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An Interpolatory Estimate and Shift Operators

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Eingereicht von:
Richard Lechner

Angefertigt am:
Institut für Analysis

Beurteilung:
a.Univ.-Prof.Dipl.-Ing.Dr. Paul F. X. Müller (Betreuung)
Univ.-Prof.Dr. Hans Georg Feichtinger (Universität Wien)

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Abstract

An Interpolatory Estimate and Shift Operators

This thesis comprises two main parts. The first part, that is Chapter 2, focuses on an interpolatory estimate for the vector–valued directional Haar projection. The scalar–valued version of this interpolatory inequality was crucial in the work of S. Müller, and was later extended by S. Müller, J. Lee and P. F. X. Müller, in order to obtain sequential weak lower semi-continuity for integrals arising in the theory of compensated compactness. In Chapter 2 we will establish a vector–valued version of this interpolatory estimate by using martingale methods suitable for UMD–spaces. With this vector-valued interpolatory key result at hand we will extend the result of S. Müller, J. Lee and P. F. X. Müller on sequential weak lower semi-continuity to vector–valued functions.

In the second part, Chapter 3, we will give a new proof for the estimates on the shift operators T_m and U_m, first established by T. Figiel. The proof of T. Figiel involves hard combinatorics and has many cases to be considered, especially for the structurally more complicated operator U_m. We will use the well-known one-third-trick to circumvent the hard combinatorics of T. Figiel and thereby reduce the estimates for T_m to the simplest case. Furthermore, we will decompose the more complex operator U_m into five parts, where each of these parts behaves like the much simpler operator T_m. So the one-third-trick in conjunction with this decomposition of U_m allows us not only to treat the operators T_m and U_m equally, but also to consider solely the simplest case for both operators.

Kurzfassung

An Interpolatory Estimate and Shift Operators

Diese Dissertation umfaßt zwei Hauptteile. Der Schwerpunkt des ersten Teils ist eine interpolatorische Ungleichung für die vektorwertige direktionale Haarprojektion, welche in Kapitel 2 behandelt wird. Die skalarwertige Version dieser interpolatorischen Ungleichung, war ausschlagebend in der Arbeit von S. Müller, welche später von S. Müller, J. Lee and P. F. X. Müller erweitert wurde, um schwache Folgenunterhalbstetigkeit für aus dem Gebiet Compensated Compactness kommende Integrale zu erhalten. In Kapitel 2 werden wir mittels für UMD Räume geeignete Martingale Methoden eine vektorwertige Version dieser interpolatorischen Ungleichung beweisen. Mit diesem vektorwertigen interpolatorischen Schlüsselergebnis werden wir das Ergebnis von S. Müller, J. Lee and P. F. X. Müller über schwache Folgenunterhalbstetigkeit auf vektorwertige Funktionen erweitern.

Im zweiten Teil, Kapitel 3, werden wir einen neuen Beweis für die Abschätzungen für die Shift Operatoren T_m und U_m, welche als erstes von T. Figiel bewiesen wurden, angeben. Der Beweis von T. Figiel, insbesondere für den strukturell komplexeren Operator U_m, beinhaltet eine Vielzahl von Fallunterscheidungen und harte Kombinatorik. Wir werden den wohlbekannten Eindrittel-Trick verwenden um die harte Kombinatorik von T. Figiel zu umgehen, und somit die Abschätzungen für T_m auf den einfachsten Fall zurückführen. Des Weiteren werden wir den komplexeren Operator U_m in fünf Teile zerlegen, die sich wie der wesentlich einfachere Operator T_m verhalten. Also ermöglicht uns der Eindrittel-Trick zusammen mit dieser Zerlegung des Operators U_m nicht nur die Operatoren T_m und U_m gleichermaßen zu behandeln, sondern auch für beide Operatoren nur den einfachsten Fall zu betrachten.

Eidestattliche Erklärung

Ich erkläre an Eides statt, dass ich die vorliegende Dissertation selbstständig und ohne fremde Hilfe verfasst, andere als die angegebenen Quellen und Hilfsmittel nicht benutzt bzw. die wörtlich oder sinngemäß entnommenen Stellen als solche kenntlich gemacht habe.

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CHAPTER 1

Preliminaries

This brief section will provide notions and tools used frequently in what follows. First, we will introduce the Haar system supported on dyadic cubes. Then the notions of Banach spaces with the UMD–property and type and cotype of Banach spaces will be discussed, briefly. We recall Kahane's contraction principle and Bourgain's version of Stein's martingale inequality. Then we turn to the shift operators T_m, $m \in \mathbb{Z}^n$.

The Haar System.
For the Haar system supported on cubes we refer the reader to [**Cie87**].
Consider the collection of dyadic intervals at scale $j \in \mathbb{Z}$ given by

$$\mathscr{D}_j = \{ \ [2^{-j}k, 2^{-j}(k+1)[: \ k \in \mathbb{Z} \ \},$$

and the collection of the dyadic intervals

$$\mathscr{D} = \bigcup_{j \in \mathbb{Z}} \mathscr{D}_j.$$

We define the L^∞–normalized Haar system by

$$h_{[0,1[}(t) = \mathbb{1}_{[0,\frac{1}{2}[}(t) - \mathbb{1}_{[\frac{1}{2},1[}(t), \qquad t \in \mathbb{R},$$

and for every $I \in \mathscr{D}$ set

$$h_I(t) = h_{[0,1[}\big(\frac{t - \inf I}{|I|}\big), \qquad t \in \mathbb{R},$$

where $\mathbb{1}_A$ denotes the characteristic function of a set A.

In arbitrary dimensions $n \geq 2$ one can obtain a basis for $L^p(\mathbb{R}^n)$ as follows. For every $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \in \{0,1\}^n$, $\varepsilon \neq 0$ define

$$h_Q^{(\varepsilon)}(t) = \prod_{i=1}^n h_{I_i}^{\varepsilon_i}(t_i),$$

where $t = (t_1, \dots, t_n) \in \mathbb{R}^n$, $Q = I_1 \times \dots \times I_n$, $|I_1| = \ldots = |I_n|$, $I_i \in \mathscr{D}$, and by $h_{I_i}^{\varepsilon_i}$ we denote the function

$$h_{I_i}^{\varepsilon_i} = \begin{cases} h_{I_i}, & \varepsilon_i = 1, \\ \mathbb{1}_{I_i}, & \varepsilon_i = 0. \end{cases}$$

We shall call the collection of all such cubes Q by $\mathscr{Q}$, so

$$\mathscr{Q} = \big\{ I_1 \times \dots \times I_n \ : \ I_i \in \mathscr{D}, \, 1 \leq i \leq n, \ |I_1| = \dots = |I_n| \big\}.$$

For a dyadic cube $Q \in \mathscr{Q}$ the side length of Q is

$$\text{sidelength}(Q) = |I_1|.$$

Finally, define the dyadic predecessor map $\pi : \mathscr{Q} \to \mathscr{Q}$, where the dyadic predecessor $\pi(Q)$ is the unique cube $M \in \mathscr{Q}$ with $M \supset Q$ and $\text{sidelength}(M) = 2 \cdot \text{sidelength}(Q)$. By π^λ, $\lambda \geq 1$ we denote the composition of the function π with itself.

Banach Spaces with the UMD–**Property.**

By $L^p(\Omega,\mu;X)$ we denote the space of functions with values in X, Bochner–integrable with respect to μ. If $\Omega=\mathbb{R}^n$ and μ is the Lebesgue measure $|\cdot|$ on $\mathbb{R}^n$, then set $L^p_X(\mathbb{R}^n)=L^p(\mathbb{R}^n,|\cdot|;X)$, if unambiguous further abbreviated as L^p_X.

We say X is a UMD space if for every X–valued martingale difference sequence $\{d_j\}_j\subset L^p(\Omega,\mu;X)$ and choice of signs $\varepsilon_j\in\{-1,1\}$ one has

$$\Big\|\sum_j \varepsilon_j\, d_j\Big\|_{L^p(\Omega,\mu;X)} \le \mathscr{U}_p(X)\cdot\Big\|\sum_j d_j\Big\|_{L^p(\Omega,\mu;X)}, \tag{0.1}$$

where $\mathscr{U}_p(X)$ does not depend on ε_j or d_j. The constant $\mathscr{U}_p(X)$ is called UMD–constant. We refer the reader to [**Bur81**].

Type and Cotype.

A Banach space X is said to be of type $\mathcal{T}$, $1<\mathcal{T}\le 2$, respectively of cotype $\mathcal{C}$, $2\le\mathcal{C}<\infty$, if there are constants $A(\mathcal{T},X)>0$ and $B(\mathcal{C},X)>0$, such that for every finite set of vectors $\{x_j\}_j\subset X$ we have

$$\int_0^1 \Big\|\sum_j r_j(t)\,x_j\Big\|_X\,\mathrm{d}t \le A(\mathcal{T},X)\cdot\Big(\sum_j\|x_j\|_X^{\mathcal{T}}\Big)^{1/\mathcal{T}}, \tag{0.2}$$

respectively

$$\int_0^1 \Big\|\sum_j r_j(t)\,x_j\Big\|_X\,\mathrm{d}t \ge B(\mathcal{C},X)\cdot\Big(\sum_j\|x_j\|_X^{\mathcal{C}}\Big)^{1/\mathcal{C}}, \tag{0.3}$$

where $\{r_j\}_j$ is an independent sequence of Rademacher functions.

It is well known that if X is a UMD–space, then for every $1<p<\infty$ the space $L^p_X(\mathbb{R}^n)$ has (non–trivial) type and cotype (see [**Mau75**], [**MP76**] and [**Ald79**]).

Kahane's Contraction Principle.

For every Banach space X, $1\le p<\infty$, finite set $\{x_j\}_j\subset X$ and bounded sequence of scalars $\{c_j\}_j$ we have

$$\Big(\int_0^1\Big\|\sum_j r_j(t)\,c_j\,x_j\Big\|_X^p\,\mathrm{d}t\Big)^{1/p} \le \sup_j|c_j|\cdot\Big(\int_0^1\Big\|\sum_j r_j(t)\,x_j\Big\|_X^p\,\mathrm{d}t\Big)^{1/p}, \tag{0.4}$$

where $\{r_j\}_j$ denotes an independent sequence of Rademacher functions. For details see [**Kah85**]. Below we give a short proof, see [**Kah85**].

Proof. By scaling inequality (0.4), we may assume $|c_j|\le 1$, for all j. We represent each c_j as the series $c_j=\sum_{k\ge1}\varepsilon_{jk}\,2^{-k}$, with suitable $\varepsilon_{jk}\in\{\pm1\}$ and observe

$$\begin{aligned}\Big(\int_0^1\Big\|\sum_j r_j(t)\,c_j\,x_j\Big\|_X^p\,\mathrm{d}t\Big)^{1/p} &\le \sum_{k\ge1}2^{-k}\Big(\int_0^1\Big\|\sum_j r_j(t)\,\varepsilon_{jk}\,x_j\Big\|_X^p\,\mathrm{d}t\Big)^{1/p}\\ &= \Big(\int_0^1\Big\|\sum_j r_j(t)\,x_j\Big\|_X^p\,\mathrm{d}t\Big)^{1/p}.\end{aligned}$$

The last equality holds true since $\sum_j r_j(t)\,\varepsilon_{jk}\,x_j$ has the same distribution as $\sum_j r_j(t)\,x_j$ for all choices of signs ε_{jk}. □

The Martingale Inequality of Stein – Bourgain's Version.

Let $(\Omega, \mathcal{F}, \mu)$ be a probability space, and let $\mathcal{F}_1 \subset \ldots \subset \mathcal{F}_m \subset \mathcal{F}$ denote an increasing sequence of σ–algebras. For every choice of $f_1, \ldots, f_m \in L^p(\Omega, \mu; X)$ let $r_1, \ldots, r_m$ denote independent Rademacher functions, then

$$\int_0^1 \Big\| \sum_{i=1}^m r_i(t)\ \mathbb{E}(f_i|\mathcal{F}_i) \Big\|_{L^p(\Omega,\mu;X)} \mathrm{d}t \le C \cdot \int_0^1 \Big\| \sum_{i=1}^m r_i(t)\, f_i \Big\|_{L^p(\Omega,\mu;X)} \mathrm{d}t, \tag{0.5}$$

where C depends only on p and X.

The Banach space X having the UMD–property assures $C < \infty$. The scalar valued version of (0.5) by E. M. Stein can be found in [**Ste70b**]. The vector valued extension is due to J. Bourgain [**Bou86**]. The proof below is taken from [**FW01**].

