set. Now let $U \subseteq X$ be open. Then by definition, for all $x \in U$ there is an $r > 0$ such that

$$B(x, r) \subseteq U.$$

Let $m(x) \in \mathbb{Z}$ and $\alpha(x) \in \mathcal{K}_m$ be such that

$$\{x\} \subseteq Q_{\alpha(x)}^{m(x)} \subseteq B(x, r).$$

This is possible if we choose $m(x) \in \mathbb{Z}$ such that $q^m < \frac{r}{2c_2K}$, where K is the constant from the quasi-triangle inequality and c_2 is the same as in Theorem 2.2.3. Then

$$U = \bigcup_{x \in U} Q_{\alpha(x)}^{m(x)}.$$

With property 6 of Theorem 2.2.3 we see that this union is countable. □

Remark. Proposition 1.3.6 and the above lemma yield that for a Banach space E we have that

$$\lim_{k \to \infty} \|\mathbb{E}(f|\mathcal{F}_k) - f\|_{L_E^p(X)} = 0$$

for all $1 \leq p < \infty$. So we get for every $f \in L_E^p(X)$ a unique series expansion

$$f = \sum_{k=0}^\infty a_k d_k, \quad a_k \in E,$$

which converges unconditionally in $L_E^p(X)$ for UMD-spaces E and $1 < p < \infty$ by definition of the UMD-property. In particular for $p = 2$ and $E = \mathbb{R}$, $(d_k)_{k \in \mathbb{N}}$ is an orthonormal basis.

2.2.4 Isotropic Basis in $L^2(X \times X)$

Next we introduce an isotropic orthogonal basis in $L^2(X \times X)$. Here, the word isotropic means that for an element $f \otimes g$ of this basis (here, $f \otimes g(x, y) := f(x)g(y)$ is the standard tensor product of two functions), the support looks like a square and not like a rectangle. Most of the notation used in the sequel was introduced in Definition 2.2.4. Let $\varepsilon \in \{0, 1\}$. For $Q \in \mathcal{E}(A)$ and $A \in \mathcal{A}$ we define

$$d_Q^{(\varepsilon)} := d_Q \quad \text{for } \varepsilon = 1, \qquad \text{and} \qquad d_Q^{(\varepsilon)} := \frac{1_A}{\sqrt{\mu(A)}} \quad \text{for } \varepsilon = 0.$$

Note that the function $d_Q^{(0)}$ is L^2-normalized as is $d_Q^{(1)}$. With these settings, we define the collection Z of functions on $X \times X$:

$$\begin{aligned} Z \;:=\; & \{1_X \otimes 1_X\} \cup \\ & \{d_Q^{(\varepsilon_1)} \otimes d_R^{(\varepsilon_2)} : Q, R \in \mathcal{E}(\mathcal{A}), \operatorname{lev} Q = \operatorname{lev} R, \varepsilon = (\varepsilon_1, \varepsilon_2) \in \{0,1\}^2 \backslash \{(0,0)\}\}. \end{aligned} \tag{2.2.7}$$

Explicitly, up to multiplicative constants that come from the L^2-normalization, the three groups in (2.2.7) have the form

$$\{d_Q \otimes d_R : A, B \in \mathcal{A}_{n+1}, Q \in \mathcal{E}(A), R \in \mathcal{E}(B), n \in -\mathbb{N}\}, \tag{2.2.8}$$

$$\{d_Q \otimes 1_B : A, B \in \mathcal{A}_{n+1}, Q \in \mathcal{E}(A), n \in -\mathbb{N}\}, \tag{2.2.9}$$

$$\{1_A \otimes d_R : A, B \in \mathcal{A}_{n+1}, R \in \mathcal{E}(B), n \in -\mathbb{N}\} \tag{2.2.10}$$

The system Z forms an orthonormal basis in $L^2(X \times X)$ and this result follows from the well known classical

Lemma 2.2.8. *If $\{e_k\}_{k=1}^\infty$ is an orthogonal basis in $L^2(X, \mathcal{F}, \mu)$, then $\{e_k \otimes e_j\}_{k,j=1}^\infty$ is an orthogonal basis in $L^2(X \times X, \mathcal{F} \otimes \mathcal{F}, \mu \otimes \mu)$.*

Proof. Let $f \in L^2(X \times X, \mathcal{F} \otimes \mathcal{F}, \mu \otimes \mu)$. Then from

$$\int_{X \times X} f(x_1, x_2) e_k(x_1) e_j(x_2) d(\mu \otimes \mu)(x_1, x_2) = 0$$

it follows with Fubini that

$$\int_X \int_X f(x_1, x_2) e_k(x_1) \; d\mu(x_1) e_j(x_2) \; d\mu(x_2) = 0.$$

Since this identity holds for all $j \in \mathbb{N}$, we get, since $\{e_j\}_{j \in \mathbb{N}}$ is a basis, for every $k \in \mathbb{N}$ that

$$g_k(x_2) := \int_X f(x_1, x_2) e_k(x_1) \; d\mu(x_1) = 0 \quad \text{for } \mu\text{-a.e. } x_2 \in X.$$

Let $N \subseteq X$ be a set with $\mu(N) = 0$ and s.t. for all $k \in \mathbb{N}$, we have $g_k \equiv 0$ on $X \backslash N$. Now, take an $x_2 \notin N$, then we infer from $g_k(x_2) = 0$ for all $k \in \mathbb{N}$ and the fact that $\{e_k\}_{k \in \mathbb{N}}$ is an orthogonal basis:

$$f(x_1, x_2) = 0 \quad \text{for } \mu\text{-a.e. } x_1 \in X. \tag{2.2.11}$$

What we have to show is that $f = 0$ $\mu \otimes \mu$−a.e. Indeed, a further use of Fubini's theorem gives

$$\begin{aligned}\mu \otimes \mu([f \neq 0]) &= \int_X \mu(\{x_1 \in X : f(x_1, x_2) \neq 0\}) d\mu(x_2) \\ &= \int_{X \setminus N} \mu(\{x_1 \in X : f(x_1, x_2) \neq 0\}) d\mu(x_2) \\ &\quad + \int_N \mu(\{x_1 \in X : f(x_1, x_2) \neq 0\}) d\mu(x_2) = 0,\end{aligned}$$

the first term being zero since the integrand is zero and the second term vanishes because of $\mu(N) = 0$. □

Lemma 2.2.9. *Z is an orthonormal basis in $L^2(X \times X)$.*

Proof. Since the verification of orthonormality is a straightforward calculation, we proceed with showing the basis property. Since we know from Lemma 2.2.8 that the set

$$\{d_S \otimes d_T : S \in \mathcal{A}_{m+1}, T \in \mathcal{A}_{n+1}, n, m \in -\mathbb{N}\} \quad \text{with} \quad d_X := 1_X$$

is an orthogonal basis in $L^2(X \times X)$, we have to show that each $d_S \otimes d_T$ can be decomposed in a finite linear combination of functions of the form (2.2.8) – (2.2.10). To do that, we need the following identities:

$$1_U = 1_{A^*(U)} + \sum_{V \in \mathcal{E}(U)} 1_V, \quad U \in \mathcal{A}_{m+1}, \tag{2.2.12}$$

$$d_R = c_1 1_{A^*(B)} + \sum_{V \in \mathcal{E}(B)} c_V 1_V, \quad R \in \mathcal{E}(B), c_1, c_V \in \mathbb{R} \text{ for } V \in \mathcal{E}(B). \tag{2.2.13}$$

We then have four cases:

1. Let $d_S = 1_X$, $d_T = 1_X$, then clearly $d_S \otimes d_T \in Z$.

2. $d_S = 1_X$, $T \in \mathcal{A}_n, n \in -\mathbb{N}$, $B \in \mathcal{A}_{n+1}$ with $T \in \mathcal{E}(B)$. Then we get recursively from (2.2.12), that 1_X is a finite linear combination of functions of the form 1_C, where $C \in \mathcal{A}_{n+1}$. With (2.2.10), we see that $1_X \otimes d_T \in \operatorname{lin} Z$.

3. Analogously, we treat the case $d_T = 1_X$ and $d_S \neq 1_X$.

4. $S \in \mathcal{A}_n, T \in \mathcal{A}_m, m, n \in -\mathbb{N}$, $S \in \mathcal{E}(A), T \in \mathcal{E}(B), A \in \mathcal{A}_{n+1}, B \in \mathcal{A}_{m+1}$. If $m = n$, we see from (2.2.8) that $d_S \otimes d_T \in Z$. Without loss of generality we now assume that $m > n$ and we decompose d_T in the form (2.2.13). Additionally, if $m > n + 1$, we proceed recursively with (2.2.12) and get from (2.2.9) that $d_S \otimes d_T \in \operatorname{lin} Z$. □

2.2.5 Dyadic Annuli

Recall that $\mathcal{A}_n$ is the set of dyadic cubes of level n for $n \in -\mathbb{N}_0$ and $\mathcal{A}_0$ consists only of the whole space X and the size of cubes *decreases* with *decreasing* index n. We now introduce the set of all pairs of dyadic cubes of the same level

$$\mathcal{C} := \{(A, B) : A, B \in \mathcal{A}_n, n \in -\mathbb{N}_0\}$$

and its decomposition into annuli $\mathcal{C} = \bigcup_{m=0}^{\infty} \mathcal{C}_m$, where

$$\mathcal{C}_m = \{(A,B) \in \mathcal{C} : q^{m-1+\operatorname{lev} A} \le d(A,B) < q^{m+\operatorname{lev} A}\} \quad \text{for } m \in \mathbb{N}$$

and

$$\mathcal{C}_0 = \{(A,B) \in \mathcal{C} : d(A,B) < q^{\operatorname{lev} A}\}.$$

Recall that $\operatorname{lev} A$ denotes the level of A (that is if $A \in \mathcal{A}_n$, then $\operatorname{lev} A = n$) and q is the constant from Theorem 2.2.3 that determines the growth factor of the cubes in each level. This definition can be interpreted in the following way: Given $A \in \mathcal{A}_n$, we draw an annulus around A with inner radius $q^{m-1+\operatorname{lev} A}$ and outer radius $q^{m+\operatorname{lev} A}$ and take all pairs (A,B) such that B has no point inside the smaller circle and B has at least one point inside the larger circle. It is crucial that the annulus grows with the size of A.

2.2.6 Extracting Rearrangements - Further Decomposition of Annuli

The aim of this section is to extract (as few as possible) subcollections $\mathcal{C}_{m,i}$ from $\mathcal{C}_m$ such that for each $(A,B) \in \mathcal{C}_{m,i}$ we have that B is uniquely determined by A and A is uniquely determined by B. The benefit of this decomposition is that on $\mathcal{C}_{m,i}$ we can define an injective mapping τ such that $B = \tau(A)$ (see Definition 2.2.12). We start with the following observation:

Lemma 2.2.10. *There exists a constant M_0 independent of n and m, such that for $A \in \mathcal{A}_n$ there are at most $M_0 q^m$ elements $B \in \mathcal{A}_n$ with $(A,B) \in \mathcal{C}_m$.*

So, roughly speaking, in an annulus of level m around A, there are at most q^m cubes of the same size as A. This lemma is easily proved using the properties of dyadic cubes in Theorem 2.2.3 and the normality of X.

Proof. Let $B \in \mathcal{A}_n$ with $(A,B) \in \mathcal{C}_m$, so $d(A,B) \in \mathcal{C}_m$. The first thing to show is that there exists $C > 0$ independent of n, m such that

$$B \subseteq B(z_A, Cq^{n+m}), \tag{2.2.14}$$

where z_A denotes the center of A as in Theorem 2.2.3. Indeed, for arbitrary $\varepsilon > 0$, let $x_A \in A, x_B \in B$ such that

$$d(x_A, x_B) < q^{m+n} + \varepsilon.$$

Then we have for $x \in B$

$$\begin{aligned} d(z_A, x) &\le K[d(z_A, x_A) + d(x_A, x)] \le Kc_2 q^n + K^2[d(x_A, x_B) + d(x_B, x)] \\ &\le Kc_2 q^n + K^2(q^{m+n} + \varepsilon) + K^3[d(x_B, z_B) + d(z_B, x)] \\ &\le Kc_2 q^n + K^2(q^{m+n} + \varepsilon) + 2K^3 c_2 q^n. \end{aligned}$$

Here, K is the constant from the triangle inequality and c_2 denotes the constant introduced in Theorem 2.2.3. Since $m \ge 0$ and $\varepsilon > 0$ is arbitrary, we obtain (2.2.14) with C independent of n, m. Let $A \in \mathcal{A}_n$ and let X_m^A be the number of cubes $B \in \mathcal{A}_n$ such that $(A, B) \in \mathcal{C}_m$. Then, since X is normal,

$$\begin{aligned} X_m^A \cdot \mu(A) &\leq X_m^A \cdot \mu(B(z_A, c_2 q^n)) \leq X_m^A \cdot b_2 c_2 q^n \\ &\leq \frac{b_2 c_2}{b_1 c_1} \sum_{B \in \mathcal{A}_n, (A,B) \in \mathcal{C}_m} \mu(B(z_B, c_1 q^n)) \leq \frac{b_2 c_2}{b_1 c_1} \mu(B(z_A, C q^{n+m})), \end{aligned}$$

by (2.2.14). Now, the conclusion of the lemma follows from

$$\mu(B(z_A, C q^{n+m})) \leq C b_2 q^{n+m} \leq \frac{C b_2 q^m}{b_1 c_1} \mu(B(z_A, c_1 q^n)) \leq \frac{C b_2 q^m}{b_1 c_1} \mu(A). \qquad \square$$

Remark. The same argument shows that for each $C > 0$ there exists a constant M_0 s.t. for $A \in \mathcal{A}_n$ we have at most M_0 elements $B \in \mathcal{A}_n$ with

$$d(A, B) \leq C q^n.$$

Proposition 2.2.11. *Let $M_1 := 2M_0$ with M_0 from Lemma 2.2.10. Then we have for all $m \in \mathbb{N}_0$ that the collection $\mathcal{C}_m \subseteq \mathcal{A} \times \mathcal{A}$ admits a decomposition as*

$$\mathcal{C}_m = \mathcal{C}_{m,1} \cup \cdots \cup \mathcal{C}_{m, M_1 q^m}$$

so that each of the collections $\mathcal{C}_{m,i}$, $1 \leq i \leq M_1 q^m$ satisfies the two conditions

1. *For $B \in \mathcal{A}$ there exists at most one $A \in \mathcal{A}$ with $(A, B) \in \mathcal{C}_{m,i}$.*
2. *For $A \in \mathcal{A}$ there exists at most one $B \in \mathcal{A}$ with $(A, B) \in \mathcal{C}_{m,i}$.*

Remark. For the applications in Section 2.3 it is important that $\mathcal{C}_m$ is decomposed in $M_1 q^m$ subcollections (and not more). For instance the estimate q^{2m} would be much simpler to obtain, but would not allow us to treat singular integral operators.

Proof. **Step 1:** Idea of the proof:

Let $Q \in \mathcal{A}$. Then we define the ring collection of Q :

$$\mathcal{O}_m(Q) := \{R \in \mathcal{A} : (Q, R) \in \mathcal{C}_m\}.$$

We will show that there exists $I(Q) \subseteq \{1, \ldots, M_1 q^m\} =: I$ and an enumeration of the dyadic cubes in $\mathcal{O}_m(Q)$ such that

$$\mathcal{O}_m(Q) = \{R_i(Q) : i \in I(Q)\}$$

and we have the following property:

$$\forall Q, Q' \in \mathcal{A},\ Q \neq Q'\ \forall j \in I(Q) \cap I(Q') : R_j(Q) \neq R_j(Q'). \tag{2.2.15}$$

Then we can define the decomposition

$$\mathcal{A}_{m,i} = \{Q \in \mathcal{A} : i \in I(Q)\} \quad \text{and} \quad \mathcal{C}_{m,i} = \{(Q, R_i(Q)) : Q \in \mathcal{A}_{m,i}\}.$$

We thus obtain

$$\mathcal{C}_m = \mathcal{C}_{m,1} \cup \cdots \cup \mathcal{C}_{m, M_1 q^m}$$

and the desired properties hold.

Step 2: Construction of the enumeration:

Let $\mathcal{A} = \{Q^{(k)} : k \in \mathbb{N}\}$ be an enumeration of all dyadic cubes. We proceed by induction over k. For $k = 1$ choose $I(Q^{(1)}) = \{1, \ldots, |\mathcal{O}_m(Q^{(1)})|\}$ and select any enumeration of the cubes $\mathcal{O}_m(Q^{(1)})$. Observe that with Lemma 2.2.10 we have that $|\mathcal{O}_m(Q^{(1)})| \leq M_0 q^m$. Now let $k \in \mathbb{N}$ and assume we have constructed

$$I(Q^{(1)}), \ldots, I(Q^{(k)})$$

with

$$\mathcal{O}_m(Q^{(l)}) = \{R_i(Q^{(l)}) : i \in I(Q^{(l)})\} \text{ for } l \leq k$$

such that the following holds

$$\forall Q, Q' \in \{Q^{(1)}, \ldots, Q^{(k)}\},\ Q \neq Q'\ \forall j \in I(Q) \cap I(Q') : R_j(Q) \neq R_j(Q').$$

We will now construct $I(Q^{(k+1)})$. To do this we first set

$$\{R^{(1)}, \ldots, R^{(M_*)}\} = \mathcal{O}_m(Q^{(k+1)}), \quad \text{where } M_* \leq M_0 q^m.$$

Step 2a: We start a second induction and begin with $R^{(1)}$. We will define the index $\operatorname{ind}_{Q^{(k+1)}} R^{(1)}$ of $R^{(1)}$ in the enumeration $\mathcal{O}_m(Q^{(k+1)})$ as follows. We put

$$V(R^{(1)}) = \{Q' \in \{Q^{(1)}, \ldots, Q^{(k)}\} : R^{(1)} \in \mathcal{O}_m(Q')\},$$

so $V(R^{(1)})$ contains the cubes Q' for which $R^{(1)}$ is in their ring collection $\mathcal{O}_m(Q')$. Now, since $V(R^{(1)}) \subseteq \mathcal{O}_m(R^{(1)})$, we have an estimate for the cardinality of $V(R^{(1)})$:

$$|V(R^{(1)})| \leq M_0 q^m. \tag{2.2.16}$$

For $Q' \in V(R^{(1)})$ we already defined the indices $\operatorname{ind}_{Q'} R^{(1)} \in I$. Next we let

$$L(R^{(1)}) = \{\operatorname*{ind}_{Q'} R^{(1)} : Q' \in V(R^{(1)})\}$$

the indices of $R^{(1)}$ in the enumeration of Q'. According to (2.2.16), we have

$$|L(R^{(1)})| \leq M_0 q^m$$

and $|I| = M_1 q^m$. For the reduced index set, defined as

$$I^{\text{red}} = I \backslash L(R^{(1)}),$$

we have

$$|I^{\text{red}}| \geq M_1 q^m - M_0 q^m.$$

In particular, we have $I^{\text{red}} \neq \emptyset$. So we select any element in I^{red} to be the index of $R^{(1)}$ for $Q^{(k+1)}$:

$$\operatorname*{ind}_{Q^{(k+1)}} R^{(1)} \in I^{\text{red}}.$$

Thus the beginning of the second induction is completed.

Step 2b: Next we fix $j < M_* \leq M_0 q^m$. We now assume that we already defined

$$\operatorname*{ind}_{Q^{(k+1)}} R^{(1)}, \dots, \operatorname*{ind}_{Q^{(k+1)}} R^{(j)},$$

so we pick $R^{(j+1)} \in \mathcal{O}_m(Q^{(k+1)})$. As in the beginning of the induction, we set

$$V(R^{(j+1)}) = \{Q' \in \{Q^{(1)}, \dots, Q^{(k)}\} : R^{(j+1)} \in \mathcal{O}_m(Q')\}.$$

We again have $V(R^{(j+1)}) \subseteq \mathcal{O}_m(R^{(j+1)})$ and thus an estimate for the cardinality

$$|V(R^{(j+1)})| \leq M_0 q^m.$$

Next let

$$L(R^{(j+1)}) = \{\operatorname*{ind}_{Q'} R^{(j+1)} : Q' \in V(R^{(j+1)})\}$$

be the indices of $R^{(j+1)}$ in the enumeration of Q'. Since $|L(R^{(j+1)})| \leq M_0 q^m$, we have for the reduced index set

$$I^{\text{red}} = I \backslash (L(R^{(j+1)}) \cup \{\operatorname*{ind}_{Q^{(k+1)}} R^{(1)}, \dots, \operatorname*{ind}_{Q^{(k+1)}} R^{(j)}\})$$

an estimate for the cardinality

$$|I^{\text{red}}| > M_1 q^m - M_0 q^m - M_* \geq (M_1 - 2M_0) q^m,$$

so we have due to the definition of M_1 that $I^{\text{red}} \neq \emptyset$. We finally select then the index $\operatorname{ind}_{Q^{(k+1)}} R^{(j+1)}$ to be any element from the reduced index set I^{red}.

Step 3: We summarize and set

$$R_i(Q^{(k+1)}) = R^{(j)} \quad \text{iff} \quad i = \operatorname*{ind}_{Q^{(k+1)}} R^{(j)}$$

and the index set

$$I(Q^{(k+1)}) = \{\operatorname*{ind}_{Q^{(k+1)}} (R^{(j)}) : R^{(j)} \in \mathcal{O}_m(Q^{(k+1)})\}.$$

It follows from the construction step 2 that the enumeration R and the index sets $I(Q^{(k)})$ have the desired property (2.2.15). □

For $1 \leq i \leq M_1 q^m$ we recall the meaning of $\mathcal{A}_{m,i} \subseteq \mathcal{A}$, which was defined in the previous proof, as

$$\mathcal{A}_{m,i} = \{A \in \mathcal{A} : \exists B \in \mathcal{A}, \text{ such that } (A, B) \in \mathcal{C}_{m,i}\}.$$

Due to Proposition 2.2.11, we can define an injective mapping τ on $\mathcal{A}_{m,i}$:

Definition 2.2.12. We define

$$\begin{gathered} \tau : \mathcal{A}_{m,i} \to \mathcal{A} \\ A \mapsto \tau(A) \end{gathered}$$

through the relation

$$\tau(A) = B \quad \text{iff} \quad (A, B) \in \mathcal{C}_{m,i}.$$

Additionally we get an inverse of τ on $\tau(\mathcal{A}_{m,i})$

$$\tau^{-1}(B) = A \quad \text{iff} \quad (A, B) \in \mathcal{C}_{m,i}.$$

2.2.7 Decomposition of $\mathcal{C}_{m,i}$ using Arithmetic Progressions

Proposition 2.2.13. *For all $C > 0$ there is a constant M that depends only on C and the space of homogeneous type X such that we have the decomposition*

$$\mathcal{C}_{m,i} = \mathcal{G}_1 \cup \cdots \cup \mathcal{G}_M,$$

with the property that for all $1 \le l \le M$, $n \in -\mathbb{N}$, and all disjoint A_1, A_2 in $\mathcal{A}_n$ with

$$(A_1, \tau(A_1)) \in \mathcal{G}_l \quad \textit{and} \quad (A_2, \tau(A_2)) \in \mathcal{G}_l$$

the following separation of these sets holds:

$$d(\tau^i(A_1), \tau^j(A_2)) > Cq^n \quad \textit{for all } i, j \in \{0, 1\}. \tag{2.2.17}$$

Here, $\tau^0(A) := A$ and $\tau^1(A) := \tau(A)$.

Proof. Let $\{(Q^{(k)}, \tau(Q^{(k)})) : k \in \mathbb{N}\}$ be an enumeration of $\mathcal{C}_{m,i}$. Initialize the collections $\mathcal{G}_1, \ldots, \mathcal{G}_M$ as empty. For $k \in \mathbb{N}$, we inductively add $(Q^{(k)}, \tau(Q^{(k)}))$ to $\mathcal{G}_r$ for

$$r := \min\{i \in \mathbb{N} : \text{ for all } (A_1, \tau(A_1)) \in \mathcal{G}_i \text{ with } \operatorname{lev} A_1 = \operatorname{lev} Q^{(k)} \\ \text{we have (2.2.17) with } A_2 \text{ replaced by } Q^{(k)}\}.$$

Thus for $(A, \tau(A)) \in \mathcal{G}_{L+1}$ we have $(A, \tau(A)) \notin \mathcal{G}_l$ for all $l \le L$ and so we have that for all $l \le L$ there exists a pair $(A_l^0, \tau(A_l^0)) \in \mathcal{G}_l$ with $\operatorname{lev} A_l^0 = \operatorname{lev} A$ such that one of the four expressions

$$d(A, A_l^0), \quad d(A, \tau(A_l^0)), \quad d(\tau(A), A_l^0), \quad d(\tau(A), \tau(A_l^0))$$

is $\le Cq^n$. According to the properties of the collection $\mathcal{C}_{m,i}$, the sets in the collection $\{A_l^0\}_{l=1}^L$ as well as the sets $\{\tau(A_l^0)\}_{l=1}^L$ are disjoint. So the remark after Lemma 2.2.10 yields that L can't be greater than $4(M_0 + 1)$ with M_0 depending only on C and on the space of homogeneous type X. This proves the proposition. □

We cannot guarantee that a dyadic A cube divides into $N(A) + 1 \ge 2$ subcubes, but nevertheless we have as a consequence of the normality of X:

Lemma 2.2.14. *There exists a constant L such that for every $l \ge L$ we have that $A \in \mathcal{A}_n$, $B \in \mathcal{A}_{n-l}$ imply that $A \neq B$.*

Proof. Assume that $A \in \mathcal{A}_n$, $B \in \mathcal{A}_{n-l}$ and $A = B$. Due to the normality of (X, d, μ) we have that

$$c_1 b_1 q^n \le \mu(A) \le c_2 b_2 q^{n-l},$$

where the constants c_1, b_1, c_2, b_2 are again from Theorem 2.2.3 and from the normality of X. Then

$$q^l \le \frac{c_2 b_2}{c_1 b_1},$$

which is impossible if l is large enough. □

We now fix $\mathcal{G} = \mathcal{G}_l$ in the decomposition of Proposition 2.2.13 for some $l \le M$ and introduce levels using arithmetic progressions. We set

$$\mathcal{A}_\mathcal{G} := \{A \in \mathcal{A}_{m,i} : (A, \tau(A)) \in \mathcal{G}\}$$

and the levels

$$\mathcal{L}_r = \mathcal{A}_\mathcal{G} \cap \bigcup_{l=0}^{\infty} \mathcal{A}_{-l\cdot L(m+1)-r}, \quad \text{where } 0 \le r \le L(m+1) - 1 \tag{2.2.18}$$

and $L \in \mathbb{N}$ is chosen in such a way that the condition of Lemma 2.2.14 is satisfied. We will later (in Section 2.2.7) give additional conditions on the constant L. Given a set $A \in \mathcal{L}_r$ we now define appropriate predecessors.

Definition 2.2.15. If $A \in \mathcal{A}_{-l\cdot L(m+1)-r}$, we define the *arithmetic predecessor*

$$\widetilde{A} \tag{2.2.19}$$

to be the unique element in $\mathcal{A}_{-(l-1)\cdot L(m+1)-r}$, such that $\widetilde{A} \supset A$.

This works only if $l \ge 1$. If $l = 0$, we simply set $\widetilde{A} := X$. We remark that for $A \in \mathcal{L}_r$ we have obviously

$$\widetilde{A} \in \bigcup_{l=0}^{\infty} \mathcal{A}_{-l\cdot L(m+1)-r} \quad \text{or} \quad \widetilde{A} = X,$$

but not necessarily that $\widetilde{A} \in \mathcal{A}_\mathcal{G}$, hence $\widetilde{A}$ need not be in $\mathcal{L}_r$.
Note that the dyadic predecessor of A, denoted $\operatorname{pre} A$, defined in (2.2.3) does not coincide with the arithmetic predecessor $\widetilde{A}$ defined above.

Definition 2.2.16. Let $\mathcal{Z}$ be a collection of sets. $\mathcal{Z}$ is said to be *nested*, if for all $A, B \in \mathcal{Z}$ we have that either

$$A \cap B = \emptyset \quad \text{or} \quad A \subseteq B \quad \text{or} \quad B \subseteq A$$

holds.

The main result of this section is the following combinatorial theorem. It is the foundation of our work in the subsequent sections. It translates into norm estimates for rearrangement and shift operators in Section 2.3.3. The significance of Theorem 2.2.17 can be seen by examining the proof of T. Figiel [Fig88]. To anticipate the notation used in the following theorem, we note that $\mathcal{H}$ will be the collection of cubes A such that $\tau(A)$ has the same arithmetic predecessor as A. $\mathcal{I}$ will be the collection of cubes A such that both A and $\tau(A)$ are well inside their arithmetic predecessors and the collection $\mathcal{J}$ consists of the rest, where we again divide into the cases where either A or $\tau(A)$ or both of them lie near the boundary of their arithmetic predecessors and call the corresponding collections $\mathcal{J}_1, \mathcal{J}_2$ and $\mathcal{J}_3$ respectively.

Theorem 2.2.17. *For $r \leq L(m+1)-1$ and $\mathcal{L} = \mathcal{L}_r$ defined by (2.2.18), then for $\mathcal{L}$ there exists a decomposition*

$$\mathcal{L} = \mathcal{H} \cup \mathcal{I} \cup \mathcal{J},$$

such that

1. *The collection*
$$\{A, \tau(A), A \cup \tau(A) : A \in \mathcal{H}\}$$
is nested.

