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Dyadic Hardy spaces, p-summing multiplication operators and postorder rearrangements of the Haar system

DISSERTATION

zur Erlangung des akademischen Grades

Doktorin der Technischen Wissenschaften

im Doktoratsstudium der

Technischen Wissenschaften

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Angefertigt am:
Institut für Analysis

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Linz, Dezember 2014

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Signatur

Abstract

This thesis comprises two main parts. In the first part (Chapter 2) we constructively determine the Pietsch measure of 2-summing multiplication operators ranging in the dyadic Hardy spaces. In particular, every $u \in H^p$, $0 < p \leq 2$, with Haar expansion $u = \sum x_I h_I$ determines a 2-summing multiplication operator

$$\mathcal{M}_u : \ell^\infty \to H^p, \quad (\varphi_I) \mapsto \sum \varphi_I x_I h_I.$$

Pietsch's theorem gives the existence of a Pietsch measure, but the existence is only guaranteed by a Hahn-Banach argument and therefore, the Pietsch measure is not given constructively. We use the atomic decomposition of $u \in H^p$ to obtain an explicit formula for the Pietsch measure of the multiplication operator $\mathcal{M}_u$. We extend our result to Triebel-Lizorkin spaces, vector-valued Hardy spaces and vector-valued Triebel-Lizorkin spaces. Our method yields a constructive proof of Pisier's decomposition of $u \in H^p$

$$|u| = |x|^{1-\theta}|y|^\theta \qquad \text{and} \qquad \|x\|_{X_0}^{1-\theta}\|y\|_{H^2}^\theta \leq C\|u\|_{H^p},$$

where X_0 is Pisier's extrapolation lattice associated to H^p and H^2. Our construction of the Pietsch measure for the multiplication operator $\mathcal{M}_u$ involves the Haar coefficients of u and its atomic decomposition.

In the second part (Chapter 3) we investigate rearrangements of the Haar system on the space BMO_N. We use a new order - the postorder - on the set of dyadic intervals in $[0, 1]$ with length greater than or equal to 2^{-N}. We define a rearrangement - the postorder rearrangement - that associates to the n^{th} interval in postorder the n^{th} interval in lexicographic order. The postorder rearrangement, denoted by τ_N, induces a rearrangement operator on BMO_N given by

$$T_{\tau_N} : h_I \to h_{\tau_N(I)},$$

where $(h_I)_{I\in\mathcal{D}}$ is the L^∞-normalised Haar system. By means of operator norms on BMO_N we prove that the postorder has maximal distance to the usual lexicographic order. We extend our result to the dyadic Hardy spaces and the Triebel-Lizorkin spaces.

Kurzfassung

Diese Dissertation besteht aus zwei Hauptteilen. Im ersten Teil (Kapitel 2) bestimmen wir konstruktiv das Pietschmaß für 2-summierende Multiplikationsoperatoren mit Bild in den dyadischen Hardyräumen. Jedes $u \in H^p$, $0 < p \leq 2$, mit Haarentwicklung $u = \sum x_I h_I$ definiert einen 2-summierenden Multiplikationsoperator

$$\mathcal{M}_u : \ell^\infty \to H^p, \quad (\varphi_I) \mapsto \sum \varphi_I x_I h_I.$$

Der Satz von Pietsch gibt die Existenz eines Pietschmaßes für diesen Multiplikationsoperator, allerdings erhält man die Existenz aus dem Satz von Hahn-Banach und somit wird das Pietschmaß nicht konstruktiv bestimmt. Wir verwenden die atomare Zerlegung von $u \in H^p$ um eine explizite Formel für das Pietschmaß des Multiplikationsoperators $\mathcal{M}_u$ zu erhalten. Wir erweitern unser Ergebnis zu Multiplikationsoperatoren mit Bild in den Triebel-Lizorkin Räumen, den vektorwertigen Hardy Räumen und den vektorwertigen Triebel-Lizorkin Räumen.

Im zweiten Teil (Kapitel 3) untersuchen wir Umordnungen des Haarsystems im Raum BMO_N. Wir führen eine neue Ordnung, die sogenannte „postorder“, auf der Menge der dyadischen Intervalle in $[0, 1]$ mit Länge größer-gleich 2^{-N} ein. Diese Ordnung definiert eine Umordnung auf der obigen Menge der dyadischen Intervalle, die „postorder“-Umordnung, die dem n-ten Intervall in „postorder“ das n-te Intervall in lexikographischer Ordnung zuordnet. Die „postorder“-Umordnung τ_N erzeugt einen Umordnungsoperator auf BMO_N:

$$T_{\tau_N} : h_I \to h_{\tau_N(I)},$$

wobei $(h_I)_{I \in \mathcal{D}}$ das L^∞-normalisierte Haarsystem ist. Wir zeigen mittels der Operatornorm auf BMO_N, dass die „postorder“ maximalen Abstand zur lexikographischen Ordnung hat. Wir erweitern unser Ergebnis auf Umordnungsoperatoren auf den dyadischen Hardy Räumen und den Triebel-Lizorkin Räumen.

Preface

The thesis covers two main topics. The first one is the construction of Pietsch measures for certain p-summing multiplication operators ranging in the dyadic Hardy spaces and the more general Triebel-Lizorkin spaces. This is based on [**MP16**]. In general, Pietsch measures are given by abstract theory (Pietsch's domination theorem, Hahn-Banach theorem). We exploit the atomic decomposition of elements in the dyadic Hardy spaces resp. the Triebel-Lizorkin spaces to determine a formula for the Pietsch measure of the multiplication operators. Invoking Maurey's theorem on p-summing operators from a $C(K)$ space into Banach spaces of non-trivial cotype and the abstract version of Pietsch measures for atoms we obtain, by using the atomic decomposition as extrapolation tool, a partly constructive formula for the Pietsch measure of multiplication operators ranging in the vector-valued Hardy spaces and the vector-valued Triebel-Lizorkin spaces.

The second topic is a theorem on the extremality with respect to operator norms of certain rearrangement operators that rearrange the Haar system on the space BMO, the dyadic Hardy spaces and the Triebel-Lizorkin spaces. This is based on [**Pen14**] We introduce a non-standard order - the postorder - on the set of dyadic intervals in $[0, 1]$ of length greater than or equal to 2^{-N}, $N \in \mathbb{N}_0$. By means of operator norms we measure the distance of the postorder to the standard lexicographic order on the dyadic intervals and show that the distance is maximal.

The organization of the thesis. In Chapter 1 (Preliminaries) we collect crucial definitions, theorems and statements that are used in the following chapters. The preliminaries are collected in four main sections. In the first section - Banach space preliminaries - we start recalling Kahane's inequality and Kahane's contraction principle and give analogous inequalities for sums of q-stable random variables in a Banach space. To this end we give the definition and properties of q-stable random variables. Furthermore, we discuss the definition of the cotype of a Banach space and the related definitions of convexity and concavity of Banach lattices. In the second section we collect definitions and statements concerning dyadic intervals and dyadic trees. We give the definition of dyadic trees and discuss the dyadic tree structure of the set of dyadic intervals. We introduce two different orders on the set of dyadic intervals, the standard lexicographic order and the postorder. For information on the origin of the postorder in computer sciences we refer to the introductory part in Chapter 3. We discuss order intervals with respect to these orders and give some geometric representation of them. In the third section we define the dyadic space of functions of bounded mean oscillation (BMO), the dyadic Hardy spaces H^p

and the Triebel-Lizorkin spaces f_p^q. For the last two we define vector-valued versions, H_X^p and $f_p^q(X)$ for some Banach space X of non-trivial cotype. We give crucial properties of these spaces, as for example convexity and concavity resp. the cotype and duality relations. Furthermore, we give the atomic decomposition of elements in these spaces. We define the finite dimensional spaces BMO_N, H_N^p, $f_{p,N}^q$ and the space $\mathcal{M}(\mathcal{C})$ of functions with Haar support in a finite and non-empty collection of dyadic intervals $\mathcal{C}$. In the fourth section we define the operators of this thesis. We start with the definition of p-summing operators and give two well-known theorems on p-summing operators, Pietsch's domination theorem and a theorem of Maurey. We introduce one special type of p-summing operators, multiplication operators into the dyadic Hardy spaces (scalar- and vector-valued) and into the Triebel-Lizorkin spaces (scalar- and vector-valued). We give an argument of Pisier that states that the multiplication operators into the scalar-valued spaces are 2-summing. Maurey's theorem gives the p-summability of the vector-valued spaces. Besides p-summing operators we introduce operators rearranging the Haar system on the spaces BMO_N, H_N^p and $f_{p,N}^q$. We give operator norm estimates and interpolation and extrapolation statements for these operators.

Chapter 2 is the first main part of the thesis. In this part we investigate the p-summing multiplication operators defined in the preliminaries. We start examining the 2-summing multiplication operators into the dyadic Hardy spaces H^p, $0 < p \leq 2$. By Pietsch's theorem they have a Pietsch measure which is apriori not given explicitly. We use the atomic decomposition of elements in the dyadic Hardy spaces to determine an explicit formula for the Pietsch measure of the multiplication operators. This happens in Theorem 2.1. By a convexification method we extend this result to the more general Triebel-Lizorkin spaces. This is the statement of Corollary 2.2. Exploiting Maurey's theorem and the abstract version of Pietsch measures for atoms we can rebuild the argument for vector-valued Hardy spaces and vector-valued Triebel-Lizorkin spaces. Again we use the atomic decomposition, which works as an extrapolation tool, to obtain at least a partially constructive formula for the Pietsch measure. This is the statement of Theorem 2.3 and Theorem 2.8. In Section 2.5 we apply our explicit formula to obtain a constructive proof of

$$H^p = (X_0)^{1-\theta}(H^2)^\theta,$$

where X_0 is Pisier's extrapolation lattice for H^p and H^2.

Chapter 3 is the second main part of the thesis. In this part we examine rearrangement operators that rearrange the Haar system. We investigate one particular rearrangement operator that is induced by the postorder rearrangement of the dyadic intervals. We prove that the postorder has maximal distance from the lexicographic order by estimating the operator norms of the rearrangement operators induced by the postorder rearrangement and its inverse. For rearrangement operators on BMO_N this happens in Theorem 3.1. For the proof of Theorem 3.1 one needs deeper insight in the rearrangement induced by the postorder. In Section 3.1 we give formulae that describe the postorder rearrangement precisely. Furthermore, we give estimates for the Carleson constant of some special order intervals of dyadic intervals that determine the operator norm estimates. By using duality relations and interpolation and extrapolation results for rearrangement operators we obtain the norm estimates for operators on the dyadic Hardy spaces. This is the statement of Corollary 3.10. By the same convexification procedure as in Chapter 2 we obtain estimates for rearrangement operators on Triebel-Lizorkin spaces (Corollary 3.13). In Section 3.5 we investigate the behaviour of the rearrangement operators on general order intervals to gain further insight into the nature of the operators. We use the geometric representation of postorder order intervals given in Chapter 1 to obtain estimates for the rearrangement operators on lexicographic order intervals.

In Chapter 4 we give the proof of an inequality that is used in course of proving the atomic decomposition of an element in the vector-valued Hardy space. This inequality was stated without proof in [**Mül12**]. Since we use this inequality repeatedly in Chapter 2 we provide the proof here.

Acknowledgements. I thank my PhD advisor P.F.X. Müller for many helpful discussions and suggestions during the preparation of this work. This research has been supported by the Austrian Science foundation (FWF) Pr.Nr. P22549.

I want to thank my parents for the support during the last years and I would particularly like to thank Herbert Enser for encouraging and supporting me the entire time.

Contents

CHAPTER 1

Preliminaries

Throughout this work we will denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ the set of non-negative integers.

1.1. Banach space preliminaries

Kahane's inequality and Kahane's contraction principle. See [**Kah85**]. Let $(r_n)_{n\in\mathbb{N}}$ denote the independent Rademacher system. For any $0 < p < q < \infty$ there is a constant $C_{p,q}$ such that for any Banach space X and for any sequence $(x_n)_{n=1}^N$ in X we have

$$\left(\int_0^1 \left\|\sum_{n=1}^N r_n(t)x_n\right\|_X^q dt\right)^{\frac{1}{q}} \le C_{p,q}\left(\int_0^1 \left\|\sum_{n=1}^N r_n(t)x_n\right\|_X^p dt\right)^{\frac{1}{p}}. \tag{1.1}$$

Let X be a Banach space and $(x_n)_{n=1}^N$ a sequence in X. Then for all sequences $(a_n)_{n=1}^N$ of real numbers and for all $1 \le p < \infty$ one has

$$\left(\int_0^1 \left\|\sum_{n=1}^N r_n(t)a_nx_n\right\|_X^p dt\right)^{\frac{1}{p}} \le \sup_{1\le n\le N}|a_n| \left(\int_0^1 \left\|\sum_{n=1}^N r_n(t)x_n\right\|_X^p dt\right)^{\frac{1}{p}}. \tag{1.2}$$

Cotype of a Banach space. See e.g. [**LT91**]. Let $2 \le q \le \infty$. Let again $(r_n)_{n\in\mathbb{N}}$ denote the independent Rademacher system. A Banach space X is called of *cotype* q (or Rademacher *cotype* q) if there is a constant C such that for all finite sequences (x_n) in X

$$\left(\sum_{n=1}^N \|x_n\|^q\right)^{\frac{1}{q}} \le C\left(\int_0^1 \left\|\sum_{n=1}^N r_n(t)x_n\right\|_X^2 dt\right)^{\frac{1}{2}}. \tag{1.3}$$

We denote by $C_q(X)$ the smallest possible constant C. A Banach space X is of *non-trivial cotype* if it is of cotype $q < \infty$.

Let $1 \le p \le \infty$. Every L^p-space is of cotype $\max(p,2)$. If X is a Banach space of cotype q, then for $1 \le r \le \infty$, L^r_X is of cotype $\max(r,q)$, cf.[**LT91**]. In particular, X is of cotype q if and only if L^2_X is of cotype q and

$$C_q(X) = C_q(L^2_X),$$

cf. [**DJT95**].

Note that in the literature there are different notions of the cotype of a Banach space, e.g. Rademacher cotype, Gaussian cotype and stable cotype, cf. [**LT91**].

Sums of q-stable random variables in a Banach space. For general information on q-stable random variables see [**LT91**].

Definition 1.1 (q-stable random variables). Let $0 < q \leq 2$. Let $(\Omega, \Sigma, \mathbb{P})$ be a probability space. A real-valued random variable θ on Ω will be called standard q-stable if

$$\int_\Omega e^{it\theta(\omega)}\, d\mathbb{P}(\omega) = e^{-\frac{|t|^q}{2}}, \text{ for all } t \in \mathbb{R}. \tag{1.4}$$

Recall that the left-hand side in (1.4) defines the characteristic function of θ. In the following we will use the terminology q-stable random variable and think of a standard q-stable random variable.

Note that q-stable random variables are symmetric, this follows immediately from (1.4).

Recall, that a real-valued random variable g on a probability space $(\Omega, \Sigma, \mathbb{P})$ is standard Gaussian, if and only if its probability density function is given by

$$\mu_g(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}$$

or equivalently if and only if its characteristic function satisfies

$$\int_\Omega e^{itg(\omega)}\, d\mathbb{P}(\omega) = e^{-\frac{t^2}{2}}, \text{ for all } t \in \mathbb{R}.$$

Therefore, 2-stable random variables are standard Gaussian random variables. However, the case $q < 2$ is very different from the case of Gaussian random variables. For a Gaussian random variable g we have $\int |g|^s < \infty$ for all $0 < s < \infty$. Whereas, if θ is a q-stable random variable for some $q < 2$, the best we can obtain is

$$\lim_{t\to\infty} t^p\, \mathbb{P}\{|\theta| > t\} = c_p,$$

where c_p is a positive constant depending only on p. Therefore, a q-stable random variable θ, $1 < q < 2$, has moments of order s for every $s < q$, i.e. $\int |\theta|^s < \infty$ for all $s < q$, and $\|\theta\|_{L^s} = c_{p,s}$, where $c_{p,s}$ depends on p and s only. Summarizing we have the following Proposition

Proposition 1.2. *Let θ be a q-stable random variable on a probability space $(\Omega, \Sigma, \mathbb{P})$. If $q = 2$, then $\theta \in L^s(\Omega, \Sigma, \mu)$ for $0 < s < \infty$. If $q < 2$, then $\theta \in L^s(\Omega, \Sigma, \mu)$ for $0 < s < q$.*

Stable random variables are characterized by their stability property: if (θ_i) is a sequence of independent q-stable random variables, for any sequence (a_i) of real numbers, $\sum_i a_i\theta_i$ has the same distribution as $(\sum_i |a_i|^q)^{\frac{1}{q}}\, \theta_1$. In particular, for any $s < q$,

$$\Big\| \sum_i a_i \theta_i \Big\|_{L^s} = c_{q,s} \Big(\sum_i |a_i|^q \Big)^{\frac{1}{q}},$$

so that the span of the sequence θ_i in L^s, $s < q$, is isometric to ℓ^q. Of course, the span of an independent sequence of Gaussian random variables in L^s, $0 < s < \infty$, is isometric to ℓ^2.

From [**HJ73**] and [**Kwa73**] we obtain the analogue to Kahane's inequality for q-stable random variables. Let $(\theta_n)_{n\in\mathbb{N}}$ denote an independent sequence of q-stable random variables on the probability space $(\Omega, \Sigma, \mathbb{P})$. For any $0 < s, r < q$ there exists a constant $C_{r,s}$ such that for any Banach space X and for any sequence $(x_n)_{n=1}^N$ in X one has

$$\left(\int_\Omega \left\| \sum_{n=1}^N \theta_n(\omega) x_n \right\|_X^r d\mathbb{P}(\omega) \right)^{\frac{1}{r}} \leq C_{r,s} \left(\int_\Omega \left\| \sum_{n=1}^N \theta_n(\omega) x_n \right\|_X^s d\mathbb{P}(\omega) \right)^{\frac{1}{s}}. \tag{1.5}$$

Of course inequality (1.5) is true for Gaussian random variables with parameters $0 < s, r < \infty$.

The analogue of Kahane's contraction principle is due to [**HJ74**]. Let again $(\theta_n)_{n\in\mathbb{N}}$ denote an independent sequence of q-stable random variables on the probability space $(\Omega, \Sigma, \mathbb{P})$. Let X be a Banach space and $(x_n)_{n=1}^N$ a sequence in X. Let $0 < s < q$, if $q < 2$ and $0 < s < \infty$, if $q = 2$. Then for all sequences $(a_n)_{n=1}^N$ of real numbers one has

$$\left(\int_\Omega \left\| \sum_{n=1}^N \theta_n(\omega) a_n x_n \right\|_X^s d\mathbb{P}(\omega) \right)^{\frac{1}{s}} \leq \sup_n |a_n| \left(\int_\Omega \left\| \sum_{n=1}^N \theta_n(\omega) x_n \right\|_X^s d\mathbb{P}(\omega) \right)^{\frac{1}{s}}. \tag{1.6}$$

Banach lattices. For general reference on Banach lattices we refer to [**LT79**] and on quasi-Banach lattices to [**Kal84**]. The following definition can be found in [**Pis79a**].

Let (Ω, Σ, μ) be a measure space and $L^0(\Omega, \Sigma, \mu)$ the space of all measurable functions with real values. Let X be a (quasi-) Banach space, whose elements form a subspace of $L^0(\Omega, \Sigma, \mu)$. We call X a (quasi-) *Banach lattice* over the measure space (Ω, Σ, μ), if for all $f \in X$ and for all $g \in L^0(\Omega, \Sigma, \mu)$ with $|g| \leq |f|$ holds that $g \in X$ and $\|g\|_X \leq \|f\|_X$.

q-convexity and q-concavity of Banach lattices. We refer to [**LT79**] and for quasi-Banach lattices to [**CT86**]. Let $0 < q \leq \infty$. A (quasi-) Banach lattice X is called *q-convex*, if there exists a constant $M > 0$ such that

$$\left\| \left(\sum_{i=1}^n |x_i|^q \right)^{\frac{1}{q}} \right\| \leq M \left(\sum_{i=1}^n \|x_i\|^q \right)^{\frac{1}{q}}, \tag{1.7}$$

for every choice of vectors $\{x_i\}_{i=1}^n$ in X. The smallest possible constant M is denoted by $M^{(q)}(X)$. A (quasi-) Banach lattice X is called *q-concave*, if there exists a constant $M > 0$ such that

$$\left(\sum_{i=1}^n \|x_i\|^q \right)^{\frac{1}{q}} \leq M \left\| \left(\sum_{i=1}^n |x_i|^q \right)^{\frac{1}{q}} \right\|, \tag{1.8}$$

for every choice of vectors $\{x_i\}_{i=1}^n$ in X. The smallest possible constant M is denoted by $M_{(q)}(X)$.

Let $0 < p \leq \infty$. The L^p spaces are r-convex for all $r \leq p$ and s-concave for all $s \geq p$ with $M^{(r)}(L^p) = M_{(s)}(L^p) = 1$.

Every q-concave Banach lattice with $q \geq 2$ is of cotype q, cf.[**LT79, DJT95**]. Hence, for $1 \leq p < \infty$, the L^p spaces are of cotype $\max(p, 2)$.

1.2. Dyadic intervals and dyadic trees

1.2.1. Dyadic intervals. An interval $I \subseteq [0,1]$ is called a *dyadic interval*, if there exist non-negative integers ℓ and k with $0 \leq k \leq 2^\ell - 1$ such that

$$I = I_{\ell,k} = \left[\frac{k}{2^\ell}, \frac{k+1}{2^\ell}\right[.$$

The length of a dyadic interval $I_{\ell,k}$ is given by $|I_{\ell,k}| = 2^{-\ell}$. We denote by $\mathcal{D}$ the set of dyadic intervals, i.e.

$$\mathcal{D} = \{I \subseteq [0,1] : I \text{ is dyadic interval}\}.$$

For every $N \in \mathbb{N}_0$ let $\mathcal{D}_N$ be the set of dyadic intervals with length greater than or equal to 2^{-N}, i.e.

$$\mathcal{D}_N = \{I \in \mathcal{D} : |I| \geq 2^{-N}\} \tag{1.9}$$

or equivalently

$$\mathcal{D}_N = \{I_{\ell,k} : 0 \leq \ell \leq N, 0 \leq k \leq 2^\ell - 1\}. \tag{1.10}$$

1.2.2. Carleson constant. Let $\mathcal{C} \subseteq \mathcal{D}$. We define the Carleson constant of $\mathcal{C}$ as follows

$$[\![\mathcal{C}]\!] = \sup_{I \in \mathcal{C}} \frac{1}{|I|} \sum_{J \subseteq I, J \in \mathcal{C}} |J|. \tag{1.11}$$

If $\mathcal{C}$ is non-empty, then $[\![\mathcal{C}]\!] \geq 1$, otherwise $[\![\mathcal{C}]\!] = 0$.

1.2.3. Blocks of dyadic intervals. Let $\mathcal{L}$ be a collection of dyadic intervals. We say that $\mathcal{C}(I) \subseteq \mathcal{L}$ is a block of dyadic intervals in $\mathcal{L}$ if the following conditions hold:

(1) The collection $\mathcal{C}(I)$ has a unique maximal interval, namely the interval I.
(2) If $J \in \mathcal{C}(I)$ and $K \in \mathcal{L}$, then

$$J \subseteq K \subseteq I \text{ implies } K \in \mathcal{C}(I).$$

1.2.4. Dyadic trees. See [**BP05, Knu05**]. A *dyadic tree* $\mathcal{T}$ consists of a set of nodes that is either empty or has the following properties:

(1) One of the nodes, say R, is designated the root node.
(2) The remaining nodes (if any) are partitioned into two disjoint subsets, called the left subtree and the right subtree, respectively, each of which is a dyadic tree.

The definition yields that every node of a tree is the root of some subtree contained in the tree $\mathcal{T}$. The root of the left resp. the right subtree described in property (2) is called the *left child* resp. *the right child* of the root R. Conversely, the root R is called the *parent* of the left (resp. right) child. We use the terminology of family trees: parent, children, descendant, etc. The nodes of a dyadic tree $\mathcal{T}$ can be partitioned into disjoint sets, called *levels*, depending on the length ℓ of the unique path from a node to the root R. The root R is at level 0. The *lowermost level* of $\mathcal{T}$ is the set of nodes, whose unique path from the node to the root $\mathcal{R}$ has maximal length within the tree $\mathcal{T}$. The *depth* of $\mathcal{T}$ is the number of levels in $\mathcal{T}$ that do not contain the root R. A dyadic tree $\mathcal{T}$ is *complete*, if every node in $\mathcal{T}$ has exactly two children, except the nodes in the lowermost level, which have exactly zero children, cf. figure 1. In the following we consider complete dyadic trees of depth N, $N \in \mathbb{N}_0$. The number of nodes in each level ℓ, $0 \leq \ell \leq N$, is given by 2^ℓ and the total number of nodes in a complete dyadic tree of depth N is given by $2^{N+1} - 1$.

1.2.5. The dyadic tree structure of the set $\mathcal{D}_N$. The set $\mathcal{D}_N$, given by equation (1.9) resp. (1.10), has a natural dyadic tree structure, cf. figure 1. The

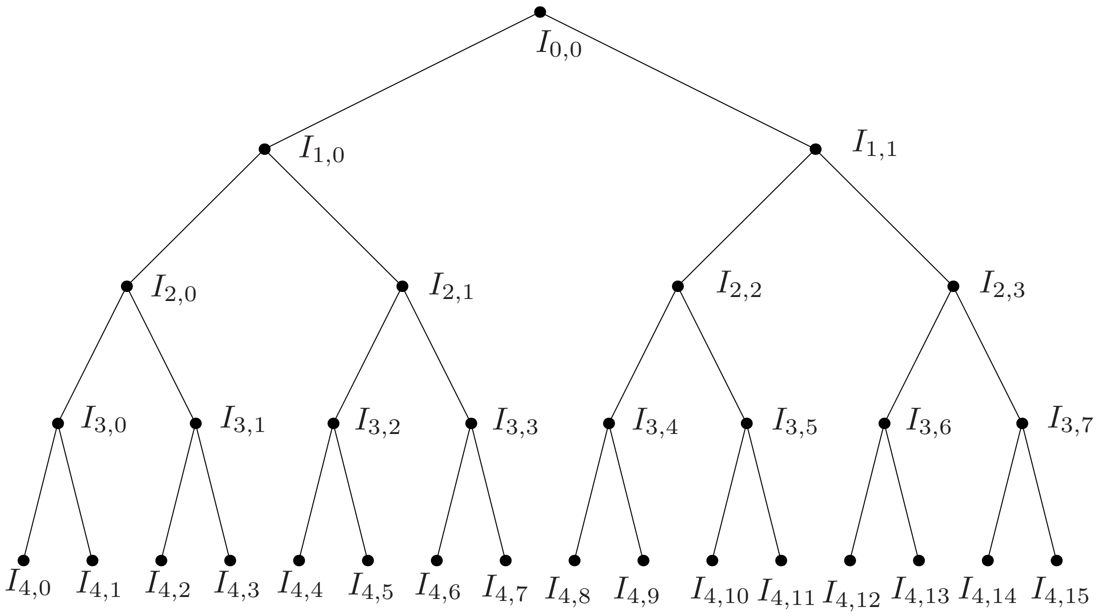

FIGURE 1. The dyadic tree structure of $\mathcal{D}_4$.

root of the complete dyadic tree $\mathcal{D}_N$ is the dyadic interval $I_{0,0}$. The depth of $\mathcal{D}_N$ is equal to N. For an interval $I_{\ell,k} \in \mathcal{D}_N$ the index ℓ denotes its level within the tree and k its position within the level. The left resp. the right child of an interval $I_{\ell,k} \in \mathcal{D}_N$ is given by

$$I_{\ell+1,2k} = \left[\frac{2k}{2^{\ell+1}}, \frac{2k+1}{2^{\ell+1}}\right[\quad \text{resp.} \quad I_{\ell+1,2k+1} = \left[\frac{2k+1}{2^{\ell+1}}, \frac{2\,(k+1)}{2^{\ell+1}}\right[. \tag{1.12}$$

1.2.6. Dyadic subtrees of $\mathcal{D}_N$. Let $I_{\ell,k} \in \mathcal{D}_N$. We denote by $\mathcal{T}^N_{\ell,k}$ the complete dyadic subtree of $\mathcal{D}_N$ with root $I_{\ell,k}$ and depth $N - \ell$. Note that

$$\mathcal{T}^N_{\ell,k} = \{I \in \mathcal{D}_N : I \subseteq I_{\ell,k}\},$$

cf. figure 2. Therefore, we get from (1.11) the Carleson constant

$$[\![\mathcal{T}^N_{\ell,k}]\!] = \frac{1}{|I_{\ell,k}|} \sum_{I \in \mathcal{T}^N_{\ell,k}} |I| = N - \ell + 1. \tag{1.13}$$

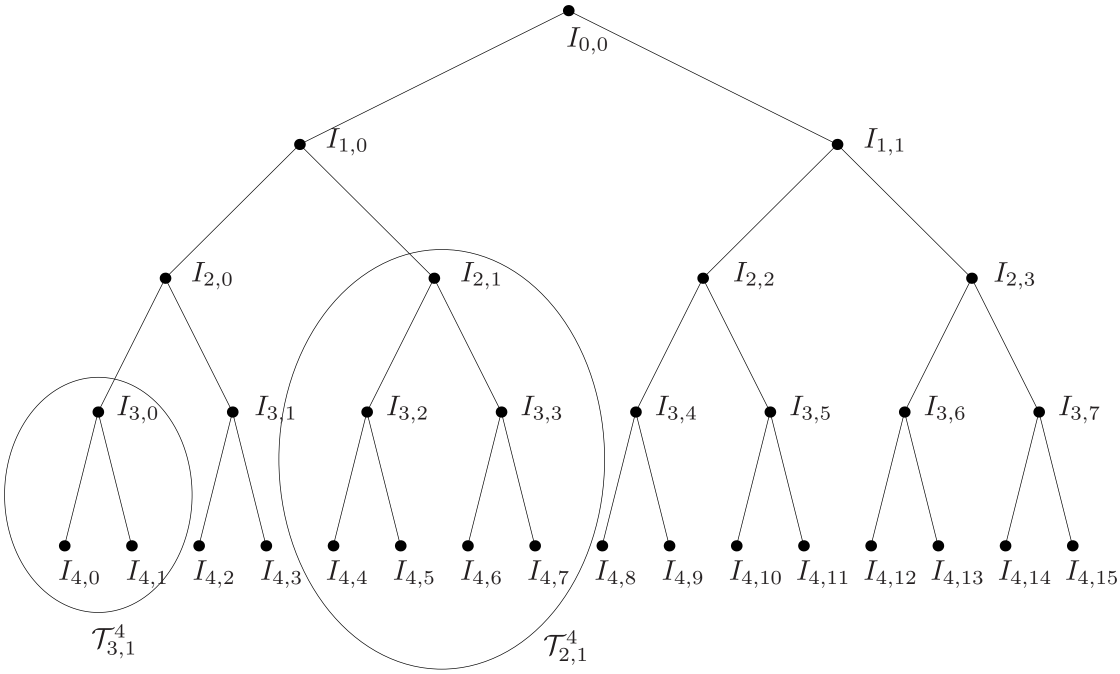

FIGURE 2. Dyadic subtrees of $\mathcal{D}_4$.

1.2.7. The order on $\mathcal{D}_N$. See [**MS97**], [**BP05**] and [**Knu05**].

DEFINITION 1.3. Let $I, J \in \mathcal{D}_N$. We say $I \preceq J$ if either I and J are disjoint and I is to the left of J, or I is contained in J.

We call "$\preceq$" the *postorder* on $\mathcal{D}_N$. In terms of the dyadic tree structure of $\mathcal{D}_N$ the postorder $\preceq$ is defined as follows: children are always smaller than their parent, the left child is always smaller than the right child and smaller than the descendants of the right child, cf. figure 3.

The natural order on the set $\mathcal{D}_N$ is the *lexicographic order*, $\leq_l$, on the set $\{(\ell, k)\}$, cf. figure 4. The postorder on $\mathcal{D}_N$, in contrast to the lexicographic order depends on the depth N. The postorder works its way up from the leftmost node in the lowermost level to the root of the dyadic tree $\mathcal{D}_N$. Therefore, it is clear from the definition that the root of the dyadic tree $\mathcal{D}_N$ has postorder ordinal number $2^{N+1}-1$, which is the total number of nodes contained in the tree $\mathcal{D}_N$.

Observe that $I_{1,0}$ is the left child and $I_{1,1}$ is the right child of the root $I_{0,0}$. Hence, the complete dyadic subtree $\mathcal{T}^N_{1,0}$ resp. $\mathcal{T}^N_{1,1}$ of $\mathcal{D}_N$ is the left resp. right subtree of the root $I_{0,0}$. The definition of the postorder yields that the left subtree contains the ordinal numbers $1, \dots, 2^N - 1$ and the right subtree the ordinal numbers $2^N, \dots, 2^{N+1} - 2$.