Proof. For all $0 \le t \le 1$ and $x \in \Omega$ we define

$$F(t,x) = \sum_{i=1}^m r_i(t)\, f_i(x),$$

and the sets

$$\mathcal{G}_n = \sigma\text{-algebra}\{r_1, \ldots, r_n\} \times \mathcal{F}_n$$

for all $1 \le n \le m$. Note that $\{\mathcal{G}_n\}_n$ is a filtration on the product space $[0,1] \times \Omega$, and observe that

$$\mathbb{E}(F|\mathcal{G}_n)(t,x) = \sum_{i \le n} r_i(t)\ \mathbb{E}(f_i|\mathcal{F}_n).$$

We define $F_n = \mathbb{E}(F|\mathcal{G}_n)$ for $1 \le n \le m$, set $F_0 = 0$, and note that $F = \sum_{n=1}^m F_n - F_{n-1}$. If we additionally define $g_{ni} = \mathbb{E}(f_i|\mathcal{F}_n) - \mathbb{E}(f_i|\mathcal{F}_{n-1})$, $2 \le n \le m$, $1 \le i \le m$, we see

$$(F_n - F_{n-1})(t,x) = r_n(t)\ \mathbb{E}(f_n|\mathcal{F}_n)(x) + \sum_{i<n} r_i(t)\, g_{ni}(x).$$

In the above equation we make use of the convention of a sum over the empty set being zero. Now we use the UMD–property on the martingale difference sequence $\{F_n\}_n$ (with respect to the filtration $\{\mathcal{G}_n\}_n$) to obtain

$$\big\|F\big\|^p_{L^p_X([0,1]\times\Omega)} \ge \mathscr{U}_p(X)^{-p} \cdot \int_\Omega \int_0^1 \Big\| \sum_{n=1}^m r_n(s) \cdot (F_n - F_{n-1})(t,x) \Big\|^p_X \mathrm{d}t\, \mathrm{d}x,$$

for all $s \in [0,1]$. First, note that

$$\sum_{n=1}^m r_n(s) \cdot (F_n - F_{n-1})(t,x) = \sum_{n=1}^m r_n(s)\, r_n(t)\ \mathbb{E}(f_n|\mathcal{F}_n)(x) + \sum_{n=1}^m \sum_{i<n} r_n(s)\, r_i(t)\, g_{ni}(x),$$

Second, with s and x fixed, consider the function

$$t \mapsto \sum_{n=1}^m r_n(t)\ \mathbb{E}(f_n|\mathcal{F}_n)(x) + \sum_{n=1}^m \sum_{i<n} r_n(s)\, r_i(s)\, r_i(t)\, g_{ni}(x)$$

and observe it has the same distribution as $\sum_{n=1}^m r_n(s) \cdot (F_n - F_{n-1})(t,x)$, hence

$$\begin{aligned}\big\|F\big\|^p_{L^p_X([0,1]\times\Omega)} \ge \mathscr{U}_p(X)^{-p} \cdot \int_\Omega \int_0^1 \Big\| &\sum_{n=1}^m r_n(t)\ \mathbb{E}(f_n|\mathcal{F}_n)(x) \\ &+ \sum_{n=1}^m \sum_{i<n} r_n(s)\, r_i(s)\, r_i(t)\, g_{ni}(x) \Big\|^p_X \mathrm{d}t\, \mathrm{d}x,\end{aligned}$$

for all $s \in [0,1]$. Integrating the last estimate with respect to s and applying Jensen's inequality yields the desired result

$$\|F\|^p_{L^p_X([0,1]\times\Omega)} \geq \mathscr{U}_p(X)^{-p} \cdot \int_\Omega \int_0^1 \|\sum_{n=1}^m r_n(t)\ \mathbb{E}(f_n|\mathcal{F}_n)(x)\|^p_X \,\mathrm{d}t\,\mathrm{d}x,$$

since $\int_0^1 r_n(s)\, r_i(s)\,\mathrm{d}s = 0$, if $i < n$. □

The Shift Operator T_m.

For every $m \in \mathbb{Z}^n$ let $\tau_m : \mathscr{Q} \to \mathscr{Q}$ denote the rearrangement given by

$$\tau_m(Q) = Q + m \cdot \text{sidelength}(Q). \tag{0.6}$$

The map τ_m induces the rearrangement operator T_m, as the linear extension of

$$T_m h_Q = h_{\tau_m(Q)}, \qquad Q \in \mathscr{Q}. \tag{0.7}$$

Let X be a UMD space, then

$$\|T_m : L^p_X(\mathbb{R}^n) \to L^p_X(\mathbb{R}^n)\| \leq C \cdot \log(2 + |m|)^\alpha, \tag{0.8}$$

where $0 < \alpha < 1$ depends on X, and $C = C(n, p, \mathscr{U}_p(X), \alpha)$; This result is due to T. Figiel, see [**Fig88**] and [**Fig90**]. An new proof of estimate (0.8) and other results concerning shift operators are provided in Chapter 3 of this thesis.

The Riesz Transform.

For all $1 \leq i_0 \leq n$ we define the Riesz transform R_{i_0} formally by

$$R_{i_0} f = K_{i_0} * f, \tag{0.9}$$

$$K_{i_0}(x) = c_n \frac{x_{i_0}}{|x|^{n+1}}, \qquad x = (x_1, \dots, x_n) \in \mathbb{R}^n. \tag{0.10}$$

Details may be found in [**Ste70a**] and [**Ste93**].

Supplementary Definitions.

Denote the standard Fourier multiplier $\langle\cdot\rangle$ by

$$\langle\xi\rangle = \left(1 + |\xi|^2\right)^{\frac{1}{2}}, \tag{0.11}$$

for all $\xi \in \mathbb{R}^n$.

The Haar spectrum of an operator $T : L^p_X(\mathbb{R}^n) \to L^p_X(\mathbb{R}^n)$, is defined by

$$\mathscr{Q} \setminus \big\{Q \in \mathscr{Q} : \langle Tu, h_Q^{(\varepsilon)}\rangle = 0, \text{ for all } u \in L^p_X(\mathbb{R}^n) \text{ and } \varepsilon \in \{0,1\}^n \setminus \{0\}\big\}. \tag{0.12}$$

Given a collection of sets $\mathscr{C}$, we denote by σ-algebra$(\mathscr{C})$ the smallest σ-algebra containing $\mathscr{C}$, i.e.

$$\sigma\text{-algebra}(\mathscr{C}) = \bigcap \big\{\mathscr{A} : \mathscr{A} \text{ is a } \sigma\text{-algebra}, \mathscr{C} \subset \mathscr{A}\big\}.$$

Given a normed vector space X, we define the vector space X^n as the cross product of n identical copies $X^n = X \times \cdots \times X$ endowed with the norm $\|(x_1, \dots, x_n)\|_{X^n} = \sum_i \|x_i\|_X$.

CHAPTER 2

An Interpolatory Estimate for the UMD–Valued Directional Haar Projection

We prove an interpolatory estimate linking the directional Haar projection $P^{(\varepsilon)}$ to the Riesz transform in the context of Bochner–Lebesgue spaces L^p_X, $1 < p < \infty$, provided X is a UMD–space. If $\varepsilon_{i_0} = 1$, the result is the following inequality

$$\|P^{(\varepsilon)}u\|_{L^p_X} \leq C \cdot \|u\|_{L^p_X}^{1/\mathcal{T}(L^p_X)} \, \|R_{i_0}u\|_{L^p_X}^{1-1/\mathcal{T}(L^p_X)}, \tag{0.1}$$

where the constant C depends only on n, p, the UMD constant of X and the Rademacher type $\mathcal{T}(L^p_X)$.

In order to obtain the interpolatory result (0.1) we analyze stripe operators S_λ, $\lambda \geq 0$ which are used as basic building blocks to dominate the directional Haar projection. The main result on stripe operators (see Theorem 2.5) is the estimate

$$\|S_\lambda u\|_{L^p_X} \leq C \cdot 2^{-\lambda/\mathcal{C}(L^p_X)} \, \|u\|_{L^p_X}, \tag{0.2}$$

where the constant C depends only on n, p, the UMD constant of X and the Rademacher cotype $\mathcal{C}(L^p_X)$. The proof of (0.2) relies on a uniform bound for the shift operators T_m, $0 \leq m < 2^\lambda$ acting on the image of S_λ (see Theorem 2.2).

Based upon inequality (0.1) we prove a vector-valued result on sequential weak lower semi-continuity of integrals of the form

$$u \mapsto \int f(u) \, \mathrm{d}x,$$

where $f : X^n \longrightarrow \mathbb{R}^+$ is separately convex satisfying $f(x) \leq C \cdot (1 + \|x\|_{X^n})^p$.

1. Main Results

1.1. A Brief History of Development.

The Calculus of Variations, in particular the theory of compensated compactness has long been a source of hard problems in harmonic analysis. One development started with the work of F. Murat and L. Tartar and especially in [**Tar78, Tar79, Tar83, Tar84, Tar90, Tar93**], and [**Mur78, Mur79, Mur81**]. Briefly, we will now review their framework below. Let $\mathcal{A}$ be a first-order linear differential operator of the form

$$\mathcal{A} = \sum_{i=1}^{n} A_i \, \partial_i,$$

where $A_i \in L(\mathbb{R}^m, \mathbb{R}^d)$, and denote its Symbol by A, given by

$$A(\xi) = \sum_{i=1}^{n} A_i \, \xi_i, \qquad \xi \in \mathbb{R}^n.$$

One of their objectives was to impose exact conditions on a given function $f : \mathbb{R}^d \to \mathbb{R}$ such that

$$\liminf_{r\to\infty} \int\limits_{\mathbb{R}^n} f(v_r(x)) \, \varphi(x) \, \mathrm{d}x \geq \int\limits_{\mathbb{R}^n} f(v(x)) \, \varphi(x) \, \mathrm{d}x, \tag{1.1}$$

for all $\varphi \in C_0^+$, $v_r \to v$ weakly in L^p and $\mathcal{A}(v_r)$ being precompact in $W^{-1,p}$. Particularly, f has to satisfy the growth condition $0 \leq f(x) \leq C\,(1 + |x|)^p$ and be $\mathcal{A}$-quasi-convex. The function f is $\mathcal{A}$-quasi-convex if

$$\int\limits_{[0,1]^n} f(a + u(x)) \, \mathrm{d}x \geq f(a)$$

for all smooth and $[0,1]^n$ periodic functions $u : \mathbb{R}^n \to \mathbb{R}^d$ having mean zero and $\mathcal{A}(u) = 0$. Another requirement we specifically want to emphasize was for $\mathcal{A}$ to be of constant rank, that is

$$\#\big\{ \operatorname{rank}(A(\xi)) \, : \, \xi \neq 0 \big\} = 1.$$

Imposing the constant rank hypothesis on $\mathcal{A}$ implicates that $\mathbb{P}(\xi) : \mathbb{R}^m \to \mathbb{R}^m$, denoting the orthogonal projection onto $\ker(A(\xi))$, is a smooth function being positively homogeneous of degree 0. As a consequence $\mathbb{P}$ is a Fourier multiplier, hence T defined in the Fourier domain by $\widehat{Tu} = \mathbb{P} \cdot \widehat{u}$ is bounded from L^p to itself. The operator T was used to decompose a given function v into

$$v = u + w, \quad \text{where} \quad u = Tv \quad \text{and} \quad w = v - Tv, \tag{1.2}$$

with u and w satisfying

$$\mathcal{A}(u_r) = 0 \qquad \text{and} \qquad \big\| w_r \big\|_{L^p} \leq C \, \big\| \mathcal{A}(v_r) \big\|_{W^{-1,p}}.$$

A reduction step allowed them to assume the sequence v_r in (1.1) is $[0,1]^n$ periodic, has mean zero and satisfies $\mathcal{A}(v_r) \to 0$ in $W^{-1,p}$. These additional assumptions combined with the decomposition $v_r = u_r + w_r$ according to (1.2) imply

$$\mathcal{A}(u_r) = 0 \qquad \text{and} \qquad \|w_r\|_p \to 0, \tag{1.3}$$

and $\int_{[0,1]^n} u_r = 0$. Using that f is $\mathcal{A}$-quasi-convex yields

$$\int\limits_{[0,1]^n} f(a + u_r(x)) \, \mathrm{d}x \geq f(a) \qquad \text{and} \qquad \|w_r\|_p \to 0, \tag{1.4}$$

from which one can obtain (1.1).

In [**Mül99**] S. Müller obtained analogous results for separately convex integrands f, for which the constant rank condition is not satisfied. The method introduced by S. Müller in [**Mül99**] consists of time–frequency localization in combination with modern Calderon–Zygmund theory. The result is the following. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be separately convex satisfying $0 \le f(z) \le C\,(1 + |z|^2)$ and let $U \subset \mathbb{R}^2$ be open and suppose that

$$\begin{aligned} u_j &\rightharpoonup u_\infty, & v_j &\rightharpoonup v_\infty, & &\text{in } L^2_{\mathrm{loc}}(U),\\ \partial_2 u_j &\rightharpoonup \partial_2 u_\infty, & \partial_1 v_j &\rightharpoonup \partial_1 v_\infty, & &\text{in } H^{-1}_{\mathrm{loc}}(U). \end{aligned}$$

Then for every $V \subset U$

$$\int_V f(u_\infty, v_\infty) \le \liminf_{j\to\infty} \int_V f(u_j, v_j)\,\mathrm{d}x. \tag{1.5}$$

The basis of the result were interpolatory estimates for the directional Haar projection $P^{(\varepsilon)}$, $\varepsilon \in \{0,1\}^n \setminus \{0\}$, defined as follows. Let $u \in L^p(\mathbb{R}^n)$, with $n \ge 2$ and $1 < p < \infty$ be fixed, then $P^{(\varepsilon)} : L^p(\mathbb{R}^n) \longrightarrow L^p(\mathbb{R}^n)$ is given by

$$P^{(\varepsilon)} u = \sum_{Q \in \mathscr{Q}} \langle u, h_Q^{(\varepsilon)} \rangle \, h_Q^{(\varepsilon)} \, |Q|^{-1},$$

where $\mathscr{Q}$ is the collection of dyadic cubes and $h_Q^{(\varepsilon)}$ denotes Haar function supported on Q having mean zero in coordinate i whenever $\varepsilon_i = 1$. For details see (3.1) in Section 3 on page 43. The crucial interpolatory estimate in [**Mül99**] was

$$\|P^{(\varepsilon)} u\|_{L^2(\mathbb{R}^2)} \le C\,\|u\|_{L^2(\mathbb{R}^2)}^{1/2}\,\|R_{i_0} u\|_{L^2(\mathbb{R}^2)}^{1-1/2}, \tag{1.6}$$

where R_{i_0} denotes the Riesz transform in direction $i_0 \in \{1,2\}$, $0 \ne (\varepsilon_1, \varepsilon_2) = \varepsilon \in \{0,1\}^2$, and $\varepsilon_{i_0} = 1$. The formal definition of R_{i_0} is supplied in Chapter 1.