2. *$\mathcal{I}$ admits a decomposition as $\mathcal{I} = \mathcal{I}_1 \cup \mathcal{I}_2$, so that the two collections*
$$\{A, \tau(A), A \cup \tau(A) : A \in \mathcal{I}_j\} \quad \text{for } j \in \{1,2\}$$
are nested.

3. *$\mathcal{J}$ admits a decomposition as $\mathcal{J} = \mathcal{J}_1 \cup \mathcal{J}_2 \cup \mathcal{J}_3$ such that we have*

   (a) *There exists an injection $\gamma_1 : \mathcal{J}_1 \cup \mathcal{J}_3 \to \mathcal{A}$ such that the collection*
$$\{A, \gamma_1(A), A \cup \gamma_1(A) : A \in \mathcal{J}_1\}$$
is nested and in addition we have for $A \in \mathcal{J}_1$
$$\gamma_1(A) \subseteq \widetilde{A}, \quad d(\gamma_1(A), \complement\widetilde{A}) \geq q^{\operatorname{lev} A} \quad \text{and} \quad d(\tau(A), \complement\widetilde{\tau(A)}) \geq q^{\operatorname{lev} A}.$$

   (b) *There exists an injection $\gamma_2 : \mathcal{J}_2 \cup \mathcal{J}_3 \to \mathcal{A}$ such that the collection*
$$\{\tau(A), \gamma_2(\tau(A)), \tau(A) \cup \gamma_2(\tau(A)) : A \in \mathcal{J}_2\}$$
is nested and in addition we have for $A \in \mathcal{J}_2$
$$\gamma_2(\tau(A)) \subseteq \widetilde{\tau(A)}, \quad d(\gamma_2(\tau(A)), \complement\widetilde{\tau(A)}) \geq q^{\operatorname{lev} A} \quad \text{and} \quad d(A, \complement\widetilde{A}) \geq q^{\operatorname{lev} A}.$$

   (c) *For $\mathcal{J}_3$ and the injections γ_1 and γ_2 defined in (a) and (b), we have for $A \in \mathcal{J}_3$*
$$d(\gamma_1(A), \complement\widetilde{A}) \geq q^{\operatorname{lev} A} \quad \text{and} \quad d(\gamma_2(\tau(A)), \complement\widetilde{\tau(A)}) \geq q^{\operatorname{lev} A}.$$
Additionally, the two collections
$$\{A, \gamma_1(A), A \cup \gamma_1(A) : A \in \mathcal{J}_3\} \quad \text{and}$$
$$\{\tau(A), \gamma_2(\tau(A)), \tau(A) \cup \gamma_2(\tau(A)) : A \in \mathcal{J}_3\}$$
are nested.

The proof of this theorem is divided into four basic steps.

Step 1 (Subsection 2.2.7) We give the definition of the decomposition of $\mathcal{L}$ into $\mathcal{H}, \mathcal{I}, \mathcal{J}$ and we further define the decomposition of $\mathcal{I}$ into $\mathcal{I}_1, \mathcal{I}_2$ and also the decomposition of $\mathcal{J}$ into $\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3$.

Step 2 (Subsection 2.2.7) We verify that $\mathcal{H}$ satisfies condition 1. of Theorem 2.2.17.

Step 3 (Subsection 2.2.7) We verify that $\mathcal{I}_1, \mathcal{I}_2$ satisfy condition 2. of Theorem 2.2.17. This involves a two-coloring of $\mathcal{I}$ and an application of the argument in Step 2.

Step 4 (Subsection 2.2.7) We first define the injections γ_1, γ_2 and verify condition 3. of the theorem. Here we use reduction to the arguments introduced in Steps 2. and 3.

Definition of the Decomposition of $\mathcal{L}$

Fix $A \in \mathcal{L}$. We make the following case distinction:

1. If $\widetilde{A} = \widetilde{\tau(A)}$, we add A to $\mathcal{H}$.
2. If $\widetilde{A} \cap \widetilde{\tau(A)} = \emptyset$, we let
   (a) $A \in \mathcal{I}$, if the values of $d(A, \complement\widetilde{A})$ and $d(\tau(A), \complement\widetilde{\tau(A)})$ are both greater or equal $q^{\operatorname{lev} A}$,
   (b) $A \in \mathcal{J}$, if one of the values $d(A, \complement\widetilde{A})$ or $d(\tau(A), \complement\widetilde{\tau(A)})$ is less than $q^{\operatorname{lev} A}$.

For the case 2a we define the following collections: Take any $A \in \mathcal{I} \cup \tau(\mathcal{I})$, define

$$\begin{aligned}\mathcal{P}(A) \quad := \quad & \{B \in \mathcal{I} : \operatorname{lev} B < \operatorname{lev} A \text{ and} \\ & [(B \cap A \neq \emptyset \wedge \tau(B) \cap \complement A \neq \emptyset) \vee (B \cap \complement A \neq \emptyset \wedge \tau(B) \cap A \neq \emptyset)]\}\end{aligned}$$

and set

$$\mathcal{R}(A) := \{J, \tau(J) : J \in \mathcal{P}(A)\}.$$

The purpose of the collection $\mathcal{P}(A)$ is that we get rid of overlappings that occur if we define a two-coloring on $\mathcal{I}$ (say with the colors black and white) and set

$$\mathcal{I}_1 := \{A \in \mathcal{I} : \text{color } A = \text{black}\}, \quad \mathcal{I}_2 := \{A \in \mathcal{I} : \text{color } A = \text{white}\}.$$

This two-coloring will have the crucial property that if $A \in \mathcal{I}$ is white, then every element in $\mathcal{P}(A)$ is black. At last, we define a decomposition of $\mathcal{J}$ and let

$$\begin{aligned}\mathcal{J}_1 &:= \{A \in \mathcal{J} : d(A, \complement\widetilde{A}) < q^{\operatorname{lev} A} \text{ and } d(\tau(A), \complement\widetilde{\tau(A)}) \geq q^{\operatorname{lev} A}\} \\ \mathcal{J}_2 &:= \{A \in \mathcal{J} : d(A, \complement\widetilde{A}) \geq q^{\operatorname{lev} A} \text{ and } d(\tau(A), \complement\widetilde{\tau(A)}) < q^{\operatorname{lev} A}\} \\ \mathcal{J}_3 &:= \mathcal{J} \backslash (\mathcal{J}_1 \cup \mathcal{J}_2).\end{aligned}$$

The Collection $\mathcal{H}$

We first analyze the collection $\mathcal{H}$, which is simpler to handle than $\mathcal{I}$ and $\mathcal{J}$.

Lemma 2.2.18. *The collection* $\{A, \tau(A), A \cup \tau(A) : A \in \mathcal{H}\}$ *is nested.*

Proof. Let $A, B \in \mathcal{H}$ with $A \neq B$. It suffices to look at the pairs $(A, B \cup \tau(B))$, $(\tau(A), B \cup \tau(B))$, $(A \cup \tau(A), B \cup \tau(B))$, since the other cases are trivial (this is the case if both elements in the pair are dyadic cubes themselves) or considered by symmetry (as for example the pair $(B, A \cup \tau(A))$). We begin with $(A, B \cup \tau(B))$: We assume

$$A \cap (B \cup \tau(B)) \neq \emptyset, \tag{2.2.20}$$

Then we have to show that either $A \subseteq B \cup \tau(B)$ or $B \cup \tau(B) \subseteq A$. We have (2.2.20) if and only if

$$A \cap B \neq \emptyset \text{ or } A \cap \tau(B) \neq \emptyset.$$

For $A \cap B \neq \emptyset$ we have the three possibilities

$$A = B \quad \text{or} \quad A \subset B \quad \text{or} \quad B \subset A,$$

where $\subset$ denotes a strict inclusion. Indeed these are the only cases that can happen, since A and B are dyadic cubes. But $A = B$ is impossible, since we assumed $A \neq B$. If $A \subset B$, we clearly have $A \subseteq B \cup \tau(B)$. If $B \subset A$, it holds also that

$$\widetilde{B} \subseteq A.$$

This yields $\tau(B) \subseteq A$, since $\widetilde{B} = \widetilde{\tau(B)}$. So $B \cup \tau(B) \subseteq A$. For the case $A \cap \tau(B) \neq \emptyset$, analogous arguments complete the analysis of the pair $(A, B \cup \tau(B))$.

The pair $(\tau(A), B \cup \tau(B))$ is then treated in the same manner.

We now come to $(A \cup \tau(A), B \cup \tau(B))$: Again, we have to consider a few cases. First we assume that

$$(A \cup \tau(A)) \cap (B \cup \tau(B)) \neq \emptyset.$$

This is the case if and only if

$$A \cap B \neq \emptyset \quad \text{or} \quad A \cap \tau(B) \neq \emptyset \quad \text{or} \quad \tau(A) \cap B \neq \emptyset \quad \text{or} \quad \tau(A) \cap \tau(B) \neq \emptyset.$$

These four cases are treated in the same way as above, if one rules out the case that either $A = \tau(B)$ or $B = \tau(A)$. This cases are excluded by the separation condition in Proposition 2.2.13. □

The Collection $\mathcal{I}$

Lemma 2.2.19. *For each* $B \in \mathcal{I} \cup \tau(\mathcal{I})$ *there exists at most one* $A \in \mathcal{I} \cup \tau(\mathcal{I})$ *such that*

$$B \in \mathcal{R}(A).$$

Proof. Let $A_1, A_2 \in \mathcal{I} \cup \tau(\mathcal{I})$ with $A_1 \neq A_2$ such that

$$B \in \mathcal{R}(A_1) \quad \text{and} \quad B \in \mathcal{R}(A_2).$$

We split the proof into two parts. Part (a) treats the case $\operatorname{lev} A_1 = \operatorname{lev} A_2$ and part (b) treats the case $\operatorname{lev} A_1 < \operatorname{lev} A_2$, which is the general case, since we can always exchange A_1 and A_2. We additionally assume $B \in \mathcal{I}$, since the argument is symmetric if we assume $B \in \tau(\mathcal{I})$.

(a) We first treat the case $\operatorname{lev} A_1 = \operatorname{lev} A_2$. Here we get from the definition of $\mathcal{I}$ and from Proposition 2.2.13 that $d(A_1, A_2) > q^{\operatorname{lev} A_1}$ and $\operatorname{lev} B \leq \operatorname{lev} A_1 - L(m+1)$. Again we distinguish two cases. In view of the fact that $B \in \mathcal{R}(A_1)$, we split to (i) $B \cap A_1 \neq \emptyset$ and (ii) $\tau(B) \cap A_1 \neq \emptyset$.

(i) With $B \cap A_1 \neq \emptyset$ it holds that $B \subset A_1$ and so $B \cap A_2 = \emptyset$. Thus we have

$$d(A_1, \tau(B)) \leq d(B, \tau(B)) \leq q^{\operatorname{lev} B + m} \leq q^{\operatorname{lev} A_1}.$$

From these facts we infer that $\tau(B) \not\subset A_2$ and that implies $\tau(B) \cap A_2 = \emptyset$, which contradicts the assumption $B \in \mathcal{R}(A_2)$.

(ii) If $\tau(B) \cap A_1 \neq \emptyset$, this leads to $\tau(B) \subset A_1$ and $\tau(B) \cap A_2 = \emptyset$. Analogously to the above we get

$$d(A_1, B) \leq d(\tau(B), B) \leq q^{\operatorname{lev} B + m} \leq q^{\operatorname{lev} A_1}.$$

This implies $B \cap A_2 = \emptyset$, which contradicts $B \in \mathcal{R}(A_2)$.

(b) Now we assume without loss of generality that $\operatorname{lev} A_1 < \operatorname{lev} A_2$. Here we consider the two cases (i) $A_1 \subset A_2$ and (ii) $A_1 \cap A_2 = \emptyset$.

(i) For $A_1 \subset A_2$ we have by definition of $\mathcal{I}$

$$d(A_1, \complement A_2) \geq d(A_1, \widetilde{\complement A_1}) > q^{\operatorname{lev} A_1}. \tag{2.2.21}$$

Like in case (a) we have to consider the two cases $B \cap A_1 \neq \emptyset$ and $\tau(B) \cap A_1 \neq \emptyset$. We proceed with $B \cap A_1 \neq \emptyset$. (The case $\tau(B) \cap A_1 \neq \emptyset$ works analogously.) So it follows that $B \subset A_1$ and so $B \subset A_2$. We have the estimate

$$d(A_1, \tau(B)) \leq d(B, \tau(B)) \leq q^{m + \operatorname{lev} B} \leq q^{\operatorname{lev} A_1}. \tag{2.2.22}$$

Now it follows from (2.2.21) and (2.2.22) that $\tau(B) \not\subset \complement A_2$, i.e. $\tau(B) \subset A_2$. This contradicts $B \in \mathcal{R}(A_2)$.

(ii) Let $A_1 \cap A_2 = \emptyset$. In that case we have

$$d(A_1, A_2) \geq d(A_1, \widetilde{\complement A_1}) > q^{\operatorname{lev} A_1}.$$

If $B \cap A_1 \neq \emptyset$ (the other case $\tau(B) \cap A_1 \neq \emptyset$ is treated analogously), it follows that $B \subset A_1$ and hence $B \subset \complement A_2$. We have

$$d(A_1, \tau(B)) \leq d(B, \tau(B)) \leq q^{m + \operatorname{lev} B} \leq q^{\operatorname{lev} A_1}.$$

Thus we get

$$\tau(B) \not\subset A_2$$

and thus $\tau(B) \cap A_2 = \emptyset$. This identity together with $B \subset \complement A_2$ contradicts $B \in \mathcal{R}(A_2)$. This finishes the proof. □

Lemma 2.2.19 allows us to introduce the announced two-coloring on $\mathcal{I} \cup \tau(\mathcal{I})$ with the colors black and white that satisfies the following three conditions:

1. For each $A \in \mathcal{I} \cup \tau(\mathcal{I})$ the collection $\mathcal{R}(A)$ is monochromatic,
2. If the color of $A \in \mathcal{I} \cup \tau(\mathcal{I})$ is already determined, then each $B \in \mathcal{R}(A)$ satisfies
$$\operatorname{color}(B) \neq \operatorname{color}(A),$$
3. For each $A \in \mathcal{I}$,
$$\operatorname{color}(A) = \operatorname{color}(\tau(A)).$$

This may be done as follows: Let $\mathcal{Z}_j = (\mathcal{I} \cup \tau(\mathcal{I})) \cap \mathcal{A}_{-j}$ for $j \geq 0$. Then define inductively for $A \in \mathcal{Z}_j$ that the color of A is black, if it is not already defined. Then for every $A \in \mathcal{Z}_j$ set the color of $\mathcal{R}(A)$ to the different color of A. This is well defined, since by Lemma 2.2.19, the collections $\mathcal{R}(A_1)$ and $\mathcal{R}(A_2)$ are disjoint for different $A_1, A_2 \in \mathcal{I} \cup \tau(\mathcal{I})$.

Based on that two-coloring of the cubes in $\mathcal{I} \cup \tau(\mathcal{I})$, define
$$\mathcal{I}_1 = \{A \in \mathcal{I} : \operatorname{color}(A) = \text{white}\} \quad \text{and} \quad \mathcal{I}_2 = \{A \in \mathcal{I} : \operatorname{color}(A) = \text{black}\}.$$

Lemma 2.2.20. *If $A \in \mathcal{I}$ and $B \notin \mathcal{P}(A)$ with $\operatorname{lev} B < \operatorname{lev} A$, then*
$$B \cup \tau(B) \subseteq A \quad \textit{or} \quad B \cup \tau(B) \subseteq \complement A.$$

Proof. This is nothing else but a logical manipulation of the definition of $\mathcal{P}(A)$. □

Lemma 2.2.21. *The two subcollections*
$$\{A, \tau(A), A \cup \tau(A) : A \in \mathcal{I}_j\} \quad \textit{for } j \in \{1, 2\}$$
are nested.

Proof. Let $A, B \in \mathcal{I}_j$ for $j \in \{1, 2\}$. We consider the three pairs (a) $(A, B \cup \tau(B))$, (b) $(\tau(A), B \cup \tau(B))$, (c) $(A \cup \tau(A), B \cup \tau(B))$.

(a) We have to show that either
$$A \cap (B \cup \tau(B)) = \emptyset \quad \text{or} \quad A \subseteq B \cup \tau(B) \quad \text{or} \quad B \cup \tau(B) \subseteq A. \tag{2.2.23}$$

We consider the three cases (i) $\operatorname{lev} A = \operatorname{lev} B$, (ii) $\operatorname{lev} A < \operatorname{lev} B$ and (iii) $\operatorname{lev} B < \operatorname{lev} A$:

(i) This is clear, since A and B are dyadic cubes.

(ii) If $A \in \mathcal{P}(B)$ then either A or B is not in $\mathcal{I}_j$; if $A \notin \mathcal{P}(B)$, then due to Lemma 2.2.20 we have
$$A \cup \tau(A) \subseteq B \quad \text{or} \quad A \cup \tau(A) \subseteq \complement B.$$

In the first case, clearly, $A \subseteq B \cup \tau(B)$. In the second case $A \subseteq \complement B$ and so
$$A \cap (B \cup \tau(B)) = A \cap \tau(B) = \begin{cases} A, & \text{if } A \cap \tau(B) \neq \emptyset \\ \emptyset, & \text{else} \end{cases}.$$

Both branches lead to one of the alternatives in (2.2.23).

(iii) Analogous to (even simpler than) case (ii).

(b) Analogous to (a).

(c) We have to show that either

$$(A\cup\tau(A))\cap(B\cup\tau(B)) = \emptyset \quad \text{or} \quad A\cup\tau(A) \subseteq B\cup\tau(B) \quad \text{or} \quad B\cup\tau(B) \subseteq A\cup\tau(A). \tag{2.2.24}$$

We consider the two cases (i) $\operatorname{lev} A = \operatorname{lev} B$, (ii) $\operatorname{lev} A < \operatorname{lev} B$:

(i) Since A and B are in a collection $\mathcal{G}$, we have that $d(\tau(A), B)$ and $d(A, \tau(B))$ are greater than $q^{\operatorname{lev} A}$, and so

$$(A \cup \tau(A)) \cap (B \cup \tau(B)) = \emptyset.$$

(ii) If $A \in \mathcal{P}(B)$ then either A or B is not in $\mathcal{I}_j$. If $A \notin \mathcal{P}(B)$ we get with Lemma 2.2.20 that either

$$A \cup \tau(A) \subseteq B \quad \text{or} \quad A \cup \tau(A) \subseteq \complement B.$$

In the first case, clearly, $A \cup \tau(A) \subseteq B \cup \tau(B)$. In the second case we get from part (b) of the Lemma that for

$$(A \cup \tau(A)) \cap \tau(B)$$

we only have the three possibilities $\emptyset$, $A \cup \tau(A)$ or $\tau(B)$. The former two lead to $(A\cup\tau(A))\cap(B\cup\tau(B)) = \emptyset$ and $A\cup\tau(A) \subseteq B\cup\tau(B)$ respectively. The third one gives

$$\tau(B) \subseteq A \cup \tau(A),$$

which is not possible (cf. Lemma 2.2.14).

□

Remark. We remark that this decomposition of $\mathcal{I}$ into $\mathcal{I}_1$ and $\mathcal{I}_2$, in particular the proof of Lemma 2.2.19, does not depend on the explicit form of $\mathcal{I}$ and, what is even more important, the corresponding injection τ. In fact, the same proof works if there exists a constant C_R such that the following conditions are satisfied:

1. τ is an injection on $\mathcal{I}$ that $\operatorname{lev} A = \operatorname{lev} \tau(A)$ whenever $A \in \mathcal{I}$,
2. for every $A \in \mathcal{I}$, $\tilde{A} \cap \widetilde{\tau(A)} = \emptyset$,
3. for $A \in \mathcal{I}$, we have that $\min(d(A, \complement\tilde{A}), d(\tau(A), \complement\widetilde{\tau(A)})) > C_R q^{\operatorname{lev} A}$,
4. for $A, B \in \mathcal{I}$ with $\operatorname{lev} A = \operatorname{lev} B$, it holds that $d(\tau^j(A), \tau^i(B)) > C_R q^{\operatorname{lev} A}$ for $i, j \in \{0, 1\}$,
5. for $A \in \mathcal{I}$, we have that $\max(d(\tilde{A}, \tau(A)), d(\widetilde{\tau(A)}, A)) \leq C_R q^{\operatorname{lev} \tilde{A}}$,
6. for two distinct sets A, B in $\mathcal{I}$ such that $\operatorname{lev} A > \operatorname{lev} B$, we have that $\operatorname{lev} A \geq \operatorname{lev} B + L(m+1)$.

The Collection $\mathcal{J}$

Lemma 2.2.22. *There exists a constant C_2 such that for all $A \in \mathcal{A}_n$ and every $l \in \mathbb{N}$, the number Y_l^A of sets B in $\mathcal{A}_{n-l}$, for which we have $B \subseteq A$, is bounded from below by*

$$C_2 q^l.$$

Proof. If we use the normality of X and point 4. of Theorem 2.2.3 the conclusion of the lemma follows from the subsequent chain of inequalities:

$$\begin{aligned} \frac{b_1 c_1}{b_2 c_2} q^n &\leq \frac{1}{b_2 c_2} \mu(A) = \frac{1}{b_2 c_2} \sum_{B \subset A, B \in \mathcal{A}_{n-l}} \mu(B) \\ &\leq \sum_{B \subset A, B \in \mathcal{A}_{n-l}} q^{n-l} = Y_l^A q^{n-l}. \quad \square \end{aligned}$$

Now, recall the definition of the boundary layer $\partial_t A$ of a cube A with level n, which we defined as

$$\partial_t A = \{x \in A : d(x, X \backslash A) \leq t q^n\}.$$

Additionally, due to Theorem 2.2.3, the measure of $\partial_t A$ admits the following upper bound

$$\mu(\partial_t A) < c_3 t^\eta \mu(A)$$

for some universal constants $c_3, \eta > 0$.

Lemma 2.2.23. *There exists a constant C_3 such that for all $A \in \mathcal{A}_n$ and every $l \in \mathbb{N}$, the number X_l^A of sets $B \in \mathcal{A}_{n-l}$ for which we have*

$$B \cap \partial_{q^{-l}} A \neq \emptyset$$

is bounded from above by

$$C_3 q^{l(1-\eta)}.$$

Proof. It is a simple consequence of the quasi-triangle inequality that there exists $d \geq 1$ depending only on X such that if $B \in \mathcal{A}_{n-l}$ we have

$$B \cap \partial_{q^{-l}} A \neq \emptyset \Rightarrow B \subset \partial_{dq^{-l}} A.$$

With this fact, the normality of X and Theorem 2.2.3, point 4. and 5., the conclusion of the lemma follows from the subsequent chain of inequalities:

$$\begin{aligned} X_l^A q^{n-l} &\leq b_2 c_2 \sum_{B \subset \partial_{dq^{-l}} A, B \in \mathcal{A}_{n-l}} q^{n-l} \leq \frac{1}{b_1 c_1} \sum_{B \subset \partial_{dq^{-l}} A, B \in \mathcal{A}_{n-l}} \mu(B) \\ &\leq \frac{1}{b_1 c_1} \mu(\partial_{dq^{-l}} A) \leq c_3 \frac{1}{b_1 c_1} d^\eta q^{-l\eta} \mu(A) \leq c_3 \frac{b_2 c_2}{b_1 c_1} d^\eta q^{-l\eta} q^n. \quad \square \end{aligned}$$

In view of the above two lemmas and Lemma 2.2.14, we can choose L in (2.2.18) large enough that for all $l \geq L$, $A \in \mathcal{A}_n$ and $B \in \mathcal{A}_{n-l}$ we don't have

$$A = B,$$

and in addition that the quotient $\frac{Y_L^A}{X_L^A}$ admits the bound

$$\frac{Y_L^A}{X_L^A} > 2.$$

This property is crucial, since it enables us to define an injection $\gamma_1 : \mathcal{J}_1 \cup \mathcal{J}_3 \to \mathcal{A}$, such that we have $\gamma_1(A) \subseteq \widetilde{A}$ and we have moved away from the boundary of $\widetilde{A}$:

$$d(\gamma_1(A), \complement\widetilde{A}) \geq q^{\operatorname{lev} A}.$$

We extend γ_1 to $\mathcal{J}_1 \cup \mathcal{J}_3 \cup \gamma_1(\mathcal{J}_1) \cup \gamma_1(\mathcal{J}_3)$ and define for $A \in \gamma_1(\mathcal{J}_1) \cup \gamma_1(\mathcal{J}_3)$ that

$$\gamma_1(A) := \gamma_1^{-1}(A).$$

It as now a straightforward consequence of the definitions that the following holds:

Lemma 2.2.24. *The collections*

$$\{A, \gamma_1(A), A \cup \gamma_1(A) : A \in \mathcal{J}_1\} \quad \textit{and} \quad \{A, \gamma_1(A), A \cup \gamma_1(A) : A \in \mathcal{J}_3\}$$

are nested.

Proof. The proof is in fact nothing else than the proof of Lemma 2.2.18 with τ replaced by γ_1, if one notices that by the property of γ_1 that it maps from the 'boundary' to the 'interior' of the arithmetic predecessor, it cannot happen that $A_1 = \gamma_1(A_2)$ for some $A_1, A_2 \in \mathcal{J}_1$. □

Analogously, we define an injection $\gamma_2 : \tau(\mathcal{J}_2) \cup \tau(\mathcal{J}_3) \to \mathcal{A}$, such that we have $\gamma_2(\tau(A)) \subseteq \widetilde{\tau(A)}$ and

$$d(\gamma_2(\tau(A)), \complement\widetilde{\tau(A)}) \geq q^{\operatorname{lev} A}$$

and extend it to $\tau(\mathcal{J}_2) \cup \tau(\mathcal{J}_3) \cup \gamma_2(\tau(\mathcal{J}_2)) \cup \gamma_2(\tau(\mathcal{J}_3))$ by defining for $A \in \gamma_2(\tau(\mathcal{J}_2)) \cup \gamma_2(\tau(\mathcal{J}_3))$:

$$\gamma_2(A) := \gamma_2^{-1}(A).$$

Again it follows as in Lemma 2.2.24 that

Lemma 2.2.25. *The collections*

$$\{\tau(A), \gamma_2(\tau(A)), \tau(A) \cup \gamma_2(\tau(A)) : A \in \mathcal{J}_2\} \quad \textit{and}$$
$$\{\tau(A), \gamma_2(\tau(A)), \tau(A) \cup \gamma_2(\tau(A)) : A \in \mathcal{J}_3\}$$

are nested.

We can now summarize our considerations and thus prove our main theorem in this section

Proof of Theorem 2.2.17. The collections $\{A, \tau(A), A \cup \tau(A) : A \in \mathcal{H}\}$ and $\{A, \tau(A), A \cup \tau(A) : A \in \mathcal{I}_i\}$ for $i \in \{1, 2\}$ are nested by the Lemmas 2.2.18 and 2.2.21 respectively. Lemmas 2.2.24 and 2.2.25 yield that the collections $\{A, \gamma_1(A), A \cup \gamma_1(A) : A \in \mathcal{J}_1\}, \{A, \gamma_1(A), A \cup \gamma_1(A) : A \in \mathcal{J}_3\}, \{\tau(A), \gamma_2(\tau(A)), \tau(A) \cup \gamma_2(\tau(A)) : A \in \mathcal{J}_2\}$ and $\{\tau(A), \gamma_2(\tau(A)), \tau(A) \cup \gamma_2(\tau(A)) : A \in \mathcal{J}_3\}$ are nested. The additional properties of the mappings γ_1 and γ_2 follow from the definition. We have thus completely proved the theorem. □

2.3 Decomposing Singular Integral Operators

In this section we decompose singular integral operators as absolutely convergent series of simple rearrangements, shifts and two paraproducts.

2.3.1 Integral Operators

We now define the integral operator K with the kernel $k : X \times X \to \mathbb{C}$, $k \in L^2(X \times X)$ by

$$K(f)(x) := \int_X k(x,y) f(y) \; d\mu(y)$$

for $f \in L^2_E(X)$ and E is a UMD Banach space. We assume structural estimates on k, in particular a strong off-diagonal decay and also a weak boundedness estimate on the diagonal. This is formalized with the following definition using the orthonormal basis from Lemma 2.2.9. First recall that q was the number with that $q^{\operatorname{lev} A}$ represents roughly the "size" of A.