1.2.8. The order intervals. Let $J_1, J_2 \in \mathcal{D}_N$. An order interval with respect to the postorder, $\preceq$, is given by

$$\mathcal{B}^N(J_1, J_2) = \{I \in \mathcal{D}_N : J_1 \preceq I \preceq J_2\}, \tag{1.14}$$

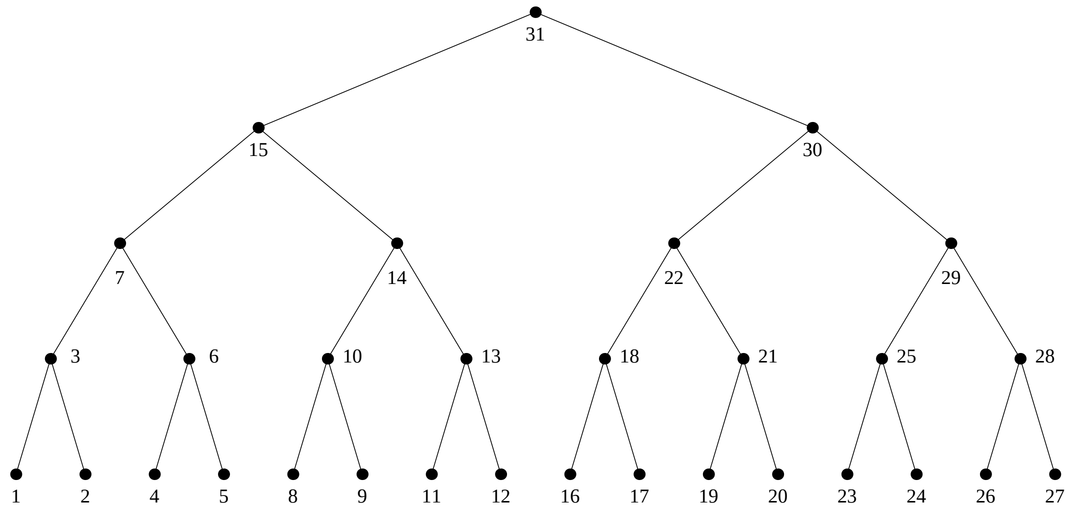

FIGURE 3. Postorder on the set $\mathcal{D}_4$.

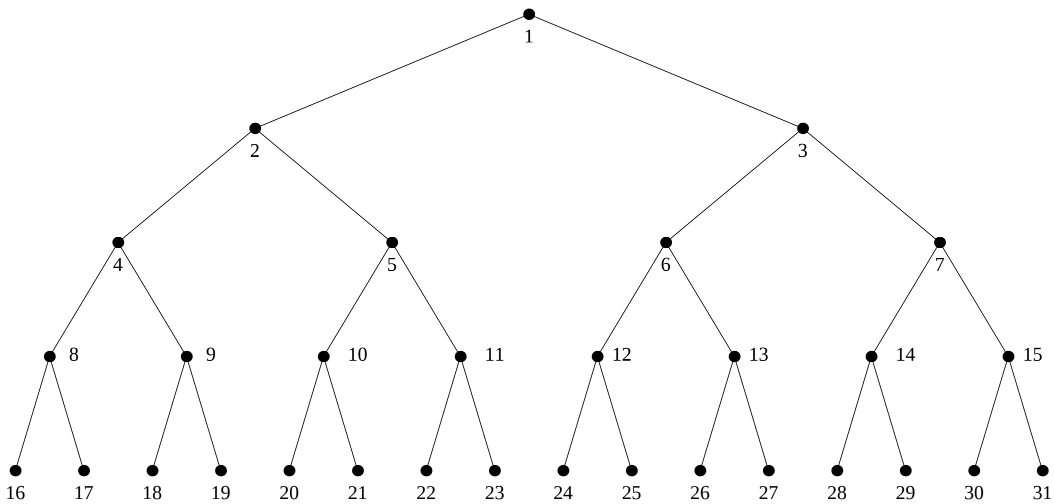

FIGURE 4. Lexicographic order on the set $\mathcal{D}_4$.

and with respect to the lexicographic order, $\leq_l$, by

$$\mathcal{E}(J_1, J_2) = \{I \in \mathcal{D}_N : J_1 \leq_l I \leq_l J_2\}. \tag{1.15}$$

The following definition and proposition is taken from [**MS97**] and describes order intervals with respect to the postorder, $\preceq$.

DEFINITION 1.4. Let $I, J \in \mathcal{D}_N$ with $I \subseteq J$.

(1) The *cone* $\mathcal{C} = \mathcal{C}(I, J)$ of dyadic intervals between I and J is the unique collection of dyadic intervals $\mathcal{C} = \{C_1, \dots, C_n\}$, where $n = \log_2 \frac{|J|}{|I|} + 1$, satisfying $C_1 = I$, $C_n = J$, $|C_i| = \frac{1}{2}|C_{i+1}|$ and $C_i \subset C_{i+1}$ for $1 \le i \le n-1$.
(2) The *right fill-up* of the cone $\mathcal{C}$ is the collection of dyadic intervals $\mathcal{R} = \mathcal{R}(I, J) = \bigcup_{i=1}^{n-1} \mathcal{U}_{i+1}$, where $\mathcal{U}_{i+1} = \{U \in \mathcal{D}_N : U \subseteq C_{i+1} \setminus C_i\}$, if C_i is the left half of C_{i+1} and $\mathcal{U}_{i+1} = \emptyset$, if C_i is the right half of C_{i+1}.

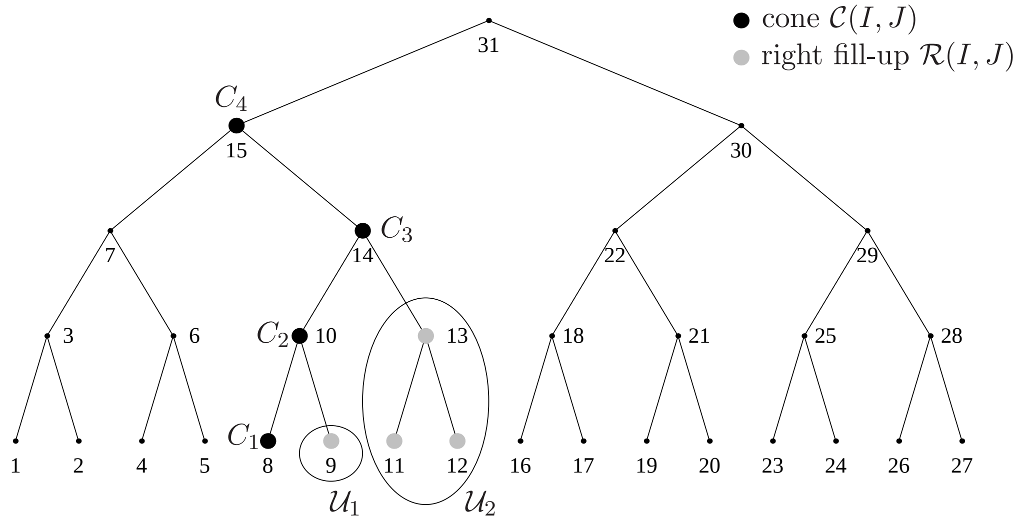

FIGURE 5. Cone and right fill-up given by the dyadic intervals $I = I_{4,4}$ and $J = I_{1,0}$ in $\mathcal{D}_4$.

PROPOSITION 1.5. *Let $J_1, J_2 \in \mathcal{D}_N$ and $J_1 \preceq J_2$. For the postorder order interval $\mathcal{B}^N(J_1, J_2)$ there exists a unique collection $\mathcal{L} = \{L_1, \dots, L_m\}$ of pairwise disjoint dyadic intervals satisfying*
(1) $|L_i| < |L_{i-1}|$, *if* $2 \le i \le m-1$;
(2) $|L_m| \le |L_{m-1}|$, *if* $m \ge 2$;
(3) L_{i+1} *lies to the right of* L_i *and the closures* $\overline{L_i}$ *and* $\overline{L_{i+1}}$ *intersect in exactly one point, the left endpoint of* $\overline{L_{i+1}}$;
(4) $J_1 \subseteq L_1$, $J_2 = L_m$ *and*

$$\mathcal{B}^N(J_1, J_2) = \mathcal{C}(J_1, L_1) \cup \mathcal{R}(J_1, L_1) \cup_{i=2}^{m} \mathcal{M}_i, \tag{1.16}$$

where $\mathcal{M}_i = \{I \in \mathcal{D}_N : I \subseteq L_i\}$.

REMARK 1.6. Note that the intervals $(L_i)_{i=1}^m$ are the maximal (with respect to inclusion) dyadic intervals in the postorder order interval $B^N(J_1, J_2)$.

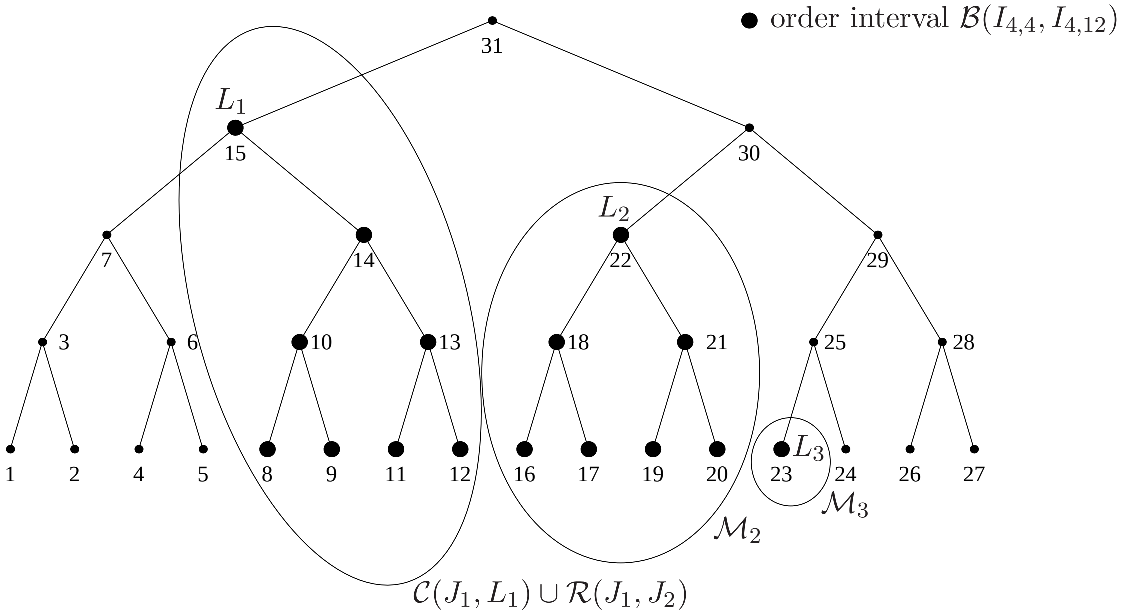

FIGURE 6. The postorder order interval $\mathcal{B}^4(I_{4,4}, I_{4,12})$ in $\mathcal{D}_4$.

1.3. The spaces

The sequence spaces $\ell^1(\mathcal{D})$ and $\ell^\infty(\mathcal{D})$. The space $\ell^1(\mathcal{D})$ is the space of all summable sequences $s = (s_I)_{I\in\mathcal{D}}$ indexed by the dyadic intervals, i.e.

$$\sum_{I\in\mathcal{D}} |s_I| < \infty,$$

equipped with the norm

$$\|s\|_1 = \sum_{I\in\mathcal{D}} |s_I|.$$

The space $\ell^\infty(\mathcal{D})$ is the space of all bounded sequences $s = (s_I)_{I\in\mathcal{D}}$ indexed by the dyadic intervals equipped with the norm

$$\|s\|_\infty = \sup_{I\in\mathcal{D}} |s_I|.$$

The Haar system. We define the L^∞-normalised *Haar system* $(h_I)_{I\in\mathcal{D}}$ indexed by the dyadic intervals $\mathcal{D}$ as follows:

$$h_I = \begin{cases} 1 & \text{on the left half of I,} \\ -1 & \text{on the right half of I,} \\ 0 & \text{otherwise.} \end{cases}$$

1.3.1. Dyadic BMO and Dyadic Hardy Spaces.

See [**Mül05, GMP05**]. Let $(x_I)_{I\in\mathcal{D}}$ be a real sequence. We define $f = (x_I)_{I\in\mathcal{D}}$ to be the real vector indexed by the dyadic intervals.

The space BMO consist of vectors $f = (x_I)_{I\in\mathcal{D}}$ for which

$$\|f\|_{\mathrm{BMO}} = \sup_{I\in\mathcal{D}} \left(\frac{1}{|I|}\sum_{J\subseteq I} |x_J|^2|J|\right)^{\frac{1}{2}} < \infty. \tag{1.17}$$

We define the square function of f as follows

$$S(f)(t) = \left(\sum_{I\in\mathcal{D}} |x_I|^2 1_I(t)\right)^{\frac{1}{2}},\ t \in [0,1]. \tag{1.18}$$

The space H^p, $0 < p < \infty$, consists of vectors $f = (x_I)_{I\in\mathcal{D}}$ for which

$$\|f\|_{H^p} = \|S(f)\|_{L^p([0,1])} < \infty. \tag{1.19}$$

For $1 \le p < \infty$, (1.19) defines a norm and H^p is a Banach space. For $0 < p < 1$, (1.19) defines a quasi-norm and the resulting Hardy spaces H^p are quasi-Banach spaces, cf.[**Woj97**].

For convenience we identify $f = (x_I)_{I\in\mathcal{D}} \in$ BMO resp. $f = (x_I)_{I\in\mathcal{D}} \in H^p$ with its formal Haar series

$$f = \sum_{I\in\mathcal{D}} x_I h_I.$$

Paley's theorem ([**Pal32**], see also [**Mül05**]) asserts that for all $1 < p < \infty$ there exists a constant A_p such that for all $f \in L^p([0,1])$ given by $f = \sum_{I\in\mathcal{D}} x_I h_I$ the following holds

$$\frac{1}{A_p}\|f\|_{L^p} \le \|S(f)\|_{L^p} \le A_p\|f\|_{L^p}. \tag{1.20}$$

This theorem identifies H^q as the dual space of H^p, where $\frac{1}{p} + \frac{1}{q} = 1$ and $1 < p < \infty$.

Fefferman's inequality ([**FS72**], see also [**Mül05**])

$$\left|\int fh\right| \le 2\sqrt{2}\,\|f\|_{H^1}\|h\|_{\mathrm{BMO}}, \tag{1.21}$$

and a theorem to the effect that every continuous linear functional $L : H^1 \to \mathbb{R}$ is necessarily of the form $L(f) = \int f\varphi$ with $\|\varphi\|_{\mathrm{BMO}} \le \|L\|$ identify BMO as dual space of H^1, (see [**FS72**],[**Gar73**],[**Mül05**]).

The H^p spaces are Banach lattices and their lattice structure is induced by the natural lattice structure on sequence spaces (cf.[**LT79**]). Therefore, they are (quasi-) Banach lattices over the dyadic intervals $\mathcal{D}$ equipped with the counting measure. For $0 < p \le 2$ the H^p spaces are 2-concave with 2-concavity constant $M_{(2)}(H^p) = 1$:

let $x^1,\dots,x^n \in H^p$, by Minkowski's inequality for $\frac{2}{p}\geq 1$ we have

$$\begin{aligned}\left(\sum_{i=1}^n \|x^i\|_{H^p}^2\right)^{\frac12} &= \left(\sum_{i=1}^n \left(\int_0^1 \left(\sum_{I\in\mathcal{D}} |x_I^i|^2 1_I(t)\right)^{\frac{p}{2}} dt\right)^{\frac{2}{p}}\right)^{\frac12}\\ &\leq \left(\int_0^1 \left(\sum_{i=1}^n\sum_{I\in\mathcal{D}} |x_I^i|^2 1_I(t)\right)^{\frac{p}{2}} dt\right)^{\frac1p}\\ &= \left\|\left(\sum_{i=1}^n |x^i|^2\right)^{\frac12}\right\|_{H^p}.\end{aligned} \tag{1.22}$$

By analogous arguments we get that H^p, $0<p<\infty$, is r-convex for all $r\leq \min(p,2)$ and s-concave for all $s\geq \max(p,2)$ with constants equal to one.

1.3.2. Discrete Triebel-Lizorkin spaces f_p^q, $0<p,q<\infty$. For general information on Triebel-Lizorkin spaces we refer to [**FJ90**][1]. Let $(x_I)_{I\in\mathcal{D}}$ be a real sequence. We define $f=(x_I)_{I\in\mathcal{D}}$ to be the real vector indexed by the dyadic intervals. We define the q-variation of f as follows

$$S_q(f)(t) = \left(\sum_{I\in\mathcal{D}} |x_I|^q 1_I(t)\right)^{\frac1q},\ t\in[0,1]. \tag{1.23}$$

The spaces f_p^q, $0<p,q<\infty$, consist of vectors $f=(x_I)_{I\in\mathcal{D}}$ for which

$$\|f\|_{f_p^q} = \|S_q(f)\|_{L^p([0,1])} < \infty. \tag{1.24}$$

For $1\leq p,q<\infty$, (1.24) defines a norm, otherwise it defines only a quasi-norm. Therefore, the Triebel-Lizorkin spaces f_p^q, $0<p,q<\infty$ are (quasi-) Banach spaces. For $0<p\leq 1$ and $p\leq q<\infty$ the following well-known triangle inequality holds

$$\|f+g\|_{f_p^q}^p \leq \|f\|_{f_p^q}^p + \|g\|_{f_p^q}^p. \tag{1.25}$$

As in the case of Hardy spaces the lattice structure on the Triebel-Lizorkin spaces is induced by the natural lattice structure on sequence spaces and therefore they are (quasi-) Banach lattices over the dyadic intervals equipped with the counting measure.

For $0<p,q<\infty$, the Triebel-Lizorkin spaces f_p^q are the $\frac{q}{2}$ - convexification of the Hardy space $H^{\frac{2p}{q}}$, meaning that f_p^q can be identified with the space of all sequences $u=(x_I)_{I\in\mathcal{D}}$ such that $|u|^{\frac{q}{2}}\in H^{\frac{2p}{q}}$ endowed with the norm $\||u|^{\frac{q}{2}}\|_{H^{\frac{2p}{q}}}^{\frac{2}{q}}$,

[1] The Triebel-Lizorkin spaces f_p^q are special cases of the discrete Triebel-Lizorkin spaces $f_p^{\alpha,q}$, defined in [**FJ90**, page 47], for the value $\alpha=-\frac12$.

cf.[**LT79, CT86**]. The absolute value is defined by $|u|^{\frac{q}{2}} = \left(|x_I|^{\frac{q}{2}}\right)_{I\in\mathcal{D}}$ and it can be identified with its formal Haar series $|u|^{\frac{q}{2}} = \sum_{I\in\mathcal{D}} |x_I|^{\frac{q}{2}} h_I$.

Recall that the (quasi-) Banach lattice H^p is s-concave for all $s \geq \max(p,2)$ and r-convex for all $r \leq \min(p,2)$. The theory of convexification ([**LT79, CT86**]) yields that the Triebel-Lizorkin spaces f_p^q are s-concave for all $s \geq \max(p,q)$ and r-convex for all $r \leq \min(p,q)$. The s-concavity and r-convexity constants $M_{(s)}(f_p^q)$ and $M^{(r)}(f_p^q)$ are equal to one.

For convenience, we sometimes identify $f = (x_I)_{I\in\mathcal{D}}$ with its formal Haar series

$$f = \sum_{I\in\mathcal{D}} x_I h_I.$$

1.3.3. Vector-valued dyadic Hardy Spaces H_X^p, $0 < p \leq 2$. [2] See [**Mül12**]. Let X be a Banach space of non-trivial cotype and $(x_I)_{I\in\mathcal{D}}$ be a sequence in X. We define $f = (x_I)_{I\in\mathcal{D}}$ to be the X-valued vector indexed by the dyadic intervals. Let $(r_I)_{I\in\mathcal{D}}$ be an enumeration of the independent Rademacher system. Then we define the square function of f as follows.

$$\mathbb{S}(f)(t) = \lim_{N\to\infty} \left(\int_0^1 \Big\| \sum_{I\in\mathcal{D}_N} r_I(s) x_I h_I(t) \Big\|_X^2 ds \right)^{\frac{1}{2}}, \tag{1.26}$$

for all $t \in [0,1]$. We say $f \in H_X^p$, if

$$\|f\|_{H_X^p} = \|\mathbb{S}(f)\|_{L^p([0,1])} < \infty. \tag{1.27}$$

For $1 \leq p \leq 2$ equation (1.27) defines a norm and H_X^p is a Banach space. For $0 < p < 1$ equation (1.27) defines a quasi-norm and H_X^p is a quasi-Banach space with the well-known triangle inequality

$$\|f+g\|_{H_X^p}^p \leq \|f\|_{H_X^p}^p + \|g\|_{H_X^p}^p. \tag{1.28}$$

As before, we identify $f = (x_I)_{I\in\mathcal{D}}$ with its formal Haar series

$$f = \sum_{I\in\mathcal{D}} x_I h_I.$$

1.3.4. Vector-valued dyadic Triebel-Lizorkin spaces $f_p^q(X)$. Let X be a Banach space of non-trivial cotype and $(x_I)_{I\in\mathcal{D}}$ a sequence in X. We define $f = (x_I)_{I\in\mathcal{D}}$ to be the X-valued vector indexed by dyadic intervals.

Let $0 < p \leq s < q < 2$. Let $(\theta_I)_{I\in\mathcal{D}}$ be a sequence of independent q-stable random variables on the probability space $(\Omega, \Sigma, \mathbb{P})$. We define the q- *variation* of f as follows

$$\mathbb{S}_q(f)(t) = \lim_{N\to\infty} \left(\int_\Omega \Big\| \sum_{I\in\mathcal{D}_N} \theta_I(\omega) x_I h_I(t) \Big\|_X^s d\mathbb{P}(\omega) \right)^{\frac{1}{s}}, \tag{1.29}$$

[2] One can extend the definition to the range $0 < p < \infty$, but it is not necessary in this work.

for all $t \in [0,1]$. Note that due to inequality (1.5) the definition (1.29) is equivalent for all $s \in [p,q)$.

Let $(g_I)_{I\in\mathcal{D}}$ be a sequence of independent standard Gaussian random variables. In the special case $q=2$ we define the square function of f as follows

$$\mathbb{S}(f)(t) = \lim_{N\to\infty}\left(\int_\Omega \Big\| \sum_{I\in\mathcal{D}_N} g_I(\omega)x_I h_I(t)\Big\|_X^2 \, d\mathbb{P}(\omega)\right)^{\frac{1}{2}}, \tag{1.30}$$

for all $t\in[0,1]$. We say $f \in f_p^q(X)$, $0<p<q\le 2$, if

$$\|f\|_{f_p^q(X)} = \|\mathbb{S}_q(f)\|_{L^p([0,1])} < \infty. \tag{1.31}$$

For $1 \le p < q$ equation (1.31) defines a norm and $f_p^q(X)$ is a Banach space. For $0<p<1$ equation (1.31) defines a quasi-norm and $f_p^q(X)$ is a quasi-Banach space with the well-known triangle inequality

$$\|f+g\|_{f_p^q(X)}^p \le \|f\|_{f_p^q(X)}^p + \|g\|_{f_p^q(X)}^p. \tag{1.32}$$

For convenience, we sometimes identify $f=(x_I)_{I\in\mathcal{D}}$ with its formal Haar series

$$f = \sum_{I\in\mathcal{D}} x_I h_I.$$

Remark 1.7. In the case $q=2$, a theorem of [**MP76**] states that for all finite sequences (x_i) in a Banach space X of non-trivial cotype r we have for all $s>r$

$$\left(\int_\Omega \Big\|\sum g_i(\omega)x_i\Big\|_X^2 d\omega\right)^{\frac{1}{2}} \le C_r(X)\|g_1\|_{L^s}\left(\int_0^1 \Big\|\sum r_i(t)x_i\Big\|_X^2 dt\right)^{\frac{1}{2}}, \tag{1.33}$$

where (g_i) is a sequence of independent Gaussian random variables on (Ω,Σ,μ) and (r_i) is the independent Rademacher system. A standard argument on symmetric random variables yields that for all finite sequences (x_i) in a Banach space X

$$\|g_1\|_{L^1}\left(\int_0^1 \Big\|\sum r_i(t)x_i\Big\|_X^2 dt\right)^{\frac{1}{2}} \le \left(\int_\Omega \Big\|\sum g_i(\omega)x_i\Big\|_X^2 d\omega\right)^{\frac{1}{2}}. \tag{1.34}$$

Hence, by inequalities (1.33) and (1.34) the definition (1.30) is equivalent to (1.26). Therefore, $f_p^2(X)$ is isomorphic to H_X^p, $0<p\le 2$, defined in the previous section.

The Haar support. Let $f\in \mathrm{BMO}$ resp. $f \in H^p$ resp. $f\in f_p^q$ be given by its formal Haar series

$$f = \sum_{I\in\mathcal{D}} x_I h_I,$$

where $(x_I)_{I\in\mathcal{D}}$ is a real sequence. The set $\{I\in\mathcal{D} : x_I \ne 0\}$ is called *Haar support* of f.

Let X be a Banach space. Let $f\in H_X^p$ resp. $f\in f_p^q(X)$ given by its Haar expansion

$$f = \sum_{I\in\mathcal{D}} x_I h_I,$$

where $(x_I)_{I\in\mathcal{D}}$ is a sequence in X. Analogously to the scalar-valued case we call the set $\{I \in \mathcal{D} : x_I \neq 0\}$ the *Haar support* of f.

1.3.5. Atomic decomposition. The following theorem states the decomposition of an element in H^p_X into absolutely summing elements with disjoint Haar support and bounded square function. The decomposition is done by a stopping time argument that may be regarded as a constructive algorithm. The decomposition originates in the work of S. Janson and P.W. Jones [**JJ82**].

THEOREM 1.8 (Atomic decomposition). *For all $0 < p \leq 2$ there exist constants a_p, A_p such that for every $u \in H^p_X$ with Haar expansion*

$$u = \sum_{J\in\mathcal{D}} x_J h_J, \quad x_J \in X$$

there exists an index set $\mathcal{N} \subseteq \mathbb{N}$ and a sequence $(\mathcal{G}_i)_{i\in\mathcal{N}}$ of blocks of dyadic intervals such that for

$$u_i = \sum_{J\in\mathcal{G}_i} x_J h_J, \quad i \in \mathcal{N}$$

the following holds:

i) *$(\mathcal{G}_i)_{i\in\mathcal{N}}$ is a disjoint partition of $\mathcal{D}$.*
ii) *$I_i := \bigcup_{J\in\mathcal{G}_i} J$ is a dyadic interval and $\mathcal{E} := \{I_i : i \in \mathcal{N}\}$ satisfies $[\![\mathcal{E}]\!] \leq 4$.*
iii)

$$a_p\|u\|^p_{H^p_X} \leq \sum_{i\in\mathcal{N}} \|u_i\|^p_{H^p_X} \leq \sum_{i\in\mathcal{N}} |I_i| \|\mathbb{S}(u_i)\|^p_\infty \leq A_p\|u\|^p_{H^p_X}. \tag{1.35}$$

The family $(u_i, \mathcal{G}_i, I_i)_{i\in\mathcal{N}}$ is called the atomic decomposition of $u \in H^p_X$.

REMARK 1.9. If we set $X = \mathbb{R}$ in the above theorem, we get the atomic decomposition of $u \in H^p$. Note that in this case $a_p = 1$ for all $0 < p \leq 2$. The scalar-valued decomposition procedure, particularly the inequalities (1.35) in the case $X = \mathbb{R}$, can be found in [**Mül05**]. The right-hand side inequality in (1.35) transfers directly from the scalar-valued case to the vector-valued case. For the left-hand side estimate in (1.35) we have to consider two cases. In the case $0 < p \leq 1$ use the well-known triangle inequality

$$\|f + g\|^p_{H^p_X} \leq \|f\|^p_{H^p_X} + \|g\|^p_{H^p_X}.$$

In the case $1 < p \leq 2$ there are some differences between the scalar- and the vector-valued case. In the scalar-valued case, the left-hand side inequality follows immediately from the disjoint decomposition of $\mathcal{D}$ into blocks of dyadic intervals. In the vector-valued case one can adapt the proof of [**GM08**, Lemma 3.3]. We include the Appendix (Chapter 4) in order to give the proof in detail.

However, note that the left-hand side inequality of (1.35) depends only on the fact that $(\mathcal{G}_i)_{i\in\mathcal{N}}$ is a sequence of disjoint blocks of dyadic intervals and that the Carleson constant of $\mathcal{E}$ is finite. Therefore, for $\varphi = (\varphi_I)_{I\in\mathcal{D}} \in \ell^\infty(\mathcal{D})$ we have

$$\Big\| \sum_{I\in\mathcal{D}} \varphi_I x_I h_I \Big\|^p_{H^p_X} \leq \frac{1}{a_p} \sum_{i\in\mathcal{N}} \Big\| \sum_{I\in\mathcal{G}_i} \varphi_I x_I h_I \Big\|^p_{H^p_X}. \tag{1.36}$$

REMARK 1.10. The atomic decomposition of an element $u \in f_p^q$, $0 < p \le q < \infty$, can be deduced from the above theorem. Note that the given restriction for the parameters p, q yields $\frac{2p}{q} \in (0, 2]$. Applying the atomic decomposition procedure to the function $|u|^{\frac{q}{2}} = \sum_{I \in \mathcal{D}} |u_I|^{\frac{q}{2}} h_I \in H^{\frac{2p}{q}}$, yields the atomic decomposition of $u = (x_I)_{I \in \mathcal{D}} \in f_p^q$, denoted by $(u_i, I_i, \mathcal{G}_i)$, where $u_i = (x_I)_{I \in \mathcal{G}_i}$ and I_i, $\mathcal{G}_i$ are as in Theorem 1.8.

To obtain the atomic decomposition for an element $u \in f_p^q(X)$, $0 < p < q \le 2$, one has to rebuild the decomposition procedure described in [**Mül05**] by replacing the square function of an element in H^p by the q-variation $\mathbb{S}_q(u)$ of $u \in f_p^q(X)$. This yields the analogous statement as in Theorem 1.8 for Triebel-Lizorkin spaces $f_p^q(X)$, $0 < p < q \le 2$. The constants a_p and A_p are then replaced by constants that depend additionally on q. Again we denote the atomic decomposition by $(u_i, \mathcal{G}_i, I_i)_{i \in \mathcal{N}}$. Note that for $X = \mathbb{R}$ one obtains the atomic decomposition of $u \in f_p^q$ described above.

1.3.6. Finite dimensional spaces. Let $N \in \mathbb{N}_0$. We define the finite dimensional BMO space denoted by BMO_N, the finite dimensional Hardy spaces H_N^p, $0 < p < \infty$, and the finite dimensional Triebel-Lizorkin spaces $f_{p,N}^q$, $0 < p, q < \infty$ as follows.

$$\mathrm{BMO}_N = \Big(\operatorname{span}\{h_I : I \in \mathcal{D}_N\},\ \|\cdot\|_{\mathrm{BMO}}\Big),$$
$$H_N^p = \Big(\operatorname{span}\{h_I : I \in \mathcal{D}_N\},\ \|\cdot\|_{H^p}\Big)$$

and

$$f_{p,N}^q = \Big(\operatorname{span}\{h_I : I \in \mathcal{D}_N\},\ \|\cdot\|_{f_p^q}\Big),$$

where $\|\cdot\|_{\mathrm{BMO}}$ is given by equation (1.17), $\|\cdot\|_{H^p}$ by equation (1.18) and (1.19) and $\|\cdot\|_{f_p^q}$ by equation (1.23) and (1.24).

Note that if one associates $f = \sum_{I \in \mathcal{D}_N} x_I h_I$ with the sequence of coefficients $(x_I)_{I \in \mathcal{D}_N}$, it is clear that BMO_N is isometrically isomorphic to a finite dimensional subspace of BMO. Analogous statements apply to the spaces H_N^p and $f_{p,N}^q$.

Let $(x_I)_{I \in \mathcal{D}_N}$ be a real sequence and let $f = \sum_{I \in \mathcal{D}_N} x_I h_I$. Recall that the Haar support of f is the set $\{I \in \mathcal{D}_N : x_I \neq 0\}$. The Haar support of f is contained in a non-empty collection of dyadic intervals $\mathcal{C} \subseteq \mathcal{D}_N$ if and only if $f = \sum_{I \in \mathcal{C}} x_I h_I$. We denote by

$$\mathcal{M}(\mathcal{C})$$

the space of all functions f that have Haar support in the non-empty collection $\mathcal{C} \subseteq \mathcal{D}_N$. Note that $\mathcal{M}(\mathcal{C}) \subseteq \operatorname{span}\{h_I : I \in \mathcal{D}_N\}$. Hence, we can equip $\mathcal{M}(\mathcal{C})$ with any one of the norms $\|\cdot\|_{\mathrm{BMO}}$, $\|\cdot\|_{H^p}$ or $\|\cdot\|_{f_p^q}$.

1.4. The operators

1.4.1. p-summing operators. See e.g. [**Pie67**]. Let X, Y be Banach spaces. $L(X, Y)$ is the space of all linear and bounded operators $T\colon X \to Y$ equipped with the operator norm

$$\|T\colon X \to Y\| = \sup_{x \in X, x \neq 0} \frac{\|Tx\|_Y}{\|x\|_X}.$$

For convenience we use the abbreviation $\|T\|_X := \|T\colon X \to X\|$. If X and Y are clear from the context we only write $\|T\|$.

Let $1 \leq p \leq \infty$. An operator $T \in L(X,Y)$ is called *p-summing* if there is a constant K so that for every choice of an integer n and vectors $(x_i)_{i=1}^n$ in X, we have

$$\left(\sum_{i=1}^{n} \|Tx_i\|^p\right)^{\frac{1}{p}} \leq K \sup_{\|x^*\|_{X^*}\leq 1} \left(\sum_{i=1}^{n} |x^*(x_i)|^p\right)^{\frac{1}{p}}. \tag{1.37}$$

The smallest possible constant K is denoted by $\pi_p(T)$. The class of all p-summing operators in $L(X,Y)$ is denoted by $\Pi_p(X,Y)$. π_p defines a norm on $\Pi_p(X,Y)$ with $\|T\| \leq \pi_p(T)$ for all $T \in \Pi_p(X,Y)$.