This inequality was extended by J. Lee, P. F. X. Müller and S. Müller in [**LMM07**] to arbitrary $1 < p < \infty$ and dimensions $n \ge 2$ to

$$\|P^{(\varepsilon)} u\|_{L^p(\mathbb{R}^n)} \le C\,\|u\|_{L^p(\mathbb{R}^n)}^{1/\min(2,p)}\,\|R_{i_0} u\|_{L^p(\mathbb{R}^n)}^{1-1/\min(2,p)}, \tag{1.7}$$

where $\varepsilon \in \{0,1\}^n \setminus \{0\}$, $\varepsilon_{i_0} = 1$. This interpolatory result was crucial in order to establish

$$\liminf_{r\to\infty} \int_{\mathbb{R}^n} f(v_r(x))\,\varphi(x)\,\mathrm{d}x \ge \int_{\mathbb{R}^n} f(v(x))\,\varphi(x)\,\mathrm{d}x, \tag{1.8}$$

for all $\varphi \in C_0^+$, $v_r \to v$ weakly in L^p and $\mathcal{A}_0(v_r)$ being precompact in $W^{-1,p}$, where

$$\big(\mathcal{A}_0(u)\big)_{i,j} = \begin{cases} \partial_i u^{(j)} & i \ne j,\\ 0 & i = j. \end{cases}$$

Note that $\mathcal{A}_0$ does not have constant rank. The function f was required to be separately convex and satisfy $0 \le f(x) \le C\,(1 + |x|)^p$. In [**Mül99**] as well as [**LMM07**] the decomposition (1.2) based on Fourier multipliers is replaced by

$$v = P(v) + (v - P(v)), \qquad v = (v^{(1)}, \dots, v^{(n)}),$$

where $Pv = (P^{(e_1)} v^{(1)}, \dots, P^{(e_n)} v^{(n)})$ and the e_k are the standard unit vectors. For the first part $P(v)$ they used the following form of Jensen's inequality

$$f\Big(\int_{[0,1]^n} P(v)\,\mathrm{d}x\Big) \le \int_{[0,1]^n} f(P(v))\,\mathrm{d}x. \tag{1.9}$$

For the second part $v - P(v)$ the interpolatory results (1.6) respectively (1.7) were used to dominate $v - P(v)$ by

$$\left\|v - P(v)\right\|_p \leq C \left\|v\right\|_p^\theta \cdot \left(\sum_i \sum_{j \neq i} \left\|R_i(v^{(j)})\right\|_p \right)^{1-\theta}, \tag{1.10}$$

for some $0 < \theta < 1$. Again, one can assume that v_r is $[0,1]^n$ periodic, has mean zero and $\mathcal{A}_0(v_r) \to 0$ in $W^{-1,p}$. Consequently, $R_i(v_r^{(j)}) \to 0$ in L^p, for all $i \neq j$, and inequality (1.10), which was deduced by the interpolatory result (1.7), implies

$$\left\|v_r - P(v_r)\right\|_p \to 0. \tag{1.11}$$

Rescaling (1.9) and using (1.11) shows

$$\int_{[0,1]^n} f(a + P(v_r)(x))\,\mathrm{d}x \geq f(a) \qquad \text{and} \qquad \|v_r - P(v_r)\|_p \to 0, \tag{1.12}$$

which are the same key properties as for the classical decomposition $v_r = u_r + w_r$, see (1.4), from which one can obtain (1.8), again.

Let us emphasize that the methods to obtain the key properties (1.4) for the decomposition $v_r = u_r + w_r$ are based on Fourier multipliers requiring the operator $\mathcal{A}$ to satisfy the constant rank hypothesis. Note that the operator $\mathcal{A}_0$ is not of constant rank, and the methods used to establish the key properties (1.12) for the decomposition $v = P(v) + v - P(v)$ are based on Jensen's inequality (1.9), and the crucial interpolatory result (1.7).

One can rewrite the interpolatory inequality (1.7) using the notion of type $\mathcal{T}(L^p(\mathbb{R}^n)) = \min(2, p)$

$$\|P^{(\varepsilon)}u\|_{L^p(\mathbb{R}^n)} \leq C\,\|u\|_{L^p(\mathbb{R}^n)}^{1/\mathcal{T}(L^p(\mathbb{R}^n))}\,\|R_{i_0}u\|_{L^p(\mathbb{R}^n)}^{1-1/\mathcal{T}(L^p(\mathbb{R}^n))}. \tag{1.13}$$

It is in this form that (1.7) can be given a vector–valued extension, see estimate (1.14). In [**Mül99**] and [**LMM07**] the proofs of (1.6) respectively (1.7) are based on two consecutive time–frequency localizations of the operator $P^{(\varepsilon)}$, based on Littlewood–Paley and wavelet expansions. The L^p–estimates in [**LMM07**] were obtained by systematically interpolating between the spaces H^1, L^2 and BMO. In this thesis we obtain vector–valued extensions of (1.13), see Theorem 1.1, working directly on L^p_X avoiding interpolation and using martingale methods, instead. Having Theorem 1.1 at our disposal allows us to extend the result of [**LMM07**] on sequential weak lower semi-continuity, that was inequality (1.8), to vector-valued functions detailed in Theorem 1.2 on the facing page.

1.2. The Main Results.

S. Müller asks in [**Mül99**] whether it is possible to obtain (1.6) in such a way that the original time–frequency decompositions are replaced by the **canonical martingale decomposition** of T. Figiel (see [**Fig90**]). This thesis provides an affirmative answer to this question. The details of the decomposition are worked out in section 3. This allows us to extend the interpolatory estimate (1.13) to the Bochner–Lebesgue space $L^p_X(\mathbb{R}^n)$, provided X satisfies the UMD–property. With this vector-valued interpolatory estimate we obtain a vector-valued extension of the result of [**LMM07**] on sequential weak lower semi-continuity, as well.

Let $1 < p < \infty$, and let X be a UMD space ([**Mau75**]) with type $\mathcal{T}(X)$. It is well known that X has non–trivial type $\mathcal{T}(X) > 1$ and cotype $\mathcal{C}(X) < \infty$ ([**Mau75**], [**MP76**] and [**Ald79**]). Consequently, $L^p_X(\mathbb{R}^n)$ has non–trivial type $\mathcal{T}(L^p_X(\mathbb{R}^n))$ and cotype given by $\min(p, \mathcal{T}(X))$ and $\max(p, \mathcal{C}(X))$, respectively (see [**LT91**, section 9.2, page 247]).

The main inequality of this thesis reads as follows.

THEOREM 1.1. *Let $1 < p < \infty$ and X be a Banach space with the* UMD*–property. Denote by $\mathcal{T}(L^p_X(\mathbb{R}^n))$ the (non–trivial) type of $L^p_X(\mathbb{R}^n)$. Let*

$$\varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \in \{0,1\}^n \quad \text{with} \quad \varepsilon_{i_0} = 1.$$

Then for every $u \in L^p_X(\mathbb{R}^n)$

$$\|P^{(\varepsilon)} u\|_{L^p_X(\mathbb{R}^n)} \leq C \cdot \|u\|^{1/\mathcal{T}(L^p_X(\mathbb{R}^n))}_{L^p_X(\mathbb{R}^n)} \, \|R_{i_0} u\|^{1-1/\mathcal{T}(L^p_X(\mathbb{R}^n))}_{L^p_X(\mathbb{R}^n)}, \tag{1.14}$$

where the constant C depends only on n, p, the UMD *constant of X and the type $\mathcal{T}(L^p_X(\mathbb{R}^n))$.*

For the proof of Theorem 1.1 see subsection 1.4 on page 23.

The L^p–estimates of Theorem 1.1 on the facing page are obtained directly from estimates of rearrangement operators avoiding the detour to the endpoint spaces H^1 and BMO. The basic tools for the proof of the above theorem are vector–valued estimates of stripe operators S_λ, developed in section 2. We also point out that the L^2–estimates for the stripe operator are obvious in the scalar case, but form the main obstacle in the vector valued case.

The vector-valued interpolatory estimate (1.14) allows us to extend the scalar valued result (1.8) on sequential weak lower semi-continuity to the vector-valued result Theorem 1.2 below. Note that the only additional hypothesis is (1.19), which is superfluous in the scalar-valued case.

THEOREM 1.2. *Let X be a Banach space with the* UMD *property and $1 < p < \infty$. Let $\mathcal{A}_0 : L^p(\mathbb{R}^n; X^n) \to W^{-1,p}(\mathbb{R}^n; X^n \times X^n)$ denote the differential operator given by*

$$\big(\mathcal{A}_0(u)\big)_{i,j} = \begin{cases} \partial_i u^{(j)} & i \neq j, \\ 0 & i = j, \end{cases} \tag{1.15}$$

where $u = (u^{(j)})_{j=1}^n$. Assume the function $f : X^n \to \mathbb{R}$ is separately convex and satisfies

$$0 \leq f(x) \leq C \cdot (1 + \|x\|_{X^n})^p, \tag{1.16}$$

for all $x \in X^n$, where $C > 0$ does not depend on x. Let the sequence $\{v_r\} \subset L(\mathbb{R}^n; X^n)$ be such that

$$v_r \to v \quad \text{weakly in } L^p(\mathbb{R}^n; X^n), \tag{1.17}$$

$$\mathcal{A}_0(v_r) \quad \text{precompact in } W^{-1,p}(\mathbb{R}^n; X^n \times X^n), \tag{1.18}$$

and

$$\big\| \big(\langle v_r^{(j)} - v^{(j)}, h_Q^{(e_j)} \rangle \big)_{j=1}^n \big\|_{X^n} \to 0 \quad \text{for all } Q \in \mathcal{Q}. \tag{1.19}$$

Then we have

$$\liminf_{r \to \infty} \int_{\mathbb{R}^n} f\big(v_r(x)\big)\, \varphi(x)\, \mathrm{d}x \geq \int_{\mathbb{R}^n} f\big(v(x)\big)\, \varphi(x)\, \mathrm{d}x, \tag{1.20}$$

for all $\varphi \in C_0^+(\mathbb{R}^n)$.

The proof of Theorem 1.2 may be found in subsection 1.5 on page 23.

1.3. The Main Inequality and Interpolation.

The interpolatory main result, Theorem 1.1, represents a result on interpolation of operators, linking the identity map, the Riesz transforms and the directional Haar projection. We would now like to give a reformulation of Theorem 1.1 which places it in the context of structure Theorems for the so called K–method of interpolation spaces. To this end, we first introduce the K–functional, cite the relevant structure theorem (Proposition 1.3) and apply it to the interpolatory inequality (1.14).

Define the K–functional

$$K(f,t) = \inf \big\{ \|g\|_{E_0} + t\, \|h\|_{E_1} \,:\, f = g + h,\ g \in E_0, h \in E_1 \big\},$$

for all $f \in E_0 + E_1$ and $t > 0$, and the interpolation space

$$(E_0, E_1)_{\theta,1} = \big\{ f \,:\, f \in E_0 + E_1,\ \|f\|_{\theta,1} < \infty \big\},$$

where $0 < \theta < 1$, endowed with the norm

$$\|f\|_{\theta,1} = \int_0^\infty t^{-\theta}\, K(f,t)\, \frac{\mathrm{d}t}{t}.$$

The following Proposition 1.3 interprets interpolatory estimates such as the ones obtained in Theorem 1.1 in terms of continuity of the identity map between interpolation spaces. The following proposition is a result of general interpolation theory (see [**BS88**, Proposition 2.10, Chapter 5]).

Proposition 1.3. *Let* (E_0, E_1) *be a compatible couple and suppose* $0 < \theta < 1$. *Then the estimate*

$$\|f\|_E \le C\, \|f\|_{\theta,1} \tag{1.21}$$

holds for some constant C *and all* f *in* $(E_0, E_1)_{\theta,1}$ *if and only if*

$$\|f\|_E \le C\, \|f\|_{E_0}^{1-\theta}\, \|f\|_{E_1}^{\theta}$$

holds for some constant C *and for all* f *in* $E_0 \cap E_1$.

In the following we will specify the spaces E, E_0 and E_1 so that the two equivalent conditions of the above proposition match precisely the assertions of Theorem 1.1.

Application of Proposition 1.3 to Theorem 1.1.

Fix $0 \neq \varepsilon \in \{0,1\}^n$, let R denote one of the Riesz transform operators

$$R_i \,:\, L^p_X(\mathbb{R}^n) \to L^p_X(\mathbb{R}^n)$$

defined in Chapter 1, where $\varepsilon_i = 1$, and abbreviate $P^{(\varepsilon)}$ by P. If we define the Banach spaces

$$\begin{aligned}
E &= L^p_X(\mathbb{R}^n)/\ker(P), & \|u + \ker(P)\|_E &= \|Pu\|_{L^p_X(\mathbb{R}^n)},\\
E_0 &= L^p_X(\mathbb{R}^n), & \|u\|_{E_0} &= \|u\|_{L^p_X(\mathbb{R}^n)},\\
E_1 &= L^p_X(\mathbb{R}^n)/\ker(R), & \|u + \ker(R)\|_{E_1} &= \|Ru\|_{L^p_X(\mathbb{R}^n)},
\end{aligned}$$

then Proposition 1.3 together with Theorem 1.1 yields

$$(E_0, E_1)_{\theta,1} \hookrightarrow E.$$

In other words, there exists a constant $C > 0$ such that

$$\|u\|_E \le C \cdot \|u\|_{\theta,1},$$

for all $u \in (E_0, E_1)_{\theta,1}$.