Definition 2.3.1. Let $k \in L^2(X \times X)$. We say that k is an *admissible kernel* if there exist $C_S > 0$ and $\delta > 0$ such that $|\langle k, 1_X \otimes 1_X \rangle| \leq C_S$ and for all $Q, R \in \mathcal{E}(\mathcal{A})$ with $\operatorname{lev} Q = \operatorname{lev} R$ we have

$$\left|\left\langle k, d_Q^{(\varepsilon_1)} \otimes d_R^{(\varepsilon_2)} \right\rangle\right| \leq C_S \left(1 + \frac{d(\operatorname{pre} Q, \operatorname{pre} R)}{q^{\operatorname{lev} Q + 1}}\right)^{-1-\delta}, \quad \varepsilon = (\varepsilon_1, \varepsilon_2) \in \{0,1\}^2 \backslash \{(0,0)\}. \tag{2.3.1}$$

In this section we provide vector valued norm estimates for integral operators defined by admissible kernels. We point out that the L^p-norm of the integral operator depends just on the structural constants C_S and δ, the value of p and the BMO-norms of $K(1), K^*(1)$. In particular, the L^2-norm of k is not present in the estimates. From now on, we work with admissible kernels k. We expand the kernel k in the isotropic orthonormal basis introduced in Section 2.2.4. The division of Z into three groups (see (2.2.7)) gives rise to the following decomposition of the kernel k. We let

$$\begin{aligned}
k_1 &:= \sum_{n=0}^{\infty} \sum_{A,B \in \mathcal{A}_{-n}} \sum_{Q \in \mathcal{E}(A)} \sum_{R \in \mathcal{E}(B)} \langle k, d_Q \otimes d_R \rangle \, d_Q \otimes d_R, \\
k_2 &:= \sum_{n=0}^{\infty} \sum_{A,B \in \mathcal{A}_{-n}} \sum_{Q \in \mathcal{E}(A)} \frac{\langle k, d_Q \otimes 1_B \rangle}{\mu(B)} d_Q \otimes 1_B, \\
k_3 &:= \sum_{n=0}^{\infty} \sum_{A,B \in \mathcal{A}_{-n}} \sum_{R \in \mathcal{E}(B)} \frac{\langle k, 1_A \otimes d_R \rangle}{\mu(A)} 1_A \otimes d_R.
\end{aligned}$$

If we decompose k into the isotropic orthonormal basis we see that

$$k = \int_X \int_X k(s,t) d\mu(s) d\mu(t) + k_1 + k_2 + k_3.$$

We note the following identities (which follow from $X = \bigcup_{A\in\mathcal{A}_{-n}} A$ for every $n \in \mathbb{N}_0$)

$$\sum_{B\in\mathcal{A}_{-n}} \langle k, d_Q \otimes 1_B\rangle = \langle K(1), d_Q\rangle\,, \qquad \sum_{A\in\mathcal{A}_{-n}} \langle k, 1_A \otimes d_R\rangle = \langle K^*(1), d_R\rangle\,.$$

Now we let $f \in L^p_E(X)$ and $g \in L^{p'}_{E'}(X)$ (for $1/p + 1/p' = 1$ and $1 < p < \infty$) be finite linear combinations of Haar functions and E be a UMD-space. Then we see that k_2 has the further decomposition

$$\langle k_2, g \otimes f\rangle = B_2(f,g) + \widetilde{B_2}(f,g), \tag{2.3.2}$$

where

$$\begin{aligned} B_2(f,g) &:= \sum_{n=0}^{\infty}\sum_{A,B\in\mathcal{A}_{-n}}\sum_{Q\in\mathcal{E}(A)} \frac{\langle k, d_Q\otimes 1_B\rangle}{\mu(B)} \left\langle d_Q \otimes \left(1_B - \frac{\mu(B)}{\mu(A)}1_A\right), g\otimes f\right\rangle \quad \text{and} \\ \widetilde{B_2}(f,g) &:= \sum_{n=0}^{\infty}\sum_{A\in\mathcal{A}_{-n}}\sum_{Q\in\mathcal{E}(A)} \frac{\langle K(1), d_Q\rangle}{\mu(A)} \langle d_Q\otimes 1_A, g\otimes f\rangle\,. \end{aligned} \tag{2.3.3}$$

We also decompose k_3 further and get the following identity, which is valid in $L^2_E(X)$

$$\int_X k_3(x,y)f(y)d\mu(y) = K_3 f(x) + \widetilde{K_3}f(x), \tag{2.3.4}$$

where

$$\begin{aligned} K_3 f(x) &:= \sum_{n=0}^{\infty}\sum_{A,B\in\mathcal{A}_{-n}}\sum_{R\in\mathcal{E}(B)} \frac{\langle k, 1_A\otimes d_R\rangle}{\mu(A)} \langle d_R, f\rangle \left(1_A(x) - \frac{\mu(A)}{\mu(B)}1_B(x)\right) \quad \text{and} \\ \widetilde{K_3}f(x) &:= \sum_{n=0}^{\infty}\sum_{B\in\mathcal{A}_{-n}}\sum_{R\in\mathcal{E}(B)} \frac{\langle K^*(1), d_R\rangle}{\mu(B)} \langle d_R, f\rangle\, 1_B(x) \end{aligned} \tag{2.3.5}$$

Furthermore we set

$$K_1 f(x) := \int_X k_1(x,y)f(y)\ d\mu(y).$$

2.3.2 Statement of the Main Theorems

Recall the definition of the σ-algebras $\mathcal{F}_k^{\text{lev}}$, which were defined to be the σ-algebras generated by the dyadic cubes of level $-k$. In this section (Section 2.3), each occurrence of BMO means the space $\text{BMO}(X, \mathcal{F}_k^{\text{lev}})$ with these σ-algebras. Further, we let E be a UMD-space (see Section 1.3.3). We now state the main result in this chapter.

Theorem 2.3.2. *Let K be the integral operator defined in the last section satisfying (2.3.1). Then K, initially defined on finite linear combinations of Haar functions, extends linearly to a unique bounded operator on L^p_E for $1 < p < \infty$, i.e. we have a constant C_K such that*

$$\|K : L^p_E(X) \to L^p_E(X)\| \le C_K$$

and C_K depends only on p, the BMO*–norms of $K(1)$ and $K^*(1)$, the constants C_S and δ coming from the structural estimate (2.3.1) and the* UMD *constant of E.*

The starting point and basic idea of the proof is the following decomposition of the bilinear form $\langle Kf, g\rangle$:

$$\langle Kf, g\rangle = \widetilde{B_2}(f,g) + \left\langle \widetilde{K_3}f, g\right\rangle + \sum_{m=0}^{\infty} \langle k_{1,m}, g\otimes f\rangle + B_{2,m}(f,g) + \langle K_{3,m}f, g\rangle \,, \quad (2.3.6)$$

where these operators are defined in (2.3.3),(2.3.5),(2.3.8),(2.3.9) and (2.3.10). Clearly, we assumed here that k has mean zero with respect to the product measure $\mu\otimes\mu$. In fact, as we will see in the proof of Theorem 2.3.3 and the proof of Theorem 2.3.4, this decomposition of the operator K can be further split as

$$P^*_{K(1)} + P_{K^*(1)} + \sum_{m=0}^{\infty}\sum_{i=1}^{M_1 q^m} \left(\sum_{j,k=1}^{N-1} T^{(j,k)}_{m,i} \circ \mathcal{M}^{(j,k)}_{m,i} + \sum_{j=1}^{N-1} W^{(j)}_{m,i} \circ \widetilde{\mathcal{M}}^{(j)}_{m,i} + \sum_{k=1}^{N-1} U^{(k)}_{m,i} \circ \mathcal{M}^{(k)}_{m,i} \right), \quad (2.3.7)$$

where $P_{K(1)}$ and $P_{K^*(1)}$ are paraproducts defined in the proof of Theorem 2.3.4, $T^{(j,k)}_{m,i}, W^{(j)}_{m,i}, U^{(k)}_{m,i}$ are shift and rearrangement operators defined in Section 2.3.3 and the operators $\mathcal{M}$ are suitable Haar multipliers. The five summands in (2.3.6) correspond to the summands in (2.3.7) in the same order.

Remark. We note explicitly that the constant C_K in the last theorem does *not* depend on the L^2-norm of $k(x,y)$, which is the crucial fact about the statement. It thus can be shown that

1. Theorem 2.3.2 yields a direct generalization of T. Figiel's $T(1)$ theorem [Fig90] to spaces of homogeneous type (see Appendix A.2) and
2. Theorem 2.3.2 yields a direct generalization of Coifman's $T(1)$ theorem (as presented in [Chr90a], for the origin of the method see also [CJS89]) to vector valued singular integral operators given by standard kernels. For a proof, that our structural bounds (2.3.1) imply that k is a 'standard kernel', see Appendix A.3.

According to the decomposition of $\mathcal{C}$ we split k_1, B_1 and K_2 further and define

$$k_{1,m} := \sum_{(A,B)\in\mathcal{C}_m}\sum_{Q\in\mathcal{E}(A)}\sum_{R\in\mathcal{E}(B)} \langle k, d_Q\otimes d_R\rangle\, d_Q\otimes d_R, \quad \text{in } L^2(X\times X), \quad (2.3.8)$$

$$B_{2,m}(f,g) := \sum_{(A,B)\in\mathcal{C}_m}\sum_{Q\in\mathcal{E}(A)} \frac{\langle k, d_Q\otimes 1_B\rangle}{\mu(B)} \left\langle d_Q \otimes \left(1_B - \frac{\mu(B)}{\mu(A)}1_A\right), g\otimes f\right\rangle, \quad (2.3.9)$$

$$K_{3,m}f := \sum_{(A,B)\in\mathcal{C}_m}\sum_{R\in\mathcal{E}(B)} \frac{\langle k, 1_A\otimes d_R\rangle}{\mu(A)} \langle d_R, f\rangle \left(1_A - \frac{\mu(A)}{\mu(B)}1_B\right), \quad \text{in } L^2_E(X). \quad (2.3.10)$$

Associated to the kernel $k_{1,m}$ we define the integral operator

$$K_{1,m}(f)(x) := \int_X k_{1,m}(x,y) f(y)\; d\mu(y).$$

In later sections we prove the following theorems, from which our main result (Theorem 2.3.2) follows. In the subsequent theorem, δ is the positive number coming from the structural estimate (2.3.1) and q is the constant appearing in Theorem 2.2.3.

Theorem 2.3.3. *For all $1 < p < \infty$ there exists a constant C_p depending only on p, X, the* UMD *constant of E and C_S from (2.3.1), such that for all $f \in L^p_E(X), g \in L^{p'}_{E'}(X)$, which are finite linear combinations of Haar functions, the operators $K_{1,m}, K_{3,m}$ and the bilinear form $B_{2,m}$ satisfy the following estimates:*

$$\|K_{1,m}(f)\|_{L^p_E(X)} \le C_p(m+1)q^{-m\delta} \|f\|_{L^p_E(X)}, \tag{2.3.11}$$

$$\|K_{3,m}(f)\|_{L^p_E(X)} \le C_p(m+1)q^{-m\delta} \|f\|_{L^p_E(X)}, \tag{2.3.12}$$

$$|B_{2,m}(f,g)| \le C_p(m+1)q^{-m\delta} \|f\|_{L^p_E(X)} \|g\|_{L^{p'}_{E'}(X)}. \tag{2.3.13}$$

Here, $p' = p/(p-1)$ denotes the conjugate exponent to p.

Remark. For this theorem, we need the L^p-boundedness of rearrangement and shift operators, which will be introduced in Section 2.3.3 and the boundedness of these operators will be proved in Sections 2.3.4 and 2.3.5.

Theorem 2.3.4. *For all $1 < p < \infty$ there exists a constant C_p, which depends only on p, X and the* UMD *constant of E such that for all $f \in L^p_E(X), g \in L^{p'}_{E'}(X)$ which are finite linear combinations of Haar functions, the operator $\widetilde{K_3}$ and the bilinear form $\widetilde{B_2}$ satisfy the estimates*

$$\left|\widetilde{B_2}(f,g)\right| \le C_p \|K(1)\|_{\mathrm{BMO}} \|f\|_{L^p_E(X)} \|g\|_{L^{p'}_{E'}(X)}, \tag{2.3.14}$$

$$\left|\left\langle \widetilde{K_3} f, g\right\rangle\right| \le C_p \|K^*(1)\|_{\mathrm{BMO}} \|f\|_{L^p_E(X)} \|g\|_{L^{p'}_{E'}(X)}. \tag{2.3.15}$$

Again, $p' = p/(p-1)$ is the conjugate exponent to p.

Proof. For the proof we use paraproduct operators which are formally given by

$$(P_a f)(x) = \sum_{n=0}^{\infty} \sum_{B \in \mathcal{A}_{-n}} \sum_{R \in \mathcal{E}(B)} \frac{\langle a, d_R\rangle}{\mu(B)} \langle d_R, f\rangle \, 1_B(x), \tag{2.3.16}$$

where $a \in \mathrm{BMO}$. Observe that P_a is the linear extension of the mapping

$$d_R \longmapsto \langle a, d_R\rangle \frac{1_B}{\mu(B)}, \quad R \in \mathcal{E}(B) \quad \text{and}$$
$$1_X \longmapsto 0,$$

so that for finite linear combinations of Haar functions f, g we have

$$\left|\widetilde{B_2}(f,g)\right| = \left|\langle P_{K(1)} g, f\rangle\right| \quad \text{and} \quad \left|\left\langle \widetilde{K_3} f, g\right\rangle\right| = \left|\langle P_{K^*(1)} f, g\rangle\right|. \tag{2.3.17}$$

Now let both $f : X \to E$ and $a : X \to \mathbb{R}$ be finite linear combinations of Haar functions. We then consider the bilinear operation (as in Chapter 1.3.7)

$$P(a,f) := \sum_k (\mathbb{E}_k f)(\Delta_{k+1} a),$$

which has an immediate connection to the paraproduct operator in (2.3.16), since we can compute for $R \in \mathcal{E}(B)$

$$\langle P(a,f), d_R \rangle = \sum_k \langle (\mathbb{E}_k f)(\Delta_{k+1} a), d_R \rangle = \langle a, d_R \rangle \frac{1}{\mu(B)} \int_B f \, d\mu = \langle f, P_a d_R \rangle .$$

Additionally, $\langle P(a,f), 1 \rangle = \langle f, P_a 1 \rangle$, since

$$\langle P(a,f), 1 \rangle = \sum_k \mathbb{E}[(\mathbb{E}_k f)\,(\Delta_{k+1} a)] = 0 = \langle f, P_a 1 \rangle , \tag{2.3.18}$$

so $P(a,\cdot)$ is the adjoint of P_a. Now we use a result from the preliminary chapter, namely the L^p_E-estimate (1.3.11) of $P(a,\cdot)$, which reads

$$\|P(a,f)\|_{L^p_E(X)} \le C \, \|a\|_{BMO} \, \|f\|_{L^p_E(X)} \tag{2.3.19}$$

for $f \in L^p_E(X)$ and $a \in$ BMO.

With the L^p_E-boundedness of $P(a,\cdot)$, (2.3.17) and the fact that P_a is the adjoint of $P(a,\cdot)$, we finally get that

$$\left| \widetilde{B_2}(f,g) \right| = |\langle P(K(1),f), g \rangle| \le C \, \|K(1)\|_{\mathrm{BMO}} \, \|f\|_{L^p_E(X)} \, \|g\|_{L^{p'}_{E'}(X)}$$

and

$$\left| \left\langle \widetilde{K_3} f, g \right\rangle \right| = |\langle f, P(K^*(1), g) \rangle| \le C \, \|K^*(1)\|_{\mathrm{BMO}} \, \|f\|_{L^p_E(X)} \, \|g\|_{L^{p'}_{E'}(X)} ,$$

since E is a UMD–space and thus reflexive. □

Proof of Theorem 2.3.2. For $1/p + 1/p' = 1$, let $f \in L^p_E(X)$ and $g \in L^{p'}_{E'}(X)$ be finite linear combinations of Haar functions, then we have

$$\langle Kf, g \rangle = \widetilde{B_2}(f,g) + \left\langle \widetilde{K_3} f, g \right\rangle + \sum_{m=0}^{\infty} \langle k_{1,m}, g \otimes f \rangle + B_{2,m}(f,g) + \langle K_{3,m} f, g \rangle ,$$

where these operators are defined in (2.3.3),(2.3.5),(2.3.8),(2.3.9) and (2.3.10). Thus we obtain from Theorems 2.3.3 and 2.3.4 that there exists a constant C_K which has only the stated dependences and we have

$$|\, \langle Kf, g \rangle \,| \le C_K \, \|f\|_{L^p_E(X)} \, \|g\|_{L^{p'}_{E'}(X)} .$$

Hence for fixed $f \in L^p_E(X)$ which is a finite linear combination of Haar functions, the functional S_f defined by

$$S_f : g \mapsto \langle Kf, g \rangle$$

is bounded on the subspace U consisting of finite linear combinations of Haar functions of $L^{p'}_{E'}(X)$. Since U is dense in $L^{p'}_{E'}(X)$, it has a unique continuous extension to the whole space $L^{p'}_{E'}(X)$. Recall now that UMD-spaces are reflexive, and so $(L^{p'}_{E'}(X))'$ is canonically identified with $L^p_E(X)$. Hence there exists $z \in L^p_E(X)$ such that

$$\langle z, g \rangle = \langle Kf, g \rangle \ \text{ for all } g \in U \quad \text{and} \quad \|z\|_{L^p_E(X)} \le C_K \, \|f\|_{L^p_E(X)} .$$

We get that $z = Kf$ since they have the same Haar coefficients, and so

$$\|Kf\|_{L^p_E(X)} \leq C_K \|f\|_{L^p_E(X)}$$

for all finite linear combinations of Haar functions f. Since again these functions are dense in $L^p_E(X)$, K has a unique bounded linear extension to all of $L^p_E(X)$ and the theorem is proved. □

The rest of Section 2.3 is now devoted to proving Theorem 2.3.3.

2.3.3 Rearrangement and Shift Operators

Definition 2.2.12 (The definition of the injection τ on $\mathcal{A}_{m,i}$) gives rise to rearrangement and shift operators, which are closely related to the integral operators K_1, K_3 and the bilinear form B_2. For $m \in \mathbb{N}_0$, $1 \leq i \leq M_1 q^m$ (see Proposition 2.2.11) we define

$$U^{(k)}_{m,i}(f) := \sum_{A \in \mathcal{A}_{m,i}} \frac{\langle d_{Q_k(\tau(A))}, f \rangle}{\sqrt{\mu(A)}} \left(1_A - \frac{\mu(A)}{\mu(\tau(A))} 1_{\tau(A)} \right) \tag{2.3.20}$$

$$T^{(j,k)}_{m,i}(f) := \sum_{A \in \mathcal{A}_{m,i}} \langle d_{Q_k(\tau(A))}, f \rangle \, d_{Q_j(A)}, \tag{2.3.21}$$

where f is a finite linear combination of Haar functions and $Q_j(A)$ is any enumeration of the elements in $\mathcal{E}(A)$. If the parameter k is greater than the number $N(A)$ of Haar functions corresponding to children of A, we simply set $d_{Q_k(A)} \equiv 0$.

Remark. We see that $U^{(k)}_{m,i}$ is the linear extension of the map

$$d_{Q_k(\tau(A))} \longmapsto \frac{1}{\sqrt{\mu(A)}} \left(1_A - \frac{\mu(A)}{\mu(\tau(A))} 1_{\tau(A)} \right), \quad \text{for } 1 \leq k \leq N(\tau(A)),$$

with $A \in \mathcal{A}_{m,i}$. Analogously the mapping $T^{(j,k)}_{m,i}(f)$ is the linear extension of

$$d_{Q_k(\tau(A))} \longmapsto d_{Q_j(A)}, \quad \text{for } 1 \leq k \leq N(\tau(A)),$$

where $A \in \mathcal{A}_{m,i}$.

In order to show Theorem 2.3.3, we prove the following L^p-bounds of the operators $U^{(k)}_{m,i}$ and $T^{(j,k)}_{m,i}$:

Proposition 2.3.5. *The operators $U^{(k)}_{m,i}$ and $T^{(j,k)}_{m,i}$ satisfy the $L^p_E(X)$-estimate* ($1 < p < \infty$)

$$\left\| U^{(k)}_{m,i} : L^p_E(X) \to L^p_E(X) \right\| \leq C_p(m+1), \tag{2.3.22}$$

$$\left\| T^{(j,k)}_{m,i} : L^p_E(X) \to L^p_E(X) \right\| \leq C_p(m+1). \tag{2.3.23}$$

for all $1 \leq j, k \leq N-1$, where C_p depends only on p, X and the UMD*–constant of E. Here, as in Section 2.2, N is the maximal number of children a dyadic cube can have.*

The rough idea of the proof of these bounds is the following: We prove a version of Proposition 2.3.5 under the constraint that we restrict the sum in (2.3.20) and (2.3.21) from $\mathcal{A}_{m,i}$ to a collection that satisfies the so called Figiel's compatibility condition. In this case we get a bound, which is independent of m. Thereafter we invoke the decomposition of $\mathcal{A}_{m,i}$ into such subcollections introduced in Section 2.2.7.

2.3.4 Figiel's Compatibility Condition

Here we review the martingale estimates of rearrangement operators that satisfy Figiel's compatibility condition. We follow [Fig88] and the expositions [FW01], [Mül05].

Definition 2.3.6. Let $\mathcal{D} \subseteq \mathcal{A}_{m,i}$ be a subset of $\mathcal{A}_{m,i}$ and $\tau : \mathcal{D} \to \mathcal{A}$ be an injective map. We say that the pair $(\tau, \mathcal{D})$ satisfies *Figiel's compatibility condition* if the collection

$$\mathcal{Z} := \{A, \tau(A), A \cup \tau(A) : A \in \mathcal{D}\}$$

is nested, $\operatorname{lev} A = \operatorname{lev} \tau(A)$ and $\tau(A) \notin \mathcal{D}$ for all $A \in \mathcal{D}$

Recall that a collection of sets $\mathcal{Z}$ is said to be *nested*, if for every choice $A, B \in \mathcal{Z}$ it holds that either

$$A \subseteq B \quad \text{or} \quad B \subseteq A \quad \text{or} \quad A \cap B = \emptyset.$$

We remark that if $(\tau, \mathcal{D})$ satisfies Figiel's compatibility condition, the pair $(\tau^{-1}, \tau(\mathcal{D}))$ also satisfies Figiel's compatibility condition. Then the following theorems concerning the boundedness of the operators T and U hold.

Theorem 2.3.7. *Let $(\tau, \mathcal{D})$ satisfy the compatibility condition. Then the operator*

$$T_{m,i}^{(j,k),\mathcal{D}} f := \sum_{A \in \mathcal{D}} \left\langle d_{Q_k(\tau(A))}, f \right\rangle d_{Q_j(A)}$$

is bounded on L_E^p for all $1 \leq j, k \leq N-1$ and satisfies the estimate

$$\left\| T_{m,i}^{(j,k),\mathcal{D}} f \right\|_{L_E^p(X)} \leq C_p \|f\|_{L_E^p(X)}$$

where C_p depends only on p, X and the UMD*–constant of E.*

Proof. We divide $\mathcal{Z}$ into generations and let $G_n(\mathcal{Z})$ for $n \in \mathbb{N}_0$ be the n-th generation. This is possible, since $\mathcal{Z}$ is nested. To $G_n(\mathcal{Z})$ we define appendant $\sigma-$algebras

$$\mathcal{F}_n := \sigma\left(\bigcup_{j=0}^{n} G_j(\mathcal{Z})\right).$$

Now for every dyadic cube $A \in \mathcal{D}$, there exists a unique index $n \in \mathbb{N}_0$ such that the set $A \cup \tau(A)$ is an atom in $\mathcal{F}_n$. Let

$$K_n := \{A \in \mathcal{Z} : A \cup \tau(A) \text{ is an atom of } \mathcal{F}_n\}.$$

We then have for every $A \in K_n$ that

$$\mathbb{E}(1_{\tau(A)}|\mathcal{F}_n) = \frac{\mu(\tau(A))}{\mu(A \cup \tau(A))}(1_A + 1_{\tau(A)}). \tag{2.3.24}$$

We let f be a finite linear combination of basis functions d_R :

$$f = \sum_{l=1}^{K} a_l d_{Q_k(\tau(A_l))}, \quad \text{where } A_l \in \mathcal{D} \text{ and } a_l \in E \text{ for all } l.$$

We assume further that for each $\tau(A_l)$ the number of direct descendants $N(\tau(A_l))$ is greater or equal k (otherwise there would not be a k-th Haar function). Then we have

$$T_{m,i}^{(j,k),\mathcal{D}} f = \sum_{l=1}^{K} a_l d_{Q_j(A_l)}.$$

In the following C denotes a constant which depends only on p, X and E and it possibly changes from estimate to estimate. Furthermore, to shorten notation, the expression $\|\cdot\|_p$ means the $L_E^p(X)$ norm. The unconditionality of the system $\{d_Q\}$ now yields

$$\left\| T_{m,i}^{(j,k),\mathcal{D}} f \right\|_p^p \leq C \int_0^1 \left\| \sum_{l=1}^{K} a_l r_l(t) d_{Q_j(A_l)} \right\|_p^p dt.$$

With Kahane's contraction principle (Theorem 1.2.1), Lemma 2.2.6 and (2.3.24) we further get

$$\begin{aligned}
\int_0^1 \left\| \sum_{l=1}^{K} a_l r_l(t) d_{Q_j(A_l)} \right\|_p^p dt &\leq C \int_0^1 \left\| \sum_{l=1}^{K} a_l r_l(t) \frac{1_{A_l}}{\sqrt{\mu(A_l)}} \right\|_p^p dt \\
&= C \int_0^1 \left\| \sum_{l=1}^{K} a_l r_l(t) \frac{c_{A_l} \mathbb{E}(1_{\tau(A_l)}|\mathcal{F}_{n(l)}) - 1_{\tau(A_l)}}{\sqrt{\mu(A_l)}} \right\|_p^p dt,
\end{aligned}$$

where we set $c_A := \frac{\mu(A \cup \tau(A))}{\mu(\tau(A))}$, $n(l)$ such that we have $A_l \in K_{n(l)}$ and as usual (r_l) are independent Rademacher functions. Using Bourgain's version of Stein's martingale inequality Theorem 1.3.5 and Kahane's contraction principle Theorem 1.2.1, we get that this is less or equal

$$C \int_0^1 \left\| \sum_{l=1}^{K} a_l r_l(t) \frac{1_{\tau(A_l)}}{\sqrt{\mu(A_l)}} \right\|_p^p dt.$$

Now let Σ_l be for $1 \leq l \leq K$ an increasing sequence of σ-algebras such that $\tau(A_l)$ is an atom in Σ_l. Then with Kahane's contraction principle we get that the last display is less or equal

$$C \int_0^1 \left\| \sum_{l=1}^{K} a_l r_l(t) \mathbb{E}(|d_{Q_k(\tau(A_l))}| \; |\Sigma_l) \right\|_p^p dt,$$

by Lemma 2.2.6. Stein's martingale inequality Theorem 1.3.5 and again Kahane's contraction principle Theorem 1.2.1 supply us then with the final estimate

$$\left\|T_{m,i}^{(j,k),\mathcal{D}} f\right\|_p^p \leq C \int_0^1 \left\|\sum_{i=1}^{l} a_{R_i} r_i(t) d_{R_i}\right\|_p^p dt,$$

and so, by the unconditionality of martingale differences, we get the assertion of the theorem. □

Theorem 2.3.8. *Let $(\tau, \mathcal{D})$ satisfy the compatibility condition. Then the operator*

$$U_{m,i}^{(k),\mathcal{D}} f := \sum_{A \in \mathcal{D}} \frac{\langle d_{Q_k(\tau(A))}, f \rangle}{\sqrt{\mu(A)}} \left(1_A - \frac{\mu(A)}{\mu(\tau(A))} 1_{\tau(A)} \right)$$

is bounded on L_E^p and satisfies the estimate

$$\left\|U_{m,i}^{(k),\mathcal{D}} f\right\|_{L_E^p(X)} \leq C_p \|f\|_{L_E^p(X)},$$

where C_p depends only on p, X and the UMD*–constant of E.*

Proof. We define $\mathcal{F}_n$ and K_n as in the previous proof. Let f be a finite linear combination of basis functions

$$f = \sum_{l=1}^{K} a_l d_{Q_k(\tau(A_l))}, \quad \text{where } A_l \in \mathcal{D}, a_l \in E \text{ for all } 1 \leq l \leq K.$$

Then we have

$$U_{m,i}^{(k),\mathcal{D}} f = \sum_{l=1}^{K} \frac{a_l}{\sqrt{\mu(A_l)}} \left(1_{A_l} - \frac{\mu(A_l)}{\mu(\tau(A_l))} 1_{\tau(A_l)} \right).$$

Since the collection $\{A, \tau(A), A \cup \tau(A) : A \in \mathcal{D}\}$ is nested, we can enumerate the collection of functions $1_{A_i} - \frac{\mu(A_i)}{\mu(\tau(A_i))} 1_{\tau(A_i)}$ where $1 \leq i \leq l$, such that they become martingale differences (observe that the mean value of these functions vanishes). We employ the same convention concerning C and $\|\cdot\|_p$ as in the proof of the foregoing theorem. Then we obtain, since martingale differences are unconditional in L_E^p

$$\left\|U_{m,i}^{(k),\mathcal{D}} f\right\|_p^p \leq C \int_0^1 \left\|\sum_{l=1}^{K} \frac{a_l}{\sqrt{\mu(A_l)}} \left(1_{A_l} - \frac{\mu(A_l)}{\mu(\tau(A_l))} 1_{\tau(A_l)} \right) r_l(t)\right\|_p^p dt,$$

where (r_l) are independent Rademacher functions. Using now identity (2.3.24) to eliminate the characteristic function 1_{A_l} and performing the same steps as in the proof of the last theorem, we obtain the conclusion. □

The proofs of the foregoing two theorems are slight modifications of the analogous results for the Haar system in $\mathbb{R}$ in [Mül05].