The following fundamental theorem on p-summing operators can be found in most of the literature on Banach space theory (see e.g. [**Woj91**]). Note that this theorem is also known as Pietsch's factorization theorem (see e.g. [**LT77**]).

THEOREM 1.11 (Pietsch's domination theorem [**Pie67**]). *Let $1 \leq p < \infty$. Let K be a compact Hausdorff space and let X, Y be Banach spaces. An operator $T \in L(X,Y)$ is p-summing with constant $\pi_p(T) < \infty$ if and only if for every isometric embedding $i : X \to C(K)$ there exists a Borel probability measure μ on K such that*

$$\|Tx\| \leq \pi_p(T) \left(\int_K |i(x)|^p d\mu\right)^{\frac{1}{p}}, \quad \textit{for } x \in X.$$

Every such a measure μ is called *Pietsch measure* or *dominating measure* for the operator T. The existence of the Pietsch measure is given by a Hahn-Banach argument (see e.g. [**Woj91**]).

$C(K)$ is the space of scalar-valued continuous functions on the compact Hausdorff space K equipped with the norm

$$\|\varphi\|_\infty = \sup_{k\in K} |\varphi(k)|.$$

We next recall Maurey's theorem on p-summing operators from a $C(K)$-space into a Banach space of non-trivial cotype. We refer to [**Mau74b**] and the exposition of Maurey's theorem in [**DJT95**].

THEOREM 1.12 (Maurey's theorem). *For all $2 \leq r < \infty$ and $r < s < \infty$ there exists a positive constant $K_{s,r}$ such that for every Banach space Y of cotype r with cotype-r constant $C_r(Y)$ and for every compact Hausdorff space K we have*

$$L(C(K),Y) = \Pi_s(C(K),Y)$$

and for every $T \in L(C(K),Y)$

$$\pi_s(T) \leq K_{s,r} C_r(Y)\|T\|.$$

The case when the target space Y is of cotype 2 allows the following strengthening of the conclusion. This is the context of the theorem in [**DPR72**].

THEOREM 1.13 ([**DPR72**]). *There exists a positive constant B such that for every Banach space Y of cotype 2 with cotype-2 constant $C_2(Y)$ and for every compact Hausdorff space K we have*

$$L(C(K), Y) = \Pi_2(C(K), Y)$$

and for every $T \in L(C(K), Y)$

$$\pi_2(T) \leq BC_2(Y)^2\|T\|.$$

1.4.2. Multiplication operators. Let $0 < p \leq 2$. Every $u \in H^p$ given by its Haar expansion $u = \sum_{I\in\mathcal{D}} x_I h_I$ defines a multiplication operator

$$\mathcal{M}_u : \ell^\infty(\mathcal{D}) \to H^p$$

by putting

$$\mathcal{M}_u(\varphi) = \sum_{I\in\mathcal{D}} \varphi_I x_I h_I.$$

For convenience we use the "lattice convention" $\mathcal{M}_u(\varphi) = \varphi \cdot u$. These multiplication operators are bounded with

$$\|\mathcal{M}_u \colon \ell^\infty \to H^p\| = \|u\|_{H^p}.$$

An argument of Pisier (see [**Pis79a**]), given below, asserts that the multiplication operator $\mathcal{M}_u$ is 2-summing with

$$\pi_2(\mathcal{M}_u) = \|u\|_{H^p}.$$

Note that H^p, $0 < p \leq 2$, is 2-concave with 2-concavity constant $M_{(2)}(H^p)$ equal to one. Let $\varphi_1, \dots, \varphi_n \in \ell^\infty(\mathcal{D})$. Using the 2-concavity of H^p we get the following

$$\begin{aligned}\Big(\sum_{i=1}^n \|\mathcal{M}_u(\varphi_i)\|_{H^p}^2\Big)^{\frac12} &\leq \Big\|\Big(\sum_{i=1}^n |u\cdot\varphi_i|^2\Big)^{\frac12}\Big\|_{H^p}\\ &\leq \|u\|_{H^p}\Big\|\Big(\sum_{i=1}^n |\varphi_i|^2\Big)^{\frac12}\Big\|_\infty \qquad (1.38)\\ &\leq \|u\|_{H^p} \sup_{\substack{\|\varphi^*\|\leq 1\\ \varphi^*\in(\ell^\infty(\mathcal{D}))^*}} \Big(\sum_{i=1}^n |\varphi^*(\varphi_i)|^2\Big)^{\frac12}.\end{aligned}$$

Since H^p, $0 < p \leq 2$, is s-concave for all $s \geq 2$ with s-concavity constant $M_{(s)}(H^p)$ equal to one, we have by the same argument that $\mathcal{M}_u$ is s-summing with $\pi_s(\mathcal{M}_u) = \|u\|_{H^p}$ for all $s \geq 2$. Pietsch's domination theorem yields $\Pi_2(C(K), H^p) \subseteq \Pi_s(C(K), H^p)$ for all $s \geq 2$. Hence, 2-summing is the best we can obtain.

REMARK 1.14. Besides the s-concavity there are other properties of the image space H^p that imply that the multiplication operator $\mathcal{M}_u : \ell^\infty(\mathcal{D}) \to H^p$ is 2-summing for $1 \leq p \leq 2$.

Grothendieck's Theorem (see e.g. [**LP68**]) asserts that every operator $T \in L(C(K), L^p)$ is 2-summing with $\pi_2(T) \leq K_G\|T\|$, where K_G is the Grothendieck constant. Note that $\ell^\infty(\mathcal{D})$ is a $C(K)$ space for some compact Hausdorff space K (Stone-Čech-compactification, see [**Car05**]). Paley's theorem (1.20) yields that H^p is isometrically isomorphic to L^p for $1 < p \leq 2$. Furthermore, $H^1 \subseteq L^1$ (Davis inequalities, [**Dav70**]). We consider the following multiplication operators

$$\mathcal{M}_u\colon \ell^\infty \to H^p \subseteq L^p$$

and get by Grothendieck's theorem that they are 2-summing with constant $\pi_2(\mathcal{M}_u) \leq K_G \|u\|_{H^p}$.

Theorem 1.13 asserts, that every operator $T \in L(C(K), Y)$, where Y is a Banach space of cotype 2 is s-summing for all $s \geq 2$. Since H^p, $1 \leq p \leq 2$, is of cotype 2 we have that the multiplication operator

$$\mathcal{M}_u\colon \ell^\infty \to H^p$$

is 2-summing with constant $\pi_2(\mathcal{M}_u) \leq BC_2(Y)^2 \|u\|_{H^p}$.

Analogous results can be obtained for the Triebel-Lizorkin spaces. Let $0 < p \leq q < \infty$. Every $u = (x_I)_{I\in\mathcal{D}} \in f_p^q$ defines a bounded multiplication operator

$$\mathcal{M}_u\colon \ell^\infty(\mathcal{D}) \to f_p^q, \;\; \varphi \mapsto (\varphi_I x_I)_{I\in\mathcal{D}}$$

with $\|\mathcal{M}_u\| = \|u\|_{f_p^q}$. The Triebel-Lizorkin spaces f_p^q are q-concave with q-concavity constant $M_{(q)}(f_p^q) = 1$. Therefore, by the same argument as in (1.38), the multiplication operator $\mathcal{M}_u\colon \ell^\infty(\mathcal{D}) \to f_p^q$ is q-summing with $\pi_q(\mathcal{M}_u) = \|u\|_{f_p^q}$.

1.4.3. Vector-valued multiplication operators. Let $0 < p \leq 2$. Let X be a Banach space of finite cotype r, $2 \leq r < \infty$. Every $u \in H_X^p$ given by its Haar expansion $u = \sum_{I\in\mathcal{D}} x_I h_I$, where $(x_I)_{I\in\mathcal{D}}$ is a sequence in the Banach space X, defines a multiplication operator

$$\mathcal{M}_u : \ell^\infty(\mathcal{D}) \to H_X^p$$

by

$$\mathcal{M}_u(\varphi) = \sum_{I\in\mathcal{D}} \varphi_I x_I h_I.$$

For convenience we will again use the notation

$$\varphi \cdot u = \sum_{I\in\mathcal{D}} \varphi_I x_I h_I.$$

By Kahane's contraction principle (1.2) these multiplication operators are bounded and $\|\mathcal{M}_u\| = \|u\|_{H_X^p}$.

Let $1 \leq p \leq 2$. Kahane's inequality (1.1) yields

$$\|f\|_{H_X^p} \lesssim \left(\int_0^1 \int_0^1 \Big\| \sum_{I\in\mathcal{D}} x_I h_I(t) r_I(s) \Big\|_X^p \, ds\, dt \right)^{\frac{1}{p}} = \Big\| \sum_{I\in\mathcal{D}} x_I h_I r_I \Big\|_{L_X^p}.$$

Therefore, H_X^p forms a subspaces of $L_X^p([0,1]\times[0,1], dsdt)$[3]. X is of cotype r. Then L_X^p is of cotype r with $C_r(X) \approx C_r(L_X^p)$ and so is H_X^p, $1 \leq p \leq 2$, with $C_r(H_X^p) \lesssim$

[3] Note that L_X^p is the Bochner-Lebesgue space. For definition see e.g. [**DU77**]

$C_r(X)$. Maurey's theorem (Theorem 1.12) asserts that the multiplication operator $\mathcal{M}_u$ is s-summing for all $s > r$ and

$$\pi_s(\mathcal{M}_u) \leq K_{s,r} C_r(X) \|\mathcal{M}_u\| \leq K_{s,r} C_r(X) \|u\|_{H^p_X}. \tag{1.39}$$

Note that by Theorem 1.13 we have that if X is of cotype 2, then $\mathcal{M}_u$ is 2-summing.

Analogous results can be obtained for vector-valued Triebel-Lizorkin spaces. Let $0 < p < q \leq 2$. Let X be a Banach space of finite cotype r_0, $2 \leq r_0 < \infty$. Every $u = (x_I)_{I \in \mathcal{D}} \in f^q_p(X)$, where $(x_I)_{I \in \mathcal{D}}$ is a sequence in the Banach space X, defines a multiplication operator

$$\mathcal{M}_u : \ell^\infty(\mathcal{D}) \to f^q_p(X)$$

by

$$\mathcal{M}_u(\varphi) = (\varphi_I x_I)_{I \in \mathcal{D}}.$$

By inequality (1.6) these multiplication operators are bounded and $\|\mathcal{M}_u\| = \|u\|_{f^q_p(X)}$.

Inequality (1.5) yields that for $0 < p < q$ we have

$$\|f\|_{f^q_p(X)} \lesssim \left(\int_0^1 \int_\Omega \Big\| \sum_{I \in \mathcal{D}} x_I h_I \theta_I(\omega) \Big\|^p_X d\omega \, dt \right)^{\frac{1}{p}} = \Big\| \sum_{I \in \mathcal{D}} x_I h_I \theta_I \Big\|_{L^p_X}.$$

Therefore, the vector-valued Triebel-Lizorkin space $f^q_p(X)$ forms a subspace of $L^p_X([0,1] \times \Omega, d\mathbb{P}dt)$. X is of cotype r_0. Hence, by the same argument as above, $f^q_p(X)$ is of finite cotype r_0 and $C_{r_0}(f^q_p(X)) \lesssim C_{r_0}(X)$. Maurey's theorem (Theorem 1.12) asserts that the multiplication operator $\mathcal{M}_u$ is r-summing for all $r > r_0$ and

$$\pi_r(\mathcal{M}_u) \leq K_{r,r_0} C_{r_0}(X) \|\mathcal{M}_u\| \leq K_{r,r_0} C_{r_0}(X) \|u\|_{f^q_p(X)}. \tag{1.40}$$

Note that by Theorem 1.13 we have that if X is of cotype 2, then $\mathcal{M}_u$ is 2-summing.

1.4.4. Rearrangements of the Haar system. Recall that $\mathcal{D}_N$ is the set of dyadic intervals with length greater than or equal to 2^{-N}. Let τ be a bijective map defined on the set $\mathcal{D}_N$. On BMO_N we study rearrangements of the L^∞-normalised Haar system $(h_I)_{I \in \mathcal{D}_N}$ given by the rearrangement operator

$$T_\tau \colon h_I \mapsto h_{\tau(I)},$$

and on H^p_N, $0 < p < \infty$, rearrangements of the L^p-normalised Haar system given by the rearrangement operator

$$T_{\tau,p} : \frac{h_I}{|I|^{\frac{1}{p}}} \mapsto \frac{h_{\tau(I)}}{|\tau(I)|^{\frac{1}{p}}}.$$

A standard argument (given below) yields the following norm estimates for rearrangement operators on BMO_N

$$\sup_{\substack{\mathcal{C} \subseteq \mathcal{D}_N \\ \text{non-empty}}} \frac{[\![\tau_N(\mathcal{C})]\!]^{\frac{1}{2}}}{[\![\mathcal{C}]\!]^{\frac{1}{2}}} \leq \|T_\tau\|_{\mathrm{BMO}} \leq (N+1)^{\frac{1}{2}}. \tag{1.41}$$

Note that the lower bound in (1.41) is always greater than or equal to one. Let $x = \sum_{I\in\mathcal{D}_N} x_I h_I$. Then

$$\|T_\tau x\|_{\mathrm{BMO}}^2 = \sup_{I\in\mathcal{D}_N} \frac{1}{|I|}\sum_{J\subseteq I}\left|x_{\tau^{-1}(J)}\right|^2|J| \le \sup_{I\in\mathcal{D}_N}|x_I|^2 [\![\mathcal{D}_N]\!] \le \|x\|_{\mathrm{BMO}}^2 [\![\mathcal{D}_N]\!].$$

Definition (1.11) yields $[\![\mathcal{D}_N]\!] = N+1$. Let $\mathcal{C}\subseteq\mathcal{D}_N$ be any non-empty collection of dyadic intervals. Let $x = \sum_{I\in\mathcal{C}} h_I$. Then

$$\|x\|_{\mathrm{BMO}} = [\![\mathcal{C}]\!]^{\frac{1}{2}} \text{ and } \|T_\tau x\|_{\mathrm{BMO}} = [\![\tau(\mathcal{C})]\!]^{\frac{1}{2}}.$$

Let $x = \sum_{I\in\mathcal{C}} x_I h_I$ for some non-empty collection of dyadic intervals $\mathcal{C}\subseteq\mathcal{D}_N$. The above argument provides the following rough upper bound

$$\|T_\tau x\|_{\mathrm{BMO}} \le \|x\|_{\mathrm{BMO}} [\![\tau(\mathcal{C})]\!]^{\frac{1}{2}}. \tag{1.42}$$

The adjoint operator of a rearrangement operator is again a rearrangement operator induced by the inverse rearrangement. By the duality of H^1 and BMO we have that the operator T_τ on BMO_N is the adjoint operator of $T_{\tau^{-1},1}$ on H^1_N with

$$\frac{1}{C_F}\|T_\tau\|_{\mathrm{BMO}_N} \le \|T_{\tau^{-1},1}\|_{H^1} \le C_F\|T_\tau\|_{\mathrm{BMO}_N}, \tag{1.43}$$

where $C_F = 2\sqrt{2}$ is the constant appearing in Fefferman's inequality (1.21). We will use the following abbreviation for equation 1.43

$$\|T_\tau\|_{\mathrm{BMO}_N} \approx_{C_F} \|T_{\tau^{-1},1}\|_{H^1}. \tag{1.44}$$

1.4.5. Interpolation and extrapolation of rearrangement operators. See [**GMP05, Mül05**]. The following interpolation resp. extrapolation theorem provides a tool that enables one to deduce norm estimates for the rearrangement operators $T_{\tau,p}$ on H^p_N for every $0 < p < 2$ from norm estimates of some rearrangement operator T_{τ,p_0} on $H^{p_0}_N$, $0 < p_0 < 2$. The left-hand side inequality corresponds to an extrapolation based on Pisier's extrapolation norm (see [**GMP05**]). The right-hand side inequality is obtained by a standard interpolation argument. Note that $\|T_{\tau,2}\|_{H^2_N} = 1$.

THEOREM 1.15. *For all* $0 < s < r < 2$ *there exists a constant* $c_{r,s} > 0$ *such that*

$$\frac{1}{c_{r,s}}\|T_{\tau,s}\|_{H^s_N}^{\frac{s}{2-s}} \le \|T_{\tau,r}\|_{H^r_N}^{\frac{r}{2-r}} \le c_{r,s}\|T_{\tau,s}\|_{H^s_N}^{\frac{s}{2-s}}.$$

The duality of H^p and H^q, $1 < q < 2$, $\frac{1}{p}+\frac{1}{q} = 1$, gives the following corollary to Theorem 1.15. Recall that the adjoint rearrangement operator on H^p coincides with the inverse rearrangement operator on H^q.

COROLLARY 1.16. *For all* $2 < p < \infty$ *there exists a constant* c_p *such that*

$$\frac{1}{c_p}\|T_{\tau,p}\|_{H^p_N} \le \|T_{\tau^{-1},1}\|_{H^1_N}^{1-\frac{2}{p}} \le c_p\|T_{\tau,p}\|_{H^p_N}.$$

Remark 1.17. Observe that by the above theorem and corollary rearrangement operators $T_{\tau,p}$ on H^p_N, $0 < p < \infty$, induced by any bijective map τ acting on $\mathcal{D}_N$, have the norm estimate

$$\|T_{\tau,p}\|_{H^p_N} \le c_p\,(N+1)^{\left|\frac{1}{p}-\frac{1}{2}\right|}.$$

CHAPTER 2

p-summing multiplication operators

The operators of this chapter are multipliers acting on the Haar system, cf. Section 1.4.2. For $u \in H^p$, $0 < p \le 2$ with Haar expansion $u = \sum_{I\in\mathcal{D}} x_I h_I$ the multiplier $\mathcal{M}_u$ is defined by

$$\mathcal{M}_u : \ell^\infty(\mathcal{D}) \to H^p$$

and

$$\mathcal{M}_u(\varphi) = \sum_{I\in\mathcal{D}} \varphi_I x_I h_I.$$

For convenience we write $\varphi \cdot u = \mathcal{M}_u(\varphi)$. $\mathcal{M}_u$ is a bounded operator and

$$\|\mathcal{M}_u \varphi\|_{H^p} \le \|u\|_{H^p} \sup |\varphi_I|. \tag{2.1}$$

Note that H^p, $0 < p \le 2$, is 2-concave as a sequence space. Hence, by an argument of Pisier ([**Pis79a**]) our multiplication operator is 2-summing, see Section 1.4.2. By the work of Pietsch [**Pie67**] (see Theorem 1.11) $\mathcal{M}_u$ has a Pietsch measure. It is easy to show (Step 2 and 3 in the proof of Theorem 2.4) that the measure has the following form. There exists $\omega = (\omega_I)_{I\in\mathcal{D}}$ with $\omega_I \ge 0$ and $\sum \omega_I \le 1$ such that we have a significant strengthening of the basic estimate (2.1):

$$\|\mathcal{M}_u \varphi\|_{H^p} \le C \|u\|_{H^p} \left(\sum_{I\in\mathcal{D}} |\varphi_I|^2 \omega_I \right)^{\frac{1}{2}}. \tag{2.2}$$

Since $\mathcal{M}_u$ is determined by $u \in H^p$, also the Pietsch measure $(\omega_I)_{I\in\mathcal{D}}$ is given by $u \in H^p$. Note, however, that the existence of Pietsch measures comes from an application of the Hahn-Banach theorem and therefore, Pietsch measures are not given constructively.

In Theorem 2.1 we give explicit formulae for the weights $\omega = (\omega_I)_{I\in\mathcal{D}}$, using as input the Haar coefficients $(x_I)_{I\in\mathcal{D}}$ of $u \in H^p$. We obtain several extensions and variants of the formulae referred to above. These include Triebel-Lizorkin spaces (Section 2.2) vector-valued dyadic Hardy spaces (Section 2.3) and vector-valued Triebel-Lizorkin spaces (Section 2.4).

Multiplier operators, as described above, arise with interpolation and extrapolation of Banach lattices. Here we refer to Pisier's proof in [**Pis79a**] of the equation

$$X = (X_0)^{1-\theta}(L^2)^\theta, \tag{2.3}$$

where X is a q-concave and q'-convex Banach lattice, $\theta = \frac{2}{q}$ and X_0 is Pisier's extrapolation lattice for X and L^2 (see Section 2.5). The equation (2.3) asserts that for $u \in X$ there is $y \in L^2(\Omega, \Sigma, \mu)$ so that

$$\left(|u||y|^{-\theta}\right)^{\frac{1}{1-\theta}} \in X_0. \tag{2.4}$$

In order to obtain $y \in L^2(\Omega, \Sigma, \mu)$ the proof in [**Pis79a**] sets up the following multiplication operator

$$\mathcal{M}_u : L^\infty(\Omega, \Sigma, \mu) \to X$$

by putting

$$\mathcal{M}_u(\varphi)(t) = u(t)\varphi(t), \quad t \in \Omega.$$

Exploiting the work of Maurey [**Mau74a**], Rosenthal [**Ros76**] and Pietsch [**Pie67**], Pisier shows in [**Pis79a**] that there exists a density $\omega \in L^1(\Omega, \Sigma, \mu)$ so that

$$\|\mathcal{M}_u\varphi\|_X \leq C\|u\|_X \left(\int_\Omega |\varphi(t)|^q \omega(t) d\mu(t)\right)^{\frac{1}{q}}.$$

To obtain (2.3) and (2.4) Pisier [**Pis79a**] puts finally

$$y(t) = \omega(t)^{\frac{1}{2}}.$$

We constructively determine for the special Banach lattice $X = H^p$ the density $\omega \in \ell^1(\mathcal{D})$ and therefore have a constructive proof of

$$H^p = (X_0)^{1-\theta}(H^2)^\theta, \tag{2.5}$$

where X_0 is Pisier's extrapolation lattice for H^p and H^2, see Theorem 2.12 in Section 2.5.
More precisely, in Section 2.5 we apply our approach to determine constructively the Calderón product

$$f_p^q = (X_0)^{1-\theta}(f_q^q)^\theta,$$

where X_0 is Pisier's extrapolation lattice for the Triebel-Lizorkin spaces f_p^q and f_q^q. This contains the result (2.5).

2.1. The main theorem - construction of Pietsch measures

The construction of the Pietsch measure of the multiplication operators is the contribution of this section: for multipliers ranging in the Hardy spaces H^p, we are able to find explicit formulae for the Pietsch measure $(\omega_I)_{I\in\mathcal{D}}$. The input for our construction is the atomic decomposition of $u \in H^p$, Theorem 1.8. The output is the equation (2.6) determining ω_I explicitly.

Recall that Theorem 1.8 asserts that $u = \sum x_I h_I \in H^p$, $0 < p \leq 2$, admits an atomic decomposition, that is a triple $(u_i, \mathcal{G}_i, I_i)_{i\in\mathcal{N}}$ so that

$$u = \sum_{i\in\mathcal{N}} u_i \qquad \text{and} \qquad u_i = \sum_{I\in\mathcal{G}_i} x_I h_I$$

satisfy (1.35), explicitly this means

$$\|u\|_{H^p}^p \leq \sum_{i\in\mathcal{N}} \|u_i\|_{H^p}^p \leq \sum_{i\in\mathcal{N}} |I_i|\|\mathbb{S}(u_i)\|_\infty^p \leq A_p\|u\|_{H^p}^p.$$

THEOREM 2.1. *Let $0 < p \le 2$. Let $u \in H^p$ with Haar expansion*

$$u = \sum_{I \in \mathcal{D}} x_I h_I$$

and atomic decomposition $(u_i, \mathcal{G}_i, I_i)_{i \in \mathcal{N}}$. Then the sequence $(\omega_I)_{I \in \mathcal{D}}$, defined by

$$\omega_I = \frac{1}{A_p} \frac{|I_i|^{1-\frac{p}{2}}}{\|u_i\|_2^{2-p}} \frac{|x_I|^2 |I|}{\|u\|_{H^p}^p}, \qquad I \in \mathcal{G}_i, \tag{2.6}$$

satisfies

$$\sum_{I \in \mathcal{D}} \omega_I \le 1 \tag{2.7}$$

and there exists a constant $C_p > 0$ such that for each $\varphi \in \ell^\infty(\mathcal{D})$

$$\|\varphi \cdot u\|_{H^p} \le C_p \|u\|_{H^p} \Big(\sum_{I \in \mathcal{D}} |\varphi_I|^2 \omega_I \Big)^{\frac{1}{2}}. \tag{2.8}$$

PROOF. From

$$\|u\|_{H^p}^p \le \sum_{i \in \mathcal{N}} \|u_i\|_{H^p}^p$$

we get the estimate

$$\|u\|_{H^p}^p \le \sum_{i \in \mathcal{N}} \|u_i\|_2^p |I_i|^{1-\frac{p}{2}}. \tag{2.9}$$

We get from

$$\sum_{i \in \mathcal{N}} |I_i| \|S(u_i)\|_\infty^p \le A_p \|u\|_{H^p}^p$$

that

$$\sum_{i \in \mathcal{N}} \|u_i\|_2^p |I_i|^{1-\frac{p}{2}} \le A_p \|u\|_{H^p}^p. \tag{2.10}$$

This follows from

$$\|u_i\|_2^p |I_i|^{1-\frac{p}{2}} \le \|S u_i\|_\infty^p |I_i|.$$

By (2.9) and equation 1.36 in Remark 1.9 we get for $\varphi \in \ell^\infty(\mathcal{D})$

$$\begin{aligned} \Big\| \sum_{I \in \mathcal{D}} \varphi_I x_I h_I \Big\|_{H^p}^p &= \Big\| \sum_{i \in \mathcal{N}} \sum_{I \in \mathcal{G}_i} \varphi_I x_I h_I \Big\|_{H^p}^p \\ &\le \sum_{i \in \mathcal{N}} \Big\| \sum_{I \in \mathcal{G}_i} \varphi_I x_I h_I \Big\|_2^p |I_i|^{1-\frac{p}{2}} \\ &= \sum_{i \in \mathcal{N}} \Big\| \sum_{I \in \mathcal{G}_i} \varphi_I \frac{x_I}{\|u_i\|_2} h_I \Big\|_2^p \|u_i\|_2^p |I_i|^{1-\frac{p}{2}}. \end{aligned} \tag{2.11}$$

With

$$\Big\| \sum_{I \in \mathcal{G}_i} \varphi_I \frac{x_I}{\|u_i\|_2} h_I \Big\|_2^p = \Big(\sum_{I \in \mathcal{G}_i} \varphi_I^2 \frac{x_I^2}{\|u_i\|_2^2} |I| \Big)^{\frac{p}{2}}$$

we get

$$\Big\| \sum_{I \in \mathcal{D}} \varphi_I x_I h_I \Big\|_{H^p}^p \leq \sum_{i \in \mathcal{N}} \Big(\sum_{I \in \mathcal{G}_i} \varphi_I^2 \frac{x_I^2}{\|u_i\|_2^2} |I| \Big)^{\frac{p}{2}} \|u_i\|_2^p |I_i|^{1-\frac{p}{2}}.$$

Applying Hölder's inequality with $\frac{p}{2} + 1 - \frac{p}{2} = 1$ to

$$\sum_{i \in \mathcal{N}} \Big(\sum_{I \in \mathcal{G}_i} \varphi_I^2 \frac{x_I^2}{\|u_i\|_2^2} |I| \|u_i\|_2^p |I_i|^{1-\frac{p}{2}} \Big)^{\frac{p}{2}} \Big(\|u_i\|_2^p |I_i|^{1-\frac{p}{2}} \Big)^{1-\frac{p}{2}}.$$

we get

$$\Big\| \sum_{I \in \mathcal{D}} \varphi_I x_I h_I \Big\|_{H^p}^p \leq \Big(\sum_{i \in \mathcal{N}} \sum_{I \in \mathcal{G}_i} \varphi_I^2 \frac{x_I^2}{\|u_i\|_2^{2-p}} |I| |I_i|^{1-\frac{p}{2}} \Big)^{\frac{p}{2}} \Big(\sum_{i \in \mathcal{N}} \|u_i\|_2^p |I_i|^{1-\frac{p}{2}} \Big)^{1-\frac{p}{2}}.$$

Applying (2.10) to the second term on the right-hand side we obtain the estimate

$$\begin{aligned} \Big\| \sum_{I \in \mathcal{D}} \varphi_I x_I h_I \Big\|_{H^p}^p &\leq A_p^{1-\frac{p}{2}} \|u\|_{H^p}^{p(1-\frac{p}{2})} \Big(\sum_{i \in \mathcal{N}} \sum_{I \in \mathcal{G}_i} \varphi_I^2 \frac{x_I^2}{\|u_i\|_2^{2-p}} |I| |I_i|^{1-\frac{p}{2}} \Big)^{\frac{p}{2}} \\ &= A_p \|u\|_{H^p}^p \Big(\sum_{i \in \mathcal{N}} \sum_{I \in \mathcal{G}_i} \varphi_I^2 \frac{x_I^2}{\|u_i\|_2^{2-p} \|u\|_{H^p}^p A_p} |I| |I_i|^{1-\frac{p}{2}} \Big)^{\frac{p}{2}}. \end{aligned}$$

Recall

$$\|u_i\|_2^2 = \sum_{I \in \mathcal{G}_i} x_I^2 |I|. \tag{2.12}$$

By (2.10) and (2.12) we obtain for the sequence $(\omega_I)_{I \in \mathcal{D}}$, defined by

$$\omega_I = \frac{|I_i|^{1-\frac{p}{2}}}{A_p \|u\|_{H^p}^p} \frac{|I| x_I^2}{\|u_i\|_2^{2-p}}, \qquad I \in \mathcal{G}_i,$$

the following estimate

$$\begin{aligned} \sum_{I \in \mathcal{D}} \omega_I &= \frac{1}{A_p \|u\|_{H^p}^p} \sum_{i \in \mathcal{N}} \sum_{I \in \mathcal{G}_i} \frac{|I_i|^{1-\frac{p}{2}} |I| x_I^2}{\|u_i\|_2^{2-p}} \\ &= \frac{1}{A_p \|u\|_{H^p}^p} \sum_{i \in \mathcal{N}} |I_i|^{1-\frac{p}{2}} \|u_i\|_2^p \\ &\leq 1. \end{aligned}$$

□

2.2. Extension to Triebel-Lizorkin spaces

We can extend the construction of the Pietsch measure to multiplication operators ranging in the Triebel-Lizorkin spaces f_p^q, $0 < p \leq q < \infty$. Recall that these multiplication operators are given as follows. Let $u = (x_I)_{I\in\mathcal{D}} \in f_p^q$, then the multiplication operator

$$\mathcal{M}_u \colon \ell^\infty \to f_p^q$$

is defined by

$$\mathcal{M}_u(\varphi) = (\varphi_I x_I)_{I\in\mathcal{D}}.$$

For convenience we write $\varphi \cdot u = \mathcal{M}_u(\varphi)$. Recall that f_p^q is the $\frac{q}{2}$-convexification of $H^{\frac{2p}{q}}$, where $\frac{2p}{q} \in (0,2]$, with

$$\|u\|_{f_p^q} = \left\| |u|^{\frac{q}{2}} \right\|_{H^{\frac{2p}{q}}}^{\frac{2}{q}}. \tag{2.13}$$

Therefore, we have for $\varphi \in \ell^\infty(\mathcal{D})$

$$\|\varphi \cdot u\|_{f_p^q} = \left\| |\varphi|^{\frac{q}{2}} \cdot |u|^{\frac{q}{2}} \right\|_{H^{\frac{2p}{q}}}^{\frac{2}{q}}. \tag{2.14}$$

We get from (2.2), (2.13) and (2.14) the following statement: Let $0 < p \leq q < \infty$. For all $u = (x_I)_{I\in\mathcal{D}} \in f_p^q$ there exists a non-negative sequence $(\omega_I)_{I\in\mathcal{D}}$ so that $\sum \omega_I \leq 1$ and for all $\varphi \in \ell^\infty(\mathcal{D})$

$$\|\varphi \cdot u\|_{f_p^q} \leq C\|u\|_{f_p^q} \left(\sum_{I\in\mathcal{D}} |\varphi_I|^q \omega_I \right)^{\frac{1}{q}}. \tag{2.15}$$

We are able to give an explicit formula for $(\omega_I)_{I\in\mathcal{D}}$ using again (2.13) and (2.14). Let $(u_i, \mathcal{G}_i, I_i)_{i\in\mathcal{N}}$ be the atomic decomposition of $u \in f_p^q$. Then $(|u_i|^{\frac{q}{2}}, \mathcal{G}_i, I_i)_{i\in\mathcal{N}}$ is the atomic decomposition of $|u|^{\frac{q}{2}} \in H^{\frac{2p}{q}}$. Theorem 2.1 asserts that there exists a constant $C_{\frac{2p}{q}}$ such that

$$\left\| |\varphi|^{\frac{q}{2}} \cdot |u|^{\frac{q}{2}} \right\|_{H^{\frac{2p}{q}}} \leq C_{\frac{2p}{q}} \Big(\sum_{I\in\mathcal{D}} |\varphi_I|^q \omega_I \Big)^{\frac{1}{2}} \left\| |u|^{\frac{q}{2}} \right\|_{H^{\frac{2p}{q}}}, \tag{2.16}$$

where

$$\omega_I = \frac{1}{A_{\frac{2p}{q}}} \frac{|I_i|^{1-\frac{p}{q}}}{\left\| |u_i|^{\frac{q}{2}} \right\|_2^{2-\frac{2p}{q}}} \frac{|x_I|^q |I|}{\left\| |u|^{\frac{q}{2}} \right\|_{H^{\frac{2p}{q}}}^{\frac{2p}{q}}}, \qquad I \in \mathcal{G}_i.$$

Summarizing we get the following statement for Triebel-Lizorkin spaces as corollary of Theorem 2.1:

Corollary 2.2. *Let $0 < p \leq q < \infty$. Let $u = (x_I)_{I\in\mathcal{D}} \in f_p^q$ with atomic decomposition $(u_i, \mathcal{G}_i, I_i)_{i\in\mathcal{N}}$. Then the sequence $(\omega_I)_{I\in\mathcal{D}}$, defined by*

$$\omega_I = \frac{1}{A_{\frac{2p}{q}}} \frac{|I_i|^{1-\frac{p}{q}}}{\|u_i\|_{f_q^q}^{q-p}} \frac{|x_I|^q |I|}{\|u\|_{f_p^q}^p}, \qquad I \in \mathcal{G}_i, \tag{2.17}$$

satisfies

$$\sum_{I\in\mathcal{D}} \omega_I \le 1 \tag{2.18}$$

and there exists a constant $C_{p,q} > 0$ *such that for each* $\varphi \in \ell^\infty(\mathcal{D})$

$$\|\varphi \cdot u\|_{f_p^q} \le C_{p,q}\|u\|_{f_p^q}\Big(\sum_{I\in\mathcal{D}} |\varphi_I|^q \omega_I\Big)^{\frac{1}{q}}, \tag{2.19}$$

where

$$\varphi \cdot u = (\varphi_I x_I)_{I\in\mathcal{D}}.$$

2.3. Extension to vector-valued Hardy spaces

Let X be a Banach space. Fix a sequence $(x_I)_{I\in\mathcal{D}}$ in X, then for $u = \sum_{I\in\mathcal{D}} x_I h_I \in H^p_X$ we define the multiplication operator

$$\mathcal{M}_u : \ell^\infty(\mathcal{D}) \to \overline{\operatorname{span}}\{x_I h_I : I \in \mathcal{D}\} \subseteq H^p_X,$$

by

$$\mathcal{M}_u(\varphi) = \sum_{I\in\mathcal{D}} \varphi_I x_I h_I.$$

By Kahane's contraction principle (1.2), for fixed $(x_I)_{I\in\mathcal{D}}$, the sequence $(x_I h_I)_{I\in\mathcal{D}}$ is an unconditional basic sequence in H^p_X. This remark links the present work on vector-valued H^p_X spaces with the lattices of the previous sections.