We summarize this brief discussion in the following

THEOREM 1.4. *Let $1 < p < \infty$, and let X be a Banach space with the* UMD–*property. Denote by $\mathcal{T}(L^p_X(\mathbb{R}^n))$ the (non–trivial) type of $L^p_X(\mathbb{R}^n)$. Furthermore, let*

$$\varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \in \{0,1\}^n \quad \textit{with} \quad \varepsilon_{i_0} = 1,$$

and define

$$\begin{aligned} E_0 &= L^p_X(\mathbb{R}^n), & \|u\|_{E_0} &= \|u\|_{L^p_X(\mathbb{R}^n)}, \\ E_1 &= L^p_X(\mathbb{R}^n)/\ker(R_{i_0}), & \|u + \ker(R_{i_0})\|_{E_1} &= \|R_{i_0} u\|_{L^p_X(\mathbb{R}^n)}, \end{aligned}$$

Then there exists a constant $C > 0$ such that

$$\|P^{(\varepsilon)} u\|_{L^p_X(\mathbb{R}^n)} \le C \cdot \|u\|_{\theta,1}, \tag{1.22}$$

for all $u \in L^p_X(\mathbb{R}^n)$, where $\theta = 1 - 1/\mathcal{T}(L^p_X)$.

The connection to general interpolation theory was pointed out by S. Geiss.

1.4. Proof of Theorem 1.1.

The subsequent proof of Theorem 1.1 merges the vector valued results of this thesis, particularly Theorem 3.6 and 3.4. Besides replacing the scalar valued estimates with our vector valued analogues we repeat the scalar valued proof in [**LMM07**].

PROOF. Within this proof we shall abbreviate $L^p_X(\mathbb{R}^n)$ by L^p_X.

First, define $M \in \mathbb{N}$ by

$$2^{M-1} \le \frac{\left\|R_{i_0} : L^p_X \to L^p_X\right\| \cdot \|u\|_{L^p_X}}{\|R_{i_0} u\|_{L^p_X}} \le 2^M. \tag{1.23}$$

Second, use decomposition (3.2) and (3.8), that is

$$P^{(\varepsilon)} = P_- + \sum_{l \ge 0} P_l^{(\varepsilon)},$$

and observe,

$$\|P^{(\varepsilon)} u\|_{L^p_X} \le \|P_-^{(\varepsilon)} R_{i_0}^{-1} R_{i_0} u\|_{L^p_X} + \sum_{l=0}^{M} \|P_l R_{i_0}^{-1} R_{i_0} u\|_{L^p_X} + \sum_{l=M}^{\infty} \|P_l u\|_{L^p_X}.$$

If we apply Theorem 3.6 on page 57 to the first two sums, and inequality (3.43) in Theorem 3.4 on page 49 to the latter sum, we get

$$\begin{aligned} \|P_-^{(\varepsilon)} R_{i_0}^{-1} R_{i_0} u\|_{L^p_X} &\lesssim \|R_{i_0} u\|_{L^p_X}, \\ \|P_l R_{i_0}^{-1} R_{i_0} u\|_{L^p_X} &\lesssim 2^{l/\mathcal{T}(L^p_X)} \cdot \|R_{i_0} u\|_{L^p_X}, \end{aligned}$$

and

$$\|P_l u\|_{L^p_X} \lesssim 2^{-l(1 - \frac{1}{\mathcal{T}(L^p_X)})} \cdot \|u\|_{L^p_X}.$$

Thus we can dominate $\|P^{(\varepsilon)} u\|_{L^p_X}$ by a constant multiple of

$$\|R_{i_0} u\|_{L^p_X} + \sum_{l=0}^{M} 2^{l/\mathcal{T}(L^p_X)} \cdot \|R_{i_0} u\|_{L^p_X} + \sum_{l=M}^{\infty} 2^{-l(1 - \frac{1}{\mathcal{T}(L^p_X)})} \cdot \|u\|_{L^p_X}.$$

Evaluating the geometric series yields

$$\|P^{(\varepsilon)} u\|_{L^p_X} \lesssim 2^{M/\mathcal{T}(L^p_X)} \|R_{i_0} u\|_{L^p_X} + 2^{-M(1 - \frac{1}{\mathcal{T}(L^p_X)})} \|u\|_{L^p_X},$$

and plugging in M concludes the proof. □

1.5. Proof of Theorem 1.2.

We will divide the proof into four steps. Define the projection $P : L^p(\mathbb{R}^n; X^n) \longrightarrow L^p(\mathbb{R}^n; X^n)$ by

$$P(v) = (P^{(e_1)}v^{(1)}, \ldots, P^{(e_n)}v^{(n)}),$$

where $v = (v^{(j)})_{j=1}^n$, and

$$P^{(\varepsilon)}u = \sum_{Q\in\mathscr{Q}} \langle u, h_Q^{(\varepsilon)}\rangle\, h_Q^{(\varepsilon)}\, |Q|^{-1},$$

for all $u \in L^p(\mathbb{R}^n; X)$, $\varepsilon \in \{0,1\}^n \setminus \{0\}$.

In the first step we will see how the interpolatory estimate (1.14) is used to obtain

$$\big\| P(w_r) - w_r \big\|_{L^p(\mathbb{R}^n;X^n)} \to 0,$$

whenever $w_r \to 0$ weakly in $L^p(\mathbb{R}^n; X^n)$ and $\big\{\mathcal{A}_0(w_r)\big\}$ is precompact in $W^{-1,p}(\mathbb{R}^n; X^n \times X^n)$.

In the second stage of the proof we will show

$$f\big(\,\mathbb{E}_M(Pv)\big) \le \mathbb{E}_M\,\big(f(Pv)\big),$$

where

$$\mathbb{E}_M\, u = \sum_{Q\in\mathscr{Q}_M} \Big(\frac{1}{|Q|}\int_Q u(x)\,\mathrm{d}x\Big)\cdot \mathbb{1}_Q$$

for all $u \in L^p(\mathbb{R}^n; X^n)$. Recall that $\mathscr{Q}_M$ is the collection of dyadic cubes having measure $2^{-M\cdot n}$.

Besides using vector-valued analogues, the first two steps are essentially the same as in the scalar-valued case (see [**Mül99**] and [**LMM07**]).

The third step of the vector-valued proof is different. This is where the additional hypothesis (1.19) enters. It is there where we obtain the weak lower semicontinuity

$$\liminf_{r\to\infty} \int_{\mathbb{R}^n} f(v_r)\,\varphi\,\mathrm{d}x \ge \int_{\mathbb{R}^n} f(v)\,\varphi\,\mathrm{d}x,$$

assuming that v is a finite sum of Haar functions, and φ has support in $(0,1)^n$.

The restrictions on v and φ will be lifted in in the last step by approximation.

Proof of Theorem 1.2. Step 1. Within this proof we shall use the abbreviations $W^{-1,p}$ for $W^{-1,p}(\mathbb{R}^n; X^n \times X^n)$ and L^p for $L^p(\mathbb{R}^n; X^n)$.

Recall that the projection $P : L^p \longrightarrow L^p$ is given by

$$P(v) = (P^{(e_1)}v^{(1)}, \ldots, P^{(e_n)}v^{(n)}),$$

where $v = (v^{(j)})_{j=1}^n$. We will show that

$$\big\| P(w_r) - w_r \big\|_{L^p} \to 0, \tag{1.24}$$

whenever $w_r \to 0$ weakly in L^p and $\big\{\mathcal{A}_0(w_r)\big\}$ is precompact in $W^{-1,p}$. Note that the operator $\mathcal{A}_0 : L^p \longrightarrow W^{-1,p}$ being bounded implies

$$\big\|\mathcal{A}_0 w_r\big\|_{W^{-1,p}} \longrightarrow 0, \qquad \text{as } r \to \infty.$$

We will prove (2.16) using the interpolatory main result Theorem 1.1 on page 20. To this end, we dominate the Riesz transform by operators T_1 and T_2, mapping $\{w_r\}_r$ into a zero-convergent sequence. Let the smooth function $\psi : \mathbb{R}^n \longrightarrow [0,1]$ satisfy $\operatorname{supp}(\psi) \subset \{\xi : |\xi| \le 1\}$ and $\psi(\xi) = 1$, if $|\xi| \le 1/2$. Then, for every $u \in L^p_X$ we have

$$R_i u = T_1\, R_i(u) + T_2\, \mathcal{F}^{-1}\big(\langle\xi\rangle^{-1}\xi_i\, \mathcal{F}u\big), \tag{1.25}$$

where $T_1(f) = \mathcal{F}^{-1}(\psi \cdot \mathcal{F} f)$, $T_2(f) = \mathcal{F}^{-1}\big(m \cdot \mathcal{F} f\big)$ and $m(\xi) = (1-\psi(\xi)) \cdot \langle \xi \rangle \cdot |\xi|^{-1}$. Observe that

$$\begin{aligned} \big|\partial_\xi^\alpha \psi(\xi)\big| &\le A_\alpha, && \text{for all } \alpha \text{ and } \xi, \\ \big|\partial_\xi^\alpha m(\xi)\big| &\le A_\alpha \cdot |\xi|^{-|\alpha|} && \text{for all } \alpha \text{ and } \xi \neq 0. \end{aligned}$$

This means m is a Fourier multiplier of order 0, and by [**McC84**, Theorem 1.1] we know that T_2 maps $L^p_X(\mathbb{R}^n)$ boundedly into itself. Let $K = \mathcal{F}^{-1}\psi$, then $T_1 f = K * f$, and by partial integration one can see

$$\big|K(x)\big| \le A_\alpha \cdot |x^{-\alpha}|,$$

for all α and $x \neq 0$. Thus, $K \in L^1_{\mathbb{C}}(\mathbb{R}^n)$, and Theorem 4.3 on page 60 implies T_1 maps $L^p_X(\mathbb{R}^n)$ compactly into itself. Now we insert $w_r^{(j)}$, $j \neq i$ in (1.25) and gain

$$\big\|R_i(w_r^{(j)})\big\|_{L^p_X} \le \big\|T_1\, R_i(w_r^{(j)})\big\|_{L^p_X} + \big\|T_2 \,:\, L^p_X \to L^p_X\big\| \cdot \big\|\partial_i w_r^{(j)}\big\|_{W_X^{-1,p}}.$$

Since $T_1\, R_i$ is compact and $w_r^{(j)} \to 0$ weakly in L^p_X, we infer $\big\|T_1\, R_i(w_r^{(j)})\big\|_{L^p_X} \to 0$. The operator T_2 being bounded and $\|\mathcal{A}_0(w_r)\|_{W_X^{-1,p}} \longrightarrow 0$ as $r \to \infty$, implies that the latter term tends to zero as well, hence

$$\big\|R_i(w_r^{(j)})\big\|_{L^p_X} \longrightarrow 0, \qquad \text{for all } i \neq j \text{ as } r \to \infty. \tag{1.26}$$

Finally, we will apply the interpolatory estimate (1.14) to $P(w_r) - w_r$, observe

$$\big\|P(w_r) - w_r\big\|_{L^p} \le C \cdot \sum_j \big\|w_r^{(j)}\big\|_{L^p_X}^{\theta} \cdot \big\|R_{j^*} w_r^{(j)}\big\|_{L^p_X}^{1-\theta}, \tag{1.27}$$

where $0 < \theta < 1$, and j^* is an arbitrary index in $\{1, \dots, n\} \setminus \{j\}$. Combining (1.26) and (1.27) yields the desired result (2.16).

Proof of Theorem 1.2. Step 2. We will prove the following version of Jensen's inequality for separately convex functions f on the range of P,

$$f\big(\mathbb{E}_M(Pv)\big) \le \mathbb{E}_M\,\big(f(Pv)\big), \tag{1.28}$$

where

$$\mathbb{E}_M\, u = \sum_{Q \in \mathcal{Q}_M} \Big(\frac{1}{|Q|} \int_Q u(x)\,\mathrm{d}x\Big) \cdot \mathbb{1}_Q$$

for all $u \in L^p$, keeping in mind that $L^p = L^p(\mathbb{R}^n; X^n)$, and $\mathcal{Q}_M$ is the collection of dyadic cubes having measure $2^{-M \cdot n}$.

First we will show

$$f\Big(\int_{[0,1]^n} P(v)\,\mathrm{d}x\Big) \le \int_{[0,1]^n} f(P(v))\,\mathrm{d}x. \tag{1.29}$$

Once we have established (3.22), rescaling and translating (3.22) yields the desired inequality (2.17).

Define the Haar projections $P_k^{(\varepsilon)} u = \sum_{j=-\infty}^{k} \sum_{Q \in \mathcal{Q}_j} \langle u, h_Q^{(\varepsilon)} \rangle\, h_Q^{(\varepsilon)}\, |Q|^{-1}$, for every $u \in L^p_X$, $k \in \mathbb{Z}$, and furthermore

$$P_k v = \big(P_k^{(e_1)} v^{(1)}, \dots, P_k^{(e_n)} v^{(n)}\big),$$

for all $v \in L^p$, $k \in \mathbb{Z}$.