Remark. If we apply this theorem to the collection $\tau(\mathcal{D})$ and the map τ^{-1}, we get that the operator

$$f \mapsto W_{m,i}^{(k),\mathcal{D}}(f) := \sum_{A\in\mathcal{D}} \frac{\langle d_{Q_k(A)}, f\rangle}{\sqrt{\mu(\tau(A))}} \left(1_{\tau(A)} - \frac{\mu(\tau(A))}{\mu(A)} 1_A\right)$$

is bounded on L^p_E. $W_{m,i}^{(k),\mathcal{D}}$ is the linear extension of the mapping

$$d_{Q_k(A)} \longmapsto \frac{1}{\sqrt{\mu(\tau(A))}} \left(1_{\tau(A)} - \frac{\mu(\tau(A))}{\mu(A)} 1_A\right),$$

for $A \in \mathcal{D}$.

2.3.5 The Boundedness of the Operators $W_{m,i}^{(k)}, U_{m,i}^{(k)}, T_{m,i}^{(j,k)}$

Using the decomposition theorems proved in Section 2.2, we are now able to reduce the general case of Proposition 2.3.5 to the special case of nested collections proved in the preceding Section 2.3.4.

Proof of Proposition 2.3.5. If we invoke the decomposition results of Chapter 2.2, we see that $\mathcal{C}_{m,i}$ splits into M collections $\mathcal{G}$, where M is constant. Further, every $\mathcal{A}_\mathcal{G}$ splits into $L(m+1)$ collections $\mathcal{L}$. $\mathcal{L}$ decomposes in $\mathcal{H}, \mathcal{I}_1, \mathcal{I}_2, \mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3$, where on $\mathcal{H}$, $\mathcal{I}_1$ and $\mathcal{I}_2$, the operators $W_{m,i}^{(k)}, U_{m,i}^{(k)}$ and $T_{m,i}^{(j,k)}$ are bounded by a constant which is independent of m. Since the collections $\mathcal{H}$, $\mathcal{I}_1$ and $\mathcal{I}_2$ satisfy Figiel's compatibility condition (with the injection τ) by Theorem 2.2.17, this follows directly from Theorems 2.3.7, 2.3.8 and the Remark after them. The collections $\mathcal{J}_i$, $i \in \{1, 2, 3\}$ need further arguments. For the following we fix an index $1 \le j \le N-1$ and define the following map on $\gamma_1(\mathcal{J}_1)$

$$\begin{aligned} \rho : \gamma_1(\mathcal{J}_1) &\to \mathcal{A} \\ A &\longmapsto \tau \circ \gamma_1(A). \end{aligned}$$

Since the mapping $(\gamma_1, \mathcal{J}_1)$ (and hence also $(\gamma_1^{-1}, \gamma_1(\mathcal{J}_1))$) satisfies Figiel's compatibility condition (note Lemma 2.2.24), we see from Theorem 2.3.7 and Theorem 2.3.8 that the linear extensions of the mappings

$$T_{\gamma_1}^{(j,k)} : d_{Q_k(A)} \longmapsto d_{Q_j(\gamma_1(A))}, \quad W_{\gamma_1}^{(k)} : d_{Q_k(A)} \longmapsto \frac{1}{\sqrt{\mu(\gamma_1(A))}} \left(1_{\gamma_1(A)} - \frac{\mu(\gamma_1(A))}{\mu(A)} 1_A\right)$$

where $A \in \mathcal{J}_1$, are bounded on L^p. In Theorem 2.2.17 we constructed a decomposition of $\mathcal{I}$ into $\mathcal{I}_1$ and $\mathcal{I}_2$ which both satisfied Figiel's compatibility condition with the injection τ. Since with γ_1 we moved sets in $\mathcal{J}_1$ away from the boundary of their arithmetic predecessors, we are in the same position for the collection $\gamma_1(\mathcal{J}_1)$ and the injection ρ, since if we again perform a decomposition like in Proposition 2.2.13 we are able to use the Remark after the proof of Lemma 2.2.21. We thus obtain from the $\mathcal{I}$-part of Theorem 2.2.17 and again from Theorems 2.3.7, 2.3.8 and the Remark after them that the linear extension of the mappings

$$T_{\rho}^{(j,k)} : d_{Q_k(A)} \longmapsto d_{Q_j(\rho(A))}, \quad W_{\rho}^{(k)} : d_{Q_k(A)} \longmapsto \frac{1}{\sqrt{\mu(\rho(A))}} \left(1_{\rho(A)} - \frac{\mu(\rho(A))}{\mu(A)} 1_A\right)$$

where $A \in \gamma_1(\mathcal{J}_1)$, are bounded on L^p_E by a constant which depends only on X. For the same reason, we may even replace ρ by ρ^{-1} and the assertions stay valid. We conclude that the composition

$$T^{(j,1)}_{\gamma_1} \circ T^{(1,k)}_{\rho^{-1}} =: T^{(j,k)}_{\tau^{-1}}$$

is bounded on L^p_E and it is the linear extension of the map

$$d_{Q_k(\tau(A))} \longmapsto d_{Q_j(A)},$$

where $A \in \mathcal{J}_1$. We remark that $T^{(j,k)}_{\tau^{-1}}$ is the shift operator $T^{(j,k),\,\mathcal{J}_1}_{m,i}$, which is thus shown to be bounded. Now we come to the linear extension of the map

$$d_{Q_k(A)} \longmapsto \frac{1}{\sqrt{\mu(\tau(A))}}\left(1_{\tau(A)} - \frac{\mu(\tau(A))}{\mu(A)} 1_A\right), \quad \text{for } A \in \mathcal{J}_1$$

which is the mapping $W^{(k),\mathcal{J}_1}_{m,i}$. For finite linear combinations of Haar functions $f = \sum_l a_l d_{Q_l}$, where $Q_l = Q_k(A_l)$,$A_l \in \mathcal{J}_1$ and $a_l \in E$, $W^{(k),\mathcal{J}_1}_{m,i}$ has the representation

$$W^{(k),\mathcal{J}_1}_{m,i} f = (W^{(1)}_{\rho} \circ T^{(1,k)}_{\gamma_1})(f) + \sum a_{Q_l} \sqrt{\frac{\mu(\tau(A_l))}{\mu(\gamma_1(A_l))}} W^{(k)}_{\gamma_1}(d_{Q_l}).$$

With the unconditionality of the $\{d_Q\}$ and Kahane's contraction principle, we conclude that $W^{(k),\mathcal{J}_1}_{m,i}$ is bounded on L^p_E. Analogously, for $f = \sum_l a_l d_{Q_l}$, where $Q_l = Q_k(\tau(A_l))$ and $A_l \in \mathcal{J}_1$, we have the representation

$$U^{(k),\mathcal{J}_1}_{m,i} f = \sum_l a_l \sqrt{\frac{\mu(A_l)}{\mu(\gamma_1(A_l))}} W^{(k)}_{\rho^{-1}}(d_{Q_l}) + (W^{(1)}_{\gamma_1^{-1}} \circ T^{(1,k)}_{\rho^{-1}})(f).$$

A similar reasoning applies to $\mathcal{J}_2$, where in this case we let

$$\gamma_2 : \tau(\mathcal{J}_2) \to \mathcal{A}$$

to move away from the boundary of the arithmetic predecessor. The mapping ρ is defined as

$$\begin{aligned} \rho : \mathcal{J}_2 &\to \mathcal{A} \\ A &\mapsto \gamma_2 \circ \tau(A). \end{aligned}$$

In the case for $\mathcal{J}_3$ we define both injections γ_1 and γ_2 from above to act on $\mathcal{J}_3$ and $\tau(\mathcal{J}_3)$ respectively and set

$$\begin{aligned} \rho : \gamma_1(\mathcal{J}_3) &\to \mathcal{A} \\ A &\longmapsto \gamma_2 \circ \tau \circ \gamma_1(A). \end{aligned}$$

If we summarize these considerations, we get a decomposition of the operators $W^{(k)}_{m,i}, U^{(k)}_{m,i}$ and $T^{(j,k)}_{m,i}$ into a sum of $C(m+1)$ bounded operators on L^p_E, where their bound depends only on p, X and the UMD–constant of E. Since so does C, we get the assertions of Proposition 2.3.5 and that $W_{m,i}$ is bounded on L^p_E by $C_p(m+1)$. □

2.3.6 The Proof of Theorem 2.3.3

Theorem 2.3.3. *For all $1 < p < \infty$ there exists a constant C_p depending only on p, X, the* UMD *constant of E and C_S from (2.3.1), such that for all $f \in L^p_E(X), g \in L^{p'}_{E'}(X)$, which are finite linear combinations of Haar functions, the operators $K_{1,m}, K_{3,m}$ and the bilinear form $B_{2,m}$ satisfy the following estimates:*

$$\|K_{1,m}(f)\|_{L^p_E(X)} \le C_p(m+1)q^{-m\delta}\,\|f\|_{L^p_E(X)}\,, \tag{2.3.11}$$

$$\|K_{3,m}(f)\|_{L^p_E(X)} \le C_p(m+1)q^{-m\delta}\,\|f\|_{L^p_E(X)}\,, \tag{2.3.12}$$

$$|B_{2,m}(f,g)| \le C_p(m+1)q^{-m\delta}\,\|f\|_{L^p_E(X)}\,\|g\|_{L^{p'}_{E'}(X)}\,. \tag{2.3.13}$$

Here, $p' = p/(p-1)$ denotes the conjugate exponent to p.

Proof. It holds that

$$K_{1,m} = \sum_{i=1}^{M_1 q^m} \sum_{j,k=1}^{N-1} T^{(j,k)}_{m,i} \circ \mathcal{M}^{(j,k)}_{m,i}\,,$$

where $T^{(j,k)}_{m,i}$ is the shift operator introduced in (2.3.21) and $\mathcal{M}^{(j,k)}_{m,i}$ is the Haar multiplication operator which maps

$$d_{Q_k(\tau(A))} \mapsto \left\langle k, d_{Q_j(A)} \otimes d_{Q_k(\tau(A))}\right\rangle d_{Q_k(\tau(A))} \quad \text{for } A \in \mathcal{A}_{m,i}.$$

Analogously we get

$$K_{3,m} = \sum_{i=1}^{M_1 q^m} \sum_{k=1}^{N-1} U^{(k)}_{m,i} \circ \mathcal{M}^{(k)}_{m,i}, \quad B_{2,m}(f,g) = \left\langle \sum_{i=1}^{M_1 q^m} \sum_{j=1}^{N-1} W^{(j)}_{m,i} \circ \widetilde{\mathcal{M}}^{(j)}_{m,i} f, g \right\rangle, \tag{2.3.25}$$

where $\mathcal{M}^{(k)}_{m,i}$ and $\widetilde{\mathcal{M}}^{(j)}_{m,i}$ are Haar multiplication operators which map

$$d_{Q_k(\tau(A))} \mapsto \frac{\left\langle k, 1_A \otimes d_{Q_k(\tau(A))}\right\rangle}{\sqrt{\mu(A)}} d_{Q_k(\tau(A))} \quad \text{and} \quad d_{Q_j(A)} \mapsto \frac{\left\langle k, d_{Q_j(A)} \otimes 1_{\tau(A)}\right\rangle}{\sqrt{\mu(\tau(A))}} d_{Q_j(A)},$$

respectively. These decompositions follow from the definition of $K_{1,m}, K_{3,m}, B_{2,m}$ in (2.3.8) – (2.3.10), Proposition 2.2.11, the definition of the shifts $T^{(j,k)}_{m,i}$ and rearrangements $U^{(k)}_{m,i}, W^{(j)}_{m,i}$ in (2.3.20), (2.3.21) and the remark after Theorem 2.3.8. Since Haar multipliers are bounded on $L^p_E(X)$ by the supremum of their coefficients, we deduce by the structural estimate (2.3.1) and Proposition 2.3.5

$$\begin{aligned}
\|K_{1,m} f\|_{L^p_E(X)} &= \left\| \sum_{i=1}^{M_1 q^m} \sum_{j,k=1}^{N-1} T^{(j,k)}_{m,i} \circ \mathcal{M}^{(j,k)}_{m,i} f \right\|_{L^p_E(X)} \\
&\le \sum_{i=1}^{M_1 q^m} \sum_{j,k=1}^{N-1} \left\| T^{(j,k)}_{m,i} \right\|_{L^p_E(X)\to L^p_E(X)} \left\| \mathcal{M}^{(j,k)}_{m,i} \right\|_{L^p_E(X)\to L^p_E(X)} \|f\|_{L^p_E(X)} \\
&\le C q^m (m+1) \sup_{\substack{1\le j,k\le N-1 \\ A\in\mathcal{A}_{m,i}}} \left|\left\langle k, d_{Q_j(A)} \otimes d_{Q_k(\tau(A))}\right\rangle\right| \|f\|_{L^p_E(X)} \\
&\le C q^m (m+1) \sup_{A\in\mathcal{A}_{m,i}} \left(1 + \frac{d(A,\tau(A))}{q^{\operatorname{lev} A}}\right)^{-1-\delta} \|f\|_{L^p_E(X)}\,,
\end{aligned}$$

where C is a constant depending only on p, X, the UMD-constant of E and C_S that possibly changes from line to line. Now we get from the definition of $\mathcal{C}_m$ in Section 2.2.5 (and hence from the corresponding property for $\mathcal{A}_{m,i}$) that the last expression in the previous display is less or equal

$$Cq^{-\delta m}(m+1) \|f\|_{L^p_E(X)} ,$$

which is the required conclusion for $K_{1,m}$. The two remaining assertions follow from similar arguments using the decompositions in (2.3.25) □

Closing Remark. For singular integral operators in the scalar valued case, the $T(1)$ theorem was extended in an important series of papers (starting with the pioneering contribution in [DM00] and [Dav98] and extended in [NTV03]) to metric measure spaces that are not necessarily of homogeneous type. In this nonhomogeneous setting there holds a UMD valued $T(1)$ theorem ([Hyt09]). To differentiate those results from our work in this chapter we note that the initial step in [Dav98, DM00, NTV03, Hyt09] is the expansion of the singular integral kernel along the *anisotropic tensor product* Haar basis in $\mathbb{R}^n \times \mathbb{R}^n$. Such an expansion leads to decompositions of integral operators that are *structurally* different from the basic building blocks studied in the present chapter.

As mentioned already in the introduction, our decomposition (2.3.7) into simple rearrangements and shifts is the central assertion of our work and it permits us to study integral operators beyond the Calderón-Zygmund class ([KM06, LMM11]).

Chapter 3

Monotonicity of the Lebesgue constant for equally spaced knots

Let $t_i = \frac{i}{n}$ for $i = 0, \ldots, n$ be equally spaces knots in the unit interval $[0, 1]$. Let V_n be the space of piecewise linear continuous functions on $[0, 1]$ with knots $\pi_n = \{t_i : 0 \leq i \leq n\}$. Then we have the orthogonal projection P_n of $L^2([0, 1])$ onto V_n. In Section 3.1 we collect a few preliminary facts about the solutions of the recurrence $f_{k-1} - 4f_k + f_{k+1} = 0$ that we need in Section 3.2 and in Chapter 4. In this chapter we show that the sequence $a_n = \|P_n\|_\infty$ of $L^\infty-$norms of these projection operators is strictly increasing.

3.1 Solutions of $f_{k-1} - 4f_k + f_{k+1} = 0$ and their Properties

This section is to define and examine a few properties of the solutions of the recurrence $f_{k-1} - 4f_k + f_{k+1} = 0$, which we will use extensively in the sequel. For an arbitrary real number x, let $A_x := \cosh(\alpha x)$ and $\sqrt{3}B_x := \sinh(\alpha x)$ with $\alpha > 0$ defined by $\cosh \alpha = 2$. For $k \in \mathbb{N}_0$, A_k and B_k can also be defined by the recurrence relations

$$A_{k+1} = 2A_k + 3B_k \quad \text{with } A_0 = 1, \tag{3.1.1}$$
$$B_{k+1} = A_k + 2B_k \quad \text{with } B_0 = 0. \tag{3.1.2}$$

This follows from the basic identities

$$\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y, \tag{3.1.3}$$
$$\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y. \tag{3.1.4}$$

The crucial fact about A_k and B_k is that they are independent solutions of the linear recursion $f_{k-1} - 4f_k + f_{k+1} = 0$ and this recursion in turn takes into account the special form of the Gram matrix for equally spaces knots (see (3.2.1)) and the Gram matrices appearing in Chapter 4. We note that it is easy to see that the inequalities

$$A_{k+1} \leq 4A_k \quad \text{for } k \in \mathbb{N}_0, \tag{3.1.5}$$
$$B_{k+1} \leq 4B_k \quad \text{for } k \in \mathbb{N} \tag{3.1.6}$$

hold. Observe also that

$$A_k = 2A_{k+1} - 3B_{k+1} \tag{3.1.7}$$
$$B_k = 2B_{k+1} - A_{k+1} \tag{3.1.8}$$

for $k \in \mathbb{N}_0$. We also have the formulae

$$A_x = \frac{1}{2}(\lambda^x + \lambda^{-x}), \quad B_x = \frac{1}{2\sqrt{3}}(\lambda^x - \lambda^{-x}), \quad x \in \mathbb{R} \tag{3.1.9}$$

with

$$\lambda = 2 + \sqrt{3}, \quad \lambda^{-1} = 2 - \sqrt{3}.$$

We remark that $\alpha = \log \lambda$. For reference, we list the first few values of both A_n and B_n :

$$(A_0, \ldots, A_4) = (1, 2, 7, 26, 97), \qquad (B_0, \ldots, B_4) = (0, 1, 4, 15, 56).$$

Lemma 3.1.1. *For $K \in \mathbb{N}_0$ we have the following formulae*

$$\sum_{k=0}^{K} B_k + B_{k+1} = A_{K+1} - 1, \quad 2\sum_{k=0}^{K} A_k = 3B_{K+1} - A_{K+1} + 1,$$
$$\sum_{k=0}^{K} A_k + A_{k+1} = 3B_{K+1}, \quad 2\sum_{k=0}^{K} B_k = A_{K+1} - B_{K+1} - 1.$$

Proof. The proof uses induction and the recurrence formulae (3.1.1),(3.1.2),(3.1.7) and (3.1.8) for A_n and B_n. □

Lemma 3.1.2. *Let $k \in \mathbb{N}_0$. Then we have*

$$-1 \leq -\lambda^{-k} = \lambda B_k - B_{k+1} \leq 0, \tag{3.1.10}$$
$$0 \leq \lambda A_k - A_{k+1} = \sqrt{3}\lambda^{-k} \leq \sqrt{3}, \tag{3.1.11}$$
$$-1 \leq \lambda^{-k} = \sqrt{3}B_k - A_k \leq 0. \tag{3.1.12}$$

Proof. This follows from (3.1.9). □

Lemma 3.1.3. *For all $n \in \mathbb{N}$ and $0 \leq k \leq n$ the following equalities hold*

$$B_k A_{n-k} + A_k B_{n-k} = B_n, \quad B_n A_{n-k} - B_{n-k} A_n = B_k,$$
$$A_k A_{n-k} + 3B_{n-k} B_k = A_n, \quad A_n A_{n-k} - 3B_n B_{n-k} = A_k.$$

Proof. This is only a different formulation of (3.1.3) and (3.1.4). □

The last auxiliary result we need is to record a few properties of a special rational function, which is used in the expression of the L^∞-norm of our projection operators.

Lemma 3.1.4. *Let $\phi : (0, \infty) \to [1/2, 1)$ be defined by*

$$t \mapsto \phi(t) = \frac{1 + t^2}{(1 + t)^2}.$$

Then

$$\phi(t) = \phi(t^{-1}), \quad \phi'(t) = \frac{2(t-1)}{(1+t)^3}, \quad \phi''(t) = \frac{4(2-t)}{(1+t)^4}$$

for all $t > 0$. So in particular ϕ is decreasing for $t < 1$ and increasing for $t > 1$ and ϕ' is increasing for $t < 2$ and decreasing for $t > 2$. Further, it holds that

$$\phi(\lambda) = \frac{2}{3}, \quad \phi(4) = \frac{17}{25}, \quad \phi(6) = \frac{37}{49}, \quad \phi'(\lambda) = \frac{\lambda^{-1}}{3\sqrt{3}},$$

where $\lambda = 2 + \sqrt{3}$.

3.2 Equally spaced knots on $[0, 1]$

We let $t_i = i/n$ for $0 \le i \le n$ and view the partition of points $\pi_n = \{t_i : i \in \{0, \dots, n\}\}$. Additionally we set $t_{-1} = 0$ and $t_{n+1} = 1$. If we define $\delta_i := t_i - t_{i-1}$ for $0 \le i \le n+1$, we get that $\delta_i = 1/n$ for $1 \le i \le n$ in this case of equally spaced knots. Furthermore, we have for the entries $(a_{i,k})$ of the inverse of the Gram matrix $(b_{i,k}) = \langle N_i, N_k \rangle$ consisting of the pairwise scalar products of the piecewise linear, continuous B-spline functions corresponding to the partition π_n (see [Cie66] or [CK04])

$$a_{i,k} = \frac{2n}{B_n}(-1)^{i+k} A_{i \wedge k} A_{n-i \wedge n-k} \quad \text{with } 0 \le i, k \le n \tag{3.2.1}$$

Observe that from formula (3.2.1) it follows that

$$\frac{|a_{i,k}|}{|a_{i-1,k}|} = \begin{cases} \frac{A_i}{A_{i-1}}, & \text{if } 1 \le i \le k, \\ \frac{A_{n-i}}{A_{n-i+1}}, & \text{if } k < i \le n. \end{cases} \tag{3.2.2}$$

Let V_n be the space of piecewise linear continuous functions with knots π_n. For the L^∞-norm (or the L^1-norm) of the projection operator $P_n : L^2([0, 1]) \to V_n$ we have the formula (see for instance [CK04]):

$$\|P_n\|_\infty = \max_{0 \le k \le n} \sum_{i=1}^{n} p_{i,k} \phi\left(\frac{|a_{i,k}|}{|a_{i-1,k}|}\right) =: \max_{0 \le k \le n} g_k(n),$$

where $\phi(t) = \frac{1+t^2}{(1+t)^2}$, as in Lemma 3.1.4 and $p_{i,k} = \frac{\delta_i}{2}(|a_{i,k}| + |a_{i-1,k}|)$.

Using the definition of $g_k(n)$, (3.2.2) and the properties of ϕ from Lemma 3.1.4, we obtain for $0 \le k \le n$

$$g_k(n) = \frac{1}{B_n}\left(A_{n-k} \sum_{j=0}^{k-1} (A_{j+1} + A_j)\phi(A_{j+1}/A_j) + A_k \sum_{j=0}^{n-k-1} (A_{j+1} + A_j)\phi(A_{j+1}/A_j)\right). \tag{3.2.3}$$

Theorem 3.2.1. *For all $n \in \mathbb{N}$, we have that $g_0(n+1) > g_0(n)$.*

Proof. The expression $Dg_0(n) := g_0(n+1) - g_0(n)$ equals

$$\sum_{j=0}^{n} \frac{1}{B_{n+1}}(A_j + A_{j+1})\phi\left(\frac{A_{j+1}}{A_j}\right) - \sum_{j=0}^{n-1} \frac{1}{B_n}(A_j + A_{j+1})\phi\left(\frac{A_{j+1}}{A_j}\right). \qquad (3.2.4)$$

Thus, we get further

$$\begin{aligned} Dg_0(n) &= \frac{A_n + A_{n+1}}{B_{n+1}}\phi\left(\frac{A_{n+1}}{A_n}\right) + \sum_{j=0}^{n-1}\left(\frac{1}{B_{n+1}} - \frac{1}{B_n}\right)(A_j + A_{j+1})\phi\left(\frac{A_{j+1}}{A_j}\right) \\ &= \frac{A_n + A_{n+1}}{B_{n+1}}\phi\left(\frac{A_{n+1}}{A_n}\right) - \left(\frac{B_{n+1} - B_n}{B_{n+1}B_n}\right)\sum_{j=0}^{n-1}(A_j + A_{j+1})\phi\left(\frac{A_{j+1}}{A_j}\right) \\ &> \phi\left(\frac{A_{n+1}}{A_n}\right)\left(\frac{A_n + A_{n+1}}{B_{n+1}} - 3\frac{B_{n+1} - B_n}{B_{n+1}}\right) \end{aligned}$$

where for the inequality, we have used Lemma 3.1.1 and the fact that for $j \le n-1$ we have $\phi\left(\frac{A_{j+1}}{A_j}\right) < \phi\left(\frac{A_{n+1}}{A_n}\right)$, since $A_{j+1}/A_j < A_{n+1}/A_n$ for $j \le n-1$. Now we use the recurrences (3.1.1) and (3.1.2) to obtain that the last display equals zero and thus the theorem is proved. □

Theorem 3.2.2 will then show that it suffices to have Theorem 3.2.1 to conclude the monotonicity of the sequence of norms of the projection operators.

Theorem 3.2.2. *For all $n \in \mathbb{N}$ and all $k \in \mathbb{N}$ with $1 \le k \le \lfloor n/2 \rfloor$, we have*

$$g_0(n) \ge g_k(n).$$

Remark. Due to symmetry, we get this inequality for all $1 \le k \le n-1$, and in fact the equality $g_0(n) = g_n(n)$.

Proof of Theorem 3.2.2. We first observe that for general k we have to consider $h_k(n) := B_n(g_0(n) - g_k(n))$ and show that this is greater or equal zero. Now by definition

$$\begin{aligned} h_k(n) &= \sum_{j=0}^{n-1}(A_j + A_{j+1})\phi\left(\frac{A_{j+1}}{A_j}\right) - A_{n-k}\sum_{j=0}^{k-1}(A_j + A_{j+1})\phi\left(\frac{A_{j+1}}{A_j}\right) \\ &\quad -A_k \sum_{j=0}^{n-k-1}(A_j + A_{j+1})\phi\left(\frac{A_{j+1}}{A_j}\right) \end{aligned}$$

Rearranging terms, this yields

$$\begin{aligned} h_k(n) &= -(A_k - 1)\sum_{j=0}^{n-k-1}(A_j + A_{j+1})\phi\left(\frac{A_{j+1}}{A_j}\right) + \sum_{j=n-k}^{n-1}(A_j + A_{j+1})\phi\left(\frac{A_{j+1}}{A_j}\right) \\ &\quad -A_{n-k}\sum_{j=0}^{k-1}(A_j + A_{j+1})\phi\left(\frac{A_{j+1}}{A_j}\right). \end{aligned}$$

Employing again the inequality $\phi(A_{j+1}/A_j) < \phi(A_{k+1}/A_k)$ for $j < k$, we get

$$\begin{aligned} h_k(n) \;\geq\; & \phi\left(\frac{A_{n-k+1}}{A_{n-k}}\right)\left(-(A_k-1)\sum_{j=0}^{n-k-1}(A_j+A_{j+1}) + \sum_{j=n-k}^{n-1}(A_j+A_{j+1}) \right. \\ & \left. -A_{n-k}\sum_{j=0}^{k-1}(A_j+A_{j+1})\right), \end{aligned}$$

since $k \leq \lfloor n/2 \rfloor$. Finally a calculation using Lemmas 3.1.1 and 3.1.3 shows that the term in the big bracket equals zero and thus the theorem is proved. □

Remark. We observe that we can use this theorem to conclude with Theorem 3.2.1 that $\|P_n\|_\infty < \|P_{n+1}\|_\infty$. We also remark that we get a simple proof that $\|P_n\|_\infty < 2$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \|P_n\|_\infty = 2$, since with formula (3.2.3) and the last theorem we get

$$\|P_n\|_\infty = g_0(n) = \frac{1}{B_n}\sum_{j=0}^{n-1}(A_j+A_{j+1})\phi\left(\frac{A_{j+1}}{A_j}\right) < 2,$$

since $\phi\left(\frac{A_{j+1}}{A_j}\right) < 2/3$ for all $j \in \mathbb{N}_0$ and $\sum_{j=0}^{n-1}(A_j+A_{j+1}) = 3B_n$ by Lemma 3.1.1. Additionally, for $\varepsilon > 0$ we choose m such that $\phi\left(\frac{A_{m+1}}{A_m}\right) > (2-\varepsilon)/3$, which is possible, since $\lim_{m\to\infty} \phi\left(\frac{A_{m+1}}{A_m}\right) = \phi(2+\sqrt{3}) = 2/3$ by Lemma 3.1.4. For $n > m$ we thus have

$$\begin{aligned} \|P_n\|_\infty \;&=\; g_0(n) \geq \frac{1}{B_n}\sum_{j=m}^{n-1}(A_j+A_{j+1})\phi\left(\frac{A_{m+1}}{A_m}\right) \\ &>\; (2-\varepsilon)\frac{B_n-B_m}{B_n} \to 2-\varepsilon \quad \text{for } n\to\infty, \end{aligned}$$

by Lemma 3.1.1.

Chapter 4

The Lebesgue Constants for the Periodic Franklin System

We identify the torus with the unit interval $[0, 1)$ and let $n, \nu \in \mathbb{N}$, $1 \leq \nu \leq n - 1$ and $N := n + \nu$. Then we define the (partially equally spaced) knots

$$t_j = \begin{cases} \frac{j}{2n}, & \text{for } j = 0, \ldots, 2\nu, \\ \frac{j-\nu}{n}, & \text{for } j = 2\nu + 1, \ldots, N - 1. \end{cases}$$

Furthermore, given n, ν we let $V_{n,\nu}$ be the space of piecewise linear continuous functions on the torus with knots $\{t_j : 0 \leq j \leq N - 1\}$. Finally, let $P_{n,\nu}$ be the orthogonal projection operator from $L^2([0, 1))$ onto $V_{n,\nu}$. The main result is

$$\lim_{n\to\infty,\nu=1} \|P_{n,\nu} : L^\infty \to L^\infty\| = \sup_{n\in\mathbb{N},0\leq\nu\leq n} \|P_{n,\nu} : L^\infty \to L^\infty\| = 2 + \frac{33 - 18\sqrt{3}}{13}.$$

This shows in particular that the Lebesgue constant of the classical Franklin orthonormal system on the torus is $2 + \frac{33-18\sqrt{3}}{13}$.