We obtain the following statement on Pietsch measures of multiplication operators ranging in the vector-valued Hardy spaces as an application of Maurey's Theorem, Pietsch's Theorem and the atomic decomposition in H^p_X. The atomic decomposition works as extrapolation tool, transferring the Pietsch measure of multiplication operators into H^2_X to Pietsch measures for multiplication operators into H^p_X, $0 < p < 2$. Here the result is only partially constructive. The formulae for the Pietsch measures for atoms are obtained by applying Theorem 2.4 below invoking Maurey's theorem and the abstract version of Pietsch's theorem.

Theorem 2.3. *Let X be a Banach space of cotype r, $2 \le r < \infty$. Let $0 < p \le 2$ and let $u \in H^p_X$ with Haar expansion*

$$u = \sum_{I\in\mathcal{D}} x_I h_I$$

and atomic decomposition $(u_i, \mathcal{G}_i, I_i)_{i\in\mathcal{N}}$ (see Theorem 1.8). Then for each $i \in \mathcal{N}$ there exists a $\mu^{(i)} \in \ell^1(\mathcal{G}_i)$ with

$$\mu^{(i)}_I \ge 0, \quad \textit{for all } I \in \mathcal{G}_i$$

and

$$\sum_{I\in\mathcal{G}_i} \mu^{(i)}_I = 1$$

so that the following holds: The sequence $(\omega_I)_{I\in\mathcal{D}}$, *defined by*

$$\omega_I = \frac{\|u_i\|_{H^2_X}^p |I_i|^{1-\frac{p}{2}}}{A_p\|u\|_{H^p_X}^p}\mu_I^{(i)}, \qquad I \in \mathcal{G}_i, \tag{2.20}$$

satisfies

$$\sum_{I\in\mathcal{D}} \omega_I \le 1 \tag{2.21}$$

and for all $s > r$ *there exists a constant* $K_{s,r,p} > 0$ *such that for each* $\varphi \in \ell^\infty(\mathcal{D})$

$$\|\varphi \cdot u\|_{H^p_X} \le K_{s,r,p} C_r(X)\|u\|_{H^p_X}\Big(\sum_{I\in\mathcal{D}} |\varphi_I|^s \omega_I\Big)^{\frac{1}{s}}. \tag{2.22}$$

We point out that the exponent s in (2.22) is determined by the cotype of X alone. In particular it is not depending on $0 < p \le 2$.

For the proof of Theorem 2.3 we need the following theorem for vector-valued multiplication operators, which follows directly from Maurey's theorem (Theorem 1.12) and from Pietsch's domination theorem (Theorem 1.11).

THEOREM 2.4. *Let* X *be a Banach space of non-trivial cotype* r, $2 \le r < \infty$. *For all* $s > r$ *there exists a constant* $K_{s,r} > 0$ *such that the following holds. For all* $f \in H^2_X$ *with Haar expansion*

$$f = \sum_{I\in\mathcal{D}} f_I h_I, \qquad f_I \in X,$$

there exists a $\mu \in \ell^1(\mathcal{D})$ *(not depending on* s*), with*

$$\mu_I \ge 0, \qquad \text{for all } I \in \mathcal{D}$$

and

$$\sum_{I\in\mathcal{D}} \mu_I = 1$$

such that for each $\varphi \in \ell^\infty(\mathcal{D})$

$$\|\varphi \cdot f\|_{H^2_X} \le K_{s,r} C_r(X)\|f\|_{H^2_X}\Big(\sum_{I\in\mathcal{D}} |\varphi_I|^s \mu_I\Big)^{\frac{1}{s}}, \tag{2.23}$$

where $C_r(X)$ *is the cotype-*r *constant of* X *and*

$$\varphi \cdot f = \sum_{I\in\mathcal{D}} \varphi_I f_I h_I.$$

PROOF. Recall Section 1.4.3. Kahane's inequality (1.1) yields that H^2_X is of cotype r with $C_r(H^2_X) \le C_r(X)$. Let $f \in H^2_X$. We consider the multiplication operator

$$\mathcal{M}_f : \ell^\infty(\mathcal{D}) \to H^2_X, \varphi \mapsto \varphi \cdot f$$

which is bounded and $\|\mathcal{M}_f\| = \|f\|_{H^2_X}$.

Step 1: We assume that f has finite Haar support $\mathcal{D}' \subset \mathcal{D}$. Then, applying Maurey's theorem (Theorem 1.12) to the multiplication operator $\mathcal{M}_f \in L(\ell^\infty(\mathcal{D}'), H^2_X)$ we obtain that $\mathcal{M}_f$ is s-summing for all $s > r$ and

$$\pi_s(\mathcal{M}_f) \leq K_{s,r} C_r(X) \|\mathcal{M}_f\| \leq K_{s,r} C_r(X) \|f\|_{H^2_X}. \tag{2.24}$$

Since $\mathcal{D}'$ is finite, we can apply Pietsch's factorization theorem (Theorem 1.11) and get the following. There exists a $\mu \in \ell^1(\mathcal{D}')$ with $\mu_I \geq 0$ for all $I \in \mathcal{D}'$ and $\sum_{I \in \mathcal{D}'} \mu_I = 1$ such that for each $\varphi \in \ell^\infty(\mathcal{D})$

$$\|\mathcal{M}_f(\varphi)\|_{H^2_X} \leq \pi_s(\mathcal{M}_f) \Big(\sum_{I \in \mathcal{D}'} |\varphi_I|^s \mu_I \Big)^{\frac{1}{s}}. \tag{2.25}$$

Step 2: Let $f \in H^2_X$ with its formal Haar series $f = \sum_{I \in \mathcal{D}} x_I h_I$, where the partial sums are listed according to the lexicographic order in $\mathcal{D}$. The conditional expectation $\mathbb{E}_N$ with respect to the σ-algebra generated by the set $\{I \in \mathcal{D} : |I| = 2^{-N}\}$ is given by

$$\mathbb{E}_N(f) = \sum_{I \in \mathcal{D}_{N-1}} x_I h_I.$$

Then we have

$$f = \mathbb{E}_N(f) + f - \mathbb{E}_N(f) \tag{2.26}$$

and for each $\varepsilon > 0$ there exists $N(\varepsilon) \in \mathbb{N}$ such that for all $N \geq N(\varepsilon)$

$$\|\mathbb{E}_N(f) - f\|_{H^2_X} \leq \varepsilon. \tag{2.27}$$

To confirm equation (2.27) we exploit the finite cotype of X and invoke Kwapien's theorem ([**DJT95**, p.255]) and Hoffmann-Jørgensen's theorem ([**DJT95**, Theorem 12.3]). We consider $f = (x_I)_{I \in \mathcal{D}}$ with its formal Haar series $f = \sum_{I \in \mathcal{D}} x_I h_I$ and define a sequence of independent, symmetric random variables with values in the Bochner-Lebesgue space $L^2_X([0,1], dt)$, cf. [**DU77**]. Let $I \in \mathcal{D}$ and

$$R_I : [0,1] \to L^2_X, \quad s \mapsto \Big(t \mapsto x_I h_I(t) r_I(s) \Big),$$

where $(r_I)_{I \in \mathcal{D}}$ is an enumeration of the Rademacher system. Since $f = \sum x_I h_I \in H^2_X$ we have that

$$\sup_{N \in \mathbb{N}} \left(\int_0^1 \int_0^1 \Big\| \sum_{I \in \mathcal{D}_N} x_I h_I(t) r_I(s) \Big\|^2_X dt ds \right)^{\frac{1}{2}} < \infty.$$

Therefore, the partial sums of random variables $\sum_{I \in \mathcal{D}_N} R_I$ are bounded in the Bochner-Lebesgue space[1] $L^2_Z([0,1], ds)$, where $Z = L^2_X([0,1], dt)$. X has finite cotype, hence L^2_X has finite cotype and Kwapien's theorem yields that there exists a limit of the partial sums given by

$$R(s) := \lim_{N \to \infty} \sum_{I \in \mathcal{D}_N} R_I(s), \quad \text{a.e.}$$

[1]For definition see e.g. [**DU77**]

Hoffmann-Jørgensen's theorem asserts that the partial sums $\sum_{I\in\mathcal{D}_N} R_I$ converge in $L^2_Z([0,1],ds)$, i.e.

$$\lim_{N\to\infty}\Big\|R-\sum_{I\in\mathcal{D}_N} R_I\Big\|_{L^2_Z}=0,$$

and hence (2.27) holds. Note that we identified $f=(x_I)_{I\in\mathcal{D}}$ with the convergent series $R=\sum_{I\in\mathcal{D}} x_I h_I\otimes r_I\in L^2_Z$, where $Z=L^2_X$.

Iterating (2.26) and (2.27) there exists a monotonically increasing sequence $(N_i)_{i\geq 0}$ of natural numbers and a sequence $(f_i)_{i\geq 0}$ in H^2_X given by

$$f_0=\mathbb{E}_{N_0}(f),\quad f_i=\mathbb{E}_{N_i}(f)-\mathbb{E}_{N_{i-1}}(f),\ i\geq 1 \tag{2.28}$$

such that

$$\|f_i\|_{H^2_X}\leq 4^{-i}\|f\|_{H^2_X}. \tag{2.29}$$

From the construction of the sequence $(f_i)_{i\geq 0}$ we get that each f_i has finite Haar support $D_i\subset\mathcal{D}$.

Step 3: We apply Step 1 to the sequence $(f_i)_{i\geq 0}$. There exists a sequence $(\mu^i)_{i\geq 0}$, $\mu^i\in\ell^1(D_i)$ with $\mu^i_I\geq 0$ for all $I\in D_i$ and $\sum_{I\in D_i}\mu^i_I=1$ such that for each $\varphi\in\ell^\infty(\mathcal{D})$

$$\|\mathcal{M}_{f_i}(\varphi)\|_{H^2_X}\leq\pi_s(\mathcal{M}_{f_i})\Big(\sum_{I\in D_i}|\varphi_I|^s\mu^i_I\Big)^{\frac{1}{s}}. \tag{2.30}$$

Step 2 yields $f=\sum_{i=0}^\infty f_i$. Therefore,

$$\|\mathcal{M}_f(\varphi)\|_{H^2_X}\leq\sum_{i=0}^\infty\|\mathcal{M}_{f_i}(\varphi)\|_{H^2_X}\leq\sum_{i=0}^\infty\pi_s(\mathcal{M}_{f_i})\Big(\sum_{I\in D_i}|\varphi_I|^s\mu^i_I\Big)^{\frac{1}{s}}.$$

Using (2.24) and (2.29) yields

$$\begin{aligned}\|\mathcal{M}_f(\varphi)\|_{H^2_X}&\leq K_{s,r}C_r(X)\sum_{i=0}^\infty\|f_i\|_{H^2_X}\Big(\sum_{I\in D_i}|\varphi_I|^s\mu^i_I\Big)^{\frac{1}{s}}\\&\leq K_{s,r}C_r(X)\|f\|_{H^2_X}\sum_{i=0}^\infty 4^{-i}\Big(\sum_{I\in D_i}|\varphi_I|^s\mu^i_I\Big)^{\frac{1}{s}}.\end{aligned}$$

Let $\frac{1}{s} + \frac{1}{s'} = 1$. Hölder's inequality yields

$$\begin{aligned}\|\mathcal{M}_f(\varphi)\|_{H^2_X} &\le K_{s,r} C_r(X) \|f\|_{H^2_X} \left(\sum_{i=0}^{\infty} 2^{-is'}\right)^{\frac{1}{s'}} \left(\sum_{i=0}^{\infty} 2^{-is} \sum_{I \in D_i} |\varphi_I|^s \mu_I^i\right)^{\frac{1}{s}} \\ &\le \overline{K}_{s,r} C_r(X) \|f\|_{H^2_X} \left(\sum_{i=0}^{\infty} 2^{-i} \sum_{I \in D_i} |\varphi_I|^s \mu_I^i\right)^{\frac{1}{s}} \\ &= \overline{K}_{s,r} C_r(X) \|f\|_{H^2_X} \left(\sum_{I \in \mathcal{D}} |\varphi_I|^s \sum_{i=0}^{\infty} 2^{-i} 1_{D_i}(I) \mu_I^i\right)^{\frac{1}{s}} \\ &= \overline{K}_{s,r} C_r(X) \|f\|_{H^2_X} \left(\sum_{I \in \mathcal{D}} |\varphi_I|^s \nu_I\right)^{\frac{1}{s}},\end{aligned}$$

where $\nu_I = \sum_{i=0}^{\infty} 2^{-i} 1_{D_i}(I) \mu_I^i$ satisfies $\nu_I \ge 0$ for all $I \in \mathcal{D}$ and

$$\sum_{I \in \mathcal{D}} \nu_I = \sum_{i=0}^{\infty} 2^{-i} \sum_{I \in D_i} \mu_I^i = \sum_{i=0}^{\infty} 2^{-i} = 2.$$

□

Remark 2.5. The proof above has a direct extension to H^p_X, $1 \le p < \infty$, where again the measure $\mu \in \ell^1(\mathcal{D})$ has its origin in the abstract version of Pietsch's theorem.

Remark 2.6. If X is of cotype 2 then we know from Theorem 1.13 that the statement of Theorem 2.4 is valid for $s \ge 2$. Especially it is valid for $s = 2$.

Proof of Theorem 2.3. Recall the inequalities in (1.35) from the atomic decomposition in Theorem 1.8. We have with

$$\|u\|^p_{H^p_X} \le \frac{1}{a_p} \sum_{i \in \mathcal{N}} \|u_i\|^p_{H^p_X}$$

the following estimate

$$\|u\|^p_{H^p_X} \le \frac{1}{a_p} \sum_{i \in \mathcal{N}} \|u_i\|^p_{H^2_X} |I_i|^{1-\frac{p}{2}}. \tag{2.31}$$

We also get from

$$\sum_{i \in \mathcal{N}} |I_i| \|\mathbb{S}(u_i)\|^p_\infty \le A_p \|u\|^p_{H^p_X}$$

that

$$\sum_{i \in \mathcal{N}} \|u_i\|^p_{H^2_X} |I_i|^{1-\frac{p}{2}} \le A_p \|u\|^p_{H^p_X}. \tag{2.32}$$

This follows from

$$\|u_i\|^p_{H^2_X} |I_i|^{1-\frac{p}{2}} \le \|\mathbb{S} u_i\|^p_\infty |I_i|.$$

By (2.31) and Remark 1.9 we get for $\varphi \in \ell^\infty(\mathcal{D})$

$$
\begin{aligned}
\Big\| \sum_{I\in\mathcal{D}} \varphi_I x_I h_I \Big\|_{H^p_X}^p &= \Big\| \sum_{i\in\mathcal{N}} \sum_{I\in\mathcal{G}_i} \varphi_I x_I h_I \Big\|_{H^p_X}^p \\
&\le \frac{1}{a_p} \sum_{i\in\mathcal{N}} \Big\| \sum_{I\in\mathcal{G}_i} \varphi_I x_I h_I \Big\|_{H^2_X}^p |I_i|^{1-\frac{p}{2}} \\
&= \frac{1}{a_p} \sum_{i\in\mathcal{N}} \Big\| \sum_{I\in\mathcal{G}_i} \varphi_I \frac{x_I}{\|u_i\|_{H^2_X}} h_I \Big\|_{H^2_X}^p \|u_i\|_{H^2_X}^p |I_i|^{1-\frac{p}{2}}.
\end{aligned} \tag{2.33}
$$

We apply Theorem 2.4 with the specification $f = u_i$. Recall that $u_i = \sum_{I\in\mathcal{G}_i} x_I h_I$. Therefore, we obtain the following statement: For all $s > r$ there exists a constant $K_{s,r}$ such that

$$
\Big\| \sum_{I\in\mathcal{G}_i} \varphi_I \frac{x_I}{\|u_i\|_{H^2_X}} h_I \Big\|_{H^2_X}^p \le K_{s,r}^p C_r(X)^p \Big(\sum_{I\in\mathcal{G}_i} |\varphi_I|^s \mu_I^{(i)} \Big)^{\frac{p}{s}},
$$

where $\mu_i \in \ell^1(\mathcal{G}_i)$ with $\mu_I^{(i)} \ge 0$ for all $I \in \mathcal{D}$ and $\sum_{I\in\mathcal{G}_i} \mu_I^{(i)} = 1$. Summing up we get from Maurey's theorem (in particular Theorem 2.4) the following statement

$$
\Big\| \sum_{I\in\mathcal{D}} \varphi_I x_I h_I \Big\|_{H^p_X}^p \le \frac{K_{s,r}^p C_r(X)^p}{a_p} \sum_{i\in\mathcal{N}} \Big(\sum_{I\in\mathcal{G}_i} |\varphi_I|^s \mu_I^{(i)} \Big)^{\frac{p}{s}} \|u_i\|_{H^2_X}^p |I_i|^{1-\frac{p}{2}}. \tag{2.34}
$$

We rewrite the sum on the right-hand side in an appropriate way and apply Hölder's inequality with $\frac{p}{s} + 1 - \frac{p}{s} = 1$:

$$
\begin{aligned}
&\sum_{i\in\mathcal{N}} \Big(\sum_{I\in\mathcal{G}_i} |\varphi_I|^s \mu_I^{(i)} \Big)^{\frac{p}{s}} \|u_i\|_{H^2_X}^p |I_i|^{1-\frac{p}{2}} \\
&\quad = \sum_{i\in\mathcal{N}} \Big(\sum_{I\in\mathcal{G}_i} |\varphi_I|^s \|u_i\|_{H^2_X}^p |I_i|^{1-\frac{p}{2}} \mu_I^{(i)} \Big)^{\frac{p}{s}} \Big(\|u_i\|_{H^2_X}^p |I_i|^{1-\frac{p}{2}} \Big)^{1-\frac{p}{s}} \\
&\quad \le \Big(\sum_{i\in\mathcal{N}} \sum_{I\in\mathcal{G}_i} |\varphi_I|^s \|u_i\|_{H^2_X}^p |I_i|^{1-\frac{p}{2}} \mu_I^{(i)} \Big)^{\frac{p}{s}} \Big(\sum_{i\in\mathcal{N}} \|u_i\|_{H^2_X}^p |I_i|^{1-\frac{p}{2}} \Big)^{1-\frac{p}{s}}.
\end{aligned} \tag{2.35}
$$

By (2.32) we get an estimate for the second term on the right-hand side and therefore

$$\begin{aligned}&\sum_{i\in\mathcal{N}}\Big(\sum_{I\in\mathcal{G}_i}|\varphi_I|^s\mu_I^{(i)}\Big)^{\frac{p}{s}}\|u_i\|^p_{H^2_X}|I_i|^{1-\frac{p}{2}}\\ &\le\Big(\sum_{i\in\mathcal{N}}\sum_{I\in\mathcal{G}_i}|\varphi_I|^s\|u_i\|^p_{H^2_X}|I_i|^{1-\frac{p}{2}}\mu_I^{(i)}\Big)^{\frac{p}{s}}A_p^{1-\frac{p}{s}}\|u\|^{p(1-\frac{p}{s})}_{H^p_X}\\ &=A_p\|u\|^p_{H^p_X}\Big(\sum_{i\in\mathcal{N}}\sum_{I\in\mathcal{G}_i}|\varphi_I|^s\frac{\|u_i\|^p_{H^2_X}|I_i|^{1-\frac{p}{2}}}{A_p\|u\|^p_{H^p_X}}\mu_I^{(i)}\Big)^{\frac{p}{s}}.\end{aligned}\tag{2.36}$$

Combining (2.34) and (2.36) yields

$$\Big\|\sum_{I\in\mathcal{D}}\varphi_Ix_Ih_I\Big\|_{H^p_X}\le K_{s,r,p}C_r(X)\|u\|_{H^p_X}\Big(\sum_{i\in\mathcal{N}}\sum_{I\in\mathcal{G}_i}|\varphi_I|^s\frac{\|u_i\|^p_{H^2_X}|I_i|^{1-\frac{p}{2}}}{A_p\|u\|^p_{H^p_X}}\mu_I^{(i)}\Big)^{\frac{1}{s}},$$

where $K_{s,r,p}$ is dependent on the constant $K_{s,r}$ from Theorem 2.4 and the constants a_p, A_p from the atomic decomposition (Theorem 1.8). Now we set

$$\omega_I=\frac{\|u_i\|^p_{H^2_X}|I_i|^{1-\frac{p}{2}}}{A_p\|u\|^p_{H^p_X}}\mu_I^{(i)},\qquad I\in\mathcal{G}_i\tag{2.37}$$

and obtain

$$\Big\|\sum_{I\in\mathcal{D}}\varphi_Ix_Ih_I\Big\|_{H^p_X}\le C_r(X)K_{s,r,p}\|u\|_{H^p_X}\Big(\sum_{I\in\mathcal{D}}|\varphi_I|^s\omega_I\Big)^{\frac{1}{s}}.$$

The sequence $(\omega_I)_{I\in\mathcal{D}}$, defined by (2.37), satisfies

$$\sum_{I\in\mathcal{D}}\omega_I=\frac{1}{A_p\|u\|^p_{H^p_X}}\sum_{i\in\mathcal{N}}\|u_i\|^p_{H^2_X}|I_i|^{1-\frac{p}{2}}\sum_{I\in\mathcal{G}_i}\mu_I^{(i)}.$$

Recall that μ_i is a probability measure on $\mathcal{G}_i$, thus we get from (2.32)

$$\begin{aligned}\sum_{I\in\mathcal{D}}\omega_I&=\frac{1}{A_p\|u\|^p_{H^p_X}}\sum_{i\in\mathcal{N}}\|u_i\|^p_{H^2_X}|I_i|^{1-\frac{p}{2}}\\ &\le 1.\end{aligned}$$

□

REMARK 2.7. If X is of cotype 2 then the statement of Theorem 2.3 is valid for all $s\ge 2$ including $s=2$, cf. Theorem 1.13 and the Remark 2.6.

2.4. Extension to vector-valued Triebel-Lizorkin spaces

Analogously to the previous section we can extend the main theorem (Theorem 2.1) to vector-valued Triebel-Lizorkin spaces $f^q_p(X)$, where X is some Banach space of non-trivial cotype. Let $u=(x_I)_{I\in\mathcal{D}}\in f^q_p(X)$. Then

$$\mathcal{M}_u\colon \ell^\infty(\mathcal{D})\to f^q_p(X)$$

is defined by

$$\mathcal{M}_u(\varphi) = (\varphi_I x_I)_{I\in\mathcal{D}}.$$

THEOREM 2.8. *Let X be a Banach space of non-trivial cotype r_0. Let $0 < p \leq s < q < 2$ and let $u = (x_I)_{I\in\mathcal{D}} \in f_p^q(X)$ with atomic decomposition $(u_i, \mathcal{G}_i, I_i)_{i\in\mathcal{N}}$, cf. Remark 1.10. Then for each $i \in \mathcal{N}$ there exists a $\mu^{(i)} \in \ell^1(\mathcal{G}_i)$ with*

$$\mu^{(i)}(I) \geq 0, \quad \text{for all } I \in \mathcal{G}_i,$$

and

$$\sum_{I\in\mathcal{G}_i} \mu^{(i)}(I) = 1$$

so that the following holds: the sequence $(\omega(I))_{I\in\mathcal{D}}$, defined by

$$\omega(I) = \frac{\|u_i\|^p_{f_s^q(X)} |I_i|^{1-\frac{p}{s}}}{A_{p,q}\|u\|^p_{f_p^q(X)}} \mu^{(i)}(I), \qquad I \in \mathcal{G}_i, \tag{2.38}$$

satisfies

$$\sum_{I\in\mathcal{D}} \omega(I) \leq 1 \tag{2.39}$$

and for all $r > r_0$ there exists a constant $K_{r,r_0,p} > 0$ such that for each $\varphi \in \ell^\infty(\mathcal{D})$

$$\|\varphi \cdot u\|_{f_p^q(X)} \leq K_{r,r_0,p} C_{r_0}(X) \|u\|_{f_p^q(X)} \left(\sum_{I\in\mathcal{D}} |\varphi_I|^r \omega(I) \right)^{\frac{1}{r}} \tag{2.40}$$

where $C_{r_0}(X)$ is the cotype-r_0 constant of X and $\varphi \cdot f = (\varphi_I x_I)_{I\in\mathcal{D}}$.

For the proof of Theorem 2.8 we need the following theorem for multiplication operators ranging in the vector-valued Triebel-Lizorkin spaces, which gives an analogous statement as in Theorem 2.4.

THEOREM 2.9. *Let $0 < s < q < 2$. Let X be a Banach space of non-trivial cotype r_0. For all $r > r_0$ there exists a constant $K_{r,r_0} > 0$ and for all $f = (x_I)_{I\in\mathcal{D}} \in f_s^q(X)$ there exists a $\mu \in \ell^1(\mathcal{D})$ with*

$$\mu(I) \geq 0, \quad \text{for all } I \in \mathcal{D}$$

and

$$\sum_{I\in\mathcal{D}} \mu(I) = 1,$$

such that for each $\varphi \in \ell^\infty(\mathcal{D})$

$$\|\varphi \cdot f\|_{f_s^q(X)} \leq K_{r,r_0} C_{r_0}(X) \|f\|_{f_s^q(X)} \left(\sum_{I\in\mathcal{D}} |\varphi_I|^r \mu(I) \right)^{\frac{1}{r}}, \tag{2.41}$$

where $C_{r_0}(X)$ is the cotype-r_0 constant of X and

$$\varphi \cdot f = (\varphi_I x_I)_{I\in\mathcal{D}}.$$

The proof is an adaption of the proof of Theorem 2.4. Therefore, we give only an outline.

PROOF SKETCH. We identify $f = (x_I)_{I\in\mathcal{D}} \in f_p^q(X)$ with its formal Haar series $\sum_{I\in\mathcal{D}} x_I h_I$ and consider the multiplication operator

$$\mathcal{M}_f : \ell^\infty(\mathcal{D}) \to f_p^q(X), \varphi \mapsto \sum_{I\in\mathcal{D}} \varphi_I x_I h_I.$$

Step 1: We suppose that f has finite Haar support $\mathcal{D}' \subset \mathcal{D}$. Analogously to Step 1 in the proof of Theorem 2.4 we get that there exists a $\mu \in \ell^1(\mathcal{D}')$ with $\mu_I \geq 0$ for all $I \in \mathcal{D}'$ and $\sum_{I\in\mathcal{D}'} \mu_I = 1$ such that for each $\varphi \in \ell^\infty(\mathcal{D})$

$$\|\mathcal{M}_f(\varphi)\|_{f_p^q(X)} \leq \pi_s(\mathcal{M}_f)\left(\sum_{I\in\mathcal{D}'} |\varphi_I|^r \mu_I\right)^{\frac{1}{r}}. \tag{2.42}$$

We know from equation (1.40) that

$$\pi_r(\mathcal{M}_u) \leq K_{r,r_0} C_{r_0}(X) \|u\|_{f_p^q(X)}.$$

Step 2: Let $f_N = (x_I)_{I\in\mathcal{D}_N}$ with corresponding Haar expansion $f_N = \sum_{I\in\mathcal{D}_N} x_I h_I$. Then we have

$$f = f_N + f - f_N \tag{2.43}$$

and for each $\varepsilon > 0$ there exists $N(\varepsilon) \in \mathbb{N}$ such that for all $N \geq N(\varepsilon)$

$$\|f_N - f\|_{f_p^q(X)} \leq \varepsilon. \tag{2.44}$$

To confirm equation (2.44) we rebuild the argument in the proof of Theorem 2.4. We define a sequence of independent, symmetric random variables with values in the Bochner-Lebesgue space $L_X^2([0,1], dt)$. Let $I \in \mathcal{D}$ and

$$R_I : \Omega \to L_X^s, \quad \omega \mapsto \Big(t \mapsto x_I h_I(t) \theta_I(\omega)\Big),$$

where $(\theta_I)_{I\in\mathcal{D}}$ is an independent sequence of q-stable random variables on the probability space (Ω, Σ, μ). Since $f = \sum_{I\in D} x_I h_I \in f_p^q(X)$ we have that

$$\sup_{N\in\mathbb{N}} \left(\int_\Omega \int_0^1 \Big\| \sum_{I\in\mathcal{D}_N} x_I h_I(t) \theta_I(\omega) \Big\|_X^s dt\, d\mathbb{P}(\omega) \right)^{\frac{1}{s}} < \infty.$$

Therefore, the partial sums $\sum_{I\in\mathcal{D}_N} R_I$ are bounded in the Bochner-Lebesgue space[2] $L_Z^s([0,1], ds)$, where $Z = L_X^s([0,1], dt)$. Analogously to Step 2 in the proof of Theorem 2.4 we exploit the finite cotype of X and invoke Kwapien's theorem ([**DJT95**, p.255]) and Hoffmann-Jørgensen's theorem ([**DJT95**, Theorem 12.3]) to obtain (2.44). Following Step 3 in the proof of Theorem 2.4 yields the statement. □

REMARK 2.10. If X is of cotype 2 then we know from Theorem 1.13 that the statement of Theorem 2.9 is valid for $r_0 \geq 2$. Especially it is valid for $r_0 = 2$.