Let $k \geq 0$, then

$$\int\limits_{[0,1]^n} f(P_k(v))\,\mathrm{d}x = \sum_{Q\in\mathscr{Q}_k|[0,1]^n} \int\limits_Q f\big((P_k^{(e_j)}(v^{(j)}))_{j=1}^n\big)\,\mathrm{d}x$$
$$= \sum_{Q\in\mathscr{Q}_k|[0,1]^n} \int\limits_Q f\big((P_{k-1}^{(e_j)}(v^{(j)}) + c_Q^{(j)}\,h_Q^{(e_j)})_{j=1}^n\big)\,\mathrm{d}x.$$

Observe that $P_{k-1}^{(e_j)}(v^{(j)}) \mid Q = a_Q^{(j)}$ is constant, and $h_Q^{(e_j)}(x) = h_Q^{(e_j)}(x_j)$, for all $x \in Q$ and $1 \leq j \leq n$. Since f is separately convex, we apply Jensen's inequality to each direction e_j, $1 \leq j \leq n$ yielding

$$\int\limits_{[0,1]^n} f(P_k(v))\,\mathrm{d}x \geq \sum_{Q\in\mathscr{Q}_k|[0,1]^n} |Q|\cdot f\Bigg(\bigg(\frac{1}{\big|I_Q^{(j)}\big|}\int\limits_{I_Q^{(j)}} a_Q^{(j)} + c_Q^{(j)}h_Q^{(e_j)}(x_j)\,\mathrm{d}x_j\bigg)_{j=1}^n\Bigg)$$
$$= \sum_{Q\in\mathscr{Q}_k|[0,1]^n} |Q|\cdot f\big((P_{k-1}^{(e_j)}(v^{(j)}))_{j=1}^n\big),$$

where $\prod_{j=1}^n I_Q^{(j)} = Q$. Hence,

$$\int\limits_{[0,1]^n} f(P_k(v))\,\mathrm{d}x \geq \int\limits_{[0,1]^n} f(P_{k-1}(v))\,\mathrm{d}x,$$

for all $k \geq 1$. Since $P_{-1}(v)$ is constant on $[0,1]^n$, we certainly have

$$\int\limits_{[0,1]^n} f(P_{-1}(v))\,\mathrm{d}x = f\bigg(\int\limits_{[0,1]^n} P_{-1}(v)\,\mathrm{d}x\bigg),$$

so by induction on $k \geq 0$ we gain

$$\int\limits_{[0,1]^n} f(P_k(v))\,\mathrm{d}x \geq f\bigg(\int\limits_{[0,1]^n} P_{k-1}(v)\,\mathrm{d}x\bigg),$$

for all $k \geq 1$. If we let $k \to \infty$, the last inequality yields estimate (3.22), from which (2.17) follows immediately by rescaling and translating (3.22), as mentioned above.

Proof of Theorem 1.2. Step 3. First, we will assume that v is a finite Haar series, and $\operatorname{supp}(\varphi) \subset (0,1)^n$.

Let $\mathscr{B} \subset \mathscr{Q}$ be a finite collection of pairwise disjoint dyadic cubes such that

$$v = \sum_{Q\in\mathscr{B}} c_Q\,\mathbb{1}_Q. \tag{1.30}$$

Now define

$$f_Q(x) = f(x + c_Q), \qquad \text{for all } Q \in \mathscr{Q} \text{ and } x \in \mathbb{R}^n. \tag{1.31}$$

Theorem 4.1 on page 58 asserts that

$$\big|f_Q(x) - f_Q(y)\big| \leq A(n,p,c_Q)\cdot\big(1 + \|x\|_{X^n} + \|y\|_{X^n}\big)^{p-1}\cdot\|x-y\|_{X^n}, \tag{1.32}$$

for all $x, y \in X^n$. We shall abbreviate $A(n,p,c_Q)$ by A. If we set $w_r = v_r - v$, then since $w_r \to 0$, weakly in L^p and $\{\mathcal{A}_0(w_r)\}_r$ is precompact in $W^{-1,p}$. We know from (2.16) in the first step of the proof that

$$\big\|P(w_r) - w_r\big\|_{L^p} \longrightarrow 0, \qquad \text{as } r \to \infty. \tag{1.33}$$

Let $Q \in \mathscr{B}$ be an arbitrary dyadic cube. A glance at (1.30) and (1.31) shows

$$\begin{aligned}\int_Q f(v_r)\,\varphi\,\mathrm{d}x &= \int_Q f_Q(w_r)\,\varphi\,\mathrm{d}x \\ &= \int_Q f_Q(Pw_r)\,\varphi\,\mathrm{d}x + \int_Q \big(f_Q(w_r) - f_Q(Pw_r)\big)\,\varphi\,\mathrm{d}x.\end{aligned}$$

In view of the Lipschitz estimate (1.32), the latter term is bounded by

$$A \cdot \big\|1 + \|w_r\|_{X^n} + \|Pw_r\|_{X^n}\big\|_{L^p(0,1)^n}^{p/q} \cdot \big\|w_r - Pw_r\big\|_{L^p},$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Since $\sup_r \big\|w_r\big\|_{L^p} \le C$, for some constant C, we gain

$$\int_Q f(v_r)\,\varphi\,\mathrm{d}x \ge \int_Q f_Q(Pw_r)\,\varphi\,\mathrm{d}x - A \cdot C \cdot \big\|w_r - Pw_r\big\|_{L^p}. \tag{1.34}$$

We introduce the conditional expectation $\mathbb{E}_M$

$$\int_Q f_Q(Pw_r)\,\varphi\,\mathrm{d}x = \int_Q f_Q(Pw_r)\,\mathbb{E}_M\,\varphi\,\mathrm{d}x + \int_Q f_Q(Pw_r)\,(\varphi - \mathbb{E}_M\,\varphi)\,\mathrm{d}x, \tag{1.35}$$

for every $M \in \mathbb{Z}$. If M is sufficiently large, then

$$\int_Q f_Q(Pw_r)\,\mathbb{E}_M\,\varphi\,\mathrm{d}x = \int_Q \mathbb{E}_M\,\big(f_Q(Pw_r)\big)\,\mathbb{E}_M\,\varphi\,\mathrm{d}x$$

and applying Jensen's inequality on the range of P, that is inequality (2.17), yields

$$\begin{aligned}\int_Q f_Q(Pw_r)\,\mathbb{E}_M\,\varphi\,\mathrm{d}x &\ge \int_Q f_Q\big(\mathbb{E}_M(Pw_r)\big)\,\mathbb{E}_M\,\varphi\,\mathrm{d}x \\ &= \int_Q f_Q(0)\,\mathbb{E}_M\,\varphi\,\mathrm{d}x + \int_Q \big(f_Q(\mathbb{E}_M(Pw_r)) - f_Q(0)\big)\,\mathbb{E}_M\,\varphi\,\mathrm{d}x\end{aligned} \tag{1.36}$$

Using the Lipschitz estimate (1.32) and the L^p boundedness of $\{w_r\}_r$ as above, we can dominate last term of (1.36) by

$$A \cdot C \cdot \big\|\,\mathbb{E}_M\,Pw_r\big\|_{L^p(0,1)^n}.$$

Combining this with (1.34), (1.35), (1.36), and using the estimate $f_Q(Pw_r) \le A(c_Q) \cdot (1 + \|Pw_r\|_{X^n})^p$, in the latter term of (1.35), we gain

$$\begin{aligned}\int_Q f(v_r)\,\varphi\,\mathrm{d}x \ge \int_Q f_Q(0)\,\mathbb{E}_M\,\varphi\,\mathrm{d}x &- A \cdot C \cdot \big\|\,\mathbb{E}_M\,Pw_r\big\|_{L^p(0,1)^n} \\ &- C \cdot \big\|\varphi - \mathbb{E}_M\,\varphi\big\|_\infty - A \cdot C \cdot \big\|w_r - Pw_r\big\|_{L^p}.\end{aligned} \tag{1.37}$$

Now let us consider

$$\begin{aligned}\mathbb{E}_M\,Pw_r = &\sum_{2^{-Mn} < |K| < 2^{Mn}} \Big(\langle P^{(e_j)} w_r^{(j)}, h_K^{(e_j)}\rangle\, h_K^{(e_j)}\,|K|^{-1}\Big)_{j=1}^n \\ &+ \sum_{|K| \ge 2^{Mn}} \Big(\langle P^{(e_j)} w_r^{(j)}, h_K^{(e_j)}\rangle\, h_K^{(e_j)}\,|K|^{-1}\Big)_{j=1}^n\end{aligned}$$

then with M fixed, the first term converges to zero as $r \to \infty$ in view of (1.19). The $L^p(0,1)^n$ norm of the latter term is dominated by

$$\sum_{\substack{|K| \geq 2^{Mn} \\ K \supset [0,1]^n}} \|w_r\|_{L^p} \, |K|^{\frac{1}{q}-1} \leq C \cdot 2^{Mn(\frac{1}{q}-1)},$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Plugging this into (1.37) and considering (3.21) we have that

$$\liminf_{r\to\infty} \int_Q f(v_r)\, \varphi \, \mathrm{d}x \geq \int_Q f_Q(0)\; \mathbb{E}_M\, \varphi \, \mathrm{d}x - C \cdot \left\|\varphi - \mathbb{E}_M\, \varphi\right\|_{L^\infty(0,1)^n} - C \cdot 2^{Mn(\frac{1}{q}-1)},$$

for all M. Letting $M \to \infty$, recalling (1.31) and (1.30) and noting $f_Q(0) = f(v(x))$, $x \in Q$, we obtain

$$\liminf_{r\to\infty} \int_Q f(v_r)\, \varphi \, \mathrm{d}x \geq \int_Q f(v)\, \varphi \, \mathrm{d}x,$$

for every $Q \in \mathscr{Q}$. Since $\mathscr{B}$ is a finite collection, summation over $Q \in \mathscr{B}$ yields

$$\liminf_{r\to\infty} \int_{\mathscr{B}^*} f(v_r)\, \varphi \, \mathrm{d}x \geq \int_{\mathscr{B}^*} f(v)\, \varphi \, \mathrm{d}x,$$

where $\mathscr{B}^* = \bigcup_{Q\in\mathscr{B}} Q$. Repeating the argument above with f_Q replaced by f shows that

$$\liminf_{r\to\infty} \int_{(\mathscr{B}^*)^c} f(v_r)\, \varphi \, \mathrm{d}x \geq \int_{(\mathscr{B}^*)^c} f(v)\, \varphi \, \mathrm{d}x.$$

Note that $w_r(x) = v_r(x)$, for all $x \in (\mathscr{B}^*)^c$. Adding the latter two estimates yields

$$\liminf_{r\to\infty} \int_{\mathbb{R}^n} f(v_r)\, \varphi \, \mathrm{d}x \geq \int_{\mathbb{R}^n} f(v)\, \varphi \, \mathrm{d}x, \tag{1.38}$$

under the additional restrictions of v being a finite Haar series and φ having support in $(0,1)^n$.

Proof of Theorem 1.2. Step 4. Consider the auxiliary operators P_k, $k \geq 1$ given by

$$P_k u = \sum_{\varepsilon\neq 0} \sum_{j\,:\,|j|\leq k} \sum_{\substack{Q\in\mathscr{Q}_j \\ Q\subset B(0,k)}} \langle u, h_Q^{(\varepsilon)}\rangle \, h_Q^{(\varepsilon)} \, |Q|^{-1},$$

where $B(0,k) = \{x \in \mathbb{R}^n \,:\, |x| \leq k\}$. Due to the UMD–property and the uniform boundedness principle

$$\left\|P_k - \mathrm{Id} \,:\, L^p \to L^p\right\| \longrightarrow 0, \quad \text{as } k \to \infty.$$

Now we will lift the restriction that v is a finite Haar series. We know from Step 1 that

$$\left\|w_r - P(w_r)\right\|_{W^{-1,p}} \longrightarrow 0,$$

where $w_r = v_r - v$. Let $k \geq 1$ and consider

$$\left\|P_k\, w_r - P\, P_k\, w_r\right\|_{L^p} \leq \left\|P_k \,:\, L^p \to L^p\right\| \cdot \left\|w_r - P(w_r)\right\|_{L^p}.$$

If we apply Step 3 to $P_k(v_r)$ in place of v_r, and $P_k v$ instead of v we gain from (1.38)

$$\liminf_{r\to\infty} \int_{\mathbb{R}^n} f(P_k v_r)\, \varphi \, \mathrm{d}x \geq \int_{\mathbb{R}^n} f(P_k v)\, \varphi \, \mathrm{d}x, \tag{1.39}$$

for all $k \geq 1$. In view of the Lipschitz estimate (1.32) and $P_k \to \mathrm{Id}$, we may lift the restriction of v being a finite Haar series.

Now we lift the restriction $\operatorname{supp}(\varphi) \subset (0,1)^n$, so let $\varphi \in C_0^+(\mathbb{R}^n)$ be arbitrary. Let $\eta_k \in C_0^+(0,1)^n$, $k \geq 1$ be functions such that $0 \leq \eta_k \leq 1$ and $\eta_k \to \mathtt{1}_{(0,1)^n}$ point-wise. Now extend η_k periodically to $\mathbb{R}^n$, and note that

$$\begin{aligned}\liminf_{r\to\infty} \int_{\mathbb{R}^n} f(v_r)\,\varphi\,\mathrm{d}x &\geq \liminf_{r\to\infty} \int_{\mathbb{R}^n} f(v_r)\,\varphi\,\eta_k\,\mathrm{d}x \\ &= \sum_{|Q|=1} \liminf_{r\to\infty} \int_{\mathbb{R}^n} f(v_r)\,\mathtt{1}_Q\,\varphi\,\eta_k\,\mathrm{d}x,\end{aligned}$$

for all $k \geq 1$. Since $\mathtt{1}_Q \cdot \varphi \cdot \eta_k \in C_0^+(Q)$, translating the integration domain of inequality (1.38) from $[0,1]^n$ to the dyadic cube Q yields

$$\liminf_{r\to\infty} \int_{\mathbb{R}^n} f(v_r)\,\varphi\,\mathrm{d}x \geq \int_{\mathbb{R}^n} f(v)\,\varphi\,\eta_k\,\mathrm{d}x,$$

for all $k \geq 1$. Letting $k \to \infty$ concludes the proof of Theorem 1.2. $\square$

2. The Stripe Operator S_λ

Here we define and study the stripe operator S_λ (defined in (2.6)), mapping h_Q, $Q \in \mathscr{Q}$ onto the blocks $g_{Q,\lambda}$, each supported on a dyadic stripe (see (2.3), (2.5) and the Figures 1 and 2). The vector–valued estimates given by

$$\|S_\lambda u\|_{L^p_X(\mathbb{R}^n)} \leq C \cdot 2^{-\lambda/\mathcal{C}(L^p_X(\mathbb{R}^n))} \|u\|_{L^p_X(\mathbb{R}^n)}, \tag{2.1}$$

constitute the main technical component of the first part of this thesis (see Theorem 2.5 on page 38).