4.1 Introduction

Let $(N_k)_{k\geq 0}$ be an orthonormal basis in $L^2[0, 1]$. The Fourier partial sums with respect to this basis are given by

$$P_N(f) = \sum_{k=0}^{N} \langle f, N_k \rangle N_k. \tag{4.1.1}$$

Clearly, every P_N is a projection onto its (finite dimensional) range and its norm as an operator from $L^\infty[0, 1]$ to $L^\infty[0, 1]$ (or as an operator from $L^1[0, 1]$ to $L^1[0, 1]$) is given by

$$L_N = \operatorname*{ess\,sup}_{s\in[0,1]} \int_0^1 |K_N(s, t)|dt,$$

where K_N is the *Dirichlet kernel*

$$K_N(s, t) = \sum_{k=0}^{N} N_k(s)N_k(t).$$

The *Lebesgue constant* of the basis $(N_k)_{k\geq 0}$ is now defined as

$$L := \sup_{N\geq 0} L_N.$$

As a particular instance of an orthonormal basis in $L^2[0,1]$, we consider the *general Franklin system* $(N_k)_{k\geq 0}$ on the torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$: That is we choose a sequence of points $\mathcal{T} = (t_k)_{k\geq 0}$ in $[0,1)$ (we identify this interval with the torus), which is dense in $[0,1)$ and with $t_0 = 0$. The space of piecewise linear and continuous functions on $\mathbb{T}$ with knots $\{t_0,\dots,t_N\}$ is denoted by $V_N(\mathcal{T})$. Then we define $f_0 \equiv 1$ on $\mathbb{T}$ and inductively, for $k \geq 1$ the k-th Franklin function corresponding to the sequence $\mathcal{T}$ is uniquely determined by the conditions

$$f_k \in V_k(\mathcal{T}), \quad f_k \perp V_{k-1}(\mathcal{T}), \quad \|f_k\|_2 = 1, \quad f_k(t_k) > 0.$$

The Franklin functions f_k are splines of degree $d = 1$. We now make a few comments about the history of calculating or estimating the Lebesgue constant of splines of degree d.

For $d = 0$ (piecewise constant functions), the projection is easily calculated and the Lebesgue constant is 1.

For $d = 1$ (piecewise linear functions), Z. Ciesielski ([Cie63]) proved that for any partition π of $[0,1]$, the L^∞-norm onto piecewise linear functions with knots π is ≤ 3. He showed this for the non-periodic case, but exactly the same argument gives the upper bound 3 in the periodic case. Moreover, P. Oswald ([Osw77]) and K. Oskolkov ([Osk79]) proved independently that in the non-periodic case, the constant 3 is optimal if one considers arbitrary partitions π. Moreover, Ciesielski ([Cie75b]) showed that in case for uniform partitions, the exact upper bound is 2. Some numerical experiments suggested that for the (classical, corresponding to dyadic knots) non-periodic Franklin system, the exact upper bound is $2+(2-\sqrt{3})^2$ ([CN77]). Several years later, P. Bechler ([Bec03]) proved that for the piecewise linear Strömberg wavelet, the Lebesgue constant is indeed $2 + (2-\sqrt{3})^2$. Then, Z. Ciesielski and A. Kamont (in [CK04]) showed that for the classical non-periodic Franklin system, the Lebesgue constant is $2 + (2-\sqrt{3})^2$, verifying the conjecture in [CN77].

For splines of higher degree ($d \geq 2$), a problem was the mere existence of a bound C_d for the $L^\infty-$norms of orthogonal projections onto splines of degree d with arbitrary knots, where C_d depends only on d and not on the partition. This was a long standing conjecture by C. de Boor solved by A. Yu Shadrin in [Sha01] (in the non-periodic case). Predating Shadrin's result, there were several results specializing in the degree (for instance [dB68] for $d = 2$ in the non-periodic case) or specializing in the sequence of points (for instance [Dom72] and [Dom76] viewing the sequence of dyadic partitions both in the non-periodic and periodic case respectively for arbitrary degree d). In the periodic case, there is a further partial result showing the existence of a bound C_2 for the $L^\infty-$norm of orthogonal projections for $d = 2$ not depending on the knots in [Ker08]. The exact values of the Lebesgue constants in the cases $d \geq 2$ are not known.

Here, we study and determine the Lebesgue constant for the periodic (classical) Franklin system (corresponding to $d = 1$). Its value is $2 + \frac{33-18\sqrt{3}}{13}$. The analysis presented in this chapter was constantly guided by extensive computer simulations (both numerically and symbolically) involving the Gram matrix and its inverse (see Section 4.3).

4.2 Formulation of the Main Theorem

Our main result concerns partially equally spaced knots on the torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. We choose special points

$$t_j = \begin{cases} \frac{j}{2n}, & \text{for } j = 0, \dots, 2\nu \\ \frac{j-\nu}{n}, & \text{for } j = 2\nu+1, \dots, N-1 \end{cases} \tag{4.2.1}$$

for arbitrary $n, \nu \in \mathbb{N}$ with $1 \leq \nu \leq n-1$ and $N := n+\nu$. We remark that for $\nu = 0$ or $\nu = n$ we would arrive at equally spaced knots. Let $V_{n,\nu}$ be the linear subspace generated by the piecewise linear, continuous functions with knots (4.2.1) and $P_{n,\nu}$ be the orthogonal projection onto $V_{n,\nu}$. The B-spline basis for $V_{n,\nu}$ with a special choice of parameters n, ν is pictured in Figure 4.1.

The main theorem now reads as follows:

Theorem 4.2.1. *For all $n \in \mathbb{N}, 0 \leq \nu \leq n$, we have the following bound for the norm of the projection operator $P_{n,\nu}$ onto $V_{n,\nu}$:*

$$\|P_{n,\nu}\|_\infty := \|P_{n,\nu} : L^\infty(\mathbb{T}) \to L^\infty(\mathbb{T})\| < 2 + \frac{33 - 18\sqrt{3}}{13} =: \gamma.$$

Furthermore, for $n \to \infty, \nu = 1$ it holds that

$$\lim_{n\to\infty} \|P_{n,1}\|_\infty = \gamma.$$

4.3 Orthogonal Projections

Let V be an N-dimensional subspace of $L^2[0,1]$ and $\{N_0, \dots, N_{N-1}\}$ a basis of V. We first look at the changes in formula (4.1.1), if the orthogonality is lost. In this case, the orthogonal projection P onto V is given by

$$Pf(s) = \sum_{j,k=0}^{N-1} a_{jk} \langle N_k, f \rangle N_j(s),$$

or equivalently as integral operator with kernel $k(s,t) = \sum_{j,k=0}^{N-1} a_{jk} N_j(s) N_k(t)$

$$Pf(s) = \int_0^1 k(s,t) f(t) dt,$$

where $A = (a_{jk})$ is the inverse of the Gram matrix $B = (b_{jk})$ with $b_{jk} = \langle N_j, N_k \rangle$. The norm of P as a mapping from $L^\infty[0,1]$ to $L^\infty[0,1]$ is

$$\|P\|_\infty = \operatorname*{ess\,sup}_{s\in[0,1]} \int_0^1 |k(s,t)| dt. \tag{4.3.1}$$

Since P is self adjoint, the norm of P as operator from $L^1[0,1]$ to $L^1[0,1]$ is the same.

We now consider periodic B-splines of degree one on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. For this let $0 = t_0 < t_1 < \cdots < t_{N-1} < 1$ with an arbitrary natural number $N \geq 2$. Further set $t_{-1} := t_{N-1} - 1$, $t_N := 1$ and $\delta_j := t_{j+1} - t_j$ for $-1 \leq j \leq N-1$. Then we let N_j for $0 \leq j \leq N-1$ be the unique continuous function on $\mathbb{T}$, which is linear on every interval (t_{k-1}, t_k) and has values $N_j(t_k) = \delta_{j,k}$ for $0 \leq k \leq N-1$. Formally we define the functions $N_j : \mathbb{T} \to [0,1]$ for $0 \leq j \leq N-1$ as

$$N_j([t]) := \begin{cases} (s - t_{j-1})/\delta_{j-1}, & \text{if } [t] = [s] \text{ for } t_{j-1} < s \leq t_j, \\ (t_{j+1} - s)/\delta_j, & \text{if } [t] = [s] \text{ for } t_j < s \leq t_{j+1}, \\ 0, & \text{else,} \end{cases} \tag{4.3.2}$$

where we denote by $[\cdot]$ the canonical surjection taking each $t \in \mathbb{R}$ onto its equivalence class in $\mathbb{T}$. From now on we identify the unit interval $[0,1)$ with $\mathbb{T}$ and furthermore, by a slight abuse of notation we consider N_j to be defined on $[0,1)$.

Figure 4.1 shows periodic B-splines of degree one defined in (4.3.2) for the points in (4.2.1) with a special choice of parameters n, ν.

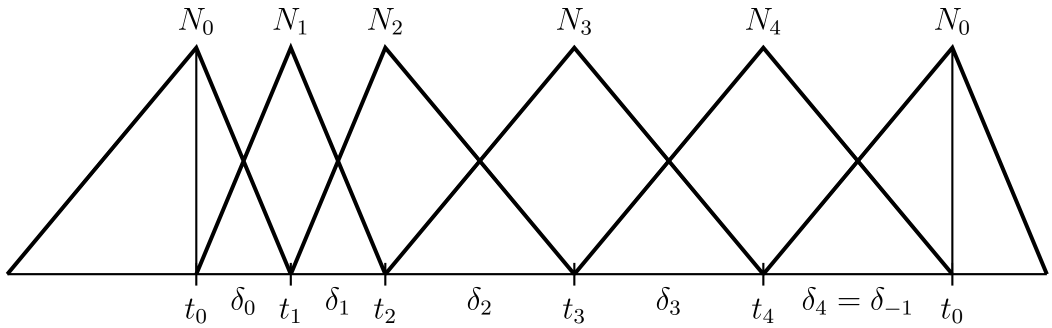

Figure 4.1: Situation for $N = 5, \nu = 1, n = N - \nu = 4$.

Let (as above) V be the (finite dimensional) subspace generated by $\{N_0, \ldots, N_{N-1}\}$ and P be the orthogonal projection of $L^2[0,1)$ onto V. Then formula (4.3.1) for the norm of P simplifies to

$$\|P\|_\infty = \max_{j=0,\ldots,N-1} \int_0^1 |k(t_j, t)| dt,$$

where the kernel k is given by $k(s,t) = \sum_{j,k=0}^{N-1} a_{j,k} N_j(s) N_k(t)$. Now recall that $(a_{j,k})$ is the inverse of the Gram matrix $(b_{j,k}) = \langle N_j, N_k \rangle$. If we let $\kappa(j) := \int_0^1 |k(t_j,t)| dt$, it can be shown by an elementary calculation that

$$\kappa(j) = \sum_{k=0}^{N-1} \frac{\delta_k}{2} \begin{cases} |a_{j,k}| + |a_{j,k+1}|, & \text{if } \operatorname{sgn} a_{j,k} = \operatorname{sgn} a_{j,k+1}, \\ \frac{a_{j,k}^2 + a_{j,k+1}^2}{|a_{j,k}| + |a_{j,k+1}|}, & \text{else,} \end{cases} \tag{4.3.3}$$

where every subindex is understood to be an index modulo N. With the rational function $\phi(t) := \frac{1+t^2}{(1+t)^2}$ this can be rewritten to

$$\kappa(j) = \sum_{k=0}^{N-1} \frac{\delta_k}{2} (|a_{j,k}| + |a_{j,k+1}|) \cdot \begin{cases} 1, & \text{if } \operatorname{sgn} a_{j,k} = \operatorname{sgn} a_{j,k+1}, \\ \phi(|a_{j,k+1}|/|a_{j,k}|), & \text{else.} \end{cases} \tag{4.3.4}$$

By (4.3.4), exact formulae for the entries of the inverse (a_{jk}) to the Gram matrix are absolutely necessary in determining the exact value of the Lebesgue constant. We will provide this information in Proposition 4.4.1 for the periodic case. In the non-periodic dyadic case, such exact formulae for the inverse of the Gram matrix were given in [Cie66] and they were used in the calculation of the corresponding Lebesgue constant in [CK04]. For the general Franklin system, there are important estimates both for the non-periodic case and for the periodic case (see [KS89] and [Ker05] respectively). To calculate the exact value of the Lebesgue constant, we supplemented these already known estimates with exact formulae.

Before we begin the proof of the main theorem 4.2.1 we emphasize that the results of Section 3.1 are essential tools in the calculations that follow.

4.4 Proof of the Main Theorem

4.4.1 Equally Spaced Knots

As a preliminary case we view the points (4.2.1) for $\nu = 0$ and $N = n$ and show that the L^∞-norm $\|P_{n,0}\|_\infty$ obeys the estimate $\|P_{n,0}\|_\infty < 2$ and that $\lim_{n\to\infty} \|P_{n,0}\|_\infty = 2$. In case for equally spaced knots we get for the Gram matrix $(b_{jk})_{0\le j,k\le N-1}$

$$(b_{jk}) = \frac{1}{6n}\begin{pmatrix} 4 & 1 & & & 1 \\ 1 & 4 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 4 & 1 \\ 1 & & & 1 & 4 \end{pmatrix}, \tag{4.4.1}$$

where the empty entries are zero. Since every row is equal up to shifts in (b_{jk}), the same must be true for the inverse (a_{jk}). We get for the first row of the inverse of the Gram matrix

$$a_{0,k} = \frac{6n(-1)^k}{D(N)} g_k,$$

where

$$g_k = B_{N-k} + (-1)^N B_k \quad \text{and} \quad D(N) = 2((-1)^{N-1} + A_N). \tag{4.4.2}$$

Since every row in (a_{jk}) is equal up to shifts, formula (4.3.4) does not depend on j in this case. So while we consider equally spaced knots, we write κ to denote the value of $\kappa(j)$ for arbitrary $0 \le j \le N-1$.

N **even.** If we let N even, we obtain for κ out of (4.3.4)

$$\kappa \;=\; 3D(N)^{-1}\sum_{k=0}^{N-1}(g_k + g_{k+1})\phi\left(\frac{g_{k+1}}{g_k}\right).$$

Using the definition of g_k and Lemma 3.1.2 we see that $\lambda^{-1} < \frac{g_{k+1}}{g_k} < \lambda$, so by Lemma 3.1.4, $\phi\left(\frac{g_{k+1}}{g_k}\right) < \phi(\lambda)$ and thus we obtain

$$\kappa < 6\phi(\lambda)D(N)^{-1}\sum_{k=0}^{N-1} B_k + B_{k+1}.$$

Lemma 3.1.1 and the fact that $\phi(\lambda) = \frac{2}{3}$ then give us

$$\kappa < 4\frac{A_N - 1}{2(A_N - 1)} = 2.$$

N **odd.** For N odd, we see that (4.3.4) becomes

$$\begin{aligned}\kappa \;&=\; 6D(N)^{-1}\Big[B_{(N+1)/2} - B_{(N-1)/2} + \\ &\sum_{j=0}^{(N-3)/2}(B_{N-j} + B_{N-j-1} - B_j - B_{j+1})\phi\left(\frac{B_{N-j} - B_j}{B_{N-j-1} - B_{j+1}}\right)\Big].\end{aligned}$$

The mean value theorem implies

$$\phi(q_j) \le \phi(\lambda) + (q_j - \lambda)\phi'(\lambda), \quad \text{where} \quad q_j := \frac{B_{N-j} - B_j}{B_{N-j-1} - B_{j+1}},$$

since $\phi'(t)$ is decreasing for $t \ge \lambda \ge 2$ and $q_j \ge \lambda$ by Lemma 3.1.2. For $q_j - \lambda$, we have again due to Lemma 3.1.2 and $0 \le j \le (N-3)/2$

$$\begin{aligned}q_j - \lambda \;&=\; \frac{B_{N-j} - \lambda B_{N-j-1} + \lambda B_{j+1} - B_j}{B_{N-j-1} - B_{j+1}} \le \frac{1 + \lambda B_{j+1}}{B_{N-j-1} - B_{j+1}} \\ &\le\; \frac{1 + \lambda B_{j+1}}{B_{N-j-1}(1 - \lambda^{-N+2j+2})} \le 2\frac{1 + \lambda B_{j+1}}{B_{N-j-1}}.\end{aligned}$$

If we use these facts and the estimates $B_{(N-1)/2} \ge \lambda^{-1}B_{(N+1)/2} - \lambda^{-1}$ (Lemma 3.1.2) and $-B_j \le 0$, we obtain for κ

$$\begin{aligned}\kappa \;&\le\; 6D(N)^{-1}\Big[(1 - \lambda^{-1})B_{(N+1)/2} + \lambda^{-1} + \\ &+ \sum_{j=0}^{(N-3)/2}(B_{N-j} + B_{N-j-1} - B_{j+1})\left(\phi(\lambda) + 2\phi'(\lambda)\frac{1 + \lambda B_{j+1}}{B_{N-j-1}}\right)\Big] \quad (4.4.3)\end{aligned}$$

We split the analysis of this expression into a few subcases and thereby introduce the notation $p = \frac{N+1}{2}$ to shorten indices.

Term I. $\sum_{j=0}^{p-2} B_{N-j} + B_{N-j-1} - B_{j+1}$
We apply Lemma 3.1.1 and get that

$$\begin{aligned}\sum_{j=0}^{p-2} B_{N-j} + B_{N-j-1} - B_{j+1} \;&=\; \frac{1}{2}(2A_N - 3A_p + B_p + 1) \\ &\le\; \frac{1}{2}(2A_N - (3\sqrt{3} - 1)B_p + 1),\end{aligned}$$

by Lemma 3.1.2.

TERM II. $II := \sum_{j=0}^{p-2} \frac{(B_{N-j}+B_{N-j-1}-B_{j+1})(1+\lambda B_{j+1})}{B_{N-j-1}}$
Since by Lemma 3.1.2, $B_{N-j} = \lambda B_{N-j-1} + \lambda^{-N+j+1}$ and $\lambda^{-N+j+1} \leq \lambda^{-N+1} B_{j+1}$, we get that

$$II \leq (1+\lambda)\sum_{j=0}^{p-2}(1+\lambda B_{j+1}) - (1-\lambda^{-N+1})\sum_{j=0}^{p-2}\frac{B_{j+1}(1+\lambda B_{j+1})}{B_{N-j-1}}.$$

But now, clearly

$$\sum_{j=0}^{p-2}\frac{B_{j+1}(1+\lambda B_{j+1})}{B_{N-j-1}} \geq \lambda\frac{B_{p-1}^2}{B_p},$$

and by Lemma 3.1.1 and Lemma 3.1.2

$$\sum_{j=0}^{p-2} 1+\lambda B_{j+1} = \frac{N-1}{2} + \frac{\lambda}{2}(A_p - B_p - 1) \leq \frac{N-1}{2} + \frac{\lambda(\sqrt{3}-1)}{2}B_p$$

so we obtain finally for II that

$$II \leq (1+\lambda)\left(\frac{N-1}{2} + \frac{\lambda(\sqrt{3}-1)}{2}B_p\right) - (1-\lambda^{-N+1})\lambda\frac{B_{p-1}^2}{B_p}$$

Out of these estimates and from (4.4.3) we now obtain, if we note that $D(N) \geq 2A_N$ and $\phi(\lambda) = 2/3$

$$\kappa \leq 2 + \frac{3}{A_N}\left[\theta B_p + \lambda^{-1} + \frac{1}{3} + 2\phi'(\lambda)\left((1+\lambda)\frac{N-1}{2} - (1-\lambda^{-N+1})\lambda\frac{B_{p-1}^2}{B_p}\right)\right],$$

where

$$\theta = (1-\lambda^{-1}) - (\sqrt{3} - \frac{1}{3}) + (1+\lambda)\lambda\phi'(\lambda)(\sqrt{3}-1) = 0.$$

Since $\frac{B_{p-1}^2}{B_p}$ dominates $(N-1)/2$ for large N, we finally get that for N sufficiently large ($N \geq 8$),

$$\kappa < 2.$$

In fact, if we look at Table 4.1 on page 83, we see that for all $N \geq 2$ we have this inequality. An analogous argument as in Section 4.4.3, Paragraph Asymptotic Behavior, finally yields that $\lim_{N\to\infty} \kappa = 2$, and this completes what we wanted to show in this section.

4.4.2 The Inverse of the Gram Matrix

We now view the points (4.2.1) in case $1 \leq \nu \leq n-1$ (i.e. the case where the knots are not equally spaced anymore). The first step is to calculate the inverse of the Gram matrix in this case, which we will do in this section. As above and in the following we understand every index concerning the Gram matrix (b_{jk}) or its inverse

(a_{jk}) as an index modulo N. The Gram matrix $(b_{jk}) = (\langle N_j, N_k\rangle)_{0\le j,k\le N-1}$ admits the following representation

$$(b_{jk}) = \frac{1}{12n}\begin{pmatrix} 6 & 1 & & & & & & & 2 \\ 1 & 4 & 1 & & & & & & \\ & \ddots & \ddots & \ddots & & & & & \\ & & 1 & 4 & 1 & & & & \\ & & & 1 & 6 & 2 & & & \\ & & & & 2 & 8 & 2 & & \\ & & & & & \ddots & \ddots & \ddots & \\ & & & & & & 2 & 8 & 2 \\ 2 & & & & & & & 2 & 8 \end{pmatrix}, \tag{4.4.4}$$

where the row with the pattern $1, 6, 2$ has the index 2ν. This leads to the following equations concerning the inverse (a_{jk}) of (b_{jk}):

$$6a_{0,k} + a_{1,k} + 2a_{N-1,k} = 12n\delta_{0,k}, \tag{4.4.5}$$
$$a_{j-1,k} + 4a_{j,k} + a_{j+1,k} = 12n\delta_{j,k} \quad \text{for } j = 1, \dots, 2\nu - 1, \tag{4.4.6}$$
$$a_{2\nu-1,k} + 6a_{2\nu,k} + 2a_{2\nu+1,k} = 12n\delta_{2\nu,k}, \tag{4.4.7}$$
$$a_{j-1,k} + 4a_{j,k} + a_{j+1,k} = 6n\delta_{j,k} \quad \text{for } j = 2\nu + 1, \dots, N - 1, \tag{4.4.8}$$

where $\delta_{j,k}$ is the Kronecker delta and $0 \le k \le N - 1$. Let

$$D(N, \nu) := 2A_N + \frac{3}{2}B_{2\nu}B_{N-2\nu} - 2(-1)^N. \tag{4.4.9}$$

We note that for fixed N and ν the entries $(a_{j,k})$ are given by

$$a_{j,k} = \frac{6n(-1)^{k+j}}{D(N,\nu)} g(N, \nu, j, k)$$

for $0 \le j, k \le N - 1$, where g is a function depending on N, ν, j, k defined through this equation and we will identify its values in a couple of case distinctions for the various parameters. In order to make notation simpler, we view N and ν fixed and additionally in the current context, j will be fixed, so we set $g_k = g(N, \nu, j, k)$. If we now plug the definition of g_k into (4.3.4), we obtain for $\kappa(j)$ with the abbreviation

$$\xi_{j,k} = \begin{cases} 1, & \text{if } \operatorname{sgn} a_{j,k} = \operatorname{sgn} a_{j,k+1}, \\ \phi(|g_{k+1}|/|g_k|), & \text{else} \end{cases}$$

that

$$\kappa(j) = D(N,\nu)^{-1}\left[\frac{3}{2}\sum_{k=0}^{2\nu-1}(|g_k| + |g_{k+1}|)\cdot\xi_{j,k} + 3\sum_{k=2\nu}^{N-1}(|g_k| + |g_{k+1}|)\cdot\xi_{j,k}\right]. \tag{4.4.10}$$

Proposition 4.4.1. *If $0 \le j \le 2\nu - 1$ we have that g_k equals*

$$\begin{cases} 2(-1)^N B_{j-k} + B_{N-j+k} + B_{2\nu-j}A_{N-2\nu+k} + B_k(A_{N-j} + 3B_{2\nu-j}B_{N-2\nu}), \\ 2(-1)^N B_{k-j} + B_{N-k+j} + B_{2\nu-k}A_{N-2\nu+j} + B_j(A_{N-k} + 3B_{2\nu-k}B_{N-2\nu}), \\ (-1)^N(B_{k-j} + A_{k-2\nu}B_{2\nu-j}) + B_{N-k+j} + B_jA_{N-k}. \end{cases}$$

if $k \le j, j \le k \le 2\nu$ or $2\nu \le k \le N-1$ respectively. If $2\nu \le j \le N-1$, we have that g_k equals

$$\begin{cases} (-1)^N(B_{j-k}+A_{j-2\nu}B_{2\nu-k})+B_{N-j+k}+A_{N-j}B_k, & \text{if } k \le 2\nu \le j, \\ (-1)^N B_{j-k}+A_{k-2\nu}B_{N-j+2\nu}+A_{N-j}B_k+\frac{3}{2}B_{k-2\nu}B_{2\nu}B_{N-j}, & \text{if } 2\nu \le k \le j, \\ (-1)^N B_{k-j}+A_{N-k}B_j+A_{j-2\nu}B_{N-k+2\nu}+\frac{3}{2}B_{2\nu}B_{N-k}B_{j-2\nu}, & \text{if } j \le k \le N-1. \end{cases}$$

Proof. If we insert these formulae for g_k into equations (4.4.5) and (4.4.6) for $0 \le j \le 2\nu-1$ and into equations (4.4.7) and (4.4.8) for $2\nu \le j \le N-1$, we see the assertion of the proposition after a few case distinctions and uses of the fact that A_n and B_n are solutions of the recurrence $f_{n-1}-4f_n+f_{n+1}=0$. Observe that for evaluating (4.4.5),(4.4.6),(4.4.7),(4.4.8) the recursions (3.1.1),(3.1.2),(3.1.7),(3.1.8) for A_n and B_n and the identities from Lemma 3.1.3 are useful. □

Remark. From the formulae in Proposition 4.4.1 we obtain that for N even, $g_k \ge 0$ for all $0 \le k \le N-1$ and for N odd it holds that $g_k \ge 0$ for $|k-j| \le \frac{N-1}{2}$ and $g_k \le 0$ for $|k-j| \ge \frac{N+1}{2}$.