[2]For definition see e.g. [**DU77**]

Proof of Theorem 2.8. Recall Remark 1.10. The inequalities (1.35) from the atomic decomposition are for $u \in f_p^q(X)$ given by

$$a_{p,q}\|u\|^p_{f_p^q(X)} \le \sum_{i\in\mathcal{N}} \|u_i\|^p_{f_p^q(X)} \le \sum_{i\in\mathcal{N}} |I_i|\|\mathbb{S}(u_i)\|^p_\infty \le A_{p,q}\|u\|^p_{f_p^q(X)}. \tag{2.45}$$

The left-hand side inequality in (2.45) implies by the application of the Hölder inequality with $1 = \frac{p}{s} + (1-\frac{p}{s})$ the estimate

$$\|u\|^p_{f_p^q(X)} \le \frac{1}{a_{p,q}} \sum_{i\in\mathcal{N}} \|u_i\|^p_{f_s^q(X)} |I_i|^{1-\frac{p}{s}}. \tag{2.46}$$

By the right-hand side inequality in (2.45) we obtain the following estimate

$$\sum_{i\in\mathcal{N}} \|u_i\|^p_{f_s^q(X)} |I_i|^{1-\frac{p}{s}} \le A_{p,q}\|u\|^p_{f_p^q(X)}. \tag{2.47}$$

This follows from the inequality

$$\|u_i\|^p_{f_s^q(X)} |I_i|^{1-\frac{p}{s}} \le \|\mathbb{S}_q u_i\|^p_\infty |I_i|.$$

From (2.46) and equation (1.36) we get for $\varphi \in \ell^\infty(\mathcal{D})$

$$\begin{aligned} \|\varphi\cdot u\|^p_{f_p^q(X)} &= \Big\| \sum_{i\in\mathcal{N}} (\varphi_I x_I)_{I\in\mathcal{G}_i} \Big\|^p_{f_p^q(X)} \\ &\le \frac{1}{a_{p,q}} \sum_{i\in\mathcal{N}} \|(\varphi_I x_I)_{I\in\mathcal{G}_i}\|^p_{f_s^q(X)} |I_i|^{1-\frac{p}{s}} \\ &= \frac{1}{a_{p,q}} \sum_{i\in\mathcal{N}} \Big\| \Big(\varphi_I \frac{x_I}{\|u_i\|_{f_s^q(X)}}\Big)_{I\in\mathcal{G}_i} \Big\|^p_{f_s^q(X)} \|u_i\|^p_{f_s^q(X)} |I_i|^{1-\frac{p}{s}}. \end{aligned} \tag{2.48}$$

We apply Theorem 2.9 with the specification $f = \frac{u_i}{\|u_i\|} \in f_s^q(X)$. Recall that $u_i = (x_I)_{I\in\mathcal{G}_i}$. Therefore, we obtain the following statement: for all $r > r_0$ there exists a constant K_{r,r_0} such that

$$\Big\| \Big(\varphi_I \frac{x_I}{\|u_i\|_{f_s^q(X)}}\Big)_{I\in\mathcal{G}_i} \Big\|^p_{f_s^q(X)} \le K^p_{r,r_0} C_{r_0}(X)^p \Big(\sum_{I\in\mathcal{G}_i} |\varphi_I|^r \mu^{(i)}(I) \Big)^{\frac{p}{r}},$$

where $\mu^{(i)} \in \ell^1(\mathcal{G}_i)$ with $\mu^{(i)}(I) \ge 0$ for all $I \in \mathcal{D}$ and $\sum_{I\in\mathcal{G}_i} \mu^{(i)}(I) = 1$. Summarizing we get the following statement

$$\|\varphi\cdot f\|^p_{f_p^q(X)} \le \frac{K^p_{r,r_0} C_{r_0}(X)^p}{a_{p,q}} \sum_{i\in\mathcal{N}} \Big(\sum_{I\in\mathcal{G}_i} |\varphi_I|^r \mu^{(i)}(I) \Big)^{\frac{p}{r}} \|u_i\|^p_{f_s^q(X)} |I_i|^{1-\frac{p}{s}}. \tag{2.49}$$

We rewrite the sum on the right-hand side in an appropriate way and apply Hölder's inequality with $\frac{p}{r}+1-\frac{p}{r}=1$:

$$\begin{aligned}
&\sum_{i\in\mathcal{N}}\Big(\sum_{I\in\mathcal{G}_i}|\varphi_I|^r\mu^{(i)}(I)\Big)^{\frac{p}{r}}\|u_i\|^p_{f^q_s(X)}|I_i|^{1-\frac{p}{s}}\\
&\quad=\sum_{i\in\mathcal{N}}\Big(\sum_{I\in\mathcal{G}_i}|\varphi_I|^r\|u_i\|^p_{f^q_s(X)}|I_i|^{1-\frac{p}{s}}\mu^{(i)}(I)\Big)^{\frac{p}{r}}\Big(\|u_i\|^p_{f^q_s(X)}|I_i|^{1-\frac{p}{s}}\Big)^{1-\frac{p}{r}}\\
&\quad\le\Big(\sum_{i\in\mathcal{N}}\sum_{I\in\mathcal{G}_i}|\varphi_I|^r\|u_i\|^p_{f^q_s(X)}|I_i|^{1-\frac{p}{s}}\mu^{(i)}(I)\Big)^{\frac{p}{r}}\Big(\sum_{i\in\mathcal{N}}\|u_i\|^p_{f^q_s(X)}|I_i|^{1-\frac{p}{s}}\Big)^{1-\frac{p}{r}}.
\end{aligned}$$

From (2.47) we get an estimate for the second term on the right-hand side and therefore,

$$\begin{aligned}
&\sum_{i\in\mathcal{N}}\Big(\sum_{I\in\mathcal{G}_i}|\varphi_I|^r\mu^{(i)}(I)\Big)^{\frac{p}{r}}\|u_i\|^p_{f^q_s(X)}|I_i|^{1-\frac{p}{s}}\\
&\quad\le\Big(\sum_{i\in\mathcal{N}}\sum_{I\in\mathcal{G}_i}|\varphi_I|^r\|u_i\|^p_{f^q_s(X)}|I_i|^{1-\frac{p}{s}}\mu^{(i)}(I)\Big)^{\frac{p}{r}}A_{p,q}^{1-\frac{p}{r}}\|u\|^{p(1-\frac{p}{r})}_{f^q_p(X)}\\
&\quad=A_{p,q}\|u\|^p_{f^q_p(X)}\Big(\sum_{i\in\mathcal{N}}\sum_{I\in\mathcal{G}_i}|\varphi_I|^r\frac{\|u_i\|^p_{f^q_s(X)}|I_i|^{1-\frac{p}{s}}}{A_{p,q}\|u\|^p_{f^q_p(X)}}\mu^{(i)}(I)\Big)^{\frac{p}{r}}.
\end{aligned}\tag{2.50}$$

Combining inequalities (2.49) and (2.50) yields

$$\|\varphi\cdot f\|_{f^q_p(X)}\le KC_{r_0}(X)\|u\|_{f^q_p(X)}\Big(\sum_{i\in\mathcal{N}}\sum_{I\in\mathcal{G}_i}|\varphi_I|^r\frac{\|u_i\|^p_{f^q_s(X)}|I_i|^{1-\frac{p}{s}}}{A_{p,q}\|u\|^p_{f^q_p(X)}}\mu^{(i)}(I)\Big)^{\frac{1}{r}},\tag{2.51}$$

where K is dependent on the constant K_{r,r_0} from Theorem 2.4 and the constants $a_{p,q}$, $A_{p,q}$ from atomic decomposition. Now we set

$$\omega(I)=\frac{\|u_i\|^p_{f^q_s(X)}|I_i|^{1-\frac{p}{s}}}{A_{p,q}\|u\|^p_{f^q_p(X)}}\mu^{(i)}(I),\qquad I\in\mathcal{G}_i,\tag{2.52}$$

and obtain

$$\|\varphi\cdot f\|_{f^q_p(X)}\le C_{r_0}(X)K\|u\|_{f^q_p(X)}\Big(\sum_{I\in\mathcal{D}}|\varphi_I|^r\omega(I)\Big)^{\frac{1}{s}}.$$

The sequence $(\omega(I))_{I\in\mathcal{D}}$, defined by (2.52), satisfies

$$\sum_{I\in\mathcal{D}}\omega(I)=\frac{1}{A_{p,q}\|u\|^p_{f^q_p(X)}}\sum_{i\in\mathcal{N}}\|u_i\|^p_{f^q_s(X)}|I_i|^{1-\frac{p}{s}}\sum_{I\in\mathcal{G}_i}\mu^{(i)}(I).$$

Recall that $\mu^{(i)}$ is a probability measure on $\mathcal{G}_i$, thus from (2.47) we get

$$\sum_{I\in\mathcal{D}}\omega(I)=\frac{1}{A_{p,q}\|u\|^p_{f^q_p(X)}}\sum_{i\in\mathcal{N}}\|u_i\|^p_{f^q_s(X)}|I_i|^{1-\frac{p}{s}}\le 1.$$

□

REMARK 2.11. If X is of cotype 2, then the statement of Theorem 2.8 is valid for all $s \geq 2$, especially for $s = 2$, cf. Remark 2.10.

2.5. Application to Pisier's extrapolation lattices

Fix $0 < \theta < 1$ and $1 < q < \infty$. Define p, r as follows

$$\frac{1}{p} = 1 - \theta + \frac{\theta}{q} \qquad \text{and} \qquad \frac{1}{r} = \frac{\theta}{q}.$$

Let X be a lattice over a measure space (Ω, Σ, μ) which we assume r-concave and p-convex (with constants one). Let $X_1 = L^q(\Omega, \Sigma, \mu)$. The lattice $X_0 \subseteq L^0(\Omega, \Sigma, \mu)$ introduced by Pisier in [**Pis79a**] and [**Pis79b**] is defined by putting $x \in X_0$ iff

$$\|x\|_{X_0} = \sup \left\{ \left\| |x|^{1-\theta} |y|^\theta \right\|_X^{\frac{1}{1-\theta}} : \|y\|_{X_1} \leq 1 \right\} < \infty. \tag{2.53}$$

Pisier's theorem, [**Pis79a**], [**Pis79b**], asserts that X_0 is a Banach lattice and

$$X = (X_0)^{1-\theta} (X_1)^\theta. \tag{2.54}$$

The lattice $(X_0)^{1-\theta}(X_1)^\theta$ is called Calderón product of the Banach lattices X_0, X_1 and is defined as follows (cf.[**Cal64**]). The lattice $(X_0)^{1-\theta}(X_1)^\theta$ is the space of those functions $u \in L^0(\Omega, \Sigma, \mu)$ such that $|u| = |x|^{1-\theta}|y|^\theta$ with $x \in X_0$ and $y \in X_1$ equipped with the norm

$$\|u\|_{(X_0)^{1-\theta}(X_1)^\theta} = \inf\{\|x\|_{X_0}^{1-\theta} \|y\|_{X_1}^\theta : |u| = |x|^{1-\theta}|y|^\theta\}. \tag{2.55}$$

Specifically, (2.54) states that given $u \in X$ there is $y \in X_1$ and $x \in X_0$ so that

$$|u| = |x|^{1-\theta}|y|^\theta \qquad \text{and} \qquad \|x\|_{X_0}^{1-\theta} \|y\|_{X_1}^\theta \leq C\|u\|_X. \tag{2.56}$$

The proof in [**Pis79a**] obtains $y \in X_1$ (and hence $x \in X_0$) by a Hahn-Banach argument, as explained in the third paragraph of the introduction. For specific examples of lattices it may however be possible to obtain $y \in X_1$ constructively.

These considerations were the stimulus for our work on multiplication operators into Hardy spaces H^p (real-valued and vector-valued) and into Triebel-Lizorkin spaces. Recall, the Triebel-Lizorkin spaces f_p^q, $1 < p \leq q < \infty$, are p-convex and q-concave Banach lattices over the dyadic intervals equipped with the counting measure. Indeed, f_p^q is r-concave for all $r \geq q$. M.Frazier and B.Jawerth showed in [**FJ90**] that $f_p^q \simeq (f_1^q)^{1-\theta}(f_q^q)^\theta$, where $0 < \theta < 1$ and $\frac{1}{p} = 1 - \theta + \frac{\theta}{q}$. Moreover, Pisier's theorem ([**Pis79a**, Theorem 2.10, Remark 2.13]) asserts that

$$f_p^q \simeq (X_0)^{1-\theta}(f_q^q)^\theta, \tag{2.57}$$

where X_0 is the lattice of all elements $f = (x_I)_{I \in \mathcal{D}}$ for which

$$\|f\|_{X_0} = \sup \left\{ \left\| (|x_I|^{1-\theta}|y_I|^\theta)_{I \in \mathcal{D}} \right\|_{f_p^q}^{\frac{1}{1-\theta}} : y = (y_I)_{I \in \mathcal{D}} \in f_q^q, \|y\|_{f_q^q} \leq 1 \right\} < \infty. \tag{2.58}$$

As we pointed out, the factorization of $u \in f_p^q$ as

$$|u| = |x|^{1-\theta}|y|^{\theta}, \quad \|x\|_{X_0}^{1-\theta}\|y\|_{f_q^q}^{\theta} \le C_{p,q}\|u\|_{f_p^q} \tag{2.59}$$

uses a Pietsch measure for the multiplication operator

$$\mathcal{M}_u : \ell^\infty(\mathcal{D}) \to f_p^q, \quad \varphi \mapsto (\varphi \cdot u). \tag{2.60}$$

Using our explicit formulae for the Pietsch measure of (2.60), we obtain the decomposition (2.59) constructively without invoking Hahn Banach theorems.

THEOREM 2.12. *Let $1 < p \le q < \infty$ and $\theta = \frac{q}{q-1}\frac{p-1}{p}$. Let X_0 be the Banach lattice defined by (2.58). Let $u = (u_I)_{I\in\mathcal{D}} \in f_p^q$ and let $\omega \in \ell^1(\mathcal{D})$ be the weight defined by (2.17). Then $y \in f_q^q$ and $x \in X_0$ defined by*

$$y = \left(\frac{\omega_I}{|I|}\right)_{I\in\mathcal{D}}^{\frac{1}{q}} \qquad \text{and} \qquad x = \left(|u_I||y_I|^{-\theta}\right)_{I\in\mathcal{D}}^{\frac{1}{1-\theta}}$$

satisfy

$$|u| = |x|^{1-\theta}|y|^{\theta} \qquad \text{and} \qquad \|x\|_{X_0}^{1-\theta}\|y\|_{f_q^q}^{\theta} \le C_{p,q}\|u\|_{f_p^q}. \tag{2.61}$$

PROOF. We have from Corollary 2.2 that

$$\|y\|_{f_q^q}^q = \sum_{I\in\mathcal{D}} \omega_I \le 1.$$

Let $z = (z_I)_{I\in\mathcal{D}} \in f_q^q$ with $\|z\| \le 1$. Since $\frac{p(q-1)}{p-1} \ge q$ we have from Hölder's inequality

$$\left(\sum_{I\in\mathcal{D}} |y_I|^{-q\theta}|z_I|^{q\theta}\omega_I\right)^{\frac{1}{q}} \le \left(\sum_{I\in\mathcal{D}} |y_I|^{-\frac{p(q-1)}{p-1}\theta}|z_I|^{\frac{p(q-1)}{p-1}\theta}\omega_I\right)^{\frac{p-1}{p(q-1)}}. \tag{2.62}$$

Using the definition of y and θ we get

$$\begin{aligned}\sum_{I\in\mathcal{D}} |y_I|^{-\frac{p(q-1)}{p-1}\theta}|z_I|^{\frac{p(q-1)}{p-1}\theta}\omega_I &= \sum_{I\in\mathcal{D}} |\omega_I|^{-\frac{p(q-1)}{q(p-1)}\theta}|I|^{\frac{p(q-1)}{q(p-1)}\theta}|z_I|^{\frac{p(q-1)}{p-1}\theta}\omega_I \\ &= \sum_{I\in\mathcal{D}} |z_I|^q|I| = \|z\|_{f_q^q}^q \le 1.\end{aligned} \tag{2.63}$$

Therefore, combining (2.62) and (2.63)

$$\left(\sum_{I\in\mathcal{D}} |y_I|^{-q\theta}|z_I|^{q\theta}\omega_I\right)^{\frac{1}{q}} \le 1. \tag{2.64}$$

We can apply Corollary 2.2 to the sequence $\varphi_I = |y_I|^{-\theta}|z_I|^{\theta}$ and get with (2.64)

$$\begin{aligned}\left\||u||y|^{-\theta}|z|^{\theta}\right\|_{f_p^q} &\le C_{p,q}\|u\|_{f_p^q}\left(\sum_{I\in\mathcal{D}} |y_I|^{-q\theta}|z_I|^{q\theta}\omega_I\right)^{\frac{1}{q}} \\ &\le C_{p,q}\|u\|_{f_p^q}.\end{aligned} \tag{2.65}$$

Recall that $x = (x_I)_{I\in\mathcal{D}}$ with $x_I = \left(|u_I||y_I|^{-\theta}\right)^{\frac{1}{1-\theta}}$. Then invoking (2.53), the defining equation for the norm in X_0, the estimate (2.65) translates into

$$\|x\|_{X_0}^{1-\theta} \le C_{p,q}\|u\|_{f_p^q}.$$

As $\|y\|_{f_q^q} \le 1$ we have

$$\|x\|_{X_0}^{1-\theta}\|y\|_{f_q^q}^{\theta} \le C_{p,q}\|u\|_{f_p^q}.$$

Since for $I \in \mathcal{D}$

$$|y_I|^{\theta}|x_I|^{1-\theta} = \left(\frac{\omega_I}{|I|}\right)^{\frac{\theta}{q}} |u_I| \left(\frac{\omega_I}{|I|}\right)^{\frac{-\theta}{q}} = |u_I|$$

we have $|u| = |y|^{\theta}|x|^{1-\theta}$. □

Remark 2.13. The uniqueness theorem of Cwickel and Nilsson ([**CNS03**]) gives the identification of the Banach lattice X_0 defined by (2.58) as f_1^q: Let $\frac{1}{p} = 1-\theta+\frac{\theta}{q}$ and $f = (x_I)_{I\in\mathcal{D}} \in f_1^q$, then there exists a constant c such that

(2.66)

$$c\|f\|_{f_1^q} \le \sup\left\{ \left\|(|x_I|^{1-\theta}|y_I|^{\theta})_{I\in\mathcal{D}}\right\|_{f_p^q}^{\frac{1}{1-\theta}} : y = (y_I)_{I\in\mathcal{D}} \in f_q^q, \|y\|_{f_q^q} \le 1 \right\} \le \|f\|_{f_1^q}.$$

Our Theorem 2.12 complements the constructive proofs for (2.66) given by [**GMP05**] and [**Bow13**]. The common denominator of Theorem 2.12, [**GMP05**] and [**Bow13**] is the use of atomic decomposition as starting point for the proof.

CHAPTER 3

Postorder rearrangement operators

The operators of this chapter are rearrangement operators that rearrange the Haar system, cf. Section 1.4.4. Let $\mathcal{D}_N$ be the set of dyadic intervals in $[0,1]$ with length greater than or equal to 2^{-N}. Let τ be any bijective map on $\mathcal{D}_N$ and $(h_I)_{I\in\mathcal{D}_N}$ the L^∞-normalised Haar system. On the space BMO_N we consider rearrangements of the Haar system induced by the map τ:

$$T_\tau : h_I \to h_{\tau(I)}.$$

In recent years boundedness criteria and extrapolation properties for rearrangement operators that rearrange the Haar system have been studied in detail. See, [**Sem78, SS81, Sch90, Mül97, GMP05, GM09, Mül12, KM13**]. We complement the cited papers by investigating in detail one particular rearrangement and its extremal nature.

The postorder on the set $\mathcal{D}_N$, introduced in Section 1.2.7, defines a bijective map τ_N on the set $\mathcal{D}_N$, called the postorder rearrangement, that maps the n^{th} interval in postorder onto the n^{th} interval in lexicographic order. Its inverse is denoted by σ_N, cf. figure 2. In Section 3.1 we show that the postorder has maximal distance to the usual lexicographic order on $\mathcal{D}_N$. We quantify the distance by the product of operator norms

$$\|T_{\tau_N} : \mathrm{BMO}_N \to \mathrm{BMO}_N\| \|T_{\sigma_N} : \mathrm{BMO}_N \to \mathrm{BMO}_N\|.$$

Particularly, we prove that within a factor of $\sqrt{2}$, on BMO_N, both the operator T_{τ_N} and its inverse T_{σ_N} reach maximal norm. We denote

$$R(\mathrm{BMO}_N) = \sup\Big\{\|T_\tau\colon \mathrm{BMO}_N \to \mathrm{BMO}_N\| : \tau\colon \mathcal{D}_N \to \mathcal{D}_N \text{ bijective}\Big\}.$$

Our main result is

Theorem. For $T = T_{\tau_N}$ and $T = T_{\sigma_N}$ we have

$$\frac{1}{\sqrt{2}} R(\mathrm{BMO}_N) \le \|T\|_{\mathrm{BMO}_N} \le R(\mathrm{BMO}_N).$$

We extend this result to rearrangement operators on the dyadic Hardy spaces H^p_N, $0 < p < \infty$, (Section 3.3) and the Triebel-Lizorkin spaces $f^q_{p,N}$, $0 < p, q < \infty$, (Section 3.4), given by

$$T_{\tau,p}\colon \frac{h_I}{|I|^{\frac{1}{p}}} \to \frac{h_{\tau(I)}}{|\tau(I)|^{\frac{1}{p}}}.$$

This work continuous the previous study of [**MS97**], who determine from a different perspective the extremal nature of the postorder and the induced rearrangement. P.F.X. Müller and G. Schechtman show that any block basis of the Haar system $(h_I)_{I\in\mathcal{D}_N}$ with respect to the postorder, $\preceq$, spans spaces that are well isomorphic to ℓ^p_k, $1 < p \neq 2 < \infty$. On the other hand it is easy to find block bases of the Haar system with respect to the lexicographic order (the Rademacher functions) whose span is well isomorphic to ℓ^2_k.

The postorder has its origin in computer sciences (see e.g. [**BP05, Knu05**]). In computer sciences, especially in the design and analysis of algorithms, dyadic trees are commonly used data structures, which enable efficient access to data. Tree traversal algorithms, which systematically walk through a tree and visit each node exactly once, enhance this efficient access. These algorithms define a specific order on the nodes of a tree. This makes it possible to talk about the node following or preceding a given one. The postorder tree traversal visits the left child, then the right child and then the node itself. Considering the dyadic tree structure of $\mathcal{D}_N$ this traversal induces exactly the postorder, $\preceq$, on $\mathcal{D}_N$. The postorder tree traversal is for example used in the mergesort algorithm, invented by von Neumann in 1945. A more basic application is deallocating memory of all nodes of a tree, i.e. deleting a tree. In calculator programs the postorder tree traversal is used to evaluate postfix notation.

The Mallat algorithm for discrete wavelet transform (DWT) (see [**Mal89, Mey93**]) determines the wavelet coefficients of a given discrete signal in a specific order which works its way up from the finest level to the coarsest. In case of the Haar transform this order is exactly our postorder, $\preceq$, cf. figure 1. We discuss the discrete Haar wavelet transform (see e.g. [**Wal08**]) now in detail. Let $N \in \mathbb{N}_0$.

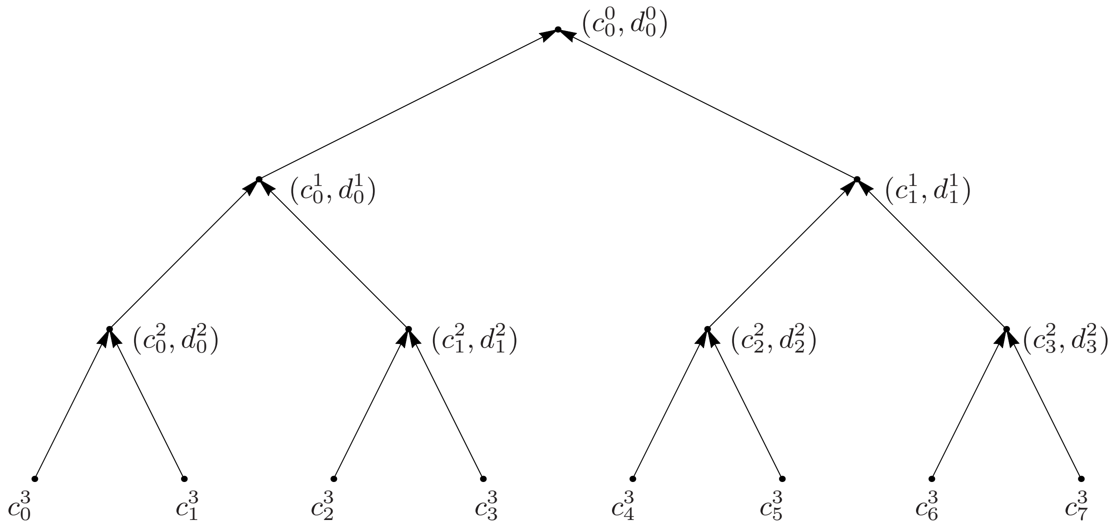

FIGURE 1. Calculation of the trend (c^j_k) and the fluctuation (d^j_k) for $0 \leq j \leq 3$.

Suppose a discrete signal on $[0, 1]$ is given by the sequence $c^N = (c^N_1, \dots c^N_{2^N-1})$. We process the signal by decomposing it into its trend (approximating coefficients) c^{N-1} and its fluctuation (detail coefficients) d^{N-1}:

$$c_k^{N-1} = \frac{1}{\sqrt{2}}\left(c_{2k}^N + c_{2k+1}^N\right) \quad \text{and} \quad d_k^{N-1} = \frac{1}{\sqrt{2}}\left(c_{2k}^N - c_{2k+1}^N\right).$$

The trend and the fluctuation are two subsignals of c^N with half of its length. The signal c^{N-1} is again decomposed into its trend c^{N-2} and its fluctuation d^{N-2}, which are again subsignals of c^{N-1} with half of its length. Successively we compute from c^j the trends c^{j-1} and the fluctuations d^{j-1}. Finally, after N steps, we have decomposed the signal c^N into the coarsest information c_0^0 and the detail coefficients $(d^j)_{j=0}^{N-1}$, where $d^j = (d_0^j, \dots, d_{2^j-1}^j)$.

The Haar system is the most basic orthonormal wavelet basis used in DWT and gives insight in more sophisticated wavelet transforms.

In the following let (unless stated otherwise) ℓ, k and N be non-negative integers such that $0 \le \ell \le N$ and $0 \le k \le 2^{\ell-1}$. Furthermore, we will denote by $\lfloor x \rfloor$ the *floor function* and by $\lceil x \rceil$ the *ceiling function* of an element $x \in \mathbb{R}$. Recall the definition

$$\lfloor x \rfloor = \max\{z \in \mathbb{Z} : z \le x\}, \quad \lceil x \rceil = \min\{z \in \mathbb{Z} : z \ge x\}.$$

3.1. The main theorem

Recall the definition of the postorder on the set of dyadic intervals $\mathcal{D}_N$ defined in Section 1.2.7. We denote by τ_N the bijective map on the dyadic intervals that associates to the n^{th} interval in postorder the n^{th} interval in lexicographic order, cf. figure 2. This rearrangement is called *postorder rearrangement*. Its inverse, which associates to the n^{th} interval in lexicographic order the n^{th} interval in postorder, is denoted by σ_N.

The rearrangements τ_N and σ_N induce rearrangement operators on BMO_N, on the H_N^p-spaces and on the Triebel-Lizorkin spaces $f_{p,N}^q$, cf. Section 1.4.4. On BMO_N we consider the rearrangement operators

$$T_{\tau_N} : h_I \mapsto h_{\tau_N(I)} \quad \text{and} \quad T_{\sigma_N} : h_I \mapsto h_{\sigma_N(I)}$$

and obtain the following norm estimates for these rearrangement operators applied to functions with Haar support in the sets

$$\mathcal{T}_{\ell,0}^N = \{I \in \mathcal{D}_N : I \subseteq I_{\ell,0}\}$$

and $\mathcal{D}_{N-\ell}$. Recall that

$$\mathcal{M}(\mathcal{T}_{\ell,0}^N) = \operatorname{span}\{h_I : I \in \mathcal{T}_{\ell,0}^N\}$$

and

$$\mathcal{M}(\mathcal{D}_{N-\ell}) = \operatorname{span}\{h_I : I \in \mathcal{D}_{N-\ell}\}.$$

THEOREM 3.1. *Let $N \in \mathbb{N}_0$ and $0 \le \ell \le N$. Let $T = T_{\tau_N}\big|_{\mathcal{M}(\mathcal{T}_{\ell,0}^N)}$ or $T = T_{\sigma_N}\big|_{\mathcal{M}(\mathcal{D}_{N-\ell})}$. Then*

$$\frac{1}{\sqrt{2}}(N-\ell+1)^{\frac{1}{2}} \le \|T\|_{BMO} \le (N-\ell+1)^{\frac{1}{2}}. \tag{3.1}$$

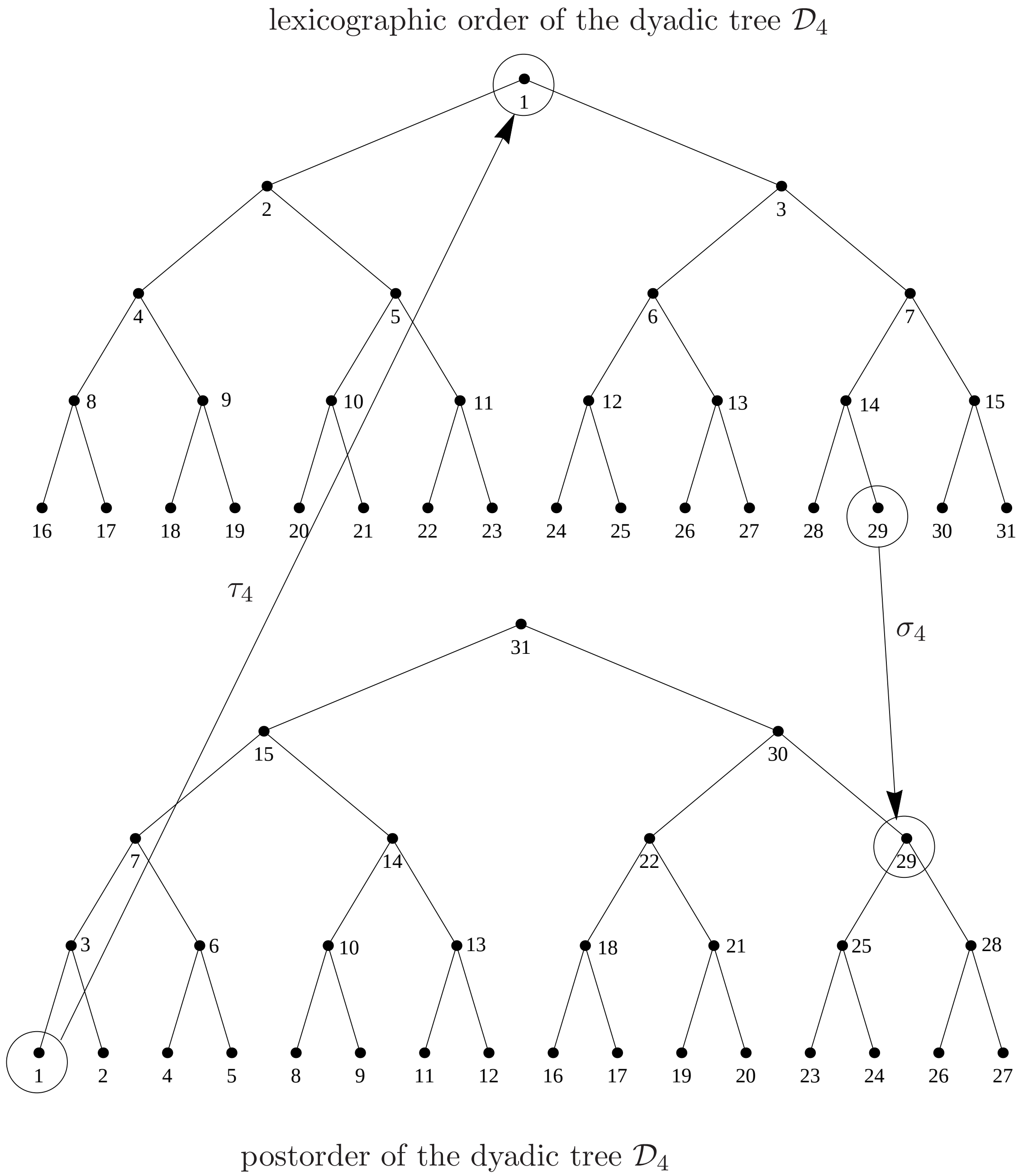

FIGURE 2. Lexicographic order and postorder of the dyadic tree $\mathcal{D}_4$. Postorder rearrangement τ_4 on $\mathcal{D}_4$ and its inverse σ_4.

This theorem in combination with the general upper bound in (1.41) reveals the extremal nature of the rearrangements τ_N and σ_N in the sense that for $T = T_{\tau_N}$ resp. $T = T_{\sigma_N}$ we have

$$\frac{1}{\sqrt{2}} R(\mathrm{BMO}_N) \leq \|T\|_{\mathrm{BMO}} \leq R(\mathrm{BMO}_N),$$

where

$$R(\mathrm{BMO}_N) = \sup\Big\{\|T_\tau\colon \mathrm{BMO}_N \to \mathrm{BMO}_N\| : \tau\colon \mathcal{D}_N \to \mathcal{D}_N \text{ bijective}\Big\}.$$

Obviously, the lower bound in (3.1) is the important one for this result and the statement of Theorem 3.1. The upper bound in (3.1) is the trivial one that originates from the depth (in the sense of dyadic trees) of the sets $\mathcal{D}_{N-\ell}$ resp. $\mathcal{T}^N_{\ell,0}$.

3.2. Proof of the main theorem

3.2.1. Parameters associated with the postorder rearrangement. For the proof of the main theorem we need formulae that describe the map τ_N precisely. Recall that τ_N maps the n^{th} dyadic interval in postorder onto the n^{th} dyadic interval in lexicographic order. First of all we give formulae that describe the assignment of postorder ordinal numbers and lexicographic ordinal numbers to dyadic intervals in $\mathcal{D}_N$. We denote by $a^\ell(k)$ the postorder ordinal number and by $b^\ell(k)$ the lexicographic ordinal number of the dyadic interval $I_{\ell,k} \in \mathcal{D}_N$.

The assignment rule for a lexicographic ordinal number to a dyadic interval $I_{\ell,k}$ is given by

$$b^\ell(k) = \left(\sum_{i=0}^{\ell-1} 2^i\right) + k + 1 = 2^\ell + k.$$

We can determine from the ordinal number $b^\ell(k)$ the level ℓ and the position k of the associated interval $I_{\ell,k}$:

$$\ell = \lfloor \log_2 b^\ell(k) \rfloor \quad \text{and} \quad k = b^\ell(k) - 2^\ell. \tag{3.2}$$

The assignment rule for postorder ordinal numbers to the dyadic intervals is more difficult than in the lexicographic case. Let $j \in \mathbb{N}$ with dyadic expansion $j = \sum \epsilon_i 2^i$. We define $m(j) = \min\{i \in \mathbb{N} : \epsilon_i \neq 0\}$.