The crucial points in the proof of (2.1) are the cotype inequality (0.3) and Corollary 2.4, that is the uniform equivalence

$$\frac{1}{C} \cdot \|S_\lambda u\|_{L^p_X(\mathbb{R}^n)} \leq \|T_{m\cdot e_1} S_\lambda u\|_{L^p_X(\mathbb{R}^n)} \leq C \cdot \|S_\lambda u\|_{L^p_X(\mathbb{R}^n)}, \tag{2.2}$$

for all $0 \leq m \leq 2^\lambda - 1$ and $u \in L^p_X(\mathbb{R}^n)$, where C does not depend on u, λ and m. In other words, the operators T_m, $0 \leq m \leq 2^\lambda - 1$ act as isomorphisms on the image of S_λ, with norm independent of m and λ. This is in contrast to the well known norm estimates $\|T_m : L^p_X(\mathbb{R}^n) \to L^p_X(\mathbb{R}^n)\| \approx \log(2+m)^\alpha$, see (0.7). More details are supplied in Section 2 in the second part of this thesis.

2.1. Preparation.

Within this section the superscripts (ε) are omitted and we generically denote by h_Q one of the functions $\{h_Q^{(\varepsilon)} : \varepsilon \in \{0,1\}^n \setminus \{0\}\}$.

For every $Q \in \mathscr{Q}$ and $\lambda \geq 0$ define the dyadic stripe

$$\mathscr{U}_\lambda(Q) = \{E \in \mathscr{Q} : \pi^\lambda(E) = Q,\ \inf_{x\in E} x_1 = \inf_{q \in Q} q_1\}, \tag{2.3}$$

where x_1 respectively q_1 denotes the orthogonal projection of $x \in \mathbb{R}^n$ respectively $q \in \mathbb{R}^n$ onto the vector $e_1 = (1, 0, \dots, 0)$. Recall that $\pi^\lambda(E)$ is the unique $Q \in \mathscr{Q}$ such that $|Q| = 2^{\lambda n}|E|$ and $Q \supset E$. The dyadic stripe $\mathscr{U}_\lambda(Q)$ is illustrated in Figure 1. Additionally, set

$$\mathscr{U}_\lambda = \bigcup_{Q \in \mathscr{Q}} \mathscr{U}_\lambda(Q). \tag{2.4}$$

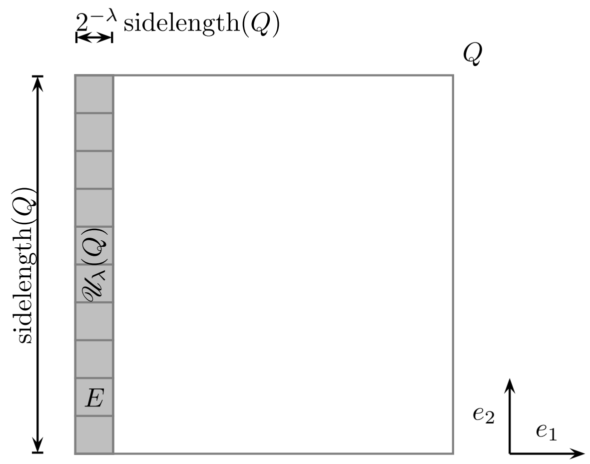

FIGURE 1. Dyadic stripe $\mathscr{U}_\lambda(Q)$ in dimension $n = 2$.

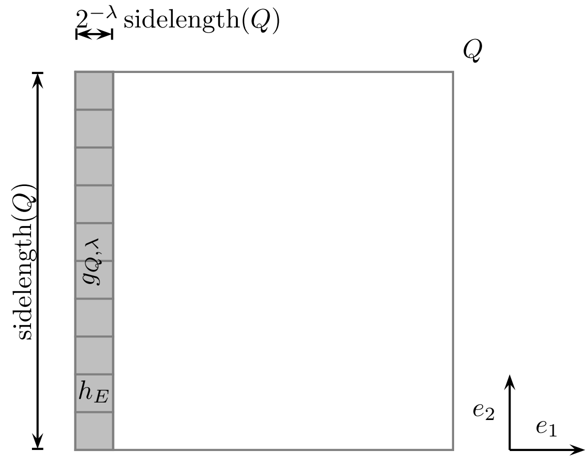

FIGURE 2. Stripe functions $g_{Q,\lambda}$ in dimension $n = 2$.

We define the stripe functions $g_{Q,\lambda}$ by

$$g_{Q,\lambda} = \sum_{E \in \mathscr{U}_\lambda(Q)} h_E, \tag{2.5}$$

and the stripe operator S_λ by

$$S_\lambda u = \sum_{Q \in \mathscr{Q}} \langle u, h_Q \rangle \, g_{Q,\lambda} \, |Q|^{-1}, \tag{2.6}$$

for all $u \in L^p_X(\mathbb{R}^n)$.

The stripe functions are visualized in Figure 2.

2.2. Shift Operators Acting on Dyadic Stripes.

In Lemma 2.1 we prove a measure estimate regarding one-dimensional dyadic stripes $\mathscr{S}_\lambda$, $\lambda \geq 1$ defined in (2.8) and the action of dyadic shift maps τ_m, $0 \leq m \leq 2^{\lambda-1}$ given by

$$\tau_m(I) = I + m \cdot |I|, \qquad I \in \mathscr{D}.$$

These estimates will then enter inequality (2.2), where we prove the uniform estimates

$$\frac{1}{C} \cdot \|u\|_{L^p_X(\mathbb{R})} \leq \|T_m u\|_{L^p_X(\mathbb{R})} \leq C \cdot \|u\|_{L^p_X(\mathbb{R})}, \tag{2.7}$$

for all u supported on $\mathscr{S}_\lambda$ and $0 \leq m \leq 2^\lambda - 1$. The constant C does not depend on λ or m. The shift operator T_m is defined in (0.7). A more detailed discussion of T_m may be found in Section 2 of Chapter 3.

The subsequent Corollary 2.4 on page 36 states that T_m acts as an isomorphism on the image of S_λ, with norm independent of m and λ, provided $0 \leq m \leq 2^\lambda - 1$.

Before we state Lemma 2.1, we build up some notation. Recall $\pi^\lambda : \mathscr{D} \to \mathscr{D}$ is given by

$$\pi^\lambda(I) = J,$$

where J is the uniquely determined $J \in \mathscr{D}$ such that $|J| = 2^\lambda |I|$ and $J \supset I$. Finally, define the one-dimensional stripe $\mathscr{S}_\lambda$ by

$$\mathscr{S}_\lambda = \{I \in \mathscr{D} : \inf I = \inf \pi^\lambda(I)\}. \tag{2.8}$$

LEMMA 2.1. *For every* $\lambda \geq 1$ *let* $0 \leq m \leq 2^{\lambda-1}$ *and*

$$\tau_m(I) = I + m \cdot |I|, \qquad I \in \mathscr{D}.$$

Let $\mathscr{B} \subset \mathscr{S}_\lambda$ such that for all $J, K \in \mathscr{B}$ with $|J| \neq |K|$ either

$$|J| \leq \frac{1}{4}|K| \quad or \quad |K| \leq \frac{1}{4}|J|.$$

Then the estimates

$$\Big| I \cap \bigcup_{d=1}^{\lambda-1} \bigcup_{\substack{J \in \mathscr{B} \\ |J| = 2^{-d}|I|}} J \cup \tau_m(J) \Big| \leq \frac{2}{3}|I|,$$

and

$$\Big| \tau_m(I) \cap \bigcup_{d=1}^{\lambda-1} \bigcup_{\substack{J \in \mathscr{B} \\ |J| = 2^{-d}|I|}} J \cup \tau_m(J) \Big| \leq \frac{2}{3}|I|,$$

hold true for all $I \in \mathscr{B}$.

Proof. First we claim that for every $I \in \mathscr{B} \cup \tau_m(\mathscr{B})$, $1 \leq d \leq \lambda - 1$ and $J, K \in \mathscr{B}$ with $|J| = |K| = 2^{-d}|I|$ holds true that whenever

$$(J \cup \tau_m(J)) \cap I \neq \emptyset \quad \text{and} \quad (K \cup \tau_m(K)) \cap I \neq \emptyset, \quad \text{then} \quad J = K. \tag{2.9}$$

We assume that the asserted implication (2.9) is incorrect. Hence we can find intervals $I \in \mathscr{B} \cup \tau_m(\mathscr{B})$, and $J, K \in \mathscr{B}$ with $J \neq K$, $|J| = |K| = 2^{-d}|I|$ where $1 \leq d \leq \lambda - 1$, such that

$$(J \cup \tau_m(J)) \cap I \neq \emptyset \quad \text{and} \quad (K \cup \tau_m(K)) \cap I \neq \emptyset.$$

Since $J \neq K$ we know from the definition of $\mathscr{B}$ that

$$\operatorname{dist}(\tau_m(J), \tau_m(K)) = \operatorname{dist}(J, K) \geq (2^\lambda - 1)|J|,$$

consequently

$$\operatorname{dist}(J \cup \tau_m(J), K \cup \tau_m(K)) \geq (2^\lambda - 1 - m)|J|.$$

We know I intersects both, $J \cup \tau_m(J)$ and $K \cup \tau_m(K)$, so

$$\begin{aligned} |I| &\geq \operatorname{dist}(J \cup \tau_m(J), K \cup \tau_m(K)) + 2|J| \\ &\geq (2^\lambda - m + 1))\, 2^{-d}|I| \\ &\geq (2^{\lambda-1} + 1)\, 2^{-d}|I| \\ &> |I|, \end{aligned}$$

which is obviously a contradiction.

Hence (2.9) holds true, which means that for all $1 \leq d \leq \lambda - 1$, every interval $I \in \mathscr{B} \cup \tau_m(\mathscr{B})$ intersects at most one element of the set

$$\{J \cup \tau_m(J) \in \mathscr{B} : |J| = 2^{-d}|I|\}.$$

If such a J exists, we denote it by $J_d(I) \in \mathscr{B}$, and define $J_d(I) = \emptyset$ otherwise. Note that for small shift widths m or small J it may happen that $J_d(I) \cup \tau_m(J_d(I)) \subset I$.

Using (2.9) we see that for every $I \in \mathscr{B} \cup \tau_m(\mathscr{B})$

$$\Big|I \cap \bigcup_{d=1}^{\lambda-1} \bigcup_{\substack{J\in\mathscr{B} \\ |J|=2^{-d}\,|I|}} J \cup \tau_m(J)\Big| \le \sum_{d=1}^{\lambda-1} \big|I \cap \big(J_d(I) \cup \tau_m(J_d(I))\big)\big|$$

$$\le \sum_{d=1}^{\lambda-1} 2 \cdot |J_d(I)|$$

$$\le 2 \cdot \sum_{d=1}^{\infty} 2^{-2d}\,|I|$$

$$= \frac{2}{3}\,|I|.$$

The last inequality follows since for $J, K \in \mathscr{B}$, if $|J| \neq |K|$, then either $|J| \le |K|/4$ or $|K| \le |J|/4$. □

For $m \in \mathbb{Z}$ the shift operator T_m is given by

$$T_m h_I = h_{\tau_m(I)}, \quad I \in \mathscr{D},$$

where $\tau_m(I) = I + m \cdot |I|$, $I \in \mathscr{D}$, (see (0.6) and (0.7)). We will now investigate the action of T_m restricted to functions supported on the dyadic stripe $\mathscr{S}_\lambda$, $\lambda \ge 0$ defined in (2.8), that was

$$\mathscr{S}_\lambda = \{I \in \mathscr{D} \,:\, \inf I = \inf \pi^\lambda(I)\}.$$

Observe that $\mathscr{S}_\lambda$ is the spectrum of the stripe operator S_λ, when it is restricted to lines in direction $(1, 0, \ldots, 0)$. This will be discussed in more detail in the subsequent Corollary 2.4 on page 36. For now, we dedicate ourselves to the one-dimensional case.

Theorem 2.2. *Let X be a Banach space with the* UMD *property and $1 < p < \infty$. For $\lambda \ge 0$ define the linear subspace Z_λ of $L^p_X(\mathbb{R})$ by*

$$Z_\lambda = \Big\{ \sum_{I\in\mathscr{S}_\lambda} u_I\, h_I\, |I|^{-1} \;:\; u_I \in X \Big\} \cap L^p_X(\mathbb{R}). \tag{2.10}$$

Then there exists a constant $C > 0$ such that for all integers λ and m satisfying $0 \le m \le 2^\lambda - 1$ we have that

$$\frac{1}{C} \cdot \|u\|_{L^p_X(\mathbb{R})} \le \|T_m\, u\|_{L^p_X(\mathbb{R})} \le C \cdot \|u\|_{L^p_X(\mathbb{R})}, \tag{2.11}$$

for all $u \in Z_\lambda$, where C depends only on p and the UMD *constant of X. In other words, T_m acts as an isomorphism on Z_λ with norm independent of m and λ.*

Proof. With $\lambda \ge 0$ fixed, we will first prove

$$\frac{1}{C} \cdot \|u\|_{L^p_X(\mathbb{R})} \le \|T_m\, u\|_{L^p_X(\mathbb{R})} \le C \cdot \|u\|_{L^p_X(\mathbb{R})}, \tag{2.12}$$

for all $0 \le m \le 2^{\lambda-1}$ and $u \in Z_\lambda$. Once we have (2.12), it is easy to see by symmetry that we also have

$$\frac{1}{C} \cdot \|T_{2^\lambda-1}\, u\|_{L^p_X(\mathbb{R})} \le \|T_m u\|_{L^p_X(\mathbb{R})} \le C \cdot \|T_{2^\lambda-1}\, u\|_{L^p_X(\mathbb{R})}, \tag{2.13}$$

for all $2^{\lambda-1} - 1 \le m \le 2^\lambda - 1$ and $u \in Z_\lambda$. Certainly, (2.12) together with (2.13) implies (2.11), since we may join (2.12) and (2.13) at the intersection of the two collections of operators

$$\{T_m \,:\, 0 \le m \le 2^{\lambda-1}\} \qquad \text{and} \qquad \{T_m \,:\, 2^{\lambda-1} - 1 \le m \le 2^\lambda - 1\},$$

that is at $m = 2^{\lambda-1}$ or at $m = 2^{\lambda-1} - 1$.