4.4.3 The Main Case $\nu = j = 1$

The first special case to analyze is the parameter choice $\nu = j = 1$. As we will see in the sequel, this is the main case in the sense that for $N \to \infty$ and $\nu = j = 1$, $\kappa := \kappa(1)$ converges to the Lebesgue constant $2 + \frac{33-18\sqrt{3}}{13}$. In this section, we set $K = N-1$, since then notation will be shorter. We get then as a special instance of Proposition 4.4.1 that $g(k) = g(N,1,1,k)$ equals

$$\begin{cases} 2\left[(-1)^N + A_K - B_K\right], & \text{if } k=0, \\ 8B_K, & \text{if } k=1, \\ 2\left[A_{N-k}+B_{N-k}+(-1)^N(A_{k-2}+B_{k-2})\right], & \text{if } 2 \le k \le N-1. \end{cases}$$

Note that $g(2) = g(0)$. Additionally it holds that

$$D(N,1) = 18B_K - 2A_K - 2(-1)^N.$$

Furthermore the use of the recurrences (3.1.1),(3.1.2),(3.1.7) and (3.1.8) for A and B yields

$$\begin{aligned} |g_1|+|g_2| &= 2(-1)^N + 6B_K + 2A_K, && (4.4.11) \\ |g_k|+|g_{k+1}| &= 4|A_{N-k}+(-1)^N A_{k-1}| \quad \text{if } k \ge 2, k \ne (N+1)/2 && (4.4.12) \end{aligned}$$

and

$$\begin{aligned} |g_{(N+1)/2}|+|g_{(N+3)/2}| &= 8A_{K/2} \quad \text{for } N \text{ even}, && (4.4.13) \\ |g_{(N+1)/2}|+|g_{(N+3)/2}| &= 8B_{K/2} \quad \text{for } N \text{ odd}. && (4.4.14) \end{aligned}$$

We recall that the indices have to be taken modulo N. The quotient of subsequent values of g has then the following special form

Lemma 4.4.2. *For $2 \le k \le N-1$ it holds that*

$$\frac{|g_{k+1}|}{|g_k|} = \frac{A_{|N/2-k|}}{A_{|N/2-k+1|}} \quad \textit{for } N \textit{ even,} \tag{4.4.15}$$

$$\frac{|g_{k+1}|}{|g_k|} = \frac{B_{|N/2-k|}}{B_{|N/2-k+1|}} \quad \textit{for } N \textit{ odd.} \tag{4.4.16}$$

Proof. Let $k \le N/2$. Then we have by (3.1.3) and (3.1.4) and the definition of A_n and B_n

$$\begin{aligned} A_{N-k-1} &= A_{N/2-k}A_{N/2-1} + 3B_{N/2-k}B_{N/2-1}, \\ B_{N-k-1} &= A_{N/2-k}B_{N/2-1} + B_{N/2-k}A_{N/2-1}, \\ A_{k-1} &= A_{N/2-1}A_{N/2-k} - 3B_{N/2-k}B_{N/2-1}, \\ B_{k-1} &= B_{N/2-1}A_{N/2-k} - A_{N/2-1}B_{N/2-k}. \end{aligned}$$

For N even, summing these four equations yields $g_{k+1}/2$ on the left hand side and $A_{N/2-k}$ times a term independent of k on the right hand side. On the other hand, for N odd, summing the first two equations and subtracting the second two gives us $|g_{k+1}|/2$ on the left hand side and $B_{N/2-k}$ times a term independent of k on the right hand side. An analogous argument for $k \ge N/2$ completes the proof of the lemma. □

Estimates for N even

For N even, we get from (4.4.10) and the fact $g(0) = g(2)$ that

$$\kappa := \kappa(1) = 3D(N,1)^{-1} \sum_{k=1}^{K} (g_k + g_{k+1}) \phi\left(\frac{g_{k+1}}{g_k}\right).$$

so inserting (4.4.11)-(4.4.14) into the expression for κ and recalling $K = N-1$ yield that κ equals

$$3D(N,1)^{-1} \left[(2 + 6B_K + 2A_K)\phi\left(\frac{1 + A_K - B_K}{4B_K}\right) + 8\sum_{k=2}^{K} A_{k-1}\phi\left(\frac{A_{|N/2-k|}}{A_{|N/2-k+1|}}\right)\right]. \tag{4.4.17}$$

Now observe that Lemma 4.4.2 and Lemma 3.1.2 imply $\lambda^{-1} < \frac{g_{k+1}}{g_k} = \frac{A_{|N/2-k|}}{A_{|N/2-k+1|}} < \lambda$ for $k \ge 2$, so using Lemma 3.1.4, the previous expression for κ is strictly less than

$$6D(N,1)^{-1} \left[(1 + 3B_K + A_K)\phi\left(\frac{1 + A_K - B_K}{4B_K}\right) + 4\phi(\lambda)\sum_{k=1}^{K-1} A_k \right].$$

If we use Lemma 3.1.1 to evaluate the sum and remark that $A_K = \sqrt{3}B_K + \lambda^{-K}$ by Lemma 3.1.2, we obtain by setting $\frac{1+A_K-B_K}{4B_K} = \eta + h$ with

$$\eta = \frac{\sqrt{3}-1}{4} \quad \text{and} \quad h = h(N) = \frac{1 + \lambda^{-K}}{4B_K}$$

the subsequent estimate for κ:

$$\kappa \leq 6D(N,1)^{-1}\left[(1+3B_K+A_K)\phi(\eta+h)+2\phi(\lambda)(3B_K-A_K-1)\right]. \qquad (4.4.18)$$

Since $\phi'(t)$ is increasing for $t \leq 2$ (Lemma 3.1.4) and $h \leq 1/2$ for $N \geq 2$, we get by applying the mean value theorem to ϕ

$$\phi(\eta+h) \leq \phi(\eta)+\phi'(\eta+\frac{1}{2})h \qquad (4.4.19)$$

Thus, using (4.4.19) in (4.4.18) we see that in order to prove $\kappa < \gamma$, it suffices to show that

$$6D(N,1)^{-1}\left[(1+3B_K+A_K)(\phi(\eta)+\phi'(\eta+1/2)h)+2\phi(\lambda)(3B_K-A_K-1)\right] < \gamma. \qquad (4.4.20)$$

If we multiply this inequality with $D(N,1)$, collect the factors for B_K and A_K and observe that

$$\theta := 6\phi(\eta)+2\gamma-12\phi(\lambda)=\frac{1}{\sqrt{3}}(18\gamma-18\phi(\eta)-36\phi(\lambda)),$$

we see that (4.4.20) is equivalent to

$$\theta(\sqrt{3}B_K-A_K-1)+6h(N)(1+3B_K+A_K)|\phi'(\eta+1/2)| > 0. \qquad (4.4.21)$$

Now we use again $A_K=\sqrt{3}B_K+\lambda^{-K}$ and insert the definition of $h(N)$ to express the left hand side of (4.4.21) as

$$(1+\lambda^{-K})\left[\frac{3}{2B_K}(1+(\sqrt{3}+3)B_K+\lambda^{-K})|\phi'(\eta+1/2)|-\theta\right].$$

Clearly, this is greater than

$$(1+\lambda^{-K})\left[\frac{3(\sqrt{3}+3)}{2}|\phi'(\eta+1/2)|-\theta\right]$$

and this is easily seen to be greater than zero. Thus we have shown for N even and $\nu = j = 1$ that $\kappa < \gamma$.

Estimates for N odd

For N odd, (4.4.10) and Remark 4.4.2 yield for κ the formula

$$\kappa = 3D(N,1)^{-1}\left[\sum_{\substack{k=1\\ k\neq(N+1)/2}}^{K}(|g_k|+|g_{k+1}|)\phi\left(\frac{|g_k|}{|g_{k+1}|}\right)+|g_{(N+1)/2}|+|g_{(N+3)/2}|\right].$$

We now use Lemma 4.4.2 and identities (4.4.11)-(4.4.14) and recall the setting $K = N-1$ to obtain after a little calculation that

$$\begin{aligned}\kappa &= 6D(N,1)^{-1}\Bigg[(3B_K+A_K-1)\phi\left(\frac{A_K-B_K-1}{4B_K}\right)\\ &\quad+4\sum_{k=2}^{K/2}(A_{N-k}-A_{k-1})\phi\left(\frac{B_{N/2-k}}{B_{N/2-k+1}}\right)+4B_{K/2}\Bigg] \qquad (4.4.22)\end{aligned}$$

We first estimate the two summands of κ separately

TERM I. $(3B_K + A_K - 1)\phi\left(\frac{A_K - B_K - 1}{4B_K}\right)$.

We have $3B_K + A_K - 1 \le (3+\sqrt{3})B_K$ by Lemma 3.1.2 and $\frac{A_K - B_K - 1}{4B_K} = \eta - h$ with

$$\eta = \frac{\sqrt{3}-1}{4} \quad \text{and} \quad h = \frac{1-\lambda^{-K}}{4B_K},$$

so the mean value theorem implies

$$\begin{aligned} (3B_K + A_K - 1)\phi\left(\frac{A_K - B_K - 1}{4B_K}\right) &\le (3+\sqrt{3})B_K\phi(\eta - h) \\ &\le (3+\sqrt{3})B_K(\phi(\eta) - \phi'(0)h) \\ &= (3+\sqrt{3})B_K(\phi(\eta) + 2h), \end{aligned}$$

since ϕ' is increasing for $t \le 2$ and $\phi'(0) = -2$.

TERM II. $II := \sum_{k=2}^{K/2}(A_{N-k} - A_{k-1})\phi\left(\frac{B_{N/2-k+1}}{B_{N/2-k}}\right)$.

Since $B_{L+1} = \lambda B_L + \lambda^{-L}$, we get with the mean value theorem and the fact that ϕ' is decreasing for $t \ge 2$

$$\phi\left(\frac{B_{N/2-k+1}}{B_{N/2-k}}\right) \le \phi(\lambda) + \phi'(\lambda)\frac{\lambda^{k-N/2}}{B_{N/2-k}}.$$

Now, if we use the identity $2\sum_{k=0}^{L} A_k = 3B_{L+1} - A_{L+1} + 1$ from Lemma 3.1.1 and simplify using the recurrences for A and B, we obtain

$$\begin{aligned} \sum_{k=2}^{K/2} A_{N-k} - A_{k-1} &= \frac{1}{2}(3B_K - A_K - 6B_{K/2} + 1) \\ &\le \frac{1}{2}((3-\sqrt{3})B_K - 6B_{K/2} + 1), \end{aligned}$$

by Lemma 3.1.2. Next, we get

$$\begin{aligned} S := \sum_{k=2}^{K/2} A_{N-k}\frac{\lambda^{k-N/2}}{B_{N/2-k}} &= \sqrt{3}\sum_{k=2}^{K/2}\frac{\lambda^{k-N/2}(\lambda^{N-k} + \lambda^{k-N})}{\lambda^{N/2-k} - \lambda^{k-N/2}} \\ &= \sqrt{3}\sum_{k=2}^{K/2}\frac{\lambda^{N-k} + \lambda^{k-N}}{\lambda^{N-2k} - 1}, \end{aligned}$$

by (3.1.9). Since $1 \le \lambda^{N-2k}/2$, we estimate

$$\begin{aligned} S &\le 2\sqrt{3}\sum_{k=2}^{K/2}\frac{\lambda^{N-k} + \lambda^{k-N}}{\lambda^{N-2k}} = 2\sqrt{3}\sum_{k=2}^{K/2}\lambda^k + \lambda^{3k-2N} \\ &= 2\sqrt{3}\left[\frac{\lambda^{K/2+1} - \lambda^2}{\lambda - 1} + \lambda^{-2N}\frac{\lambda^{3(K/2+1)} - \lambda^6}{\lambda^3 - 1}\right] \\ &\le 2\sqrt{3}\left[\frac{\lambda^{K/2+1}}{\lambda - 1} + \lambda^{-2N}\frac{\lambda^{3(K/2+1)}}{\lambda - 1}\right] \\ &= 4\sqrt{3}\frac{A_{K/2}}{1-\lambda^{-1}} \le 4\sqrt{3}\frac{\sqrt{3}B_{K/2} + 1}{1 - \lambda^{-1}}. \end{aligned}$$

If we summarize all estimates, we get for the whole sum

$$II \le \frac{\phi(\lambda)}{2}((3-\sqrt{3})B_K - 6B_{K/2}+1) + 4\sqrt{3}\phi'(\lambda)\frac{\sqrt{3}B_{K/2}+1}{1-\lambda^{-1}}.$$

Let us now return to (4.4.22). If we use the estimate $h \le \frac{1}{4B_K}$, we obtain by collecting everything together that

$$\kappa \le 6D(N,\nu)^{-1}(\sigma B_K - \tau B_{K/2} + \vartheta),$$

with $\sigma = (3+\sqrt{3})\phi(\eta) + 2\phi(\lambda)(3-\sqrt{3})$, $\tau = 12\phi(\lambda) - 4 - \frac{48\phi'(\lambda)}{1-\lambda^{-1}} > 0$ and $\vartheta = \frac{3+\sqrt{3}}{2} + 2\phi(\lambda) + \frac{16\sqrt{3}\phi'(\lambda)}{1-\lambda^{-1}}$. Now recall that $D(N,1) = 18B_K - 2A_K + 2 \ge (18-2\sqrt{3})B_K$ by Lemma 3.1.2, so in order to prove $\kappa < \gamma$, it suffices to show

$$\frac{\gamma(18-2\sqrt{3})B_K}{6} > \sigma B_K - \tau B_{K/2} + \vartheta.$$

Since $\sigma = \frac{\gamma}{6}(18-2\sqrt{3})$, this is equivalent with

$$\tau B_{K/2} - \vartheta > 0$$

and this is true for $N \ge 7$. For $N < 7$ we get the desired bound for κ from Table 4.1 on page 83.

Asymptotic Behavior

In this section, we will calculate the limit of κ with $N \to \infty$ for $\nu = j = 1$. In the following, the symbol $\sim$ will denote asymptotic equality for $N \to \infty$. If we remark $A_N \sim \sqrt{3}B_N$, $A_{N+1} \sim \lambda A_N$ (by Lemma 3.1.2) and recall the definition of $D(N,1) = 18B_K - 2A_K - 2$ (where as above, $K = N-1$) we get for N even out of (4.4.17) that κ equals

$$6D(N,1)^{-1}\left[(1+3B_K+A_K)\phi\left(\frac{1+A_K-B_K}{4B_K}\right) + 4\sum_{k=2}^{K} A_{k-1}\phi\left(\frac{A_{|N/2-k|}}{A_{|N/2-k+1|}}\right)\right]$$

This is asymptotically equal

$$\frac{6}{(18-2\sqrt{3})B_K}\left[(3+\sqrt{3})B_K\phi\left(\frac{\sqrt{3}-1}{4}\right) + 4\phi(\lambda)\sum_{k=3N/4}^{K-1} A_k\right]$$

With the identity $2\sum_{k=0}^{L} A_k = 3B_{L+1} - A_{L+1} + 1$ from Lemma 3.1.1, we get further

$$\begin{aligned}
\kappa &\sim \frac{6}{(18-2\sqrt{3})B_K}\left[(3+\sqrt{3})B_K\phi\left(\frac{\sqrt{3}-1}{4}\right) + 2\phi(\lambda)(3-\sqrt{3})B_K\right] \\
&= \frac{6}{18-2\sqrt{3}}\left[(3+\sqrt{3})\phi\left(\frac{\sqrt{3}-1}{4}\right) + 2\phi(\lambda)(3-\sqrt{3})\right] \\
&= 2 + \frac{33-18\sqrt{3}}{13} = \gamma.
\end{aligned}$$

If on the other hand N is odd, we obtain out of (4.4.22)

$$\kappa \sim \frac{6}{(18-2\sqrt{3})B_K}\left[(3+\sqrt{3})B_K\phi\left(\frac{\sqrt{3}-1}{4}\right)+4\sum_{k=2}^{K/4}A_{N-k}\phi\left(\frac{B_{N/2-k}}{B_{N/2-k+1}}\right)\right]$$

Again, with the identity $2\sum_{k=0}^{L}A_k = 3B_{L+1} - A_{L+1} + 1$ and $B_{N+1} \sim \lambda B_N$ we see in the same way as above

$$\kappa \sim \gamma.$$

Thus if we combine the estimates of this section (Section 4.4.3) with the numerical results from Table 4.1 on page 83 we have shown that for $\nu = j = 1$, we have $\kappa < \gamma$ and $\lim_{N\to\infty}\kappa = \gamma$. We will see in the next section, that this is the critical case, since we will show that for all other values of ν and j we have $\kappa < \gamma$.

4.4.4 Estimating $\kappa(j)$

In this section we will derive bounds for $\kappa(j)$ for all remaining values of ν, j, which will allow us to deduce that for all $n, \nu \in \mathbb{N}$, $0 \le \nu \le n$, we have $\|P_{n,\nu}\|_\infty < \gamma$. In order to derive these estimates for $\kappa(j)$ we first need some for the quotients of subsequent values of g. This is the content of the following two lemmas.

Lemma 4.4.3. *Let N be even. Then it holds that*

$$6^{-1} \le \frac{g_{k+1}}{g_k} \le 6 \quad \text{for } k = 0 \text{ or } k = 2\nu - 1, \tag{4.4.23}$$

$$4^{-1} \le \frac{g_{k+1}}{g_k} \le 4 \quad \text{for } k \ne 0 \text{ and } k \ne 2\nu - 1. \tag{4.4.24}$$

For $j = 0, k = 0$, we have the better estimate

$$4^{-1} \le \frac{g_{k+1}}{g_k} \le 4.$$

We get analogous estimates for N odd, but we have to add a further restriction to the domain of validity of the inequalities

Lemma 4.4.4. *Let $N \ge 7$ be odd and $|k-j| \le \frac{N-5}{2}$ or $|k-j| \ge \frac{N+5}{2}$ Then we have*

$$6^{-1} \le \frac{|g_{k+1}|}{|g_k|} \le 6 \quad \text{for } k = 0 \text{ or } k = 2\nu - 1$$

$$4^{-1} \le \frac{|g_{k+1}|}{|g_k|} \le 4 \quad \text{for } k \ne 0 \text{ and } k \ne 2\nu - 1.$$

Additionally, for $j = 0, k = 0$ we have the better estimate

$$4^{-1} \le \frac{|g_{k+1}|}{|g_k|} \le 4.$$

For a proof of Lemma 4.4.3 or parts of a proof of Lemma 4.4.4, see Appendix A. We note that in the following, we only treat the case N even. In fact, as we will show later (in Section 4.4.4), the case N odd will nonetheless follow out of these estimates. Combining thus formula (4.4.10) with Remark 4.4.2 yields for N even

$$\kappa(j) = D(N,\nu)^{-1}\left[\frac{3}{2}\sum_{k=0}^{2\nu-1}(g_k+g_{k+1})\phi\left(\frac{g_{k+1}}{g_k}\right)+3\sum_{k=2\nu}^{N-1}(g_k+g_{k+1})\phi\left(\frac{g_{k+1}}{g_k}\right)\right]. \tag{4.4.25}$$

We split three cases and view $j = 0, 1 \leq j \leq 2\nu-1, 2\nu \leq j \leq N-1$.

Case $j = 0$

Invoking Lemma 4.4.3, we get a bound for $\kappa(0)$:

$$D(N,\nu)\kappa(j) \leq \frac{3}{2}\phi(6)I_1 + \frac{3}{2}\phi(4)I_2 + 3\phi(4)I_3 =: J,$$

where

$$I_1 = g_{2\nu-1}+g_{2\nu}, \quad I_2 = \sum_{k=0}^{2\nu-2} g_k + g_{k+1}, \quad I_3 = \sum_{k=2\nu}^{N-1} g_k + g_{k+1}.$$

Proposition 4.4.5. *We have for $j = 0$*

$$\begin{aligned} I_1 &= 2(B_{2\nu-1}+B_{2\nu}) + A_{N-2\nu+1}, \\ I_2 &= 2A_{2\nu-1} - 2 + A_N - A_{N-2\nu+1} + A_{N-2\nu}(A_{2\nu}-2), \\ I_3 &= 2A_N - A_{2\nu} + A_{N-2\nu} - A_{2\nu}A_{N-2\nu} - 1. \end{aligned}$$

Proof. Insert the formulae from Proposition 4.4.1, use the recurrences (3.1.7),(3.1.8) for A and B and Lemmas 3.1.1 and 3.1.3. □

With this proposition and the identity $A_N = A_{N-2\nu}A_{2\nu} + 3B_{N-2\nu}B_{2\nu}$ (Lemma 3.1.3) we see that

$$\begin{aligned} J &= \frac{3}{2}\phi(6)\left[2(B_{2\nu-1}+B_{2\nu}) + A_{N-2\nu+1}\right] \\ &\quad + \frac{3}{2}\phi(4)\left[4A_N - 4 + 3B_{2\nu}B_{N-2\nu} - A_{N-2\nu+1} + 2(A_{2\nu-1}-A_{2\nu})\right]. \end{aligned}$$

Now recall that $D(N,\nu) = 2A_N + \frac{3}{2}B_{2\nu}B_{N-2\nu} - 2$. If we then use the recurrences (3.1.7), (3.1.8) for $A_{2\nu-1}$ and $B_{2\nu-1}$ and set $s := \frac{3}{2}(\phi(6)-\phi(4)) = \frac{138}{1225}$ it follows with $\phi(4) = \frac{17}{25}$

$$J = \frac{51}{25}D(N,\nu) + s(6B_{2\nu} - 2A_{2\nu} + A_{N-2\nu+1}). \tag{4.4.26}$$

If we plug in the estimate for $B_{2\nu}$ from Lemma 3.1.2 and remark that $2\nu \leq N-1$ and $N - 2\nu + 1 \leq N-1$, we get

$$J \leq \frac{51}{25}D(N,\nu) + s(2\sqrt{3}-1)A_{N-1}. \tag{4.4.27}$$

Using again Lemma 3.1.2 on A_{N-1}, we obtain

$$J \leq \frac{51}{25} D(N,\nu) + (A_N + \sqrt{3}) \frac{s}{\lambda}(2\sqrt{3}-1).$$

Finally, the definition of $D(N,\nu)$ and the fact that the function $\nu \mapsto B_{2\nu}B_{N-2\nu}$ is concave for $1 \leq \nu \leq (N-1)/2$ and therefore attains its minimum at the border for $2\nu = N-1$ yield

$$A_N + \sqrt{3} \leq \frac{D(N,\nu)}{2} = A_N + \frac{3}{4} B_{2\nu}B_{N-2\nu} - 1 \quad \text{for } N \geq 3.$$

Thus, $\kappa(0)$ admits the bound

$$\kappa(0) \leq \frac{51}{25} + \frac{s}{2\lambda}(2\sqrt{3}-1) \approx 2.07719 \quad \text{for } N \geq 3.$$

For $N < 3$, see Table 4.1 on page 83.

Case $1 \leq j \leq 2\nu - 1$

Like for $j = 0$, Lemma 4.4.3 yields a bound for $\kappa(j)$:

$$D(N,\nu)\kappa(j) \leq \frac{3}{2}\phi(6) I_1 + \frac{3}{2}\phi(4) I_2 + 3\phi(4) I_3 =: J,$$

where now

$$I_1 = g_0 + g_1 + g_{2\nu-1} + g_{2\nu}, \quad I_2 = \sum_{k=1}^{2\nu-2} g_k + g_{k+1}, \quad I_3 = \sum_{k=2\nu}^{N-1} g_k + g_{k+1}.$$

Proposition 4.4.6. *We have for* $1 \leq j \leq 2\nu - 1$

$$\begin{aligned}
I_1 &= 2(B_j + B_{j-1} + B_{2\nu-j} + B_{2\nu-j-1}) \\
&\quad +3B_{N-2\nu+1}(B_j + B_{2\nu-j}) + A_{N-j+1} + A_{N-2\nu+j+1} \\
I_2 &= 2D(N,\nu) - 3(B_{2\nu-j} + B_j)(B_{N-2\nu+1} + 2B_{N-2\nu}) \\
&\quad +2(A_{j-1} - A_{N-j} + A_{2\nu-j-1} - A_{N-2\nu+j}) - A_{N-j+1} - A_{N-2\nu+1+j} \\
I_3 &= A_{N-j} + A_{N-2\nu+j} - A_{2\nu-j} - A_j + 3B_{N-2\nu}(B_j + B_{2\nu-j})
\end{aligned}$$

Proof. As in the case for $j = 0$, it suffices to insert the formulae from Proposition 4.4.1, to use Lemmas 3.1.1 and 3.1.3 and employ the recurrence formulae (3.1.1),(3.1.2),(3.1.7) and (3.1.8) for A and B. □

Now recall that we defined $s = \frac{3}{2}(\phi(6) - \phi(4)) = \frac{138}{1225}$ and that $\phi(4) = \frac{17}{25}$; thus inserting Proposition 4.4.6 into the definition of J and using recursions (3.1.7) and (3.1.8) for $A_{j-1}, B_{j-1}, A_{2\nu-j-1}, B_{2\nu-j-1}$ yield

$$\begin{aligned}
J &= \frac{51}{25} D(N,\nu) + 2s(3B_j - A_j + 3B_{2\nu-j} - A_{2\nu-j}) \\
&\quad +s(A_{N-j+1} + A_{N-2\nu+j+1} + 3B_{N-2\nu+1}(B_{2\nu-j} + B_j)) =: J_1 + J_2 + J_3.
\end{aligned}$$

With Lemma 3.1.2 we deduce

$$J_2 \leq 2s(3-\sqrt{3})(B_j + B_{2\nu-j}).$$

Since the functions $x \mapsto A_x + A_{K-x}$ and $x \mapsto B_x + B_{K-x}$ are convex for $K > 0$ and $0 \leq x \leq K$, we see that the maximum is attained at the border, so we get

$$J \leq \frac{51}{25}D(N,\nu) + 2s(3-\sqrt{3})(1+B_{2\nu-1}) + s(A_N + A_{N-2\nu+2} + 3B_{N-2\nu+1}(1+B_{2\nu-1})).$$

We now require $\nu \geq 2$. Since we are in the case $1 \leq j \leq 2\nu - 1$, we see that the only case missing is $\nu = 1, j = 1$ which was treated above in Section 4.4.3. If we now use the estimates

i. $B_{2\nu-1} \leq \lambda^{-1}B_{2\nu} \leq \lambda^{-1}B_{2\nu}B_{N-2\nu}$ (Lemma 3.1.2),
ii. $A_{N-2\nu+2} \leq A_{N-2} \leq \lambda^{-2}A_N + \frac{\sqrt{3}}{\lambda}(1+\lambda^{-1})$ (Lemma 3.1.2),
iii. $3B_{N-2\nu+1}B_{2\nu-1} \leq \frac{A_N}{2}$ (Lemmas 3.1.2 and 3.1.3),
iv. $3B_{N-2\nu+1} \leq 3B_{N-3} \leq 3\lambda^{-3}B_N \leq \sqrt{3}\lambda^{-3}A_N$ (Lemma 3.1.2),

we get

$$\begin{aligned} J - \frac{51}{25}D(N,\nu) &\leq s(a_1 + a_2A_N + a_3B_{2\nu}B_{N-2\nu}) \\ &= s\left(a_1 + a_2 + \frac{a_2}{2}D(N,\nu) - (\frac{3a_2}{4} - a_3)B_{2\nu}B_{N-2\nu}\right) \end{aligned}$$

with $a_1 = 2(3-\sqrt{3}) + \frac{\sqrt{3}}{\lambda}(1+\lambda^{-1})$, $a_2 = \frac{3}{2} + \lambda^{-2} + \sqrt{3}\lambda^{-3}$ and $a_3 = \frac{2}{\lambda}(3-\sqrt{3})$. Since the function $\nu \mapsto B_{2\nu}B_{N-2\nu}$ is concave and therefore attains its minimum for $2\nu = N-1$ we conclude with the fact that $\frac{3a_2}{4} - a_3 \geq 0$ and the exact value of this constant that

$$J \leq D(N,\nu)\left[\frac{51}{25} + \frac{sa_2}{2}\right] \quad \text{for } N \geq 4.$$

Thus we obtain finally

$$\kappa(j) \leq \frac{51}{25} + \frac{sa_2}{2} \leq 2.130411 \quad \text{for } N \geq 4. \tag{4.4.28}$$

Once again, out of Table 4.1 on page 83 we get that for $N < 4$ we have the same bound for κ.

Case $2\nu \leq j \leq N-1$

In that case, we get again with Lemma 4.4.3 an upper bound for κ

$$D(N,\nu)\kappa(j) \leq \frac{3}{2}\phi(6)I_1 + \frac{3}{2}\phi(4)I_2 + 3\phi(4)I_3 =: J,$$

where

$$I_1 = g_0 + g_1 + g_{2\nu-1} + g_{2\nu}, \quad I_2 = \sum_{k=1}^{2\nu-2} g_k + g_{k+1}, \quad I_3 = \sum_{k=2\nu}^{N-1} g_k + g_{k+1}.$$

Proposition 4.4.7. *We have for* $2\nu \le j \le N-1$

$$\begin{aligned} I_1 &= (1+B_{2\nu}+B_{2\nu-1})(A_{j-2\nu}+A_{N-j})+B_j+B_{j-1}+B_{N-j}+B_{N-j+1} \\ &\quad +B_{j-2\nu}+B_{j-2\nu+1}+B_{N-j+2\nu}+B_{N-j+2\nu-1}, \\ I_2 &= A_{j-1}-A_{j-2\nu+1}+(A_{j-2\nu}+A_{N-j})(A_{2\nu-1}-2)+A_{N-j+2\nu-1}-A_{N-j+1}, \\ I_3 &= D(N,\nu)+(1-A_{2\nu})(A_{j-2\nu}+A_{N-j})-\frac{3}{2}B_{2\nu}(B_{N-j}+B_{j-2\nu}) \end{aligned}$$

Proof. Insert the formulae for g from Proposition 4.4.1 and use Lemmas 3.1.1, 3.1.3 and the recurrences (3.1.1),(3.1.2),(3.1.7) and (3.1.8) for A and B. □

If we apply the recurrences (3.1.1),(3.1.2),(3.1.7) and (3.1.8) for A and B, Lemma 3.1.3 and Proposition 4.4.7 to J, we see that it simplifies to (recall that $s=\frac{3}{2}(\phi(6)-\phi(4))=\frac{138}{1225}$ and $\phi(4)=\frac{17}{25}$)

$$\begin{aligned} J &= \frac{51}{25}D(N,\nu)+s\left[3B_j-A_j+(A_{j-2\nu}+A_{N-j})(3B_{2\nu}-A_{2\nu})\right. \\ &\quad \left.+3B_{N-j+2\nu}-A_{N-j+2\nu}+A_{N-j+1}+A_{j-2\nu+1}\right]. \end{aligned}$$

Remember that $2\nu \le j \le N-1$. Since the functions $j \mapsto A_{N-j+1}+A_{j-2\nu+1}$, $j \mapsto 3B_j-A_j+3B_{N-j+2\nu}-A_{N-j+2\nu}$, $j \mapsto A_{j-2\nu}+A_{N-j}$ are convex, they attain their maximum at the border, in our case for $j=2\nu$, so it holds that

$$J \le \frac{51}{25}D(N,\nu)+s\left[6B_{2\nu}-2A_{2\nu}+3B_N-A_N+A_{N-2\nu}(3B_{2\nu}-A_{2\nu})+2+A_{N-2\nu+1}\right].$$

For $2\nu = N-1$, we see with an estimate utilizing Lemma 3.1.2 and the recurrences for A and B that $\kappa(j) \le \frac{J}{D(N,\nu)} \le \frac{51}{25}+\frac{3}{4}s \approx 2.1245$ for $N \ge 4$. If $2\nu \le N-2$, we use the estimates

i. $\sqrt{3}B_{2\nu} \le A_{2\nu}$ (Lemma 3.1.2),
ii. $3B_N \le \sqrt{3}A_N$ (Lemma 3.1.2),
iii. $A_{N-2\nu+1} \le A_{N-1}$,
iv. $A_{N-2\nu} \le \sqrt{3}B_{N-2\nu}+1$ (Lemma 3.1.2),
v. $3B_{N-2\nu}B_{2\nu} \le A_N/2$ (Lemmas 3.1.2 and 3.1.3),
vi. $A_{N-1} \le \lambda^{-1}(A_N+\sqrt{3})$ (Lemma 3.1.2),
vii. $B_{2\nu} \le \frac{B_{2\nu}B_{N-2\nu}}{4}$ $(2\nu \le N-2)$

and obtain further

$$\begin{aligned} J-\frac{51}{25}D(N,\nu) &\le s\left[a_1+a_2A_N+a_3B_{2\nu}B_{N-2\nu}\right] \\ &= s\left(a_1+a_2+\frac{a_2}{2}D(N,\nu)-(\frac{3a_2}{4}-a_3)B_{2\nu}B_{N-2\nu}\right) \end{aligned}$$

with $a_1 = 2+\sqrt{3}\lambda^{-1}$, $a_2=\frac{3}{2}(\sqrt{3}-1)+\lambda^{-1}$, $a_3=\frac{3}{4}(3-\sqrt{3})$. Since $\frac{3}{4}a_2-a_3>0$, we conclude that

$$\kappa(j) = \frac{J}{D(N,\nu)} \le \frac{51}{25}+\frac{sa_2}{2} \approx 2.117 \quad \text{for } N \ge 5.$$

For $N<5$, see Table 4.1 on page 83.