Lemma 3.2. *Let $N \in \mathbb{N}_0$, $0 \le \ell \le N$ and $0 \le k \le 2^\ell - 1$. The postorder ordinal number of the dyadic interval $I_{\ell,k} \in \mathcal{D}_N$ is given by*

$$a^\ell(k) = (k+1)\,(2^{N-\ell+1} - 1) + \sum_{j=1}^{k} m(j). \tag{3.3}$$

Proof. Let $1 \le j \le 2^\ell - 1$ and let

$$t^\ell(j) = a^\ell(j) - a^\ell(j-1) - 1, \tag{3.4}$$

where $a^\ell(j-1)$ and $a^\ell(j)$ are the postorder ordinal numbers of two successive dyadic intervals in level ℓ. This gives the recursive formula $a^\ell(j) = a^\ell(j-1) + t^\ell(j) + 1$ and thereby the assignment rule for the postorder ordinal number:

$$a^\ell(k) = a^\ell(0) + k + \sum_{j=1}^{k} t^\ell(j). \tag{3.5}$$

The definition of the postorder and the dyadic tree structure of $\mathcal{D}_N$ yield

$$a^\ell(0) = 2^{N-\ell+1} - 1, \quad \text{for all } 0 \le \ell \le N. \tag{3.6}$$

In the following we determine a formula for $t^\ell(j)$, $1 \le j \le 2^\ell - 1$. To this end, we give formulae that associate the postorder ordinal number of a dyadic interval with the postorder ordinal number of its parent. We consider the dyadic interval $I_{\ell,k} \in \mathcal{D}_N$ with the postorder ordinal number $a^\ell(k)$ and its children $I_{\ell+1,2k}$ and $I_{\ell+1,2k+1}$ with the postorder ordinal numbers $a^{\ell+1}(2k)$ and $a^{\ell+1}(2k+1)$. By the definition of the postorder we have $a^{\ell+1}(2k) < a^{\ell+1}(2k+1) < a^\ell(k)$. Furthermore, $a^{\ell+1}(2k)$ is smaller and $a^{\ell+1}(2k+1)$ is greater than the ordinal numbers of the descendants of $I_{\ell+1,2k+1}$. The number of descendants of $I_{\ell+1,2k+1}$ is $2^{N-\ell} - 2$. Hence, the definition of the postorder yields the following recursions:

$$a^\ell(k) = a^{\ell+1}(2k+1) + 1 \quad \text{and} \quad a^\ell(k) = a^{\ell+1}(2k) + 2^{N-\ell}, \tag{3.7}$$

where $0 \le \ell \le N$ and $0 \le k \le 2^\ell - 1$. Induction shows that for $1 \le i < \ell$

$$a^{\ell-i}(s-1) = a^\ell(2^i s - 1) + i \quad \text{and} \quad a^{\ell-i}(s) = a^\ell(2^i s) + 2^{N-\ell+1}(2^i - 1), \tag{3.8}$$

where $1 \le s \le 2^{\ell-i} - 1$.

Now we can determine an explicit formula for $t^\ell(j)$. If j is odd, then the formulae in (3.7) yield $a^{\ell-1}\left(\frac{j-1}{2}\right) = a^\ell(j) + 1$ and $a^{\ell-1}\left(\frac{j-1}{2}\right) = a^\ell(j-1) + 2^{N-\ell+1}$. Therefore, by equation (3.4)

$$t^\ell(j) = 2^{N-\ell+1} - 2, \qquad \text{if } j \text{ is odd.}$$

If j is even, then there exists an integer i, $1 \le i < \ell$, given by $i = m(j)$, and an odd integer s, $1 \le s \le 2^{\ell-i} - 1$ such that $j = 2^i s$. Equation (3.4) and the formulae in (3.8) yield

$$\begin{aligned} t^\ell(s2^i) &= a^\ell(2^i s) - a^\ell(2^i s - 1) - 1 \\ &= a^{\ell-i}(s) - 2^{N-\ell+1}(2^i - 1) - a^{\ell-i}(s-1) + i - 1. \end{aligned} \tag{3.9}$$

Note that s is odd. The formulae in (3.7) yield $a^{\ell-i}(s) = a^{\ell-i-1}\left(\frac{s-1}{2}\right) - 1$ and $a^{\ell-i}(s-1) = a^{\ell-i-1}\left(\frac{s-1}{2}\right) - 2^{N-\ell+i+1}$. Therefore, by equation (3.9) we have

$$t^\ell(s2^i) = 2^{N-\ell+1} - 2 + i.$$

Summarizing the above we have for all $1 \le j \le 2^\ell - 1$

$$t^\ell(j) = m(j) + 2^{N-\ell+1} - 2. \tag{3.10}$$

Note that $m(j) = 0$, if j is odd. Putting this into equation (3.5) yields the statement. □

Given the ordinal numbers of a dyadic interval with respect to both the postorder and the lexicographic order on $\mathcal{D}_N$ we can describe the postorder rearrangement τ_N as follows. Let $I_{\ell,k} \in \mathcal{D}_N$ and $a^\ell(k)$ the corresponding postorder ordinal number. Let L and K be non-negative integers such that $a^\ell(k) = 2^L + K$. Recall that $2^L + K$ is the lexicographic ordinal number of the dyadic interval $I_{L,K} \in \mathcal{D}_N$. The postorder rearrangement τ_N is then the bijective map on $\mathcal{D}_N$ that maps the dyadic interval $I_{\ell,k}$ onto the dyadic interval $I_{L,K}$, cf. figure 2.

In the following section we describe the determination of L and K such that $a^\ell(k) = 2^L + K$. In the following we use the notation

$$\mathrm{Level}(a^\ell(k)) = L \quad \text{and} \quad \mathrm{Pos}(a^\ell(k)) = K.$$

According to (3.2) we have

$$\mathrm{Level}(a^\ell(k)) = \lfloor \log_2 (a^\ell(k)) \rfloor \quad \text{and} \quad \mathrm{Pos}(a^\ell(k)) = a^\ell(k) - 2^{\mathrm{Level}(a^\ell(k))}. \tag{3.11}$$

The following two Lemmata give formulae for $\mathrm{Level}(a^\ell(k))$ and $\mathrm{Pos}(a^\ell(k))$, which do not involve the postorder ordinal number $a^\ell(k)$ but only the level ℓ and the position k of the corresponding dyadic interval $I_{\ell,k}$.

LEMMA 3.3. *Let $N \in \mathbb{N}_0$ and $0 \le \ell \le N$. For all $0 \le k \le 2^\ell - 1$ we have*

$$\mathrm{Level}(a^\ell(k)) = \lceil \log_2 (k+1) \rceil + N - \ell. \tag{3.12}$$

PROOF. The definition of the postorder yields $a^\ell(0) = 2^{N-\ell+1} - 1$. Therefore, by equation (3.11) we have $\mathrm{Level}(a^\ell(0)) = N - \ell$. Now we show that for all $1 \le s \le \ell$

$$\mathrm{Level}\,(a^\ell(2^{s-1})) = \cdots = \mathrm{Level}\,(a^\ell(2^s - 1)) = s + N - \ell. \tag{3.13}$$

By Lemma 3.2 we have

$$a^\ell(2^{s-1}) = 2^{N-\ell+s} + 2^{N-\ell+1} - 2^{s-1} - 1 + \sum_{j=1}^{2^{s-1}} m(j). \tag{3.14}$$

Recall that for $j \in \mathbb{N}$ given by its dyadic expansion $j = \sum \epsilon_i \, 2^i$ we have $m(j) = \min\{i \in \mathbb{N} : \epsilon_i \neq 0\}$. Hence, $m(j) = 0$ for all odd integers j. We can split the sum on the right-hand side of equation (3.14) as follows

$$\sum_{j=1}^{2^{s-1}} m(j) = \sum_{j=1}^{2^{s-2}} m(2j) = \sum_{j_0=1}^{s-1} m(2^{j_0}) + \sum_{j_1=1}^{s-2} \sum_{j_2=1}^{j_1-1} m(2^{j_1} + 2^{j_2}) + \cdots$$
$$\cdots + \sum_{j_1=1}^{s-2} \cdots \sum_{j_{s-2}=1}^{j_{s-3}-1} m(2^{j_1} + \cdots + 2^{j_{s-2}}).$$

By definition, $m(2^{j_0}) = j_0$ and $m(2^{j_1} + \cdots + 2^{j_i}) = j_i$ for $2 \le i \le s-2$. Hence,

$$\begin{aligned} \sum_{j=1}^{2^{s-1}} m(j) &= \sum_{j_0=1}^{s-1} j_0 + \sum_{j_1=1}^{s-2} \sum_{j_2=1}^{j_1-1} j_2 + \cdots + \sum_{j_1=1}^{s-2} \cdots \sum_{j_{s-2}=1}^{j_{s-3}-1} j_{s-2} \\ &= \sum_{k=1}^{s-1} \binom{s-1}{k} = 2^{s-1} - 1. \end{aligned} \tag{3.15}$$

Putting this into formula (3.14) we get $a^\ell(2^{s-1}) = 2^{N-\ell+1} + 2^{N-\ell+s} - 2$. Lemma 3.2 yields

$$a^\ell(2^s - 1) = 2^{N-\ell+s+1} - 2^s - m(2^s) + \sum_{j=1}^{2^s} m(j).$$

Note that $m(2^s) = s$. By equation (3.15) we have $a^\ell(2^s - 1) = 2^{N-\ell+s+1} - s - 1$. Equation (3.11) yields

$$\text{Level}(a^\ell(2^{s-1})) = \lfloor \log_2 (2^{N-\ell+1} + 2^{N-\ell+s} - 2) \rfloor = N - \ell + s,$$
$$\text{Level}(a^\ell(2^s - 1)) = \lfloor \log_2 (2^{N-\ell+1+s} - s - 1) \rfloor = N - \ell + s.$$

Note that the map $k \mapsto \text{Level}(a^\ell(k))$ is monotonically increasing for all $0 \le k \le 2^\ell - 1$. Therefore, (3.13) is proven.

Let $0 \le k \le 2^\ell - 1$ and $s = \lceil \log_2 (k+1) \rceil$. Then $2^{s-1} \le k \le 2^s - 1$ and equation (3.13) yields $\text{Level}(a^\ell(k)) = s + N - \ell$. □

The next Lemma describes the determination of $K = \text{Pos}\,(a^\ell(k))$. As stated previously, $\text{Pos}\,(a^\ell(k))$ depends on $L = \text{Level}\,(a^\ell(k))$.

Recall that for $j \in \mathbb{N}$ with dyadic expansion $j = \sum \epsilon_i 2^i$ we have $m(j) = \min\{i \in \mathbb{N} : \epsilon_i \ne 0\}$.

Lemma 3.4. *Let $N \in \mathbb{N}_0$ and $0 \le \ell \le N$. Then $\text{Pos}(a^\ell(0)) = 2^{N-\ell} - 1$ and for all $0 < k \le 2^\ell - 1$*

$$\text{Pos}(a^\ell(k)) = (k+1)(2^{N-\ell+1} - 1) + 2^{L-N+\ell-1} - 2^L - 1 + \sum_{j=2^{L-N+\ell-1}+1}^{k} m(j).$$

Proof. Recall that $a^\ell(0) = 2^{N-\ell+1} - 1$ and $\text{Level}(a^\ell(0)) = N - \ell$. Therefore, by equation (3.11) we have $\text{Pos}(a^\ell(0)) = 2^{N-\ell} - 1$.

Fix one dyadic interval $I_{\ell,k}$, $k > 0$ with corresponding postorder ordinal number $a^\ell(k)$. Let $L = \text{Level}\,(a^{\ell(k)})$. Lemma 3.3 states that $\text{Level}\,(a^\ell(j)) = L$, for all $2^{L-N+\ell-1} \le j \le 2^{L-N+\ell} - 1$. Recall from the proof of Lemma 3.2 that $a^\ell(j) = a^\ell(j-1) + 1 + t^\ell(j)$, where $t^\ell(j) = m(j) + 2^{N-\ell+1} - 2$. Hence, by equation (3.11) we have the following recursive formula

$$\text{Pos}\,(a^\ell(j)) = \text{Pos}\,(a^\ell(j-1)) + 1 + t^\ell(j), \quad 2^{L-N+\ell-1} < j \le 2^{L-N+\ell} - 1$$

and therefore,

$$\text{Pos}\,(a^\ell(k)) = \text{Pos}\,(a^\ell(2^{L-N+\ell-1})) + k - 2^{L-N+\ell-1} + \sum_{j=2^{L-N+\ell-1}+1}^{k} t^\ell(j). \tag{3.16}$$

Since $t^\ell(j) = m(j) + 2^{N-\ell+1} - 2$, it follows that

$$\sum_{j=2^{L-N+\ell-1}+1}^{k} t^\ell(j) = k\,(2^{N-\ell+1} - 2) + 2^{L-N+\ell} - 2^L + \sum_{j=2^{L-N+\ell-1}+1}^{k} m(j). \tag{3.17}$$

Lemma 3.2 and equation (3.15) yield $a^\ell(2^{L-N+\ell-1}) = 2^L + 2^{N-\ell+1} - 2$ and by equation (3.11) we have

$$\text{Pos}\,(a^\ell(2^{L-N+\ell-1})) = 2^{N-\ell+1} - 2. \tag{3.18}$$

Putting equation (3.17) and (3.18) into equation (3.16) yields the statement. □

3.2.2. Dyadic subtrees and their lowermost level in $\mathcal{D}_N$. In this section we examine the behaviour of the postorder rearrangement τ_N on complete dyadic subtrees in $\mathcal{D}_N$ given by

$$\mathcal{T}^N_{\ell,k} = \{I \in \mathcal{D}_N : I \subseteq I_{\ell,k}\} \tag{3.19}$$

and on their lowermost level in $\mathcal{D}_N$ given by

$$\mathcal{E}^N_{\ell,k} = \{I \in \mathcal{D}_N : \, I \subseteq I_{\ell,k}, \, |I| = 2^{-N}\}. \tag{3.20}$$

Note that $\mathcal{E}^N_{\ell,k}$ is a collection of disjoint dyadic intervals and hence, $[\![\mathcal{E}^N_{\ell,k}]\!] = 1$. We know from (1.13) that $[\![\mathcal{T}^N_{\ell,k}]\!] = N - \ell + 1$. We measure the behaviour of the rearrangement by the Carleson constants $[\![\tau_N(\mathcal{T}^N_{\ell,k})]\!]$ and $[\![\tau_N(\mathcal{E}^N_{\ell,k})]\!]$. The following two theorems and the corresponding proofs reveal a remarkable phenomenon of the postorder rearrangement τ_N. A complete dyadic subtree as well as its lowermost level in $\mathcal{D}_N$ is mapped under τ_N onto collections of dyadic intervals with large Carleson constant, if it contains the leftmost interval $I_{N,0}$, cf. Theorem 3.5. Otherwise, it is mapped under τ_N onto a collection of disjoint dyadic intervals of equal length, cf. Theorem 3.9.

THEOREM 3.5. *Let $N \in \mathbb{N}_0$ and $0 \le \ell \le N$. Then*

$$[\![\tau_N(\mathcal{T}^N_{\ell,0})]\!] = N - \ell + 1 \quad \textit{and} \quad [\![\tau_N(\mathcal{E}^N_{\ell,0})]\!] \ge \frac{N - \ell + 1}{2}.$$

PROOF. Recall that τ_N maps the n^{th} interval in postorder onto the n^{th} interval in lexicographic order. The definition of the postorder yields that the dyadic intervals in $\mathcal{T}^N_{\ell,0}$ have the corresponding postorder ordinal numbers $1, \dots, 2^{N-\ell+1} - 1$, cf. Section 1.2.7. These are exactly the lexicographic ordinal numbers of the dyadic intervals in $\mathcal{D}_{N-\ell}$. Hence, $\tau_N(\mathcal{T}^N_{\ell,0}) = \mathcal{D}_{N-\ell}$ and $[\![\tau_N(\mathcal{T}^N_{\ell,0})]\!] = N - \ell + 1$.

The lowermost level of $\mathcal{T}^N_{\ell,0}$ in $\mathcal{D}_N$ is given by

$$\mathcal{E}^N_{\ell,0} = \{I_{N,r} : \, 0 \le r \le 2^{N-\ell} - 1\}.$$

By the characterisation of the postorder rearrangement τ_N in Section 3.2.1 we have

$$\tau_N(\mathcal{E}^N_{\ell,0}) = \Big\{ I_{L,K} : \, L = \mathrm{Level}(a^N(r)), \, K = \mathrm{Pos}(a^N(r)), \, 0 \le r \le 2^{N-\ell} - 1 \Big\}.$$

Lemma 3.3 and Lemma 3.4 give that $\mathrm{Level}\,(a^N(0)) = 0$ and $\mathrm{Pos}\,(a^N(0)) = 0$. Therefore, $I_{0,0} \in \tau_N(\mathcal{E}^N_{\ell,0})$ and by definition (1.11)

$$[\![\tau_N(\mathcal{E}^N_{\ell,0})]\!] \ge \frac{1}{|I_{0,0}|} \sum_{\substack{J \subseteq I_{0,0}, \\ J \in \tau_N(\mathcal{E}^N_{\ell,0})}} |J| = \sum_{J \in \tau_N(\mathcal{E}^N_{\ell,0})} |J|. \tag{3.21}$$

Note that $\tau_N(\mathcal{E}^N_{\ell,0}) \subseteq \mathcal{D}_{N-\ell}$. We split the sum on the right hand side into levels and get

$$[\![\tau_N(\mathcal{E}^N_{\ell,0})]\!] \ge \sum_{m=0}^{N-\ell} 2^{-m} |\mathcal{B}(m)|, \tag{3.22}$$

where $\mathcal{B}(m)$ is the set of dyadic intervals in the collection $\tau_N(\mathcal{E}^N_{\ell,0})$ that have length 2^{-m}. We denote by $\mathcal{A}(m)$ the set of postorder ordinal numbers corresponding to $\mathcal{B}(m)$. Then $|\mathcal{B}(m)| = |\mathcal{A}(m)|$ and

$$\mathcal{A}(m) = \{a^N(r) : 0 \le r \le 2^{N-\ell} - 1, \text{Level}(a^N(r)) = m\}.$$

Obviously, $|\mathcal{A}(0)| = 1$. By Lemma 3.3 we have $|\mathcal{A}(m)| = 2^{m-1}$ for all $1 \le m \le N-\ell$. Hence,

$$[\![\tau_N(\mathcal{E}^N_{\ell,0})]\!] \ge 1 + \sum_{m=1}^{N-\ell} 2^{-1} = \frac{N-\ell}{2} + 1 \ge \frac{N-\ell+1}{2}.$$

□

REMARK 3.6. Let $N \in \mathbb{N}_0$. An easy computation shows that

$$[\![\tau_N(\mathcal{E}^N_{\ell,0})]\!] = 1 + \frac{N-\ell}{2}, \quad \ell \in \{N-1, N\}.$$

Obviously, for $0 \le \ell \le N-2$ we have the upper bound

$$[\![\tau_N(\mathcal{E}^N_{\ell,0})]\!] \le N - \ell + 1.$$

CONJECTURE 3.7. Let $N \in \mathbb{N}$, $N \ge 2$ and $0 \le \ell \le N-2$. The supremum in definition (1.11) for the Carleson constant $[\![\tau_N(\mathcal{E}^N_{\ell,0})]\!]$ is attained for the interval $I_{1,0}$. This gives the following formula

$$[\![\tau_N(\mathcal{E}^N_{\ell,0})]\!] = \frac{N-\ell}{2} + \frac{3}{2} - 2^{N-\ell+1}. \tag{3.23}$$

We give the argument to obtain formula (3.23). We need the following lemma.

LEMMA 3.8. *Let $N, \ell, L, r, m \in \mathbb{N}_0$, $N \ge 2$ and $0 \le \ell \le N-2$. We consider the collection of dyadic intervals*

$$\mathcal{E}^N_{\ell,0} = \{I \in \mathcal{D}_N : I \subseteq I_{\ell,0}, |I| = 2^{-N}\}.$$

Let $0 \le L \le N-\ell$, $0 \le r \le 2^L - 1$ and $L \le m \le N$. Let

$$\mathcal{B}^\ell_{L,r}(m) = \{I \in \mathcal{D}_N : I \in \tau_N(\mathcal{E}^N_{\ell,0}), I \subseteq I_{L,r}, |I| = 2^{-m}\}.$$

Then

$$\left|\mathcal{B}^\ell_{L,r}(m)\right| = 0 \quad \textit{for all } m > N-\ell, \tag{3.24}$$

$$\left|\mathcal{B}^\ell_{L,0}(L)\right| = 1 \quad , \tag{3.25}$$

and for all $L < m \le N-\ell$

$$\left|\mathcal{B}^\ell_{L,0}(m)\right| = \begin{cases} 2^{m-1} & L = 0, \\ 2^{m-L-1} + 1, & L > 0. \end{cases} \tag{3.26}$$

PROOF OF LEMMA 3.8. We know from the proof of Theorem 3.5 that

$$\tau_N(\mathcal{T}^N_{\ell,0}) = \mathcal{D}_{N-\ell}.$$

Since $\mathcal{E}^N_{\ell,0} \subseteq \mathcal{T}^N_{\ell,0}$ we have $\tau_N(\mathcal{E}^N_{\ell,0}) \subseteq \mathcal{D}_{N-\ell}$. This yields (3.24).

We associate the set $\mathcal{E}^N_{\ell,0}$ with the set of postorder ordinal numbers

$$\{a^N(k) : 0 \le k \le 2^{N-\ell} - 1\}.$$

The image under the postorder rearrangement τ_N is given by

$$\tau_N(\mathcal{E}^N_{\ell,0}) = \{I_{Lev,K} :\ Lev = \mathrm{Level}(a^N(k)),\ K = \mathrm{Pos}(a^N(k)),\ 0 \le k \le 2^{N-\ell} - 1\}.$$

We know from the above proof that $I_{0,0} \in \tau_N(\mathcal{E}^N_{\ell,0})$. Let $L > 0$. Then Lemma 3.3 yields Level$\,(a^N(2^{L-1})) = L$ and Lemma 3.4 yields Pos$\,(a^N(2^{L-1})) = 0$. Therefore, $I_{L,0} \in \tau_N(\mathcal{E}^N_{\ell,0})$ and $\left|\mathcal{B}^\ell_{L,0}(L)\right| = 1$ for all $0 \le L \le N - \ell$. This yields 3.25.

We associate the set $\mathcal{B}^\ell_{L,0}(m)$ with the set of postorder ordinal numbers

$$\{a^N(k) : 0 \le k \le 2^{N-\ell} - 1, \mathrm{Level}(a^N(k)) = m, 0 \le \mathrm{Pos}(a^N(k)) \le 2^{m-L} - 1\}.$$

We denote this set by $\mathcal{A}^\ell_{L,0}(m)$. Then $\left|\mathcal{B}^L_{L,0}(m)\right| = |\mathcal{A}_{L,0}(m)|$. Lemma 3.3 yields that Level$\,(a^N(k)) = m$ for all $2^{m-1} \le k \le 2^m - 1$. Hence,

$$\mathcal{A}^\ell_{L,0}(m) = \{a^N(k) : 2^{m-1} \le k \le 2^m - 1,\ 0 \le \mathrm{Pos}(a^N(k)) \le 2^{m-L} - 1\}. \tag{3.27}$$

Observe that $\mathcal{A}^\ell_{L,0}(m) = \emptyset$ for all $m > N - \ell$. Let $L = 0$. Then

$$\left|\mathcal{A}^\ell_{0,0}(m)\right| = \left|\{a^N(k) : 2^{m-1} \le k \le 2^m - 1\}\right| = 2^{m-1}.$$

Let $L > 0$. Then Level$\,(a^N(2^{m-1}))$ = Level$\,(a^N(2^{m-1} + 2^{m-L-1})) = m$. Lemma 3.4 yields

$$\mathrm{Pos}(a^N(2^{m-1} + 2^{m-L-1})) = \mathrm{Pos}(a^N(2^{m-1})) + 2^{m-L-1} + \sum_{j=2^{m-1}+1}^{2^{m-1}+2^{m-L-1}} t^N(j)$$

and $\mathrm{Pos}(a^N(2^{m-1})) = 0$. The definition of $t^N(j)$ yields that $t^N(2^{m-1} + j) = t^N(j)$, for all $1 \le j \le 2^{m-L-1}$. Therefore, we get for the sum on the right-hand side, analogously to the proof of Lemma 3.3,

$$\sum_{j=2^{m-1}+1}^{2^{m-1}+2^{m-L-1}} t^N(j) = \sum_{j=1}^{2^{m-L-1}} t^N(2^{m-1} + j) = \sum_{j=1}^{2^{m-L-1}} t^N(j) = 2^{m-L-1} - 1.$$

Summarizing we get

$$\mathrm{Pos}(a^N(2^{m-1})) = 0 \quad \text{and} \quad \mathrm{Pos}(a^N(2^{m-1} + 2^{m-L-1})) = 2^{m-L} - 1. \tag{3.28}$$

The map $k \mapsto \mathrm{Pos}(a^N(k))$ is monotonically increasing for $2^{m-1} \le k \le 2^{m-1} + 2^{m-L-1}$, therefore we get from (3.27)

$$\left|\mathcal{B}^L_{L,0}(m)\right| = \left|\mathcal{A}^\ell_{L,0}(m)\right| = \left|\{a^N(k) : 2^{m-1} \le k \le 2^{m-1} + 2^{m-L-1}\}\right| = 2^{m-L-1} + 1.$$

□

We conjecture that for all $0 \le r \le 2^L - 1$

$$\left|\mathcal{B}^\ell_{L,r}(m)\right| \le \left|\mathcal{B}^\ell_{L,0}(m)\right|. \tag{3.29}$$

Using Lemma 3.8 and conjecture (3.29) yields formula (3.23). For convenience we give the argument. The Carleson constant of $\tau_N(\mathcal{E}^N_{\ell,0})$ is defined as follows

$$[\![\tau_N(\mathcal{E}^N_{\ell,0})]\!] = \sup_{I\in\tau_N(\mathcal{E}^N_{\ell,0})} \frac{1}{|I|} \sum_{\substack{J\subseteq I\\ J\in\tau_N(\mathcal{E}^N_{\ell,0})}} |J|. \tag{3.30}$$

We can rewrite the sum on the right-hand side as follows

$$[\![\tau_N(\mathcal{E}^N_{\ell,0})]\!] = \sup\Big\{\frac{1}{|I_{L,r}|} \sum_{\substack{J\subseteq I_{L,r}\\ J\in\tau_N(\mathcal{E}^N_{\ell,0})}} |J|\Big\}, \tag{3.31}$$

where the supremum is taken over the set

$$\{(L,r)\in\mathbb{N}_0^2 : 0\le L\le N-\ell,\ 0\le r\le 2^L-1,\ I_{L,r}\in\tau_N(\mathcal{E}^N_{\ell,0})\}.$$

We can rewrite the sum on the right hand side as follows

$$[\![\tau_N(\mathcal{E}^N_{\ell,0})]\!] = \sup\Big\{2^L\sum_{m=L}^{N} \big|\mathcal{B}^\ell_{L,r}(m)\big|2^{-m}\Big\}, \tag{3.32}$$

where the supremum is taken over the same set as above.

Lemma 3.8 yields $\big|\mathcal{B}^\ell_{L,r}(m)\big| = 0$ for all $m > N-\ell$ and conjecture 3.29 yields $\big|\mathcal{B}^\ell_{L,r}(m)\big| \le \big|\mathcal{B}^\ell_{L,0}(m)\big|$. Hence, we have

$$[\![\tau_N(\mathcal{E}^N_{\ell,0})]\!] = \sup\Big\{2^L\sum_{m=L}^{N-\ell} \big|\mathcal{B}^\ell_{L,0}(m)\big|2^{-m}\Big\}, \tag{3.33}$$

where the supremum is taken over the set $\{L\in\mathbb{N}_0 : 0\le L\le N-\ell\}$. Note that $I_{L,0}\in\tau_N(\mathcal{E}^N_{\ell,0})$ for all $0\le L\le N-\ell$.

Let $f(L) = 2^L\sum_{m=L}^{N-\ell}\big|\mathcal{B}^\ell_{L,0}(m)\big|2^{-m}$. Then by Lemma 3.8

$$f(0) = 1 + \frac{N-\ell}{2}$$

and for $1\le L\le N-\ell$

$$f(L) = \frac{3}{2} + \frac{N-\ell-L+1}{2} - 2^{-N+\ell+L}.$$

$f(L)$ is monotonically increasing for $1\le L\le N-\ell$ and $f(1)\ge f(0)$. Hence, $f(1)\ge f(L)$ for all $0\le L\le N-\ell$ and

$$[\![\tau_N(\mathcal{E}^N_{\ell,0})]\!] = f(1) = \frac{3}{2} + \frac{N-\ell}{2} - 2^{-N+\ell+1}.$$

This gives equation 3.23.

Now we consider those dyadic trees in $\mathcal{D}_N$ that are mapped under the postorder rearrangement τ_N onto collections of disjoint dyadic intervals.

THEOREM 3.9. *Let $N \in \mathbb{N}$, $0 < \ell \leq N$ and $0 < k \leq 2^{\ell-1}$. Then*

$$\llbracket \tau_N(\mathcal{T}_{\ell,k}^N) \rrbracket = \llbracket \tau_N(\mathcal{E}_{\ell,k}^N) \rrbracket = 1. \tag{3.34}$$

PROOF. The complete dyadic subtree $\mathcal{T}_{\ell,k}^N$ is a postorder order interval, cf. Section 1.2.8, given by the dyadic intervals $I_{N,k2^{N-\ell}}$ and $I_{\ell,k}$, i.e.

$$\mathcal{T}_{\ell,k}^N = \mathcal{B}^N(I_{N,k2^{N-\ell}}, I_{\ell,k}).$$

The postorder rearrangement τ_N maps postorder order intervals onto lexicographic order intervals, cf. Section 1.2.8, given by the dyadic intervals $\tau_N(I_{N,k2^{N-\ell}})$ and $\tau_N(I_{\ell,k})$, i.e.

$$\tau_N(\mathcal{T}_{\ell,k}^N) = \mathcal{E}(\tau_N(I_{N,k2^{N-\ell}}), \tau_N(I_{\ell,k})).$$

We associate the intervals $I_{N,k2^{N-\ell}}$ and $I_{\ell,k}$ with their postorder ordinal numbers $a^N(k2^{N-\ell})$ and $a^\ell(k)$. Lemma 3.3 yields that

$$\operatorname{Level} a^N(k2^{N-\ell}) = \operatorname{Level} a^\ell(k) = \lceil \log_2 (k+1) \rceil + N - \ell.$$

Hence, $\tau_N(I_{N,k2^{N-\ell}}) = I_{s+N-\ell,K_1}$ and $\tau_N(I_{\ell,k}) = I_{s+N-\ell,K_2}$, where $s = \lceil \log_2 (k+1) \rceil$, $K_1 = \operatorname{Pos}(a^N(k2^{N-\ell}))$ and $K_2 = \operatorname{Pos} a^\ell(k)$. Summarizing we have

$$\tau_N(\mathcal{T}_{\ell,k}^N) = \mathcal{E}(I_{s+N-\ell,K_1}, I_{s+N-\ell,K_2}). \tag{3.35}$$

By the definition of the lexicographic order interval we have for all $I \in \tau_N(\mathcal{T}_{\ell,k}^N)$ that $|I| = \lceil \log_2 (k+1) \rceil + N - \ell$. Therefore, $\tau_N(\mathcal{T}_{\ell,k}^N)$ is a collection of disjoint dyadic intervals with $\llbracket \tau_N(\mathcal{T}_{\ell,k}^N) \rrbracket = 1$. Since $\mathcal{E}_{\ell,k}^N \subseteq \mathcal{T}_{\ell,k}^N$, it follows that $\tau_N(\mathcal{E}_{\ell,k}^N) \subseteq \tau_N(\mathcal{T}_{\ell,k}^N)$. Therefore, $\tau_N(\mathcal{E}_{\ell,k}^N)$ is also a collection of disjoint dyadic intervals with $\llbracket \tau_N(\mathcal{E}_{\ell,k}^N) \rrbracket = 1$. □

3.2.3. The proof of Theorem 3.1. Finally, we have all ingredients that we need to prove the statement of Theorem 3.1. For convenience we recall the statement. The rearrangement operators $T = T_{\tau_N}\big|_{\mathcal{M}(\mathcal{T}_{\ell,0}^N)}$ and $T = T_{\sigma_N}\big|_{\mathcal{M}(\mathcal{D}_{N-\ell})}$ satisfy the following norm estimates

$$\frac{1}{\sqrt{2}}(N - \ell + 1)^{\frac{1}{2}} \leq \|T\|_{\mathrm{BMO}_N} \leq (N - \ell + 1)^{\frac{1}{2}}.$$

Recall that $\mathcal{M}(\mathcal{T}_{\ell,0}^N) = \operatorname{span}\{h_I : I \in \mathcal{T}_{\ell,0}^N\}$ and $\mathcal{M}(\mathcal{D}_{N-\ell}) = \operatorname{span}\{h_I : I \in \mathcal{D}_{N-\ell}\}$. The proof uses the norm estimates for rearrangement operators on BMO_N given in Section 1.4.4 and the estimates for Carleson constants given in Theorem 3.5 and Theorem 3.9.