We begin the proof of (2.12) by defining the four collections

$$\mathscr{B}^0_{\text{odd}} = \bigcup_{j\in\mathbb{Z}} \bigcup_{\substack{k=0\\ k \text{ odd}}}^{\lambda-1} \mathscr{S}_\lambda \cap \mathscr{D}_{2j\lambda+k}, \qquad \mathscr{B}^0_{\text{even}} = \bigcup_{j\in\mathbb{Z}} \bigcup_{\substack{k=0\\ k \text{ even}}}^{\lambda-1} \mathscr{S}_\lambda \cap \mathscr{D}_{2j\lambda+k},$$

$$\mathscr{B}^1_{\text{odd}} = \bigcup_{j\in\mathbb{Z}} \bigcup_{\substack{k=0\\ k \text{ odd}}}^{\lambda-1} \mathscr{S}_\lambda \cap \mathscr{D}_{(2j+1)\lambda+k}, \qquad \mathscr{B}^1_{\text{even}} = \bigcup_{j\in\mathbb{Z}} \bigcup_{\substack{k=0\\ k \text{ even}}}^{\lambda-1} \mathscr{S}_\lambda \cap \mathscr{D}_{(2j+1)\lambda+k}.$$

Let $\mathscr{B}$ denote one of those four collections. The collection $\mathscr{B}$ consists of λ consecutive levels, followed by a gap of λ levels, followed by λ consecutive levels and so on. Within this proof we shall refer to intervals located in the same λ consecutive levels of $\mathscr{B}$ as block.

We claim the existence of a filtration $\{\mathcal{F}_j\}_j$, such that for every $j \in \mathbb{Z}$ and $I \in \mathscr{B} \cap \mathscr{D}_j$ exists an atom $A(I)$ of $\mathcal{F}_j$ satisfying the inequalities

$$|A(I)| \leq 2\,|I|, \qquad |I \cap A(I)| \geq \frac{1}{3}\,|I|, \qquad |\tau_m(I) \cap A(I)| \geq \frac{1}{3}\,|I|. \tag{2.14}$$

Now, for each $I \in \mathscr{B}$ we will define atoms inductively, beginning at the finest level of a block. More precisely, fix an arbitrary $b \in \mathbb{Z}$ such that for all $I, I' \in \mathscr{B}$ with $|I| = 2^{-b}$ and $|I'| < |I|$ follows $|I'| \leq 2^{-\lambda}\,|I|$. Initially, define

$$A(I) = I \cup \tau_m(I), \tag{2.15}$$

for $I \in \mathscr{B} \cap \mathscr{D}_b$. Assume we already constructed atoms $A(J)$ if $2^{-b} \leq |J| \leq 2^{-j}$. Then define for every $I \in \mathscr{B} \cap \mathscr{D}_{j-1}$ the atom $A(I)$ by

$$A(I) = \big(I \cup \tau_m(I)\big) \setminus \Big(\bigcup_{k=j}^{b} \bigcup_{J \in \mathscr{B}\cap\mathscr{D}_k} A(J)\Big). \tag{2.16}$$

Applying Lemma 2.1 on page 30 to the atoms $A(I) \subset I \cup \tau_m(I)$ inside the block $b, b-1, \dots, b-(\lambda-1)$ we gain

$$|I \cap A(I)| = |I| - |I \cap \bigcup_{k=b-(\lambda-1)}^{b} \bigcup_{J\in\mathscr{B}\cap\mathscr{D}_k} A(J)| \geq \frac{1}{3}\,|I|,$$

and analogously

$$|\tau_m(I) \cap A(I)| \geq \frac{1}{3}\,|I|,$$

which yields (2.14). Finally we define the collection

$$\mathscr{A}_j = \big\{A(I) \,:\, I \in \mathscr{B} \cap \mathscr{D}_j\big\}, \tag{2.17}$$

and the filtration

$$\mathcal{F}_j = \sigma\text{-algebra}\Big(\bigcup_{i\leq j} \mathscr{A}_i\Big). \tag{2.18}$$

What is left to show is that every $A \in \mathscr{A}_j$ is an atom for the σ–algebra $\mathcal{F}_j$.

To see this we reason as follows. First note that each two atoms are either localized in the same block, or are separated by at least λ levels. If atoms $A(I)$ and $A(I')$ are in the same block, then they do not intersect per construction (see (2.15) and (2.16)). Whenever $A(I)$ and $A(I')$ intersect and $|I'| \leq 2^{-\lambda}\,|I|$, then since

$$A(I') \subset (I' \cup \tau_m(I')) \subset \pi^\lambda(I')$$

we have

$$\pi^\lambda(I') \cap A(I) \neq \emptyset.$$

Clearly, $A(I)$ comprises of cubes K which are at least as big as $\pi^\lambda(I')$, so note $|\,\pi^\lambda(I')| \le |K|$, hence
$$A(I') \subset A(I).$$
This means that $\bigcup_j \mathscr{A}_j$ is a nested collections of sets, thus every $A \in \mathscr{A}_j$ is an atom for the σ–algebra $\mathcal{F}_j$.

Now we are prepared to estimate the shift operator T_m. To this end we let $u \in Z_\lambda$ be fixed throughout the rest of the proof. Having (2.14) at hand and knowing that the collection $\mathscr{A}_j$ comprises of atoms of $\mathcal{F}_j$, observe
$$\mathbb{1}_I \le 18 \cdot \mathbb{E}\left(\mathbb{E}\left(\mathbb{1}_{\tau_m(I)}\,|\,\mathcal{F}_j\right)\,\big|\,\mathscr{D}_j\right), \tag{2.19}$$
and analogously
$$\mathbb{1}_{\tau_m(I)} \le 18 \cdot \mathbb{E}\left(\mathbb{E}\left(\mathbb{1}_I\,|\,\mathcal{F}_j\right)\,\big|\,\mathscr{D}_j\right). \tag{2.20}$$

The UMD property and Kahane's contraction principle applied to $|h_I| \le \mathbb{1}_I$ yields
$$\|u\|^p_{L^p_X(\mathbb{R})} \approx \int_0^1 \big\|\sum_{j\in\mathbb{Z}} r_j(t)\ \mathbb{I}(u)_j\big\|^p_{L^p_X(\mathbb{R})} \mathrm{d}t,$$
where $(\,\cdot\,)_j$ denotes the restriction of the Haar expansion to intervals in $\mathscr{D}_j$, and $\mathbb{I}\,h_I = \mathbb{1}_I$. More precisely, if
$$u = \sum_{j\in\mathbb{Z}}\sum_{I\in\mathscr{D}_j} u_I\,h_I\,|I|^{-1},$$
then
$$\mathbb{I}(\,u\,)_j = \sum_{I\in\mathscr{D}_j} u_I\,\mathbb{1}_I\,|I|^{-1}.$$
So applying Kahane's contraction principle in view of (2.19) yields
$$\|u\|^p_{L^p_X(\mathbb{R})} \lesssim \int_0^1 \big\|\sum_{j\in\mathbb{Z}} r_j(t)\ \mathbb{E}\left(\mathbb{E}\left(\mathbb{I}(T_m u)_j\,|\,\mathcal{F}_j\right)\,\big|\,\mathscr{D}_j\right)\big\|^p_{L^p_X(\mathbb{R})}.$$
Using Stein's martingale inequality (0.5) with respect to the filtration $\{\mathscr{D}_j\}_j$ gives
$$\|u\|^p_{L^p_X(\mathbb{R})} \lesssim \int_0^1 \big\|\sum_{j\in\mathbb{Z}} r_j(t)\ \mathbb{E}\left(\mathbb{I}(T_m u)_j\,|\,\mathcal{F}_j\right)\big\|^p_{L^p_X(\mathbb{R})}.$$
Now we apply Stein's martingale inequality with respect to the filtration $\{\mathcal{F}_j\}_j$ and gain
$$\|u\|^p_{L^p_X(\mathbb{R})} \lesssim \int_0^1 \big\|\sum_{j\in\mathbb{Z}} r_j(t)\ \mathbb{I}(T_m u)_j\big\|^p_{L^p_X(\mathbb{R})}.$$
Subsequently, we apply Kahane's contraction principle to $\mathbb{1}_{\tau_m(I)} \le |h_{\tau_m(I)}|$ and make use of the UMD property to dispose of the Rademacher functions and obtain
$$\|u\|^p_{L^p_X(\mathbb{R})} \lesssim \|T_m u\|^p_{L^p_X(\mathbb{R})}.$$

Repeating this argument with the roles of u and $T_m u$ reversed, and using (2.20) instead of (2.19) we get the converse inequality
$$\|T_m u\|^p_{L^p_X(\mathbb{R})} \lesssim \|u\|^p_{L^p_X(\mathbb{R})}.$$
A fortiori, we proved (2.12), that was
$$\frac{1}{C}\cdot\big\|u\big\|_{L^p_X(\mathbb{R})} \le \big\|T_m\,u\big\|_{L^p_X(\mathbb{R})} \le C\cdot\big\|u\big\|_{L^p_X(\mathbb{R})},$$
for all $\lambda \ge 0$, $0 \le m \le 2^{\lambda-1}$ and $u \in Z_\lambda$, where C depends only on p and the UMD constant of X.

Observe that due to symmetry we may use the same argument for the operators T_m, $2^{\lambda-1} \le m \le 2^\lambda - 1$, if we reverse the sign of the shift operation and replace u by $T_{2^\lambda-1}u$. Therefore inequality (2.13) holds true as well. That was

$$\frac{1}{C} \cdot \left\|T_{2^\lambda-1}\, u\right\|_{L^p_X(\mathbb{R})} \le \left\|T_m u\right\|_{L^p_X(\mathbb{R})} \le C \cdot \left\|T_{2^\lambda-1}\, u\right\|_{L^p_X(\mathbb{R})},$$

for all $2^{\lambda-1} - 1 \le m \le 2^\lambda - 1$ and $u \in Z_\lambda$, where C depends only on p and the UMD constant of X.

Joining the last two inequalities via $T_{2^{\lambda-1}}$ (or $T_{2^{\lambda-1}-1}$) as indicated above concludes the proof. □

Remark 2.3. The central difficulty of the proof was finding the filtration $\{\mathcal{F}_j\}_j$, given by (2.18), such that each collection $\mathcal{A}_j$, given by (2.17), comprises of atoms $A(I)$ of $\mathcal{F}_j$. This was achieved by subtracting the atoms $A(J)$ succeeding $A(I)$ within a block (see (2.15) and (2.16)). Note that the measure estimates in Lemma (2.1) implied inequality (2.14). As a consequence, we obtained inequality (2.19) and (2.20), which enabled us to shift h_I to $h_{\tau_m(I)}$ by means of Kahane's contraction principle and Bourgain's version of Stein's martingale inequality.

For a detailed exposition and the development of a method how one can estimate rearrangement operators that admit a supporting tree, we refer the reader to [**KM09**] and [**MS91**]. Given a rearrangement τ such that $|\tau(I)| = |I|$, a supporting tree is essentially the existence of a filtration having the properties of $\{\mathcal{F}_j\}_j$ listed above, with τ_m replaced by τ.

In order to shift an essential portion of h_I to $h_{\tau_m(I)}$, one can replace Bourgain's version of Stein's martingale inequality by the martingale transforms used in [**Fig88**, Proposition 2, Step 0]. To this end, we need additional symmetry properties (see (2.21)), which were not required for the first proof. We will refine the above construction of the filtration $\{\mathcal{F}_j\}_j$ for our purposes. The details are given in the proof below.