Summary What we have shown up to now is that in particular for $N \geq 5$ even, for all $1 \leq \nu \leq \frac{N-1}{2}$ and all $0 \leq j \leq N-1$ (*except* the case $\nu = 1, j = 1$)

$$\kappa(j) \leq 2.130411, \quad \text{(see (4.4.28))} \tag{4.4.29}$$

$\kappa(j)$ for N odd

Now let N be odd. We recall the formula (4.4.10) for $\kappa(j)$

$$\kappa(j) = D(N,\nu)^{-1}\left[\frac{3}{2}\sum_{k=0}^{2\nu-1}(|g_k| + |g_{k+1}|)\cdot\xi_{j,k} + 3\sum_{k=2\nu}^{N-1}(|g_k| + |g_{k+1}|)\cdot\xi_{j,k}\right],$$

where

$$\xi_{j,k} = \begin{cases} 1, & \text{if } \operatorname{sgn} a_{j,k} = \operatorname{sgn} a_{j,k+1}, \\ \phi(|g_{k+1}|/|g_k|), & \text{else} \end{cases}.$$

If we write formula (4.4.10) in the form $\kappa(j) = \sum_{k=0}^{N-1} s_k$, every summand s_k admits the (trivial) bound

$$s_k \leq \frac{3(|g_k| + |g_{k+1}|)}{D(N,\nu)},$$

since $\phi(t) \leq 1$ for all $t \geq 0$. We now call $D^e(N,\nu)$ and g_k^e the formulae for $D(N,\nu)$ and g_k respectively, but for N even. That is, write 1 instead of $(-1)^N$ in formula (4.4.9) and the expressions for g_k in Proposition 4.4.1, no matter if N is even or odd. Then we get further

$$s_k \leq \frac{3(g_k^e + g_{k+1}^e)}{D^e(N,\nu)} \tag{4.4.30}$$

Easy estimates for the expressions for g_k^e and $D^e(N,\nu)$ supply us now with

$$\frac{3(g_k^e + g_{k+1}^e)}{D^e(N,\nu)} \leq 10^{-3}, \tag{4.4.31}$$

provided $\frac{N-3}{2} \leq |k-j| \leq \frac{N+3}{2}$ and $N \geq 19$. So, let $N \geq 19$. Then we get for κ out of (4.4.10), with the index set $\Lambda = \left\{\frac{N-3}{2}, \frac{N-1}{2}, \frac{N+1}{2}, \frac{N+3}{2}\right\}$

$$\kappa(j) = \sum_{k=0}^{N-1} s_k = \sum_{k\notin\Lambda} s_k + \sum_{k\in\Lambda} s_k.$$

We obtain further that $\sum_{k\notin\Lambda} s_k$ equals

$$D(N,\nu)^{-1}\left[\frac{3}{2}\sum_{\substack{k=0\\k\notin\Lambda}}^{2\nu-1}(|g_k| + |g_{k+1}|)\phi(|g_{k+1}|/|g_k|) + 3\sum_{\substack{k=2\nu\\k\notin\Lambda}}^{N-1}(|g_k| + |g_{k+1}|)\phi(|g_{k+1}|/|g_k|)\right]$$

and by the above considerations this is less or equal

$$D^e(N,\nu)^{-1}\left[\frac{3}{2}\sum_{\substack{k=0\\k\notin\Lambda}}^{2\nu-1}(g_k^e + g_{k+1}^e)\phi(|g_{k+1}|/|g_k|) + 3\sum_{\substack{k=2\nu\\k\notin\Lambda}}^{N-1}(g_k^e + g_{k+1}^e)\phi(|g_{k+1}|/|g_k|)\right]$$

We apply Lemma 4.4.4 and see that the terms $\phi(|g_{k+1}|/|g_k|)$ admit the same bounds as for the case N even. Thus, if we first apply the estimate and then omit the restriction $k \notin \Lambda$ for the summation scope, we arrive in estimating the same sum as for the case N even. Since for the case N even, we got the bound (4.4.29) (except for $\nu = j = 1$), we obtain finally

$$\sum_{k \notin \Lambda} s_k \leq 2.130411.$$

It remains to estimate $\sum_{k \in \Lambda} s_k$, but out of (4.4.30) and (4.4.31) we see

$$\sum_{k \in \Lambda} s_k \leq 4 \cdot 10^{-3},$$

so, if we summarize, we get

$$\kappa(j) \leq 2.134411$$

for all $N \geq 19, \nu, j$ (no matter if N is odd or even) *except* the case $\nu = j = 1$.

Summary Thus if we combine the present section (Section 4.4.4) with Sections 4.4.1 and 4.4.3, we have now shown that for all $N \geq 19$, $0 \leq \nu \leq n$ and $0 \leq j \leq N-1$, we have the bound

$$\kappa(j) < \gamma.$$

Looking at the numerical results of Table 4.1 on page 83, we get the first assertion of our main theorem (i.e. that $\|P_{n,\nu} : L^\infty(\mathbb{T}) \to L^\infty(\mathbb{T})\| < \gamma$ for all $n \in \mathbb{N}, 0 \leq \nu \leq n$). The asymptotic value γ for $\|P_{n,1} : L^\infty(\mathbb{T}) \to L^\infty(\mathbb{T})\|$ (as $n \to \infty$) was already identified in Section 4.4.3. So, the proof of the main theorem, Theorem 4.2.1 is complete.

$\nu \rightarrow$ / $N \downarrow$	0	1	2	3	4	5	6	7	8	9
2	1.66666667									
3	1.77777778	1.84444444								
4	1.88888889	2.00000000								
5	1.94696970	2.06951872	1.99530864							
6	1.96835017	2.09951691	2.03615841							
7	1.98631436	2.12227384	2.05943912	2.03242817						
8	1.99137719	2.12904795	2.06731688	2.05587710						
9	1.99637151	2.13550178	2.07350359	2.06635304	2.04175181					
10	1.99767916	2.13721808	2.07535577	2.06916162	2.06184314					
11	1.99903054	2.13897416	2.07709926	2.07093598	2.06877403	2.04423294				
12	1.99937738	2.13942680	2.07756643	2.07147842	2.07063242	2.06343762				
13	1.99974043	2.13989929	2.07804184	2.07192928	2.07140616	2.06942343	2.04489705			
14	1.99983312	2.14002005	2.07816469	2.07206051	2.07160716	2.07106530	2.06386325			
15	1.99993046	2.14014679	2.07829271	2.07218617	2.07173511	2.07157865	2.06959951	2.04507495		
16	1.99995528	2.14017911	2.07832546	2.07221958	2.07177393	2.07171306	2.07118008	2.06397716		
17	1.99998137	2.14021308	2.07835981	2.07225375	2.07180634	2.07176873	2.07162518	2.06964688	2.04512262	
18	1.99998802	2.14022174	2.07836857	2.07226256	2.07181575	2.07178315	2.07174416	2.07121066	2.06400767	
19	1.99999501	2.14023084	2.07837778	2.07227176	2.07182477	2.07179234	2.07178110	2.07163781	2.06965959	2.04513539
20	1.99999679	2.14023316	2.07838012	2.07227411	2.07182717	2.07179513	2.07179076	2.07175241	2.07121883	2.06401584

Table 4.1: Values of $\|P_{n,\nu}\|$ for different values of $\nu, N = n + \nu$ obtained with MATHEMATICA. The numbers are rounded to the last digit. We have with the same precision $\gamma \approx 2.14023734$.

Appendix A

Supplementary results

A.1 Orthogonalization of characteristic functions

Let (X, d, μ) be a space of homogeneous type which satisfies the standard assumptions and μ a probability measure. We employ the notation of Section 2.2.2. Then we have $X = Q_1^0$, $\mathcal{K}_0 = \{1\}, \mathcal{A}_0 = \{X\}$ and $\mathcal{K}_n = \emptyset$ for all $n \in \mathbb{N}$. We use then the dyadic cubes to build a basis in $L^2(X, \mu)$ consisting of martingale differences. Fix $n \in -\mathbb{N}$, $A \in \mathcal{A}_{n+1}$ and enumerate the elements in $\mathcal{E}(A)$ in the way that $\mathcal{E}(A) = \{Q_1, \ldots, Q_{N-1}\}$. The vector space of functions on A that are constant on $Q_1, \ldots, Q_{N-1}$ and $A^*(A)$ has the family

$$\{1_A\} \cup \{1_{Q_l} : 1 \leq l \leq N-1\}$$

as a basis. We apply the Gram-Schmidt orthogonalization procedure and call the corresponding basis functions d_{Q_l}. We normalize in such a way that

$$\|d_Q\|_2 = 1, \quad \operatorname{supp} d_Q \subseteq A \text{ for } Q \in \mathcal{E}(A). \tag{A.1.1}$$

We are now able to deduce an explicit construction of the basis functions. Fix $A \in \mathcal{A}_{n+1}$ for $n \in -\mathbb{N}$. Then denote

$$\begin{aligned} &\mu_0 := 1, \quad \mu_l := \mu(Q_l), \quad \mu_N := \mu(A^*(A)), \quad \mu := \mu(A), \\ &d_0 := 1_A, \quad d_k := d_{Q_k}, \text{ for } k \geq 1. \end{aligned}$$

Further we denote by d_k also the vector of values of d_k in $Q_1, \ldots, Q_{N-1}, A^*(A)$ respectively and set

$$d_k = (d_k^1, \ldots, d_k^N).$$

Then we have the representation formula

Theorem A.1.1. *In the notation introduced above, we have up to a multiplicative constant*

$$d_k^i = \begin{cases} 0, & \text{if } 1 \leq i \leq k-1 \\ \sum_{j=k+1}^N \mu_j, & \text{if } i = k \\ -\mu_k & \text{if } N \geq i \geq k+1 \end{cases}.$$

For that Theorem we need a few simple facts, which we collect in the following

Lemma A.1.2. *We have*

1. $\|d_0\|_2^2 = \mu$
2. *We have for* $k \geq 1$: $\|d_k\|_{L^2}^2 = \sum_{l=1}^N (d_k^l)^2 \mu_l$,
3. *For* $j \geq 1$, $k \geq 1$ *we have:* $\langle 1_{Q_k}, d_j \rangle = d_j^k \mu_k$
4. *If the representation for* d_k^i, $k \geq 1$, *announced in Theorem A.1.1, holds, we have*
$$\|d_k\|_2^2 = \mu_k \left(\sum_{l=k+1}^N \mu_l \right) \left(\sum_{l=k}^N \mu_l \right),$$
5. *For* $i \leq N$ *we have*
$$\sum_{j=1}^{i-1} \frac{\mu_j}{\left(\sum_{l=j+1}^N \mu_l\right)\left(\sum_{l=j}^N \mu_l\right)} = \frac{1}{\sum_{l=i}^N \mu_l} - \frac{1}{\mu}.$$

Proof. The first three points are clear, for 4. observe that

$$\begin{aligned}\|d_k\|_2^2 = \int_A (d_k)^2 \, d\mu(x) &= \left(\sum_{j=k+1}^N \mu_j \right)^2 \mu_k + \sum_{l=k+1}^N \mu_k^2 \mu_l \\ &= \mu_k \left(\sum_{l=k+1}^N \mu_l \right) \left(\sum_{l=k}^N \mu_l \right).\end{aligned}$$

To prove 5., we note the simple formula

$$\sum_{j=1}^{i-1} \frac{b_j}{a_j(a_j + b_j)} = \sum_{j=1}^{i-1} \frac{1}{a_j} - \frac{1}{a_j + b_j}.$$

If we apply this formula to $a_j = \left(\sum_{l=j+1}^N \mu_l\right)$ and $b_j = \mu_j$, we see that this telescopes to

$$\frac{1}{\sum_{l=i}^N \mu_l} - \frac{1}{\mu}.$$

□

We continue with the

Proof of Theorem A.1.1. We proceed by induction on k. First let's observe that due to the orthogonalization, we have the formula

$$d_k = 1_{Q_k} - \sum_{j=0}^{k-1} \langle 1_{Q_k}, d_j \rangle \frac{d_j}{\|d_j\|_2^2}, \quad \text{for } k \geq 1.$$

Due to Lemma A.1.2 this leads to

$$d_k = 1_{Q_k} - \sum_{j=0}^{k-1} d_j^k \mu_k \frac{d_j}{\sum_{l=1}^N (d_j^l)^2 \mu_l}.$$

We begin the induction and set $k = 1$: In that case we have

$$d_1 = 1_{Q_1} - d_0^1 \mu_1 \frac{d_0}{\mu} = 1_{Q_1} - \frac{\mu_1}{\mu} 1_A.$$

This leads to

$$d_1^1 = 1 - \frac{\mu_1}{\mu}, \quad d_1^i = -\frac{\mu_1}{\mu}, \text{ for } i \geq 2.$$

If we multiply d_1^1 by μ we see that

$$\mu d_1^1 = \mu - \mu_1 = \sum_{j=2}^{N} \mu_j,$$

so the representation formula holds for $k = 1$.
Now assume we have the representation formula for $1, 2, \ldots, k-1$ and so we have in particular part 3 of Lemma A.1.2 for these indices. We have to show that

$$d_k^i = \delta_i^k - \sum_{j=0}^{k-1} d_j^k \mu_k \frac{d_j^i}{\|d_j\|_2^2} = \begin{cases} 0, & \text{if } 1 \leq i \leq k-1 \\ \sum_{j=k+1}^{N} \mu_j, & \text{if } i = k \\ -\mu_k & \text{if } N \geq i \geq k+1 \end{cases}.$$

Observe that in the sum, we can replace $\|d_j\|_2^2$ with $\mu_j \left(\sum_{l=j+1}^{N} \mu_l\right) \left(\sum_{l=j}^{N} \mu_l\right)$.
We have

$$\begin{aligned} d_k^i &= \delta_i^k - \frac{\mu_k}{\mu} + \sum_{j=1}^{k-1} \frac{\mu_j \mu_k}{\mu_j \left(\sum_{l=j+1}^{N} \mu_l\right) \left(\sum_{l=j}^{N} \mu_l\right)} d_j^i \\ &= \delta_i^k - \frac{\mu_k}{\mu} + \mu_k \sum_{j=1}^{k-1} \frac{d_j^i}{\left(\sum_{l=j+1}^{N} \mu_l\right) \left(\sum_{l=j}^{N} \mu_l\right)}. \end{aligned}$$

We now consider the three cases $1 \leq i \leq k-1$, $i = k$, $i \geq k+1$. Let's begin with the first case $1 \leq i \leq k-1$: Here we have, if we use the induction hypothesis

$$d_k^i = -\frac{\mu_k}{\mu} + \mu_k \left(\sum_{j=1}^{i-1} - \frac{\mu_j}{\left(\sum_{l=j+1}^{N} \mu_l\right) \left(\sum_{l=j}^{N} \mu_l\right)} + \frac{\sum_{l=i+1}^{N} \mu_l}{\left(\sum_{l=i+1}^{N} \mu_l\right) \left(\sum_{l=i}^{N} \mu_l\right)} \right).$$

In order to show $d_k^i = 0$, we have to show that the term in the big bracket equals $1/\mu$. With Lemma A.1.2, part 5, we get

$$\sum_{j=1}^{i-1} - \frac{\mu_j}{\left(\sum_{l=j+1}^{N} \mu_l\right) \left(\sum_{l=j}^{N} \mu_l\right)} = \frac{1}{\mu} - \frac{1}{\sum_{l=i}^{N} \mu_l},$$

so $d_k^i = 0$ for $1 \leq i \leq k-1$.
Now let $i = k$: Then we get

$$d_k^k = 1 - \frac{\mu_k}{\mu} + \mu_k \sum_{j=1}^{k-1} - \frac{\mu_j}{\left(\sum_{l=j+1}^{N} \mu_l\right) \left(\sum_{l=j}^{N} \mu_l\right)}.$$

Lemma A.1.2, part 5, leads to

$$d_k^k = \frac{1}{\sum_{l=k}^N \mu_l} \sum_{l=k+1}^N \mu_l.$$

Finally, we come to the third case $i \geq k+1$: We thus get

$$d_k^i = -\frac{\mu_k}{\mu} + \mu_k \sum_{j=1}^{k-1} -\frac{\mu_j}{\left(\sum_{l=j+1}^N \mu_l\right)\left(\sum_{l=j}^N \mu_l\right)}$$

and this leads to

$$d_k^i = \frac{1}{\sum_{l=k}^N \mu_l}(-\mu_k),$$

so we obtain the assertion of the theorem. □

A.2 Singular Integral Operators

In this section, we show, how our main theorem 2.3.2 can be used in the setting of Figiel [Fig90]. To do that, we first recall what we need of the abstract scheme introduced in [Fig90] and try to stick close to the notation there. Consider Haar functions d_Q for $Q \in \mathcal{E}(\mathcal{A})$ and recall the definition of the collection of our isotropic basis Z in (2.2.7), which was

$$\{1_X \otimes 1_X\} \cup \{d_Q^{(\varepsilon_1)} \otimes d_R^{(\varepsilon_2)} : Q, R \in \mathcal{E}(\mathcal{A}), \operatorname{lev} Q = \operatorname{lev} R, \varepsilon = (\varepsilon_1, \varepsilon_2) \in \{0,1\}^2 \backslash \{(0,0)\}\}.$$

For the meaning of the various symbols, see the definitions preceding (2.2.7) and Definitions 2.2.4 and 2.2.5. We let

$$\mathcal{B} := L(L_E^2(X), L_E^2(X))$$

be the bounded linear operators from $L_E^2(X)$ into itself. We define the space of weakly bounded elements WB as the bounded sequences on Z

$$\mathrm{WB} := l^\infty(Z).$$

Furthermore, we introduce an injective map $\Upsilon : \mathcal{B} \to \mathrm{WB}$ by setting

$$\Upsilon(T)(d_Q^{(\varepsilon_1)} \otimes d_R^{(\varepsilon_2)}) := \langle T d_R^{(\varepsilon_2)}, d_Q^{(\varepsilon_1)} \rangle.$$

Now, we let $J \in \mathrm{WB}$ be such that the standard bounds (2.3.1) are satisfied, i.e.

$$\left| J(d_Q^{(\varepsilon_1)} \otimes d_R^{(\varepsilon_2)}) \right| \leq C_S \left(1 + \frac{d(\operatorname{pre} Q, \operatorname{pre} R)}{q^{\operatorname{lev}(Q)+1}}\right)^{-1-\delta} \tag{A.2.1}$$

for all $Q, R \in \mathcal{E}(\mathcal{A})$ with $\operatorname{lev} Q = \operatorname{lev} R$, $\varepsilon = (\varepsilon_1, \varepsilon_2) \in \{0,1\}^2 \backslash \{(0,0)\}$. We now do the following calculation for $T \in \mathcal{B}$ to obtain the Haar coefficients of $T1$ resp. T^*1. Let S be a dyadic cube in $\mathcal{E}(\mathcal{A})$, then

$$\langle T1, d_S \rangle = \sum_{\operatorname{lev} B = \operatorname{lev}(S)+1} \langle T1_B, d_S \rangle = \sum_{\operatorname{lev} B = \operatorname{lev}(S)+1} \sqrt{\mu(B)}\, \Upsilon(T) \left(d_S \otimes \frac{1_B}{\sqrt{\mu(B)}}\right). \tag{A.2.2}$$

Analogously

$$\langle T^*1, d_S\rangle = \sum_{\text{lev } B=\text{lev}(S)+1} \langle Td_S, 1_B\rangle = \sum_{\text{lev } B=\text{lev}(S)+1} \sqrt{\mu(B)}\Upsilon(T)\left(\frac{1_B}{\sqrt{\mu(B)}} \otimes d_S\right). \tag{A.2.3}$$

Since $(A.2.2)$ and $(A.2.3)$ are finite sums, we may as well insert elements of a sequence $J \in \mathrm{WB}$ and define

$$\Theta_x(J)(d_S) := \sum_{\text{lev } B=\text{lev}(S)+1} \sqrt{\mu(B)}J\left(d_S \otimes \frac{1_B}{\mu(B)}\right),$$

$$\Theta_y(J)(d_S) := \sum_{\text{lev } B=\text{lev}(S)+1} \sqrt{\mu(B)}J\left(\frac{1_B}{\mu(B)} \otimes d_S\right).$$

Since if $f \in L^2$ has the Haar expansion $f = \sum_Q f_Q d_Q$ it holds that

$$\|f\|_{BMO} = \sup_{Q\in\mathcal{A}} \left(\frac{1}{\mu(Q)} \sum_{R\subset Q, R\in\mathcal{E}(\mathcal{A})} |f_R|^2\right)^{1/2},$$

we define a sequence $a \in l^\infty(\mathcal{E}(\mathcal{A}))$ to be in $\mathrm{bmo}(\mathcal{E}(\mathcal{A}))$ if

$$\|a\|_{\mathrm{bmo}(\mathcal{E}(\mathcal{A}))} := \sup_{Q\in\mathcal{A}} \left(\frac{1}{\mu(Q)} \sum_{R\subset Q, R\in\mathcal{E}(\mathcal{A})} |a(R)|^2\right)^{1/2} < \infty. \tag{A.2.4}$$

We then have the following

Theorem A.2.1. *Suppose that the sequence J satisfies the bound* $(A.2.1)$ *and that*

$$\Theta_x(J), \Theta_y(J) \in \mathrm{bmo}(\mathcal{E}(\mathcal{A})).$$

Then we have that J is induced by a bounded linear operator T, that is we have $J \in \Upsilon(\mathcal{B})$.

Proof. Let $p' = p/(p-1)$ be the conjugate exponent to p and let $f \in L^p_E(X)$ and $g \in L^{p'}_{E'}(X)$ be finite linear combinations of Haar functions

$$f(x) = \sum_{Q\in\mathcal{I}} f_Q d_Q(x) \quad \text{and} \quad g(y) = \sum_{Q\in\mathcal{J}} g_Q d_Q(y)$$

with $\mathcal{I}, \mathcal{J} \subset \mathcal{E}(\mathcal{A})$ finite and $f_Q \in E, g_Q \in E'$. If we set

$$\mathcal{H} := \{Q \in \mathcal{E}(\mathcal{A}) : \text{lev } Q \geq \min\{\text{lev } R : R \in \mathcal{I} \cup \mathcal{J}\}\},$$

we can define an L^2-kernel k depending on f and g that is

$$k(x, y) := \sum_{\substack{Q,R\in\mathcal{H} \\ \varepsilon\in\{0,1\}^2\setminus\{(0,0)\}}} J(d_Q^{(\varepsilon_1)} \otimes d_R^{(\varepsilon_2)})d_Q^{(\varepsilon_1)} \otimes d_R^{(\varepsilon_2)}(x, y),$$

where the summation is meant in the sense that every isotropic function $d_Q^{(\varepsilon_1)} \otimes d_R^{(\varepsilon_2)}$ occurs only once. If we call the corresponding integral operator K, we clearly have $|\langle Kf, g\rangle| \le \|Kf\|_{L^p_E} \|g\|_{L^{p'}_{E'}}$. Now it follows from Theorem 2.3.2 that

$$\|Kf\|_{L^p_E} \le C_K \|f\|_{L^p_E}, \tag{A.2.5}$$

where C_K depends only on p, the constant C_S coming from the structural estimate (A.2.1) and the BMO-norms of $K(1)$ and $K^*(1)$ for which we have

$$\|K(1)\|_{\mathrm{BMO}(X)} \le \|\Theta_x(J)\|_{\mathrm{bmo}(\mathcal{E}(\mathcal{A}))} \quad \text{and} \quad \|K^*(1)\|_{\mathrm{BMO}(X)} \le \|\Theta_y(J)\|_{\mathrm{bmo}(\mathcal{E}(\mathcal{A}))}.$$

Now define a linear functional on $L^{p'}_{E'}$, which depends on f and we set for $g \in \mathrm{lin}\{d_Q\}$

$$S_f g := \langle Kf, g\rangle,$$

K depending on f and g. Due to estimate $(A.2.5)$, we have that S_f is a continuous linear functional on $\mathrm{lin}\{d_Q\}$. So, S_f has a unique continuous linear extension to $L^{p'}_{E'}$, so there exists an $h \in L^p_E$, such that

$$\langle h, g\rangle = S_f g = \langle Kf, g\rangle$$

for all $g \in \mathrm{lin}\{d_Q\}$. So we define

$$Tf := h$$

so that we have $\|Tf\|_{L^p_E} \le \sup_{\substack{g \in L^{p'}_{E'} \\ \|g\|=1}} \langle h, g\rangle \le C_K \|f\|_{L^p_E}$. Therefore we have a bounded linear operator T on $\mathrm{lin}\{d_Q\}$ that has a unique extension to a bounded linear operator on L^p_E and the isotropic Haar coefficients $\langle T d_Q^{(\varepsilon_1)}, d_R^{(\varepsilon_2)}\rangle$ coincide with $J(d_Q^{(\varepsilon_1)} \otimes d_R^{(\varepsilon_2)})$. □

A.3 Standard kernels and Singular Integral Operators

We give an extension of Coifman's version of the $T(1)$ theorem to UMD valued operators. In this section we show that Singular Integral Operators that are given in terms of a standard kernel (see Definition A.3.1) satisfy the structural estimate (2.3.1) provided they are weakly bounded. The following terminology is taken from [Chr90a]. Recall that we assume that the underlying space of homogeneous type X is a probability space and normal (see Definition 2.2.2). Additionally we suppose that the Hölder condition of the form (2.2.2) is satisfied with the exponent β which we take fixed for this section.

Definition A.3.1. Let $D := \{(x, x) \in X \times X\}$. A kernel $k : X \times X \backslash D \to \mathbb{C}$ is called a *standard kernel*, if there exist $\delta, \kappa > 0$ and $C < \infty$ such that

$$|k(x, y)| \le \frac{C}{d(x, y)} \tag{S1}$$

and for $d(x,w) < \kappa d(x,y)$ we have:

$$|k(x,y) - k(w,y)| + |k(y,x) - k(y,w)| \leq C \frac{d(x,w)^\delta}{d(x,y)^{1+\delta}}. \tag{S2}$$

Definition A.3.2. We let Λ_s for $s \in (0,1]$ be the space of all bounded functions on X that are Hölder continuous with exponent s, i.e. for $f \in \Lambda_s$ we have

$$\|f\|_{\Lambda_s} := \|f\|_s := \|f\|_\infty + \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x,y)^s} < \infty. \tag{A.3.1}$$

Additionally we define for $f \in \Lambda_s$

$$|f|_s := \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x,y)^s}.$$

Λ_s is a Banach space with the given norm. We denote the dual space of Λ_s by Λ_s'. We fix a number $s_0 \in (0,1]$ such that $s_0 \leq \beta$ and $\Lambda_{s_0} \neq \emptyset$.

Definition A.3.3. A continuous linear operator $T : \Lambda_s \to \Lambda_s'$ is said to be *associated* to a kernel k, if k is locally integrable away from the diagonal and

$$\langle Tf, g \rangle = \int_X \int_X k(x,y) f(y) g(x) \; d\mu(y) d\mu(x)$$

for all $f, g \in \Lambda_s$ whose supports are separated by a positive distance. Here $\langle \cdot, \cdot \rangle$ denotes the dual pairing of Λ_s and Λ_s'.

Remark. For given $T : \Lambda_s \to \Lambda_s'$ we define $T^* : \Lambda_s \to \Lambda_s'$ through

$$\langle T^* f, g \rangle := \langle f, Tg \rangle .$$

T^* is thus a bounded linear operator. Furthermore, if T is associated to k, then T^* is associated to $\tilde{k}(x,y) := k(y,x)$

Definition A.3.4. A *singular integral operator* T is a continuous linear operator $T : \Lambda_s \to \Lambda_s'$ for some $s \in (0, s_0]$, which is associated to a standard kernel.

Definition A.3.5. For $s \in (0,1]$, $x \in X$ and $r > 0$ we define the set $A(s,x,r)$ to consist of all functions $\varphi \in \Lambda_s$, supported in $B(x,r)$ with $\|\varphi\|_\infty \leq 1$ and

$$|\varphi(y) - \varphi(z)| \leq r^{-s} d(y,z)^s$$

whenever $y, z \in X$. In other words, $|\varphi|_s \leq r^{-s}$.

Definition A.3.6. A singular integral operator $T : \Lambda_s \to \Lambda_s'$ is said to be *weakly bounded*, if there exists $C < \infty$ such that for all $x \in X$, $r > 0$ and $\varphi, \psi \in A(s,x,r)$ we have

$$|\langle T\varphi, \psi \rangle| \leq C\mu(B(x,r)).$$

In the following we assume that T is a weakly bounded singular integral operator as defined in Definitions A.3.4 and A.3.6.