PROOF OF THEOREM 3.1. Let $x \in \mathcal{M}(\mathcal{T}_{\ell,0}^N)$. The norm estimate (1.42) and the statement of Theorem 3.5 yield

$$\|T_{\tau_N} x\|_{\mathrm{BMO}}^2 \leq \llbracket \tau_N(\mathcal{T}_{\ell,0}^N) \rrbracket \|x\|_{\mathrm{BMO}}^2 \leq (N - \ell + 1)\|x\|_{\mathrm{BMO}}^2.$$

This gives the upper bound

$$\left\| T_{\tau_N}\big|_{\mathcal{M}(\mathcal{T}_{\ell,0}^N)} \right\|_{\mathrm{BMO}_N} \leq (N - \ell + 1)^{\frac{1}{2}}.$$

Equation (1.41) gives the lower bound

$$\left\| T_{\tau_N}\big|_{\mathcal{M}(\mathcal{T}^N_{\ell,0})} \right\|_{\mathrm{BMO}_N} \geq \sup_{\substack{\mathcal{C}\subseteq\mathcal{T}^N_{\ell,0},\\ \text{non-empty}}} \frac{[\![\tau_N(\mathcal{C})]\!]^{\frac{1}{2}}}{[\![\mathcal{C}]\!]^{\frac{1}{2}}}.$$

We consider the lowermost level $\mathcal{E}^N_{\ell,0}$ of the complete dyadic subtree $\mathcal{T}^N_{\ell,0}$ in $\mathcal{D}_N$. Obviously, $\mathcal{E}^N_{\ell,0} \subseteq \mathcal{T}^N_{\ell,0}$. We know that $[\![\mathcal{E}^N_{\ell,0}]\!] = 1$. Hence, by the statement of Theorem 3.5 we have

$$\left\| T_{\tau_N}\big|_{\mathcal{M}(\mathcal{T}^N_{\ell,0})} \right\|^2_{\mathrm{BMO}_N} \geq [\![\tau_N(\mathcal{E}^N_{\ell,0})]\!] \geq \frac{1}{2}(N-\ell+1).$$

Let $x \in \mathcal{M}(\mathcal{D}_{N-\ell})$. The norm estimate (1.42) yields

$$\|T_{\sigma_N} x\|^2_{\mathrm{BMO}} \leq [\![\sigma_N(\mathcal{D}_{N-\ell})]\!] \|x\|^2_{\mathrm{BMO}}.$$

By the proof of Theorem 3.5 it follows that $\tau_N(\mathcal{T}^N_{\ell,0}) = \mathcal{D}_{N-\ell}$. Since $\sigma_N = \tau_N^{-1}$, we have $\sigma_N(\mathcal{D}_{N-\ell}) = \mathcal{T}^N_{\ell,0}$ and $[\![\sigma_N(\mathcal{D}_{N-\ell})]\!] = [\![\mathcal{T}^N_{\ell,0}]\!] = N-\ell+1$. Hence,

$$\left\| T_{\sigma_N}\big|_{\mathcal{M}(\mathcal{D}_{N-\ell})} \right\|_{\mathrm{BMO}_N} \leq (N-\ell+1)^{\frac{1}{2}}.$$

Equation (1.41) gives the lower bound

$$\left\| T_{\sigma_N}\big|_{\mathcal{M}(\mathcal{D}_{N-\ell})} \right\|_{\mathrm{BMO}_N} \geq \sup_{\substack{\mathcal{C}\subseteq\mathcal{D}_{N-\ell},\\ \text{non-empty}}} \frac{[\![\sigma_N(\mathcal{C})]\!]^{\frac{1}{2}}}{[\![\mathcal{C}]\!]^{\frac{1}{2}}}.$$

Let $\ell < N$. The proof of Theorem 3.9 asserts that $\tau_N(\mathcal{T}^N_{\ell+1,1}) \subseteq \mathcal{D}_{N-\ell}$ and $[\![\tau_N(\mathcal{T}^N_{\ell+1,1})]\!] = 1$. Hence, we have the lower bound

$$\left\| T_{\tau_N}\big|_{\mathcal{M}(\mathcal{D}_{N-\ell})} \right\|^2_{\mathrm{BMO}_N} \geq [\![\mathcal{T}^N_{\ell+1,1}]\!] = N-\ell \geq \frac{1}{2}(N-\ell+1).$$

The case $\ell = N$ is trivial. □

3.3. Postorder rearrangement operators on dyadic Hardy spaces

The duality relation of H^1 and BMO, in particular the norm equivalence in equation (1.43), and the interpolation resp. extrapolation procedure in Theorem 1.15 and Corollary 1.16 give equivalent norm estimates as in Theorem 3.1 for the rearrangement operators on H^p_N, $0 < p < \infty$, given by

$$T_{\tau_N,p}\colon \frac{h_I}{|I|^{\frac{1}{p}}} \mapsto \frac{h_{\tau_N(I)}}{|\tau_N(I)|^{\frac{1}{p}}} \quad \text{resp.} \quad T_{\sigma_N,p}\colon \frac{h_I}{|I|^{\frac{1}{p}}} \mapsto \frac{h_{\sigma_N(I)}}{|\sigma_N(I)|^{\frac{1}{p}}}.$$

In the following corollaries we give norm estimates for these rearrangement operators applied to functions with Haar support in the sets

$$\mathcal{T}^N_{\ell,0} = \{I \in \mathcal{D}_N : I \subseteq I_{\ell,0}\}$$

and $\mathcal{D}_{N-\ell}$. Recall that

$$\mathcal{M}(\mathcal{T}^N_{\ell,0}) = \operatorname{span}\{h_I : I \in \mathcal{T}^N_{\ell,0}\}$$

and
$$\mathcal{M}(\mathcal{D}_{N-\ell}) = \operatorname{span}\{h_I : I \in \mathcal{D}_{N-\ell}\}.$$

COROLLARY 3.10. *Let $N \in \mathbb{N}_0$ and $0 \le \ell \le N$. Let $T = T_{\sigma_N,1}\big|_{\mathcal{M}(\mathcal{D}_{N-\ell})}$ or $T = T_{\tau_N,1}\big|_{\mathcal{M}(\mathcal{T}^N_{\ell,0})}$. Then*

$$\frac{1}{C_F}\frac{1}{\sqrt{2}}(N-\ell+1)^{\frac{1}{2}} \le \|T\|_{H^1_N} \le C_F(N-\ell+1)^{\frac{1}{2}}, \tag{3.36}$$

where $C_F = 2\sqrt{2}$ is the constant appearing in Fefferman's inequality (1.21).

PROOF. Recall that the adjoint of a rearrangement operator is again a rearrangement operator induced by the inverse rearrangement. Let τ be any bijective map on $\mathcal{D}_N$, then for every non-empty subset $\mathcal{C} \subseteq \mathcal{D}_N$ the operator $T_\tau\big|_{\mathcal{M}(\mathcal{C})}$ on BMO_N is the adjoint operator of $T_{\tau^{-1},1}\big|_{\mathcal{M}(\tau(\mathcal{C}))}$ on H^1_N. By equation (1.43) we have the following norm equivalences

$$\frac{1}{C_F}\Big\|T_\tau\big|_{\mathcal{M}(\mathcal{C})}\Big\|_{\mathrm{BMO}_N} \le \Big\|T_{\tau^{-1},1}\big|_{\mathcal{M}(\tau(\mathcal{C}))}\Big\|_{H^1_N} \le C_F\Big\|T_\tau\big|_{\mathcal{M}(\mathcal{C})}\Big\|_{\mathrm{BMO}_N}, \tag{3.37}$$

where $C_F = 2\sqrt{2}$ is the constant appearing in Fefferman's inequality (1.21).

Let $\tau = \tau_N$ and $\mathcal{C} = \mathcal{T}^N_{\ell,0}$. The proof of Theorem 3.5 asserts that $\tau_N(\mathcal{T}^N_{\ell,0}) = \mathcal{D}_{N-\ell}$. Since σ_N is the inverse of τ_N we have by equation (3.37)

$$\frac{1}{C_F}\Big\|T_{\tau_N}\big|_{\mathcal{M}(\mathcal{T}^N_{\ell,0})}\Big\|_{\mathrm{BMO}_N} \le \Big\|T_{\sigma_N,1}\big|_{\mathcal{M}(\mathcal{D}_{N-\ell})}\Big\|_{H^1_N} \le C_F\Big\|T_{\tau_N}\big|_{\mathcal{M}(\mathcal{T}^N_{\ell,0})}\Big\|_{\mathrm{BMO}_N}.$$

Let $\tau = \sigma_N$ and $\mathcal{C} = \mathcal{D}_{N-\ell}$. Since $\sigma_N = \tau_N^{-1}$ we have $\sigma_N(\mathcal{D}_{N-\ell}) = \mathcal{T}^N_{\ell,0}$ and equation (3.37) yields

$$\frac{1}{C_F}\Big\|T_{\sigma_N}\big|_{\mathcal{M}(\mathcal{D}_{N-\ell})}\Big\|_{\mathrm{BMO}_N} \le \Big\|T_{\tau_N,1}\big|_{\mathcal{M}(\mathcal{T}^N_{\ell,0})}\Big\|_{H^1_N} \le C_F\Big\|T_{\sigma_N}\big|_{\mathcal{M}(\mathcal{D}_{N-\ell})}\Big\|_{\mathrm{BMO}_N}.$$

The estimates in Theorem 3.1 yield equation (3.36). □

COROLLARY 3.11. *For all $0 < p < \infty$ there exists a constant C_p such that for all $N \in \mathbb{N}_0$, $0 \le \ell \le N$ and $T = T_{\tau_N,p}\big|_{\mathcal{M}(\mathcal{T}^N_{\ell,0})}$ or $T = T_{\sigma_N,p}\big|_{\mathcal{M}(\mathcal{D}_{N-\ell})}$ the following holds*

$$\frac{2^{-\left|\frac{1}{p}-\frac{1}{2}\right|}}{C_p}(N-\ell+1)^{\left|\frac{1}{p}-\frac{1}{2}\right|} \le \|T\|_{H^p_N} \le C_p\,(N-\ell+1)^{\left|\frac{1}{p}-\frac{1}{2}\right|}. \tag{3.38}$$

PROOF OF COROLLARY 3.11. **Case 1:** $0 < p < 2$. Let $T = T_{\sigma_N,p}\big|_{\mathcal{M}(\mathcal{D}_{N-\ell})}$ or $T = T_{\tau_N,p}\big|_{\mathcal{M}(\mathcal{T}^N_{\ell,0})}$. Let $s = 1$ in Theorem 1.15. Then by the upper estimate in (3.36) we have for $1 < p < 2$

$$\|T\|_{H^p_N} \le c_p\, C_F^{\frac{2}{p}-1}(N-\ell+1)^{\frac{1}{p}-\frac{1}{2}},$$

where c_p is the constant appearing in Theorem 1.15. By the lower estimate in (3.36) we get

$$\|T\|_{H_N^p} \geq \frac{1}{C_F^{\frac{2}{p}-1} c_p} 2^{\frac{1}{2}-\frac{1}{p}} (N+1)^{\frac{1}{p}-\frac{1}{2}}.$$

Let $r = 1$ in Theorem 1.15. Then again by equation (3.36) we have for $0 < p < 1$ the following estimates

$$\frac{2^{\frac{1}{2}-\frac{1}{p}}}{(c_p C_F)^{\frac{2}{p}-1}} (N-\ell+1)^{\frac{1}{p}-\frac{1}{2}} \leq \|T\|_{H_N^p} \leq (c_p C_F)^{\frac{2}{p}-1} (N-\ell+1)^{\frac{1}{p}-\frac{1}{2}}. \tag{3.39}$$

Summarizing we get that for every $0 < p < 2$ there exists a constant C_p such that for $T = T_{\sigma_N,p}\big|_{\mathcal{M}(\mathcal{D}_{N-\ell})}$ or $T = T_{\tau_N,p}\big|_{\mathcal{M}(\mathcal{T}_{\ell,0}^N)}$

$$\frac{2^{\frac{1}{2}-\frac{1}{p}}}{C_p} (N-\ell+1)^{\frac{1}{p}-\frac{1}{2}} \leq \|T\|_{H_N^p} \leq C_p (N-\ell+1)^{\frac{1}{p}-\frac{1}{2}}. \tag{3.40}$$

Case 2: $2 < p < \infty$. Theorem 1.15 restricts the interpolation/extrapolation range to $0 < p < 2$. Therefore, we have to use a duality argument in order to obtain norm estimates for the rearrangement operators on H_N^p, $2 < p < \infty$. The operator $T_{\sigma_N,p}\big|_{\mathcal{M}(\mathcal{D}_{N-\ell})}$ on H_N^p, $2 < p < \infty$, is the adjoint operator of $T_{\tau_N,p'}\big|_{\mathcal{M}(\sigma_N(\mathcal{D}_{N-\ell}))}$ on $H_N^{p'}$, where $\frac{1}{p} + \frac{1}{p'} = 1$. Paley's theorem asserts that

$$\frac{1}{A_p}\Big\|T_{\sigma_N,p}\big|_{\mathcal{M}(\mathcal{D}_{N-\ell})}\Big\|_{H_N^p} \leq \Big\|T_{\tau_N,p'}\big|_{\mathcal{M}(\sigma_N(\mathcal{D}_{N-\ell}))}\Big\|_{H_N^{p'}} \leq A_p \Big\|T_{\sigma_N,p}\big|_{\mathcal{M}(\mathcal{D}_{N-\ell})}\Big\|_{H_N^p}.$$

Recall that $\sigma_N(\mathcal{D}_{N-\ell}) = \mathcal{T}_{\ell,0}^N$. Hence, the norm estimates in equation (3.40) yield for $2 < p < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$:

$$\frac{2^{\frac{1}{p}-\frac{1}{2}}}{A_p C_{p'}} (N-\ell+1)^{\frac{1}{2}-\frac{1}{p}} \leq \Big\|T_{\sigma_N,p}\big|_{\mathcal{M}(\mathcal{D}_{N-\ell})}\Big\|_{H_N^p} \leq A_p C_{p'} (N-\ell+1)^{\frac{1}{2}-\frac{1}{p}}.$$

The procedure for the rearrangement operators $T_{\tau_N,p}\big|_{\mathcal{M}(\mathcal{T}_{\ell,0}^N)}$, $2 < p < \infty$, proceeds analogously. Consider the adjoint operators $T_{\sigma_N,p'}\big|_{\mathcal{M}(\mathcal{D}_{N-\ell})}$ on $H_N^{p'}$, $\frac{1}{p} + \frac{1}{p'} = 1$, and use again the norm estimates (3.40). Summarizing we get the statement of Corollary 3.11. □

Remark 3.12. Note that the constant C_p in Corollary 3.11 consists of the constant $C_F = 2\sqrt{2}$ appearing in Fefferman's inequality (1.21), the constant A_p from Paley's theorem and the interpolation/extrapolation constant c_p appearing in Theorem 1.15. The case H_N^1 is obtained by the H^1-BMO duality (Corollary 3.10) without invoking H^p-duality, interpolation or extrapolation. Therefore, $C_1 = C_F$. The definition of rearrangement operators $T_{\tau,2}$ on H_N^2 yields $\|T_{\tau,2}\|_{H_N^2} = 1$ for any rearrangement τ. Hence, $C_2 = 1$ in the theorem above.

Corollary 3.11 gives, considering the general upper bound in Remark 1.17, the same extremality statement for the rearrangement operators $T = T_{\tau_N,p}$ resp. $T =$

$T_{\sigma_N,p}$ on the spaces H_N^p, $0<p<\infty$. For all $0<p<\infty$ there exists a constant B_p such that

$$\frac{2^{-\left|\frac{1}{p}-\frac{1}{2}\right|}}{B_p} R(H_N^p) \le \|T\|_{H_N^p} \le R(H_N^p), \tag{3.41}$$

where $R(H_N^p)=\sup\Big\{\|T_\tau\colon H_N^p\to H_N^p\| : \tau\colon \mathcal{D}_N\to\mathcal{D}_N \text{ bijective}\Big\}$.

3.4. Postorder rearrangement operators on Triebel-Lizorkin spaces

By convexification of Hardy spaces we can extend the norm estimates for rearrangement operators on Hardy spaces H_N^p to the more general Triebel-Lizorkin spaces $f_{p,N}^q$.

Let $0<p,q<\infty$ and $f\in f_{p,N}^q$ with Haar expansion $f=\sum_{I\in\mathcal{D}_N} a_I h_I$. Recall from Section 1.3.2 that f_p^q is the $\frac{q}{2}$-convexification of the Hardy space $H^{\frac{2p}{q}}$. Hence, we have the following identification

$$\|f\|_{f_p^q}=\Big\||f|^{\frac{q}{2}}\Big\|_{H^{\frac{2p}{q}}}^{\frac{2}{q}}, \tag{3.42}$$

where

$$|f|^{\frac{q}{2}}=\sum_{I\in\mathcal{D}_N}|a_I|^{\frac{q}{2}}h_I. \tag{3.43}$$

Let τ be an arbitrary bijective map on $\mathcal{D}_N$. On $f_{p,N}^q$ we consider again rearrangements of the L^p-normalised Haar system, given by the rearrangement operator

$$T_{\tau,p}: \frac{h_I}{|I|^{\frac{1}{p}}}\mapsto\frac{h_{\tau(I)}}{|\tau(I)|^{\frac{1}{p}}}.$$

Note that $\|h_I\|_{f_p^q}=|I|^{-\frac{1}{p}}$. Using equation (3.43) we have

$$|T_{\tau,p}(f)|^{\frac{q}{2}}=\sum_{I\in\mathcal{D}_N}|a_I|^{\frac{q}{2}}\frac{|I|^{\frac{q}{2p}}}{|\tau(I)|^{\frac{q}{2p}}}h_{\tau(I)}$$

and

$$T_{\tau,\frac{2p}{q}}(|f|^{\frac{q}{2}})=\sum_{I\in\mathcal{D}_N}|a_I|^{\frac{q}{2}}\frac{|I|^{\frac{q}{2p}}}{|\tau(I)|^{\frac{q}{2p}}}h_{\tau(I)}.$$

Hence,

$$|T_{\tau,p}(f)|^{\frac{q}{2}}=T_{\tau,\frac{2p}{q}}(|f|^{\frac{q}{2}}). \tag{3.44}$$

Using the identification (3.42) and equation (3.44) yields

$$\|T_{\tau,p}(f)\|_{f_p^q}=\Big\||T_{\tau,p}(f)|^{\frac{q}{2}}\Big\|_{H^{\frac{2p}{q}}}^{\frac{2}{q}}=\Big\|T_{\tau,\frac{2p}{q}}(|f|^{\frac{q}{2}})\Big\|_{H^{\frac{2p}{q}}}^{\frac{2}{q}} \tag{3.45}$$

and

$$\|T_{\tau,p}\|_{f_{p,N}^q}=\|T_{\tau,\frac{2p}{q}}\|_{H_N^{\frac{2p}{q}}}^{\frac{2}{q}}. \tag{3.46}$$

Applying these identifications to Corollary 3.11 yields the following statement for norm estimates of rearrangement operators on $f^q_{p,N}$.

COROLLARY 3.13. *For all* $0 < p, q < \infty$ *there exists a constant* $C_{p,q}$ *such that for all* $N \in \mathbb{N}_0$ *and* $0 \le \ell \le N$ *and for* $T = T_{\sigma_N,p}\big|_{\mathcal{M}(\mathcal{D}_{N-\ell})}$ *or* $T = T_{\tau_N,p}\big|_{\mathcal{M}(\mathcal{T}^N_{\ell,0})}$ *the following holds*

$$\frac{2^{-\left|\frac{1}{p}-\frac{1}{q}\right|}}{C_{p,q}}(N-\ell+1)^{\left|\frac{1}{p}-\frac{1}{q}\right|} \le \|T\|_{f^q_{p,N}} \le C_{p,q}(N-\ell+1)^{\left|\frac{1}{p}-\frac{1}{q}\right|}. \tag{3.47}$$

REMARK 3.14. By the above convexification procedure applied to the norm estimate in Remark 1.17 we have that for rearrangement operators T_τ induced by any rearrangement τ acting on $\mathcal{D}_N$ we have the upper bound

$$\|T_{\tau,p}\|_{f^q_{p,N}} \le C_{p,q}(N+1)^{\left|\frac{1}{p}-\frac{1}{q}\right|}.$$

Corollary 3.13 and the general upper bound in Remark 3.14 gives the extremality statement for the rearrangement operators $T = T_{\tau_N,p}$ resp. $T = T_{\sigma_N,p}$ on the spaces $f^q_{p,N}$. For all $0 < p, q < \infty$ there exists a constant $c_{p,q}$ such that for all $N \in \mathbb{N}_0$ and for $T = T_{\tau_N,p}$ or $T = T_{\sigma_N,p}$ we have

$$\frac{2^{-\left|\frac{1}{p}-\frac{1}{q}\right|}}{c_{p,q}} R(f^q_{p,N}) \le \|T\|_{f^q_{p,N}} \le R(f^q_{p,N}). \tag{3.48}$$

3.5. Postorder rearrangement operators on order intervals

Theorem 3.15 provides a tool that enables one to gain insight into the rearrangement operators T_{σ_N} applied to spaces of functions with Haar support in a lexicographic order interval $\mathcal{E} \subseteq \mathcal{D}_N$. In Theorem 3.1 we have already seen that on the lexicographic order interval $\mathcal{D}_{N-\ell}$, for some small ℓ, the operator has very large norm. Theorem 3.15 provides the possibility to determine canonical collections of dyadic intervals on which the rearrangement operator has small norm. The significance of the upper bound in Theorem 3.15 is given by the fact that $\log_2 \frac{1}{|L_1|}$ is able to compensate the term N. In order to obtain the upper bound in Theorem 3.15 we use a geometric representation of order intervals with respect to the postorder, $\preceq$. Hence, one can read off the upper bound from the tree representation of $\mathcal{D}_N$, cf. figure 6.

Recall that

$$\mathcal{M}(\mathcal{E}) = \operatorname{span}\{h_I : I \in \mathcal{E}\}.$$

THEOREM 3.15. *Let* $N \in \mathbb{N}_0$. *Let* $\mathcal{E} = \mathcal{E}(E_1, E_2)$ *be the lexicographic order interval given by the dyadic intervals* $E_1, E_2 \in \mathcal{D}_N$ *with* $E_1 \le_l E_2$. *Then*

$$\left\| T_{\sigma_N}\big|_{\mathcal{M}(\mathcal{E})} \right\|^2_{BMO_N} \le N - \log_2 \frac{1}{|L_1|} + 2, \tag{3.49}$$

where L_1 *is the maximal (with respect to inclusion) dyadic interval in the postorder order interval* $\mathcal{B}^N(\sigma_N(E_1), \sigma_N(E_2))$ *that contains the left endpoint* $\sigma_N(E_1)$.

Lexicographic order intervals $\mathcal{E}(E_1, E_2)$ with large Carleson constant are given by endpoints E_1, E_2 that satisfy the property that $\log_2 \frac{1}{|E_1|}$ is much smaller than $\log_2 \frac{1}{|E_2|}$, cf. equation (3.63). The upper bound in Theorem 3.15 depends for these order intervals only on the right endpoint E_2. Particularly, the upper bound is then given by

$$\left\| T_{\sigma_N}\big|_{\mathcal{M}(\mathcal{E})} \right\|^2_{\mathrm{BMO}_N} \leq \log_2 \frac{1}{|E_2|} + 2.$$

Proof of Theorem 3.15. As mentioned above, the proof of Theorem 3.15 uses a geometric representation of order intervals with respect to the postorder, $\preceq$. This geometric representation is given in Proposition 1.5 and Definition 1.4 as follows. For every postorder order interval

$$\mathcal{B}^N(I_1, I_2) = \{I \in \mathcal{D}_N : I_1 \preceq I \preceq I_2\},$$

there exists a collection of maximal intervals $\mathcal{L} = \{L_1, \ldots, L_m\}$ such that

$$\mathcal{B}^N(I_1, I_2) = \mathcal{C}(I_1, L_1) \cup \mathcal{R}(I_1, L_1) \cup_{i=2}^m \mathcal{M}_i,$$

where $\mathcal{C}(I_1, L_1)$ is the cone of dyadic intervals between I_1 and L_1, $\mathcal{R}(I_1, L_1)$ is the right fill-up of the cone and $\mathcal{M}_i$ is the complete dyadic subtree with root L_i given by $\mathcal{M}_i = \{I \in \mathcal{D}_N : I \subseteq L_i\}$.

For the norm estimate in Theorem 3.15 we need an estimate for the Carleson constant $[\![\mathcal{B}^N(I_1, I_2)]\!]$. By the geometric representation of $\mathcal{B}^N$ given above we have that the Carleson constant $[\![\mathcal{B}^N(I_1, I_2)]\!]$ is related to the Carleson constant of the cone and the right fill-up. Therefore, we start examining the Carleson constant $[\![\mathcal{C}(I, J) \cup \mathcal{R}(I, J)]\!]$ for two non-disjoint dyadic intervals $I, J \in \mathcal{D}_N$.

THEOREM 3.16. *Let $N \in \mathbb{N}_0$. Let $I, J \in \mathcal{D}_N$ and $I \subseteq J$. If $\mathcal{R}(I, J) \neq \emptyset$, then*

$$N - \log_2 \frac{1}{|I|} + 1 \leq [\![\mathcal{C}(I, J) \cup \mathcal{R}(I, J)]\!] \leq N - \log_2 \frac{1}{|J|} + 2. \tag{3.50}$$

PROOF. The definition of the Carleson constant (1.11) yields

$$[\![\mathcal{R}(I, J)]\!] \leq [\![\mathcal{C}(I, J) \cup \mathcal{R}(I, J)]\!] \leq [\![\mathcal{R}(I, J)]\!] + [\![\mathcal{C}(I, J)]\!]. \tag{3.51}$$

Recall that the cone $\mathcal{C}(I, J)$ is a collection of dyadic intervals $\mathcal{C} = \{C_1, \ldots, C_n\}$, where $n = \log_2 \frac{|J|}{|I|} + 1$, that satisfies the following properties: $C_1 = I$, $C_n = J$, $|C_i| = \frac{1}{2}|C_{i+1}|$ and $C_i \subset C_{i+1}$ for $1 \leq i \leq n - 1$. This yields

$$[\![\mathcal{C}(I, J)]\!] = \sup_{i=1,\ldots,n} \frac{1}{|C_i|} \sum_{J \subseteq C_i,\, J \in \mathcal{C}} |J| = \sup_{i=1,\ldots,n} \frac{1}{|C_i|} \sum_{s=1}^{i} |C_s|.$$

Since $|C_i| = 2^{i-1}|C_1|$, it follows that $[\![\mathcal{C}(I, J)]\!] \leq 2$.
The right fill-up $\mathcal{R}(I, J)$ of the cone is the collection of dyadic intervals $\bigcup_{i=1}^{n-1} \mathcal{U}_{i+1}$, where $\mathcal{U}_{i+1} = \emptyset$, if C_i is the right half of C_{i+1} and $\mathcal{U}_{i+1} = \{U \in \mathcal{D}_N : U \subseteq C_{i+1} \setminus C_i\}$, if C_i is the left half of C_{i+1}. Note that by definition $\mathcal{U}_i \cap \mathcal{U}_j = \emptyset$ for every $i \neq j$. Therefore,

$$[\![\mathcal{R}(I, J)]\!] = \sup_{i=1,\ldots,n-1} [\![\mathcal{U}_{i+1}]\!]. \tag{3.52}$$

If $\mathcal{U}_{i+1} \neq \emptyset$, then $\mathcal{U}_{i+1}$ is a dyadic subtree of $\mathcal{D}_N$ with root $C_{i+1} \setminus C_i$ and depth $N - \log_2 \frac{1}{|C_{i+1} \setminus C_i|}$. We know that $|C_{i+1} \setminus C_i| = |C_i|$ and $|C_i| = 2^{i-1}|I|$. Therefore, by equation (1.13) we have $[\![\mathcal{U}_{i+1}]\!] = N - \log_2 \frac{2^{1-i}}{|I|} + 1 = N + i - \log_2 \frac{1}{|I|}$. Hence, by equation (3.52)

$$N + 1 - \log_2 \frac{1}{|I|} \leq [\![\mathcal{R}(I, J)]\!] \leq N + n - 1 - \log_2 \frac{1}{|I|}.$$

Recall that $n = \log_2 \frac{|J|}{|I|} + 1$. This gives the upper bound

$$[\![\mathcal{R}(I, J)]\!] \leq N - \log_2 \frac{1}{|J|}.$$

Summarizing we have (3.50). □

The statement of Proposition 1.5 and the estimates from Theorem 3.16 yield the following estimates for the Carleson constant $[\![\mathcal{B}^N(I_1, I_2)]\!]$.

THEOREM 3.17. *Let $N \in \mathbb{N}_0$ and $\mathcal{B}^N(I_1, I_2)$ the postorder order interval given by $I_1, I_2 \in \mathcal{D}_N$ with $I_1 \preceq I_2$. Let L_1 be the maximal interval in $\mathcal{B}^N(I_1, I_2)$ such that $I_1 \subseteq L_1$. Then*

$$N - \log_2 \frac{1}{|I_1|} + 1 \leq [\![\mathcal{B}^N(I_1, I_2)]\!] \leq N - \log_2 \frac{1}{|L_1|} + 2. \tag{3.53}$$

PROOF. Let $\mathcal{L} = \{L_1, \ldots, L_m\}$ be the maximal (with respect to inclusion) elements of $\mathcal{B}^N(I_1, I_2)$, as given in Proposition 1.5 . Since

$$\mathcal{B}^N(I_1, I_2) = \mathcal{C}(I_1, L_1) \cup \mathcal{R}(I_1, L_1) \cup_{i=2}^m \mathcal{M}_i,$$

where $\mathcal{M}_i = \{I \in \mathcal{D}_N : I \subseteq L_i\}$, and since $\mathcal{M}_i \cap \mathcal{M}_j = \emptyset$ for all $i \neq j$ and $\mathcal{M}_i \cap (\mathcal{C}(I_1, L_1) \cup \mathcal{R}(I_1, L_1)) = \emptyset$ for all i, we have

$$[\![\mathcal{B}^N(I_1, I_2)]\!] = \max\{[\![\mathcal{C}(I_1, L_1) \cup \mathcal{R}(I_1, L_1)]\!], \max_{i=2,\ldots,m} [\![\mathcal{M}_i]\!]\}. \tag{3.54}$$

If $I_1 \subseteq I_2$, then there is only one maximal interval $L_1 = I_2$. Hence, $m = 1$ and $I \in \mathcal{B}^N(I_1, I_2)$ if and only if $I \in \mathcal{C}(I_1, I_2) \cup \mathcal{R}(I_1, I_2)$, cf. proof of Proposition 1.5 in [**MS97**]. Therefore, we have

$$[\![\mathcal{B}^N(I_1, I_2)]\!] = [\![\mathcal{C}(I_1, I_2) \cup \mathcal{R}(I_1, I_2)]\!]. \tag{3.55}$$

Theorem 3.16 yields the statement.

If $I_1 \cap I_2 = \emptyset$, then there exist maximal intervals $\mathcal{L} = \{L_1, \ldots, L_m\}$, $m \geq 2$. For $2 \leq i \leq m$, $\mathcal{M}_i$ is a dyadic subtree of $\mathcal{D}_N$ with root L_i and depth $N - \log_2 \frac{1}{|L_i|}$. Equation (1.13) gives the Carleson constant $[\![\mathcal{M}_i]\!] = N - \log_2 \frac{1}{|L_i|} + 1$. Proposition 1.5 yields $|L_m| \leq |L_{m-1}| < \cdots < |L_2|$. Hence, $[\![\mathcal{M}_2]\!] = \max_{i=2,\ldots,m} [\![\mathcal{M}_i]\!]$ and

$$[\![\mathcal{B}^N(I_1, I_2)]\!] = \max\{[\![\mathcal{C}(I_1, L_1) \cup \mathcal{R}(I_1, L_1)]\!], [\![\mathcal{M}_2]\!]\}.$$

By Theorem 3.16 we have the following lower and upper bound.

$$[\![\mathcal{B}^N(I_1, I_2)]\!] \geq \max\{N - \log_2 \frac{1}{|I_1|} + 1,\, N - \log_2 \frac{1}{|L_2|} + 1\} \tag{3.56}$$

and

$$[\![\mathcal{B}^N(I_1, I_2)]\!] \leq \max\{N - \log_2 \frac{1}{|L_1|} + 2,\ N - \log_2 \frac{1}{|L_2|} + 1\}. \tag{3.57}$$

$|L_2| \leq |L_1|$ by Proposition 1.5. Therefore, (3.57) yields the upper bound in (3.53). □

Now we have all ingredients for the proof of Theorem 3.15. For convenience we give the statement of the Theorem. We have the following operator norm estimate for the rearrangement operator T_{σ_N} acting on lexicographic order intervals $\mathcal{E}(E_1, E_2)$ given by the endpoints $E_1, E_2 \in \mathcal{D}_N$ with $E_1 \leq_l E_2$:

$$\left\| T_{\sigma_N}\big|_{\mathcal{M}(\mathcal{E})} \right\|^2_{\mathrm{BMO}_N} \leq N - \log_2 \frac{1}{|L_1|} + 2,$$

where L_1 is the maximal (with respect to inclusion) dyadic interval in the postorder order interval $B^N(\sigma_N(E_1), \sigma_N(E_2))$ that contains the left endpoint $\sigma_N(E_1)$. Recall that

$$\mathcal{M}(\mathcal{E}) = \operatorname{span}\{h_I : I \in \mathcal{E}\}.$$

Proof of Theorem 3.15. Let $x \in \mathcal{M}(\mathcal{E})$. The estimates of rearrangement operators on BMO_N in Section 1.4.4 give the upper bound

$$\|T_{\sigma_N} x\|^2_{\mathrm{BMO}} \leq [\![\sigma_N(\mathcal{E})]\!]\, \|x\|^2_{\mathrm{BMO}}. \tag{3.58}$$

σ_N is the bijective map on $\mathcal{D}_N$ that maps lexicographic order intervals onto postorder order intervals. Hence, for every lexicographic order interval $\mathcal{E} = \mathcal{E}(E_1, E_2)$ there exists a unique postorder order interval $\mathcal{B} = \mathcal{B}^N(\sigma_N(E_1), \sigma_N(E_2))$ so that $\sigma_N(\mathcal{E}) = \mathcal{B}$. Hence, by equation (3.58) and Theorem 3.17 we have

$$\|T_{\sigma_N} x\|^2_{\mathrm{BMO}} \leq [\![\mathcal{B}]\!]\, \|x\|^2_{\mathrm{BMO}} \leq (N - \log_2 \frac{1}{|L_1|} + 2)\, \|x\|^2_{\mathrm{BMO}}, \tag{3.59}$$

where L_1 is the maximal interval in $\mathcal{B}^N(\sigma_N(E_1), \sigma_N(E_2))$ with $\sigma_N(E_1) \subseteq L_1$. □

Theorem 3.18 gives insight into the rearrangement operators T_{τ_N} applied to spaces of functions with Haar support in a postorder order interval $\mathcal{B} \subseteq \mathcal{D}_N$.

Recall that the postorder rearrangement τ_N maps postorder order intervals onto lexicographic order intervals. Hence, the upper norm estimate in Theorem 3.18 is only dependent on the characteristics of the corresponding lexicographic order interval. Obviously, the right-hand side does not depend on the depth N of the set of dyadic intervals $\mathcal{D}_N$. Recall that

$$\mathcal{M}(\mathcal{B}) = \operatorname{span}\{h_I : I \in \mathcal{B}\}.$$

Theorem 3.18. *Let $N \in \mathbb{N}_0$. Let $\mathcal{B} = \mathcal{B}^N(I_1, I_2)$ be the postorder order interval associated to the dyadic intervals $I_1, I_2 \in \mathcal{D}_N$ with $I_1 \preceq I_2$. Then*

$$\left\| T_{\tau_N}\big|_{\mathcal{M}(\mathcal{B})} \right\|^2_{BMO_N} \leq \log_2 \frac{|I_2|}{|I_1|} + \log_2 \left(\frac{k_2 + 1}{k_1 + 1} \right) + 2, \tag{3.60}$$

where k_1 resp. k_2 is the position of I_1 resp. I_2 within the dyadic tree $\mathcal{D}_N$.

Remark 3.19. Note that due to the relation $I_1 \preceq I_2$ the right-hand side in Theorem 3.18 is always greater than or equal to one. Theorem 3.21 gives postorder order intervals $\mathcal{B}$ such that the operator norm in (3.60) can be bounded by one.

Note that the proof of Theorem 3.18 gives a slightly better upper bound

$$\left\| T_{\tau_N}\big|_{\mathcal{M}(\mathcal{B})} \right\|_{\mathrm{BMO}_N}^2 \leq \log_2 \frac{|I_2|}{|I_1|} + \lceil \log_2 (k_2+1) \rceil - \lceil \log_2 (k_1+1) \rceil + 1. \tag{3.61}$$

Example 3.20. Let $N \in \mathbb{N}_0$ and $0 \leq \ell \leq N$. Let

$$\mathcal{B} = \mathcal{T}_{\ell,0}^N = \{I \in \mathcal{D}_N : I \subseteq I_{\ell,0}\}.$$

Then $I_1 = I_{N,0}$ and $I_2 = I_{\ell,0}$. Hence, $k_1 = k_2 = 0$ and by equation (3.60) we have

$$\left\| T_{\tau_N}\big|_{\mathcal{M}(\mathcal{B})} \right\|_{\mathrm{BMO}_N}^2 \leq \log_2 \frac{|I_2|}{|I_1|} + \log_2 \left(\frac{k_2+1}{k_1+1} \right) + 2 = N - \ell + 2.$$

Equation (3.61) yields

$$\left\| T_{\tau_N}\big|_{\mathcal{M}(\mathcal{B})} \right\|_{\mathrm{BMO}_N}^2 \leq N - \ell + 1.$$

This is the same result as in Theorem 3.1.

In Theorem 3.1 we have already seen that on the postorder order interval $\mathcal{T}_{\ell,0}^N$, for some small ℓ, the operator T_{τ_N} has very large norm. In Theorem 3.21 we determine postorder order intervals $\mathcal{B} \subseteq \mathcal{D}_N$ with large Carleson constants such that the functions with Haar support in $\mathcal{B}$ minimize the operator norm of the rearrangement operator T_{τ_N} on BMO_N.

Theorem 3.21. *Let $N \in \mathbb{N}_0$. Let $\mathcal{B}$ be a postorder order interval in $\mathcal{D}_N$ such that $\mathcal{B} \subseteq \mathcal{T}_{\ell+1,1}^N \cup I_{\ell,0}$ for some integer $0 \leq \ell \leq N$. Then*

$$\left\| T_{\tau_N}\big|_{\mathcal{M}(\mathcal{B})} \right\|_{BMO_N} = 1. \tag{3.62}$$

Proof of Theorem 3.18 and 3.21. Observe that the postorder rearrangement τ_N maps postorder order intervals onto lexicographic order intervals. Hence, the image $\tau_N(\mathcal{B}^N(I_1, I_2))$ is a lexicographic order interval $\mathcal{E}(E_1, E_2)$ given by the dyadic intervals $E_1 = \tau_N(I_1)$ and $E_2 = \tau_N(I_2)$. Obviously, if $|E_1| = |E_2|$, then $[\![\mathcal{E}(E_1, E_2)]\!] = 1$. If $|E_1| > |E_2|$ it is easy to see that

$$\log_2 \frac{|E_1|}{|E_2|} \leq [\![\mathcal{E}(E_1, E_2)]\!] \leq \log_2 \frac{|E_1|}{|E_2|} + 1. \tag{3.63}$$

In the following two theorems we give conditions on the dyadic intervals I_1 and I_2 such that $\tau_N(\mathcal{B}^N(I_1, I_2))$ forms on the one hand a disjoint collection of dyadic intervals (Theorem 3.22) and does not on the other hand (Theorem 3.23). Then we use the estimates for the Carleson constant of lexicographic order intervals and convert them into estimates for $[\![\tau_N(\mathcal{B}^N(I_1, I_2))]\!]$.

Recall from Section 3.2.2 that

$$\mathcal{T}_{\ell,k}^N = \{I \in \mathcal{D}_N : I \subseteq I_{\ell,k}\}.$$

THEOREM 3.22. *Let $N \in \mathbb{N}_0$. Let $\mathcal{B}$ be a postorder order interval in $\mathcal{D}_N$. Then $[\![\tau_N(\mathcal{B})]\!] = 1$ if and only if there exists an integer $0 \leq \ell \leq N$ such that $\mathcal{B} \subseteq \mathcal{T}^N_{\ell+1,1} \cup I_{\ell,0}$.*

PROOF. Let $\mathcal{B} \subseteq \mathcal{T}^N_{\ell+1,1} \cup I_{\ell,0}$ for some integer $0 \leq \ell \leq N$. Then $\tau_N(\mathcal{B}) \subseteq \tau_N(\mathcal{T}^N_{\ell+1,1} \cup I_{\ell,0})$. The proof of Theorem 3.9 yields

$$\tau_N(\mathcal{T}^N_{\ell+1,1}) = \mathcal{E}(I_{N-\ell,K_1}, I_{N-\ell,K_2}),$$

where $K_1 = \mathrm{Pos}(a^N(2^{N-\ell-1}))$ and $K_2 = \mathrm{Pos}(a^{\ell+1}(1))$. Lemma 3.4 asserts that

$$\mathrm{Pos}\left(a^N(2^{N-\ell-1})\right) = 0 \text{ and } \mathrm{Pos}\left(a^{\ell+1}(1)\right) = 2^{N-\ell} - 2.$$

Hence,

$$\tau_N(\mathcal{T}^N_{\ell+1,1}) = \mathcal{E}(I_{N-\ell,0}, I_{N-\ell,2^{N-\ell}-2}) = \{I_{N-\ell,k} : 0 \leq k \leq 2^{N-\ell-2}\}. \tag{3.64}$$

Lemma 3.3 and Lemma 3.4 yield $\mathrm{Level}\left(a^\ell(0)\right) = N - \ell$ and $\mathrm{Pos}\left(a^\ell(0)\right) = 2^{N-\ell} - 1$. Hence, $\tau_N(I_{\ell,0}) = I_{N-\ell,2^{N-\ell}-1}$ and

$$\tau_N(\mathcal{T}^N_{\ell+1,1} \cup I_{\ell,0}) = \left\{I_{N-\ell,k} : 0 \leq k \leq 2^{N-\ell} - 1\right\} = \mathcal{D}_{N-\ell} \setminus \mathcal{D}_{N-\ell-1}.$$

Note that $\mathcal{D}_{-1} = \emptyset$. Then we have $\tau_N(\mathcal{B}) \subseteq \mathcal{D}_{N-\ell} \setminus \mathcal{D}_{N-\ell-1}$ and therefore, $\tau_N(\mathcal{B})$ is a collection of disjoint dyadic intervals with $[\![\tau_N(\mathcal{B})]\!] = 1$.

On the other hand, let $[\![\tau_N(\mathcal{B})]\!] = 1$. Then $\tau_N(\mathcal{B})$ is either a disjoint collection of dyadic intervals or contains only one element. In both cases there exists an integer $0 \leq \ell \leq N$ such that $\tau_N(\mathcal{B}) \subseteq \mathcal{D}_{N-\ell} \setminus \mathcal{D}_{N-\ell-1}$. By the above we have $\mathcal{B} \subseteq \mathcal{T}^N_{\ell+1,1} \cup I_{\ell,0}$ for some $0 \leq \ell \leq N$. □

THEOREM 3.23. *Let $N \in \mathbb{N}_0$. Let $\mathcal{B}^N(I_1, I_2)$ be a postorder order interval in $\mathcal{D}_N$ given by the dyadic intervals $I_1, I_2 \in \mathcal{D}_N$ with $I_1 \preceq I_2$. If there exists an integer $0 \leq \ell < N$ such that $I_1 \in \mathcal{T}^N_{\ell+1,0}$ and $I_2 \in \mathcal{T}^N_{\ell+1,1} \cup I_{\ell,0}$, then*

$$\log_2 \frac{|I_2|}{|I_1|} + \log_2\left(\frac{k_2+1}{k_1+1}\right) - 1 \leq [\![\tau_N(\mathcal{B}^N(I_1, I_2))]\!] \leq \log_2 \frac{|I_2|}{|I_1|} + \log_2\left(\frac{k_2+1}{k_1+1}\right) + 2, \tag{3.65}$$

where k_i is the position of I_i in the dyadic tree $\mathcal{D}_N$.

PROOF. As mentioned before, the postorder rearrangement τ_N maps postorder order intervals $\mathcal{B}^N(I_1, I_2)$ into lexicographic order intervals $\mathcal{E}(E_1, E_2)$, where $E_1 = \tau_N(I_1)$ and $E_2 = \tau_N(I_2)$. We rewrite equation (3.63) in terms of the dyadic intervals I_1 and I_2, given by $E_1 = \tau_N(I_1)$ and $E_2 = \tau_N(I_2)$.

At first we show that $|\tau_N(I_1)| > |\tau_N(I_2)|$ if and only if there exists an integer $0 \leq \ell < N$ such that $I_1 \in \mathcal{T}^N_{\ell+1,0}$ and $I_2 \in \mathcal{T}^N_{\ell+1,1} \cup I_{\ell,0}$. We know from the proof of Theorem 3.22 that

$$\tau_N(\mathcal{T}^N_{\ell+1,1} \cup I_{\ell,0}) = \mathcal{D}_{N-\ell} \setminus \mathcal{D}_{N-\ell-1}$$

and from the proof of Theorem 3.5 that

$$\tau_N(\mathcal{T}^N_{\ell+1,0}) = \mathcal{D}_{N-\ell-1}.$$

Let $I_1 \in \mathcal{T}^N_{\ell+1,0}$ and $I_2 \in \mathcal{T}^N_{\ell+1,1} \cup I_{\ell,0}$, then $\tau_N(I_1) \subseteq \mathcal{D}_{N-\ell-1}$ and

$$\tau_N(I_2) \subseteq \mathcal{D}_{N-\ell} \setminus \mathcal{D}_{N-\ell-1}.$$

Hence, $|\tau_N(I_1)| > |\tau_N(I_2)|$. On the other hand, if $|\tau_N(I_1)| > |\tau_N(I_2)|$, then $\tau_N(I_1) \in \mathcal{D}_{L_1} \setminus \mathcal{D}_{L_1-1}$ and $\tau_N(I_2) \in \mathcal{D}_{L_2} \setminus \mathcal{D}_{L_2-1}$ with $L_1 < L_2$. Particularly, $L_i = \log_2 \frac{1}{|\tau_N(I_i)|}$, $i \in \{1, 2\}$. Hence, by the above $I_i \in \mathcal{T}^N_{N-L_i+1,1} \cup I_{N-L_i}$, $i \in \{1, 2\}$. Since $L_1 < L_2$, we have $\mathcal{T}^N_{N-L_1+1,1} \cup I_{N-L_1} \subseteq \mathcal{T}^N_{N-L_2+1,0}$.

The dyadic intervals $\tau_N(I_i)$ given by $\tau_N(I_i) = I_{L_i,K_i}$, $i \in \{1, 2\}$ can be determined from the dyadic intervals I_i as follows. Let $I_i = I_{\ell_i,k_i}$ and $a^{\ell_i}(k_i)$ its postorder ordinal number. Then $L_i = \text{Level}(a^{\ell_i}(k_i))$ and $K_i = \text{Pos}(a^{\ell_i}(k_i))$, defined in Section 3.2.1. Hence, by equation (3.63) and by the identification $\tau_N(\mathcal{B}^N(I_1, I_2)) = \mathcal{E}(\tau_N(I_1), \tau_N(I_2))$ we have

$$L_2 - L_1 \le [\![\tau_N(\mathcal{B}(I_1, I_2))]\!] \le L_2 - L_1 + 1, \qquad \text{if } |\tau_N(I_1)| > |\tau_N(I_2)|. \tag{3.66}$$

By Lemma 3.3, L_i, $i \in \{1, 2\}$ is given by

$$L_i = \text{Level}(a^{\ell_i}(k_i)) = \lceil \log_2 (k_i + 1) \rceil + N - \ell_i$$

and therefore, by the identification $\ell_2 = \log_2 \frac{1}{|I_2|}$ and $\ell_1 = \log_2 \frac{1}{|I_1|}$ we have

$$L_2 - L_1 = \log_2 \frac{|I_2|}{|I_1|} + \lceil \log_2 (k_2 + 1) \rceil - \lceil \log_2 (k_1 + 1) \rceil . \tag{3.67}$$

Since,

$$\log_2 (k_i + 1) \le \lceil \log_2 (k_i + 1) \rceil \le \log_2 (k_i + 1) + 1$$

equation (3.67) and (3.66) yield the statement 3.65. □

Now we can give the proof of Theorem 3.18 and Theorem 3.21. For convenience we give the statements. Let $\mathcal{B} = \mathcal{B}^N(I_1, I_2)$ be the postorder order interval associated to the dyadic intervals $I_1, I_2 \in \mathcal{D}_N$ with $I_1 \preceq I_2$. Theorem 3.18 states that the following norm estimate holds.

$$\left\| T_{\tau_N}\big|_{\mathcal{M}(\mathcal{B})} \right\|^2_{\text{BMO}_N} \le \log_2 \frac{|I_2|}{|I_1|} + \log_2 \left(\frac{k_2 + 1}{k_1 + 1} \right) + 1,$$

where k_1 resp. k_2 is the position of I_1 resp. I_2 within the dyadic tree $\mathcal{D}_N$. Theorem 3.21 states that if $\mathcal{B} \subseteq \mathcal{T}^N_{\ell+1,1} \cup I_{\ell,0}$ for some integer $0 \le \ell \le N$, then

$$\left\| T_{\tau_N}\big|_{\mathcal{M}(\mathcal{B})} \right\|_{\text{BMO}_N} = 1.$$

Recall that

$$\mathcal{M}(\mathcal{B}) = \text{span}\,\{h_I : I \in \mathcal{B}\}.$$

Proof of Theorem 3.18. Let $x \in \mathcal{M}(\mathcal{B})$. The estimates for rearrangement operators on BMO_N in Section 1.4.4 give the upper bound

$$\|T_{\tau_N} x\|^2_{\text{BMO}} \le [\![\tau_N(\mathcal{B})]\!] \, \|x\|^2_{\text{BMO}}. \tag{3.68}$$

Hence, by Theorem 3.23 and 3.21 we have

$$\|T_{\tau_N} x\|^2_{\text{BMO}} \le \left(\log_2 \frac{|I_2|}{|I_1|} + \log_2 \left(\frac{k_2 + 1}{k_1 + 1} \right) + 1 \right) \|x\|^2_{\text{BMO}}. \tag{3.69}$$

Note that the right-hand side in equation (3.65) is always greater than or equal to 1. □

Proof of Theorem 3.21. Let $\mathcal{B} \subseteq \mathcal{T}^N_{\ell+1,1} \cup I_{\ell,0}$ for some integer $0 \leq \ell \leq N$. We know from Section 1.4.4 that rearrangement operators on BMO_N induced by the postorder τ_N satisfy the following norm estimates.

$$1 \leq \left\| T_{\tau_N}\big|_{\mathcal{M}(\mathcal{B})} \right\|_{\mathrm{BMO}_N} \leq [\![\tau_N(\mathcal{B})]\!]^{\frac{1}{2}}. \tag{3.70}$$

Hence, Theorem 3.22 gives the statement. □

CHAPTER 4

Appendix to Chapter 2

Proof of the left-hand side of inequality (1.35). The left-hand side inequality of (1.35) was stated without proof in [**Mül12**]. Since we use this inequality repeatedly in this paper we provide the proof here. It uses the ideas of [**GM08**, Lemma 3.3], who in turn exploit the ideas of [**JO74**, Theorem 1] and [**Joh76**, p.336].

For the following definitions and statements we refer to [**Mül05**] and [**GM08**]. Let $\mathcal{E} \subseteq \mathcal{D}$ be a non-empty collection of dyadic intervals. We denote by $\mathcal{E}^*$ the set covered by $\mathcal{E}$, i.e. $\mathcal{E}^* = \bigcup_{I \in \mathcal{E}} I$. In the following we define consecutive generations of $\mathcal{E}$. We define $\mathcal{G}_0(\mathcal{E})$ to be the maximal dyadic intervals of $\mathcal{E}$, where maximal refers to inclusion. Note that the maximal intervals of a collection $\mathcal{E}$ are pairwise disjoint intervals and that $\mathcal{G}_0(\mathcal{E})$ covers the same set as $\mathcal{E}$. Suppose that we have already defined the generations $\mathcal{G}_0(\mathcal{E}), \dots, \mathcal{G}_{n-1}(\mathcal{E})$, then we define

$$\mathcal{G}_n(\mathcal{E}) = \mathcal{G}_0(\mathcal{E} \setminus (\mathcal{G}_0(\mathcal{E}) \cup \dots \cup \mathcal{G}_{n-1}(\mathcal{E}))).$$

Given $I \in \mathcal{D}$, let $I \cap \mathcal{E} = \{J \in \mathcal{E} : J \subseteq I\}$ and put

$$\mathcal{G}_\ell(I, \mathcal{E}) = \mathcal{G}_\ell(I \cap \mathcal{E}), \quad \text{for } \ell \in \mathbb{N}.$$

We fix pairwise disjoint blocks of dyadic intervals $\{\mathcal{C}(I) : I \in \mathcal{E}\}$ so that (4.1)-(4.3) hold:

$$[\![\mathcal{E}]\!] = \sup_{I \in \mathcal{E}} \frac{1}{|I|} \sum_{J \in \mathcal{E}, J \subseteq I} |J| < \infty, \tag{4.1}$$

$$\mathcal{C}(I)^* = I, \tag{4.2}$$

(4.3) The sigma algebra generated by $\{h_J : J \in \mathcal{C}(I)\}$ is purely atomic. We denote by $\mathcal{B}(I)$ the set of atoms.

For every $I \in \mathcal{E}$ we have

$$|\mathcal{G}_\ell^*(I, \mathcal{E})| \le 4 \cdot 2^{-\frac{2\ell}{4[\![\mathcal{E}]\!]+1}} |I|. \tag{4.4}$$

Therefore, by the properties above we get for every $I \in \mathcal{E}$ and every $B \in \mathcal{B}(I)$

$$|B \cap \mathcal{G}_\ell^*(I, \mathcal{E})| \le 4 \cdot 2^{-\frac{2\ell}{4[\![\mathcal{E}]\!]+1}} |B|. \tag{4.5}$$

Let $x_J \in X$, $J \in \mathcal{D}$ and put for $I \in \mathcal{E}$

$$u_I = \sum_{J \in \mathcal{C}(I)} x_J h_J.$$

Note that by property (4.3) u_I is constant on every atom $B \in \mathcal{B}(I)$.

Let $1 \le p < \infty$. Then for $u = \sum_{I\in\mathcal{E}} u_I$ we claim

$$\|u\|^p_{H^p_X} \le c_p \sum_{I\in\mathcal{E}} \|u_I\|^p_{H^p_X}. \tag{4.6}$$

In order to prove inequality (4.6) we define for each $I \in \mathcal{E}$ the set

$$A_I = I \setminus \bigcup_{J\in\mathcal{G}_1(I,\mathcal{E})} J.$$

Note that by construction $\{A_I : I \in \mathcal{E}\}$ is a collection of pairwise disjoint and measurable sets such that $\bigcup_{I\in\mathcal{E}} A_I = \mathcal{E}^*$, where $\mathcal{E}^*$ is the set covered by $\mathcal{E}$, cf.[**Mül12**, Proposition 1]. Therefore,

$$\Big\|\sum_{I\in\mathcal{E}} u_I\Big\|_{H^p_X} = \Big(\int_0^1 \mathbb{S}^p\Big(\sum_{I\in\mathcal{E}} u_I\Big)(t)dt\Big)^{\frac{1}{p}} = \Big(\sum_{K\in\mathcal{E}}\int_{A_K} \mathbb{S}^p\Big(\sum_{I\in\mathcal{E}} u_I\Big)(t)dt\Big)^{\frac{1}{p}}. \tag{4.7}$$

By the definition of A_I we get

$$\Big(\sum_{K\in\mathcal{E}}\int_{A_K} \mathbb{S}^p\Big(\sum_{I\in\mathcal{E}} u_I\Big)(t)dt\Big)^{\frac{1}{p}} = \Big(\sum_{K\in\mathcal{E}}\int_{A_K} \mathbb{S}^p\Big(\sum_{\substack{I\in\mathcal{E}\\ I\supseteq K}} u_I\Big)(t)dt\Big)^{\frac{1}{p}}. \tag{4.8}$$

We know that $\mathcal{G}_0(K,\mathcal{E}) = K$. There exists a shortest dyadic interval $\mathcal{G}_{-1}(K,\mathcal{E}) \in \mathcal{E}$ such that K is strictly contained in $\mathcal{G}_{-1}(K,\mathcal{E})$. Then there exists a shortest dyadic interval $\mathcal{G}_{-2}(K,\mathcal{E}) \in \mathcal{E}$ such that $\mathcal{G}_{-1}(K,\mathcal{E})$ is strictly contained in $\mathcal{G}_{-2}(K,\mathcal{E})$. We continue this pattern $n(K)$ steps until $\mathcal{G}_{-n(K)}(K,\mathcal{E})$ is a maximal interval in $\mathcal{E}$ and therefore not contained in any interval in $\mathcal{E}$. We have

$$K = \mathcal{G}_0(K,\mathcal{E}) \subset \mathcal{G}_{-1}(K,\mathcal{E}) \subset \mathcal{G}_{-2}(K,\mathcal{E}) \subset \cdots \subset \mathcal{G}_{-n(K)}(K,\mathcal{E}).$$

Thus

$$\sum_{I\in\mathcal{E},\, I\supseteq K} u_I = \sum_{\ell=0}^{n(K)} u_{\mathcal{G}_{-\ell}(K,\mathcal{E})}. \tag{4.9}$$

Summarizing the equations (4.7), (4.8) and (4.9) we have

$$\Big\|\sum_{I\in\mathcal{E}} u_I\Big\|_{H^p_X} = \Big(\sum_{K\in\mathcal{E}}\int_{A_K} \mathbb{S}^p\Big(\sum_{\ell=0}^{\infty} 1_{[0,n(K)]}(\ell) u_{\mathcal{G}_{-\ell}(K,\mathcal{E})}\Big)(t)dt\Big)^{\frac{1}{p}}. \tag{4.10}$$

Put $\mathcal{C}_{\ell,K} = \mathcal{C}(\mathcal{G}_{-\ell}(K,\mathcal{E}))$, then $u_{\mathcal{G}_{-\ell}(K,\mathcal{E})} = \sum_{J\in\mathcal{C}_{\ell,K}} x_J h_J$ and we have

$$\Big\|\sum_{I\in\mathcal{E}} u_I\Big\|_{H^p_X} = \Big(\sum_{K\in\mathcal{E}}\int_{A_K} \Big(\int_0^1 \Big\|\sum_{\ell=0}^{\infty} 1_{[0,n(K)]}(\ell)\sum_J x_J h_J(t) r_J(s)\Big\|_X^2 ds\Big)^{\frac{p}{2}} dt\Big)^{\frac{1}{p}}. \tag{4.11}$$

We define the sequence $(a_\ell)_{\ell=0}^\infty$ of elements in $L^p_{L^2_X}(A_K \times \mathcal{E}, dtdz)$, where dz is the counting measure on $\mathcal{E}$, as follows: $a_\ell(t,K) = 1_{[0,n(K)]}(\ell)\sum_J x_J h_J(t) r_J$. By the triangle inequality we get

$$\begin{aligned}\left(\sum_{K\in\mathcal{E}}\int_{A_K}\Big\|\sum_{\ell=0}^{\infty}a_\ell(t,K)\Big\|_{L^2_X}^{p}dt\right)^{\frac{1}{p}} &\le \sum_{\ell=0}^{\infty}\left(\sum_{K\in\mathcal{E}}\int_{A_K}\Big\|a_\ell(t,K)\Big\|_{L^2_X}^{p}dt\right)^{\frac{1}{p}}\\ &= \sum_{\ell=0}^{\infty}\left(\sum_{K\in\mathcal{E}}1_{[0,n(K)]}(\ell)\int_{A_K}\left(\int_0^1\Big\|\sum_{J\in\mathcal{C}_{\ell,K}}x_Jh_J(t)r_J(s)\Big\|_X^{2}ds\right)^{\frac{p}{2}}dt\right)^{\frac{1}{p}}\\ &= \sum_{\ell=0}^{\infty}\left(\sum_{K\in\mathcal{E}}1_{[0,n(K)]}(\ell)\int_{A_K}\mathbb{S}^p\left(u_{\mathcal{G}_{-\ell}(K,\mathcal{E})}\right)(t)dt\right)^{\frac{1}{p}}.\end{aligned}\tag{4.12}$$

Combining inequality (4.11) and (4.12) we obtain

$$\Big\|\sum_{I\in\mathcal{E}}u_I\Big\|_{H^p_X}\le\sum_{\ell=0}^{\infty}\left(\sum_{K\in\mathcal{E}}1_{[0,n(K)]}(\ell)\int_{A_K}\mathbb{S}^p\left(u_{\mathcal{G}_{-\ell}(K,\mathcal{E})}\right)(t)dt\right)^{\frac{1}{p}}.\tag{4.13}$$

By the definition of $\mathcal{G}_{-\ell}(K,\mathcal{E})$ we know that for each $\ell\in[0,n(K)]$ there exists a unique $I\in\mathcal{E}$ such that $\mathcal{G}_{-\ell}(K,\mathcal{E})=I$. Therefore,

$$\sum_{K\in\mathcal{E}}1_{[0,n(K)]}(\ell)\int_{A_K}\mathbb{S}^p\left(u_{\mathcal{G}_{-\ell}(K,\mathcal{E})}\right)(t)dt=\sum_{K\in\mathcal{E}}\sum_{\substack{I\in\mathcal{E}\\ \mathcal{G}_{-\ell}(K,\mathcal{E})=I}}\int_{A_K}\mathbb{S}^p\left(u_I\right)(t)dt.\tag{4.14}$$

Since the set $\{I,K\in\mathcal{E}:I=\mathcal{G}_{-\ell}(K,\mathcal{E})\}$ contains the same elements as $\{I,K\in\mathcal{E}: K\in\mathcal{G}_\ell(I,\mathcal{E})\}$ we can rewrite the sums in (4.14) and obtain with inequality (4.13):

$$\Big\|\sum_{I\in\mathcal{E}}u_I\Big\|_{H^p_X}\le\sum_{\ell=0}^{\infty}\left(\sum_{I\in\mathcal{E}}\sum_{K\in\mathcal{G}_\ell(I,\mathcal{E})}\int_K\mathbb{S}^p(u_I)(t)dt\right)^{\frac{1}{p}}.\tag{4.15}$$

We used above that $A_K\subseteq K$. For $\ell=0$ we obtain for the right-hand side in (4.15):

$$\left(\sum_{I\in\mathcal{E}}\int_I\mathbb{S}^p(u_I)(t)dt\right)^{\frac{1}{p}}=\left(\sum_{I\in\mathcal{E}}\|u_I\|_{H^p_X}^p\right)^{\frac{1}{p}}.\tag{4.16}$$

For $\ell\ge1$ we can rewrite the right-hand side in (4.15) as follows:

$$\sum_{K\in\mathcal{G}_\ell(I,\mathcal{E})}\int_K\mathbb{S}^p(u_I)(t)dt=\sum_{B\in\mathcal{B}(I)}\sum_{\substack{K\in\mathcal{G}_\ell(I,\mathcal{E})\\ K\subseteq B}}\int_K\mathbb{S}^p(u_I)(B)dt.\tag{4.17}$$

By (4.3) u_I is constant on each atom $B\in\mathcal{B}(I)$ and the term $\mathbb{S}^p(u_I)(B)$ is well-defined. Therefore, we get

$$\sum_{B\in\mathcal{B}(I)}\sum_{\substack{K\in\mathcal{G}_\ell(I,\mathcal{E})\\ K\subseteq B}}\int_K\mathbb{S}^p(u_I)(B)dt=\sum_{B\in\mathcal{B}(I)}\mathbb{S}^p(u_I)(B)\sum_{\substack{K\in\mathcal{G}_\ell(I,\mathcal{E})\\ K\subseteq B}}|K|.\tag{4.18}$$

By inequality (4.5) we have

$$\sum_{\substack{K\in\mathcal{G}_\ell(I,\mathcal{E})\\ K\subseteq B}} |K| = |B\cap \mathcal{G}^*_\ell(I,\mathcal{E})| \le 4\cdot 2^{-\frac{2\ell}{4[\![\mathcal{E}]\!]+1}}|B|.$$

We get the following estimate for the right-hand side in (4.18)

$$\begin{aligned}\sum_{B\in\mathcal{B}(I)} \mathbb{S}^p(u_I)(B) \sum_{\substack{K\in\mathcal{G}_\ell(I,\mathcal{E})\\ K\subseteq B}} |K| &\le 4\cdot 2^{-\frac{2\ell}{4[\![\mathcal{E}]\!]+1}} \sum_{B\in\mathcal{B}(I)} \mathbb{S}^p(u_I)(B)|B|\\ &= 4\cdot 2^{-\frac{2\ell}{4[\![\mathcal{E}]\!]+1}} \int_0^1 \mathbb{S}^p(u_I)(t)dt \qquad (4.19)\\ &= 4\cdot 2^{-\frac{2\ell}{4[\![\mathcal{E}]\!]+1}} \|u_I\|^p_{H^p_X}.\end{aligned}$$

Combining inequalities (4.15), (4.16), (4.17), (4.18) and (4.19) we obtain

$$\Big\|\sum_{I\in\mathcal{E}} u_I\Big\|_{H^p_X} \le \Big(1+4^{\frac{1}{p}}\sum_{l=1}^{\infty} 2^{-\frac{2\ell}{p(4[\![\mathcal{E}]\!]+1)}}\Big)\Big(\sum_{I\in\mathcal{E}} \|u_I\|^p_{H^p_X}\Big)^{\frac{1}{p}}. \qquad (4.20)$$

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Curriculum vitae

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Born 1987 in Vöcklabruck, Upper Austria, Austria.

Education

2006 - 2009: undergraduate studies in Technical Mathematics at the Johannes Kepler University in Linz, Upper Austria; Bachelor's degree on Sept. 30th, 2009

2009 - 2012: studies in Technical Mathematics at the Johannes Kepler University in Linz, Upper Austria; Diploma on Febr. 8th, 2012

since 2012: PhD student at the "Institut für Analysis" (J. Kepler University), supported by the FWF project P22549

Research Interests

Banach space theory, p-summing operators, Factorization theorems, Hardy spaces, rearrangements of the Haar system.

List of Publications

(1) p-summing multiplication operators, dyadic Hardy spaces and atomic decomposition, with P.F.X. Müller, Houston Journ. Math., to appear 2016.
`http://arxiv.org/pdf/1310.4312v2.pdf`

(2) Postorder rearrangement operators, preprint, 2014.
`http://arxiv.org/pdf/1410.2712v1.pdf`

List of Scientific works

(1) Diploma thesis: "Komplexe Interpolation, Eigenfunktionen und Hyperkontraktivität" (Complex interpolation, eigenfunctions and hypercontractivity);

(2) Collected topics of seminars in harmonic analysis, spectral theory, Sobolev spaces, calculus of variation and stochastic methods in analysis.

Talks

(1) **May 2014** Christian-Albrechts-Universität zu Kiel, Germany, p-summing multiplication operators, dyadic Hardy spaces and atomic decomposition.

(2) **October 2014** University of Vienna, Austria, p-summing multiplication operators, dyadic Hardy spaces and atomic decomposition.

Conferences and Workshops

(1) Spring School of Analysis, Conference Centre of the Institute of Mathematical Sciences in Bedlewo, Poland, April 2013
(2) Alexander Pełczyński Memorial Conference, Conference Centre of the Institute of Mathematical Sciences in Bedlewo, Poland, July 2014.