Lemma A.3.7. *Let T be a singular integral operator, $B \in \mathcal{A}_n, n \in -\mathbb{N}_0$, $R \in \mathcal{E}(B)$ and $x \in X$ with $d(x,B) \geq c_2\kappa^{-1}q^n$. Then there exists $C < \infty$ (independent of x, R) such that*

$$|T(d_R)(x)| := \left| \int_X k(x,y) d_R(y)\ d\mu(y) \right| \leq C\sqrt{\mu(B)} \frac{q^{n\delta}}{d(x,B)^{1+\delta}}.$$

Proof. Since x and B have positive distance from each other, $T(d_R)$ is given by the kernel and we have

$$I(x) := |T(d_R)(x)| = \left| \int_X k(x,y) d_R(y)\ d\mu(y) \right| = \left| \int_X [k(x,y) - k(x,z_B)]\, d_R(y)\ d\mu(y) \right|,$$

where z_B is the center (see construction of the dyadic cubes) of B. The last equality is due to the fact that $\mathbb{E} d_R = 0$. Further, using (S2) and the assumption $d(x,B) \geq c_2\kappa^{-1}q^n$

$$\begin{aligned} I(x) &\leq \int_X |k(x,y) - k(x,z_B)||d_R(y)|\ d\mu(y) \leq \frac{C}{\sqrt{\mu(R)}} \int_B |k(x,y) - k(x,z_B)|\ d\mu(y) \\ &\leq \frac{C}{\sqrt{\mu(R)}} \int_B \frac{d(y,z_B)^\delta}{d(x,y)^{1+\delta}} d\mu(y). \end{aligned}$$

With the properties of dyadic cubes (Theorem 2.2.3) we get

$$I(x) \leq \frac{C}{\sqrt{\mu(R)}} \int_B \frac{q^{n\delta}}{d(x,B)^{1+\delta}} d\mu(y) \leq C\sqrt{\mu(B)} \frac{q^{n\delta}}{d(x,B)^{1+\delta}}.$$

□

An immediate consequence of this lemma is the following corollary, that tells us that if two cubes are separated by a distance about the size of them, we have our structural estimate (2.3.1).

Corollary A.3.8. *Let T be a singular integral operator, $A, B \in \mathcal{A}_n$. If $d(A,B) \geq c_2\kappa^{-1}q^n$ and $Q \in \mathcal{E}(A)$, $R \in \mathcal{E}(B)$, we have*

$$|\langle T d_R, d_Q \rangle| \leq C\mu(B) \frac{q^{n\delta}}{d(A,B)^{1+\delta}}.$$

Remark. In the notation of the previous corollary, we get analogous estimates for

$$|\langle T d_R, 1_A \rangle| \quad \text{and} \quad |\langle T 1_B, d_Q \rangle|,$$

provided A and B are well separated. For the first expression, we can repeat the argument given and for the second, we have to apply the arguments to T^*. Now we treat the case where two cubes are next to each other, but are not equal.

Theorem A.3.9. *Let T be a singular integral operator, $A, B \in \mathcal{A}_n$, $d(A,B) < c_2\kappa^{-1}q^n$ and $A \neq B$. We have*

$$|\langle T 1_A, 1_B \rangle| \leq C\mu(B)$$

Proof. We can use the kernel to define $\langle T1_A, 1_B\rangle$, since

$$|\langle T1_A, 1_B\rangle| \leq \int_B \int_A d(x,y)^{-1}\, d\mu(y)d\mu(x) \tag{A.3.2}$$

and we will show that $\int_B \int_A d(x,y)^{-1}\, d\mu(y)d\mu(x) \leq C\mu(B)$. First of all we divide B into margin layers and set

$$E_j := \{x \in B : q^{n+j-1} < d(x, \complement B) \leq q^{n+j}\}.$$

Depending on $x \in E_j$, we define $F_i(x)$ to be the points in A which are in an annulus around x, so

$$F_i(x) := \{y \in A : q^{n+i-1} < d(x,y) \leq q^{n+i}\}.$$

Observe that for $i \leq j-1$ and $x \in E_j$ the set $F_i(x)$ is empty and there exists a constant D, which depends only on X, such that for all $i > D$ we have that the sets $F_i(x)$ and E_i are empty. So we split the integral over A for $x \in E_j$ into

$$\begin{aligned}\int_A d(x,y)^{-1}\, d\mu(y) &= \sum_{i=j}^{D} \int_{F_i(x)} d(x,y)^{-1}\, d\mu(y)\\ &\leq \sum_{i=j}^{D} q^{-n-i+1}\mu(F_i(x)) \leq \sum_{i=j}^{D} q \leq q(D-j) \leq qD + q|j| \ .\end{aligned}$$

We thus get for the double integral (A.3.2)

$$\begin{aligned}\int_B \int_A d(x,y)^{-1}\, d\mu(y)d\mu(x) &\leq \sum_{j=-\infty}^{D} \int_{E_j} qD + q|j|\, d\mu(x)\\ &\leq \sum_{j=-\infty}^{D} \mu(E_j)(qD + q|j|).\end{aligned}$$

The regularity of the boundary of B (see Point 5 of Theorem 2.2.3) implies now

$$\mu(E_j) \leq q^{j\eta}\mu(B) \qquad \text{for some } \eta > 0$$

and hence with

$$\int_B \int_A d(x,y)^{-1}\, d\mu(y)d\mu(x) \leq C\mu(B)$$

the assertion of the theorem. □

Remark. Analogously the terms (for $Q \in \mathcal{E}(A)$, $R \in \mathcal{E}(B)$, $d(A,B) \leq c_2\kappa^{-1}q^n$, $A \neq B$ and where A and B are of the same level)

$$\langle Td_Q, 1_B\rangle\, , \langle T1_A, d_R\rangle \ \text{ and } \ \langle Td_Q, d_R\rangle$$

can be defined, since each of them satisfies a similar estimate.

It remains to define $\langle T1_A, 1_A\rangle$ for $A \in \mathcal{A}_n$. For this, we utilize a partition of unity, corresponding to a Whitney-like decomposition of A. The following lemma and the theorem afterwards may be found in [MS79a].

Lemma A.3.10. *Let Ω be an open set strictly contained in X. Let $r(x) := (10K^2)^{-1} d(x, \complement\Omega)$. Then there exist $M \in \mathbb{N}$ and a sequence of points $\{x_k\}$ in X such that (if we denote $r(x_k)$ by r_k) the following properties are fulfilled*

1. *The balls $B(x_k, (4K)^{-1}r_k)$ are pairwise disjoint,*
2. *$\Omega = \cup_k B(x_k, r_k)$,*
3. *$B(x_k, 5Kr_k) \subseteq \Omega$ for all $k \in \mathbb{N}$,*
4. *For every $k \in \mathbb{N}, x \in B(x_k, 5Kr_k)$ implies that*
$$5Kr_k \leq d(x, \complement\Omega) \leq 15K^3 r_k,$$
5. *For every $k \in \mathbb{N}$, there exists $y_n \in \complement\Omega$ such that $d(x_n, y_n) < 15K^2 r_n$,*
6. *For every $k \in \mathbb{N}$, the number of balls $B(x_j, 5Kr_j)$ whose intersections with $B(x_k, 5Kr_k)$ are nonempty is at most M.*

Theorem A.3.11. *Let Ω be an open set strictly contained in X. Consider the sequences $\{x_k\}$ and $\{r_k\}$ given by the last Lemma. Then, there exists a sequence $\{\phi_k\}$ of non-negative functions satisfying*

1. $\operatorname{supp} \phi_k \subseteq B(x_k, 2r_k)$,
2. *$\phi_k \geq M^{-1}$ on $B(x_k, r_k)$,*
3. *There exists c such that for every $k \in \mathbb{N}$ we have $|\phi_k|_\beta \leq c r_k^{-\beta}$ and*
$$\sum_k \phi_k(x) = 1_\Omega(x)$$
for $x \in X$.

Now we may apply this partition of unity to A, since we may assume that A is open (see the Remark after Theorem 2.2.3) and $A \neq X$:

$$\sum_{k=1}^{\infty} \phi_k(x) = 1_A(x)$$

for all $x \in X$, where we assume that $r_k \downarrow 0$ and so also $d(x_k, \complement A) \downarrow 0$ as $k \to \infty$.

We then *define*

$$\langle T1_A, 1_A \rangle := \sum_{k=1}^{\infty} \langle T\phi_k, 1_A \rangle .$$

After estimating the right hand side one is able to deduce (essentially repeating the argument) that this definition does not depend on the choice of the partition of unity $\{\phi_k\}$. Now, for every $k \in \mathbb{N}$ we split 1_A into the sum of two functions ξ_1 and ξ_2 such that $\|\xi_1\|_s \leq c r_k^{-s}$, $\xi_1 + \xi_2 \equiv 1_A$ and $\operatorname{supp} \xi_1 \subseteq B(x_k, 4r_k)$, $\xi_1 = 1$ on $B(x_k, 3r_k)$. In order to do this we take any C^∞ function ψ on $[0, \infty)$ which satisfies $\operatorname{supp} \psi \subseteq [0, 4]$ and $\psi = 1$ on $[0, 3]$ and define

$$\xi_1(x) := \psi(d(x, x_k)/r_k).$$

Then ξ_1 has the desired Hölder property. We thus get for ξ_1, using the weak boundedness of T that (observe that $\operatorname{supp}\phi_k \cup \operatorname{supp}\xi_1 \subseteq B(x_k, 4r_k)$)

$$|\langle T\phi_k, \xi_1\rangle| \leq C\mu(B(x_k, 4r_k)) \leq \tilde{C}r_k. \tag{A.3.3}$$

$\xi_2 = 1_A - \xi_1$ satisfies $\operatorname{supp}\xi_2 \subseteq \complement B(x_k, 3r_k)$. For the following, set

$$X_m := \{k \in \mathbb{N} : q^{-m+\operatorname{lev} A} \leq d(x_k, \complement A) < q^{-m+1+\operatorname{lev} A}\} \quad \text{for} \quad m \in \mathbb{N}$$

and assume without loss of generality (there are at most finitely many negative values for m such that X_m is not empty and this number depends only on the space of homogeneous type X) that for every $k \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $k \in X_m$. Furthermore we define

$$F_j(x_k) := \{x \in A : 3q^j r_k \leq d(x, x_k) < 3q^{j+1} r_k\}$$

for $j \in \mathbb{N}_0$. Observe that there exists $c > 0$ depending only on X such that $F_j(x_k) = \emptyset$ for $j \geq m + c$ and $k \in X_m$ since then we have that the expressions $r_k, d(\complement A, x_k), q^{-m+\operatorname{lev} A}$ are comparable and the diameter of A is $q^{\operatorname{lev} A}$. So we get for $k \in X_m$, using the kernel $k(x, y)$ associated to T and the properties of the functions ϕ_k, ξ_2:

$$\begin{aligned}
|\langle T\phi_k, \xi_2\rangle| &\leq \sum_{j=0}^{m+c} \int_{B(x_k,2r_k)} \int_{F_j(x_k)} d(x, y)^{-1} \, d\mu(y) d\mu(x) \\
&\leq \sum_{j=0}^{m+c} \int_{B(x_k,2r_k)} \int_{F_j(x_k)} 3^{-1} q^{-j} r_k^{-1} \, d\mu(y) d\mu(x) \\
&\leq \sum_{j=0}^{m+c} \int_{B(x_k,2r_k)} 3^{-1} q^{-j} r_k^{-1} \mu(F_j(x_k)) \, d\mu(x) \\
&\leq C \sum_{j=0}^{m+c} \int_{B(x_k,2r_k)} d\mu(x) \leq C r_k (m + c),
\end{aligned}$$

where in the last line we used that X is normal. So, combining this calculation with (A.3.3) we have the estimate $|\langle T\phi_k, 1_A\rangle| \leq C r_k(m + c)$ for all $k \in X_m$. To conclude the calculation and estimate $\sum |\langle T\phi_k, 1_A\rangle|$ we need two simple lemmas, the first verifying that for $k \in X_m$, the corresponding ball $B(x_k, r_k)$ is contained in some appropriate boundary layer of the cube A and the second one giving an estimate about the cardinality of X_m depending on m.

Lemma A.3.12. *There exists $C < \infty$ depending only on X such that for all $k \in X_m$*

$$B(x_k, r_k) \subseteq \partial_{Cq^{-m}} A.$$

Proof. Let $k \in X_m, x \in B(x_k, r_k)$ and $\varepsilon > 0$. Then we choose $a \in \complement A$ with $d(x_k, a) < d(x_k, \complement A) + \varepsilon$ and estimate

$$\begin{aligned}
d(x, \complement A) &\leq d(x, a) \leq K(d(x, x_k) + d(x_k, a)) \\
&\leq K r_k + K d(x_k, \complement A) + \varepsilon,
\end{aligned}$$

by definition. Now, by the properties of our Whitney decomposition, we are able to estimate

$$r_k \leq (10K^2)^{-1} d(x_k, \complement A).$$

Using the fact that $k \in X_m$, we obtain further $d(x_k, \complement A) \leq q^{-m+1+\operatorname{lev} A}$. Combining these estimates yields

$$d(x, \complement A) \leq (K + \frac{1}{10K}) q^{-m+1+\operatorname{lev} A} + \varepsilon,$$

holding for any $\varepsilon > 0$. Thus, with an appropriate choice of C, the lemma is proved. □

Lemma A.3.13. *There exists a constant $D < \infty$, depending only on X, such that*

$$|X_m| \leq D q^{m(1-\eta)} \quad \textit{for all } m \in \mathbb{N}.$$

Proof. Using the definition of X_m and the properties of the Whitney decomposition Lemma A.3.10, we estimate

$$|X_m| \frac{1}{40K^3} q^{-m+\operatorname{lev} A} \leq |X_m| \min_{k \in X_m} \frac{r_k}{4K} \leq \sum_{k \in X_m} \frac{r_k}{4K}. \tag{A.3.4}$$

With the same constants as in Theorem 2.2.3 and in (2.2.1) and using the normality of X, we estimate this first by

$$\sum_{k \in X_m} \frac{r_k}{4K} \leq b_1^{-1} \sum_{k \in X_m} \mu(B(x_k, \frac{r_k}{4K})).$$

The last lemma and the property that the balls $B(x_k, \frac{r_k}{4K})$ are disjoint, yield further

$$b_1^{-1} \sum_{k \in X_m} \mu(B(x_k, \frac{r_k}{4K})) \leq b_1^{-1} \mu(\partial_{Cq^{-m}} A).$$

Since by Theorem 2.2.3, we have an estimate for $\mu(\partial_{Cq^{-m}} A)$ stating that

$$b_1^{-1} \mu(\partial_{Cq^{-m}} A) \leq b_1^{-1} c_3 C^\eta q^{-m\eta} \mu(A) \leq b_1^{-1} c_3 c_2 b_2 C^\eta q^{-m\eta + \operatorname{lev} A},$$

where in the last inequality, we used that the measure of A is about $q^{\operatorname{lev} A}$ and again the normality of X. Combining this with what we started in (A.3.4), we obtain the conclusion of our lemma. □

If we now use these two lemmas and the above estimate for $|\langle T\phi_k, 1_A \rangle|$, we get

$$\begin{aligned}
\sum_{k=1}^{\infty} |\langle T\phi_k, 1_A \rangle| &= \sum_{m=1}^{\infty} \sum_{k \in X_m} |\langle T\phi_k, 1_A \rangle| \leq C \sum_{m=1}^{\infty} q^{m(1-\eta)} \max_{k \in X_m} \{r_k (m+c)\} \\
&\leq C \sum_{m=1}^{\infty} q^{m(1-\eta)} q^{-m+1+\operatorname{lev} A} (m+c) \\
&= C\mu(A) \sum_{m=1}^{\infty} q^{-m\eta} (m+c) \leq C\mu(A),
\end{aligned}$$

and thus what we wanted.

Summary. We summarize this section and observe that given a singular integral operator T which is associated to a standard kernel allows us to deduce the structural estimates (2.3.1). So the boundedness Theorem 2.3.2 may be applied to T, provided T satisfies certain BMO–estimates (cf. Section A.2).

A.4 The Proofs of Lemmas 4.4.3 and 4.4.4

Proof of Lemma 4.4.3. In order to prove (4.4.23) and (4.4.24) we recall the bounds (3.1.5) and (3.1.6), which read

$$B_{l+1} \le 4B_l \text{ for } l \ge 1, \quad A_{l+1} \le 4A_l \text{ for } l \ge 0. \tag{A.4.1}$$

We view several cases:

CASE I. $0 \le j \le 2\nu - 1$

CASE I.a. $k = 0, j \neq 0$
If we note (A.4.1) and the formula for g_k from Proposition 4.4.1, we get the inequality $6g_1 - g_0 \ge 0$ immediately. For the reversed one we get, since we assumed $j \ge 1$

$$\begin{aligned} 6g_0 - g_1 &= (12B_j - 2B_{j-1}) + (6B_{N-j} - B_{N-j+1}) \\ &\quad + B_{2\nu-j}(6A_{N-2\nu} - A_{N-2\nu+1}) - A_{N-j} - 3B_{2\nu-j}B_{N-2\nu} \\ &\ge 10B_j + 2B_{N-j} + 2B_{2\nu-j}A_{N-2\nu} - A_{N-j} - 3B_{2\nu-j}B_{N-2\nu} \end{aligned}$$

by (A.4.1). If we now additionally observe that $2B_{N-j} \ge A_{N-j}$ (for $N - j \ge 1$, which is satisfied) and $A_{N-2\nu} \ge \sqrt{3}B_{N-2\nu}$, we see that this is ≥ 0.

CASE I.b. $1 \le k \le j - 1$
Again, with (A.4.1) and the assumption $k \le j - 1$ we get the first inequality $4g_k - g_{k+1} \ge 0$ immediately. The second inequality is only critical for $k = j - 1$ and in this case we get (with (A.4.1))

$$\begin{aligned} 4g_{k+1} - g_k &= -2 + 4B_N - B_{N-1} + (\text{positive term}) \\ &\ge 3B_N - 2 \ge 0 \quad \text{for } N \ge 1. \end{aligned}$$

CASE I.c. $j \le k \le 2\nu - 2$
For the first inequality $4g_{k+1} - g_k \ge 0$, it suffices to argue with (A.4.1), so it does for the second one $4g_k - g_{k+1} \ge 0$ in the case $k \neq j$. For $k = j$ it holds that

$$4g_k - g_{k+1} = -2 + 4B_N - B_{N-1} + (\text{positive term}) \ge 0 \quad \text{for } N \ge 1.$$

CASE I.d. $k = 2\nu - 1$
An analogous distinction between the cases $k = j$ and $k > j$ as in CASE I.c. supplies us with the estimate $6g_k - g_{k+1} \ge 0$. On the other hand (recall that $k = 2\nu - 1, j \le 2\nu - 1$)

$$\begin{aligned} 6g_{k+1} - g_k &= (12B_{2\nu-j} - 2B_{2\nu-1-j}) + (6B_{N-2\nu+j} - B_{N-2\nu+1+j}) \\ &\quad - A_{N-2\nu+j} + (6B_jA_{N-2\nu} - B_jA_{N-2\nu+1}) - 3B_jB_{N-2\nu} \\ &\ge 0 + 2B_{N-2\nu+j} - A_{N-2\nu+j} + 2B_jA_{N-2\nu} - 3B_jB_{N-2\nu}, \end{aligned}$$

by (A.4.1). The inequalities $2B_{N-2\nu+j} \geq A_{N-2\nu+j}$ (observe $N-2\nu+j \geq 1$) and $A_{N-2\nu} \geq \sqrt{3}B_{N-2\nu}$ then yield $6g_{k+1} - g_k \geq 0$.

CASE I.e. $2\nu \leq k \leq N-1$

Since $k > j$ in the current case, an application of (A.4.1) suffices for $4g_k - g_{k+1} \geq 0$. The same reasoning provides us with $4g_{k+1} - g_k \geq 0$ in the case $k \neq N-1 \vee j \neq 0$ and for $k = N-1, j = 0$ we have

$$\begin{aligned} 4g_{k+1} - g_k &= (4B_N - B_{N-1}) + B_{2\nu}(4A_{N-2\nu} - A_{N-1-2\nu}) - 1 \\ &\geq 3B_N - 1 \geq 0 \quad \text{for } N \geq 1. \end{aligned}$$

CASE II. $2\nu \leq j \leq N-1$

CASE II.a. $k = 0$

Again, the estimate $6g_{k+1} - g_k \geq 0$ is a trivial consequence of (A.4.1). Furthermore, by (A.4.1),

$$\begin{aligned} 6g_k - g_{k+1} &= 6B_{N-j} - B_{N-j+1} - A_{N-j} + \text{(positive terms)} \\ &\geq 2B_{N-j} - A_{N-j} \geq 0. \end{aligned}$$

CASE II.b. $1 \leq k \leq 2\nu - 2$

Here, both inequalities $4g_k - g_{k+1} \geq 0$ and $4g_{k+1} - g_k \geq 0$ are a consequence of (A.4.1).

CASE II.c. $k = 2\nu - 1$

The bound $6g_k - g_{k+1} \geq 0$ follows from (A.4.1). For the converse we get

$$\begin{aligned} 6g_{k+1} - g_k &= (6B_{j-2\nu} - B_{j-2\nu+1}) - A_{j-2\nu} \\ &\quad + (6B_{N-j+2\nu} - B_{N-j+2\nu-1}) + \text{(positive term)}. \end{aligned}$$

If $j > 2\nu$, we have $6B_{j-2\nu} - B_{j-2\nu+1} \geq 2B_{j-2\nu}$, which is greater than $A_{j-2\nu}$; if $j = 2\nu$, $6g_{k+1} - g_k \geq -2 + 5B_N \geq 0$.

CASE II.d. $2\nu \leq k \leq j-1$

For $k > 2\nu$, $4g_k - g_{k+1} \geq 0$ is a consequence of (A.4.1). If $k = 2\nu$, we have

$$4g_k - g_{k+1} = 2B_{N-j+2\nu} - \frac{3}{2}B_{2\nu}B_{N-j} + \text{(positive term)}.$$

Since $2B_{N-j+2\nu} \geq A_{N-j+2\nu}$ and $3B_{2\nu}B_{N-j} = A_{N-j+2\nu} - A_{2\nu}A_{N-j} \leq A_{N-j+2\nu}$, we get $4g_k - g_{k+1} \geq 0$. The converse estimate $4g_{k+1} - g_k$ follows once more out of (A.4.1) provided $k < j-1$. If on the other hand we have $k = j-1$, we see for instance

$$4g_{k+1} - g_k = -1 + A_{N-j}(4B_{k+1} - B_k) + \text{(positive term)} \geq 0,$$

since $k = j - 1 \geq 2\nu \geq 2$.

CASE II.e. $j \leq k \leq N-1$

The estimate $4g_k - g_{k+1} \geq 0$ follows from (A.4.1) if $k > j$, as does $4g_{k+1} - g_k \geq 0$ for $k < N-1$. For the critical values $k = j$ resp. $k = N-1$, similar calculations as in CASE II.d. conclude the statement of the lemma. □

Proof of Lemma 4.4.4. The proof consists of similar estimates as the proof of Lemma 4.4.3 but with twice as many case distinctions, since one has to consider the cases $|k-j| \le \frac{N-5}{2}$ and $|k-j| \ge \frac{N+5}{2}$ separately. We pick out one special case and omit all the others since they involve very similar arguments to the presented case or even to the proof of Lemma 4.4.3. We will treat values of ν, k, j where $2\nu \le k \le j-1$ and view the two cases mentioned above:

CASE I. $|j-k| \le \frac{N-5}{2}$
We obtain from Proposition 4.4.1 and Remark 4.4.2 that

$$\begin{aligned} |g_k| &= -B_{j-k} + A_{k-2\nu}B_{N-j+2\nu} + A_{N-j}B_k + \frac{3}{2}B_{k-2\nu}B_{2\nu}B_{N-j}, \\ |g_{k+1}| &= -B_{j-k-1} + A_{k+1-2\nu}B_{N-j+2\nu} + A_{N-j}B_{k+1} \\ &\quad + \frac{3}{2}B_{k+1-2\nu}B_{2\nu}B_{N-j}. \end{aligned}$$

The inequality $4|g_k| - |g_{k+1}| \ge 0$ for $k = 2\nu$ is a simple consequence of Lemmas 3.1.2 and 3.1.3. Utilizing Lemma 3.1.2, we get for $k \ge 2\nu + 1$ that

$$4|g_k| - |g_{k+1}| \ge -4B_{j-k} + (4-\lambda)A_{k-2\nu}B_{N-j+2\nu}. \tag{A.4.2}$$

Since $N - j + 2\nu \ge 3$, $A_3 = 26$ and $2\nu \le k$ we see with Lemma 3.1.3 that

$$A_{k-2\nu} \le \frac{A_{k-2\nu}A_{N-j+2\nu}}{A_3} = \frac{A_{k-2\nu}A_{N-j+2\nu}}{26} \le \frac{A_{N-j+k}}{26}.$$

This estimate, the definition of the recurrences A and B and Lemmas 3.1.2 and 3.1.3 yield

$$\begin{aligned} A_{k-2\nu}B_{N-j+2\nu} &\ge \frac{1}{\sqrt{3}}(A_{k-2\nu}A_{N-j+2\nu} - A_{k-2\nu}) \\ &\ge \frac{1}{2\sqrt{3}}(A_{N-j+k} - 2A_{k-2\nu}) \\ &\ge \frac{2\sqrt{3}}{13}A_{N-j+k}. \end{aligned}$$

Thus, this estimate and (A.4.2) imply

$$\begin{aligned} 4|g_k| - |g_{k+1}| &\ge (4-\lambda)\frac{2\sqrt{3}}{13}A_{N-j+k} - 4B_{j-k} \\ &\ge (4-\lambda)\frac{6}{13}B_{N-j+k} - 4B_{j-k} \\ &\ge (\lambda^5(4-\lambda)\frac{6}{13} - 4)B_{(N-5)/2} \ge 0, \end{aligned}$$

if we use Lemma 3.1.2 in conjunction with our hypothesis $|j-k| \le \frac{N-5}{2}$. The estimate $4|g_{k+1}| - |g_k| \ge 0$ follows analogously.

Case II. $|j-k| \geq \frac{N+5}{2}$
We obtain from Proposition 4.4.1 and Remark 4.4.2 that

$$\begin{aligned} |g_k| &= B_{j-k} - A_{k-2\nu}B_{N-j+2\nu} - A_{N-j}B_k - \frac{3}{2}B_{k-2\nu}B_{2\nu}B_{N-j}, \\ |g_{k+1}| &= B_{j-k-1} - A_{k+1-2\nu}B_{N-j+2\nu} - A_{N-j}B_{k+1} - \frac{3}{2}B_{k+1-2\nu}B_{2\nu}B_{N-j}. \end{aligned}$$

If we employ Lemma 3.1.2 three times, we obtain

$$4|g_k|-|g_{k+1}| \geq 3B_{j-k}-(4-\lambda)[B_{N-j+2\nu}A_{k-2\nu}+B_kA_{N-j}+\frac{3}{2}B_{2\nu}B_{N-j}B_{k-2\nu}]$$

Since by Lemma 3.1.3 every summand in the squared bracket is majorized by B_{N-j+k}, we finally get

$$4|g_k| - |g_{k+1}| \geq 3(B_{j-k} - (4-\lambda)B_{N-j+k}) \geq 0,$$

by the hypothesis $|j-k| \geq \frac{N+5}{2}$. For the inequality $4|g_{k+1}| - |g_k| \geq 0$, we first omit some positive terms to get

$$\begin{aligned} 4|g_{k+1}| - |g_k| \geq\ & 4B_{j-k-1} - B_{j-k} - 4A_{k+1-2\nu}B_{N-j+2\nu} \\ & -4A_{N-j}B_{k+1} - 6B_{k+1-2\nu}B_{2\nu}B_{N-j}. \end{aligned}$$

As above, Lemmas 3.1.3 and 3.1.2 respectively yield

$$\begin{aligned} 4|g_{k+1}| - |g_k| &\geq 4B_{j-k-1} - B_{j-k} - 10B_{N-j+k+1} \\ &\geq (4-\lambda)B_{j-k-1} - 1 - 10B_{N-j+k+1}. \end{aligned}$$

But now we employ again Lemma 3.1.2 and the fact that $|j-k| \geq \frac{N+5}{2}$ to get

$$4|g_{k+1}| - |g_k| \geq (\lambda^3(4-\lambda) - 10)B_{(N-3)/2} - 1 \geq 0,$$

and so the desired inequality. □

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Curriculum vitae

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10.2002 - 9.2005 and 10.2006 - 11.2008 studies in Technical Mathematics at the Johannes Kepler University in Linz;
Diploma on Nov. 17th, 2008

10.2005 - 9.2006 civil service at "Lebenshilfe Oberösterreich" in Hofkirchen im Traunkreis, Upper Austria

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Research Interests

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2. Banach space theory
   - UMD-spaces
3. Harmonic analysis
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List of Publications

1. (with P.F.X. Müller) A Decomposition Theorem for Singular Integral Operators on Spaces of Homogeneous Type, Preprint 2011, arXiv:1001.4926v2
2. Monotonicity of the Lebesgue constant for equally spaced knots, Preprint 2011, arXiv:1103.1949v1
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