Alternative Proof of Theorem 2.2. We modify the construction of the above collections $\mathscr{B}$, by taking only every fourth level instead of every second level, and denote each of those collections by $\mathscr{C}$. Hence, for all $J, K \in \mathscr{C}$, if $|J| \ne |K|$ we have either

$$|J| \le \frac{1}{16}\,|K| \quad \text{or} \quad |K| \le \frac{1}{16}\,|J|.$$

Considering the proof of Lemma 2.1 on page 30 we see that

$$\left| I \cap \bigcup_{d=1}^{\lambda-1} \bigcup_{\substack{J\in\mathscr{C}\\ |J|=2^{-d}\,|I|}} J \cup \tau_m(J) \right| \le \frac{2}{15}\,|I|,$$

and

$$\left| \tau_m(I) \cap \bigcup_{d=1}^{\lambda-1} \bigcup_{\substack{J\in\mathscr{C}\\ |J|=2^{-d}\,|I|}} J \cup \tau_m(J) \right| \le \frac{2}{15}\,|I|.$$

So if we construct the atoms $A(I)$ according to (2.15) and (2.16) (with $\mathscr{B}$ replaced by $\mathscr{C}$), we gain instead of (2.14) the inequalities

$$|A(I)| \le 2\,|I|, \qquad |I \cap A(I)| \ge \frac{13}{15}\,|I|, \qquad |\tau_m(I) \cap A(I)| \ge \frac{13}{15}\,|I|.$$

In what follows we denote the left and right dyadic successor of I by I_0 and I_1, respectively. To be more precise, $I_0, I_1 \in \mathscr{D}$, $|I_0| = |I_1| = |I|/2$, and $\inf I_0 = \inf I$, $\sup I_1 = \sup I$. Consequently, if we define

$$\begin{aligned} B(I) = &\big(A(I) \cap (A(I) \cap I_1 - |I|/2)\big) \cup \big(A(I) \cap (A(I) \cap I_0 + |I|/2)\big) \\ &\cup \big(A(I) \cap (A(I) \cap \tau_m(I)_1 - |I|/2)\big) \cup \big(A(I) \cap (A(I) \cap \tau_m(I)_0 + |I|/2)\big) \end{aligned}$$

and furthermore

$$C(I) = \big(B(I) \cap (B(I) - m \cdot |I|)\big) \cup \big(B(I) \cap (B(I) + m \cdot |I|)\big),$$

we see that

$$|C(I)| \leq 2\,|I|, \qquad |I \cap C(I)| \geq \frac{7}{15}\,|I|, \qquad |\tau_m(I) \cap C(I)| \geq \frac{7}{15}\,|I|.$$

Since $C(I) \subset A(I)$, the $C(I)$, $I \in \mathscr{C}$ do not intersect inside a block. Retracing our steps, we may replace $A(I)$ by $C(I)$ in the above proof. Observe that additionally we have the following identities at our disposal

$$\begin{aligned} C(I) \cap \tau_m(I) &= C(I) \cap I + m \cdot |I|, \\ C(I) \cap I_1 &= (C(I) \cap I_0) + |I|/2, \end{aligned} \tag{2.21}$$

which allow us to use the martingale transform in the proof of [**Fig88**, Proposition 2, Step 0]. Elaborating on this martingale transform we define

$$d_{I,1} = \frac{1}{2}\big(h_I + h_{\tau_m(I)}\big) \cdot \mathbb{1}_{C(I)}, \qquad \text{and} \qquad d_{I,2} = \frac{1}{2}\big(h_I - h_{\tau_m(I)}\big) \cdot \mathbb{1}_{C(I)}, \tag{2.22}$$

and due to (2.21) we see that $\{d_{I,1}, d_{I,2} \,:\, I \in \mathscr{C}\}$ is a martingale difference sequence. Furthermore, note that

$$\{h_I \cdot \mathbb{1}_{C(I)} \,:\, I \in \mathscr{C}\} \quad \text{and} \quad \{h_{\tau_m(I)} \cdot \mathbb{1}_{C(I)} \,:\, I \in \mathscr{C}\}$$

are martingale difference sequences, as well. Observe

$$d_{I,1} + d_{I,2} = h_I \cdot \mathbb{1}_{C(I)} \quad \text{and} \quad d_{I,1} - d_{I,2} = h_{\tau_m(I)} \cdot \mathbb{1}_{C(I)}, \tag{2.23}$$

thus we can swap $h_I \cdot \mathbb{1}_{C(I)}$ with $h_{\tau_m(I)} \cdot \mathbb{1}_{C(I)}$, according to [**Fig88**, Lemma 2].

So we shifted $h_I \cdot \mathbb{1}_{C(I)}$ to $h_{\tau_m(I)} \cdot \mathbb{1}_{C(I)}$ by means of the martingale transformation given by (2.23) instead of applying Bourgain's version of Stein's martingale inequality for this purpose. ☐

The following Corollary 2.4 connects the one-dimensional Theorem 2.2 on page 32 with the multidimensional stripe operators S_λ. In Figure 3 on the next page the action of the shift operators T_m, $0 \leq m \leq 2^\lambda - 1$ on the image of S_λ is visualized.

COROLLARY 2.4. *Let X be a* UMD *space. Let $1 < p < \infty$, $n \in \mathbb{N}$ and denote by e_1 the unit vector $(1, 0, \ldots, 0) \in \mathbb{R}^n$.*

Then there exists a constant $C > 0$, such that for all integers λ and m satisfying $0 \leq m \leq 2^\lambda - 1$ and every $u \in L^p_X(\mathbb{R}^n)$

$$\frac{1}{C} \cdot \big\|S_\lambda u\big\|_{L^p_X(\mathbb{R}^n)} \leq \big\|T_{m \cdot e_1}\, S_\lambda u\big\|_{L^p_X(\mathbb{R}^n)} \leq C \cdot \big\|S_\lambda u\big\|_{L^p_X(\mathbb{R}^n)}, \tag{2.24}$$

where C depends only on n, p and the UMD *constant of X. In other words, T_m acts as an isomorphism on the image of S_λ with norm independent of m and λ.*

PROOF. We recall the definitions (2.3) and (2.4), that was

$$\mathscr{U}_\lambda = \big\{E \in \mathscr{Q} \,:\, \pi^\lambda(E) = Q,\ \inf_{x \in E} x_1 = \inf_{q \in Q} q_1\big\},$$

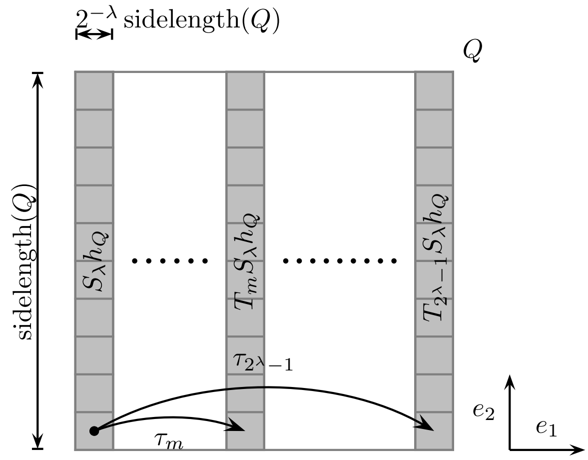

FIGURE 3. Shifting the image of a stripe operator S_λ in dimension $n = 2$.

and by $\cdot_1$ we denoted the projection onto the first coordinate. Observe that due to the definitions (2.5) and (2.6) we have

$$\operatorname{image}(S_\lambda) \subset \Big\{ \sum_{Q \in \mathscr{U}_\lambda} u_Q\, h_Q\, |Q|^{-1} \;:\; u_Q \in X \Big\} \cap L^p_X(\mathbb{R}^n).$$

With this in mind we will apply Theorem 2.2 on page 32 to every line in direction e_1.

Fix $u \in L^p_X$, define $v = S_\lambda u$ and denote by v_x the function $v(\cdot, x)$, for all $x \in \mathbb{R}^{n-1}$. Observe that for all $x \in \mathbb{R}^{n-1}$ and $t \in \mathbb{R}$ we have the identity

$$\big(T_{m\cdot e_1} v\big)(t,x) = (T_m v_x)(t),$$

hence

$$\big\|T_{m\cdot e_1} v\big\|^p_{L^p_X(\mathbb{R}^n)} = \int\limits_{\mathbb{R}^{n-1}} \int\limits_{\mathbb{R}} \big\|\big(T_m v_x\big)(t)\big\|^p_X \,\mathrm{d}t\,\mathrm{d}x = \int\limits_{\mathbb{R}^{n-1}} \big\|T_m v_x\big\|^p_{L^p_X(\mathbb{R})}\,\mathrm{d}x.$$

Note $v_x \in Z_\lambda$, for almost every $x \in \mathbb{R}^n$, so we may use Theorem 2.2 to get

$$\int\limits_{\mathbb{R}^{n-1}} \big\|T_m v_x\big\|^p_{L^p_X(\mathbb{R})}\,\mathrm{d}x \approx \int\limits_{\mathbb{R}^{n-1}} \big\|v_x\big\|^p_{L^p_X(\mathbb{R})}\,\mathrm{d}x = \big\|v\big\|^p_{L^p_X(\mathbb{R}^n)}.$$

Substituting $v = S_\lambda u$ finishes the proof of the Corollary. □

2.3. Estimates for the Stripe Operator.

Before we formulate and prove the main result on stripe operators S_λ we will recapitulate the definition of S_λ (see (2.6)). The dyadic stripe $\mathscr{U}_\lambda(Q)$ (for details see (2.3)) was the collection

$$\big\{E \in \mathscr{Q} \;:\; \pi^\lambda(E) = Q,\ \inf_{x\in E} x_1 = \inf_{q \in Q} q_1\big\},$$

where $\pi^\lambda(E)$ is the unique $Q \in \mathscr{Q}$ such that $|Q| = 2^{\lambda\, n}\,|E|$ and $Q \supset E$. Furthermore, x_1 respectively q_1 denotes the orthogonal projection of $x \in \mathbb{R}^n$ respectively $q \in \mathbb{R}^n$ onto the vector $e_1 = (1, 0, \ldots, 0)$. Then the stripe operator S_λ is given by the linear extension of

$$S_\lambda h_Q = g_{Q,\lambda},$$

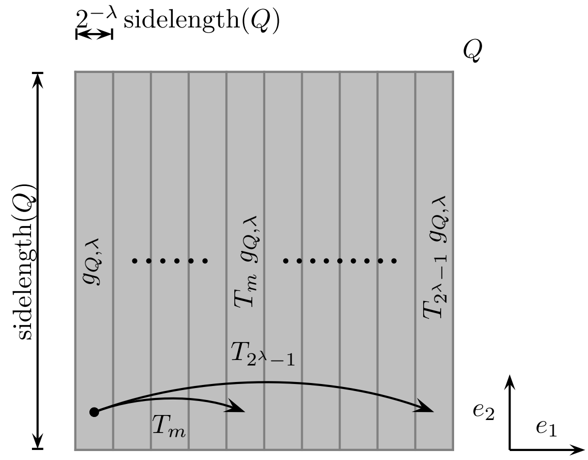

FIGURE 4. High frequency cover of the cube Q obtained by shifts of the stripe functions $g_{Q,\lambda}$.

and the stripe functions were in (2.5) defined by

$$g_{Q,\lambda} = \sum_{E \in \mathscr{U}_\lambda(Q)} h_E.$$

Having verified Corollary 2.4 on page 36 we will now present our main theorem on stripe operators.

THEOREM 2.5. *Let X be a* UMD *space, $1 < p < \infty$ and $n \in \mathbb{N}$. For $\lambda \geq 0$ let S_λ denote the stripe operator given by*

$$S_\lambda u = \sum_{Q \in \mathscr{Q}} \langle u, h_Q \rangle\, g_{Q,\lambda} |Q|^{-1},$$

for all $u \in L^p_X(\mathbb{R}^n)$. If $L^p_X(\mathbb{R}^n)$ has cotype $\mathcal{C}(L^p_X(\mathbb{R}^n))$, then there exists a constant $C > 0$ such that for every $u \in L^p_X(\mathbb{R}^n)$ and $\lambda \geq 0$

$$\|S_\lambda u\|_{L^p_X(\mathbb{R}^n)} \leq C \cdot 2^{-\lambda/\mathcal{C}(L^p_X(\mathbb{R}^n))}\, \|u\|_{L^p_X(\mathbb{R}^n)}, \tag{2.25}$$

where the constant C depends only on n, p, the UMD *constant of X and the cotype $\mathcal{C}(L^p_X(\mathbb{R}^n))$.*

PROOF. The UMD property and Kahane's contraction principle shows that the estimate holds true if we restrict λ to $0 \leq \lambda \leq 1$.

So from now on we may assume $\lambda \geq 2$. The definition of the dyadic stripe $\mathscr{U}_\lambda$ (see (2.3) and (2.4)) implies that

$$\tau_{k \cdot e_1}(\mathscr{U}_\lambda) \cap \tau_{m \cdot e_1}(\mathscr{U}_\lambda) = \emptyset, \tag{2.26}$$

if $0 \leq k < m \leq 2^\lambda - 1$. Furthermore one has the high frequency cover of $Q \in \mathscr{Q}$ given by

$$\bigcup_{m=0}^{2^\lambda-1} \tau_{m \cdot e_1}\big(\mathscr{U}_\lambda(Q)\big) = \big\{E \in \mathscr{Q} \;:\; \pi^\lambda(E) = Q\big\},$$

thus

$$|h_Q| = \Big|\sum_{m=0}^{2^\lambda-1} T_{m \cdot e_1}\, g_{Q,\lambda}\Big| \tag{2.27}$$

by the definition of $g_{Q,\lambda}$ (see Figure 4).

Now let $u \in L^p_X(\mathbb{R}^n)$ be fixed. For the rest of the proof will abbreviate $L^p_X(\mathbb{R}^n)$ by L^p_X and $\mathcal{C}(L^p_X)$ by $\mathcal{C}$. We want to bound $\|u\|_{L^p_X}$ from below by the means of the stripe operator S_λ.

First, the UMD property allows us to introduce Rademacher means

$$\|u\|_{L^p_X} \approx \int_0^1 \Big\| \sum_j r_j(t) \sum_{Q\in\mathcal{Q}_j} u_Q \, h_Q \, |Q|^{-1} \Big\|_{L^p_X} \mathrm{d}t.$$

Second, Kahane's contraction principle applied to (2.27) on the right hand side yields

$$\|u\|_{L^p_X} \approx \int_0^1 \Big\| \sum_j r_j(t) \sum_{Q\in\mathcal{Q}_j} u_Q \sum_{m=0}^{2^\lambda-1} T_{m\cdot e_1} \, g_{Q,\lambda} \, |Q|^{-1} \Big\|_{L^p_X} \mathrm{d}t. \tag{2.28}$$

Third, if we set

$$d_{(j,m)} = T_{m\cdot e_1} \sum_{Q\in\mathcal{Q}_j} g_{Q,\lambda} \quad \text{if } j\in\mathbb{Z} \text{ and } 0\le m\le 2^\lambda-1,$$

and define the lexicographic ordering relation

$$(j,m) < (j',m') \quad \text{iff} \quad \begin{cases} j<j', \text{ or} \\ j=j' \text{ and } m<m', \end{cases}$$

then $\big\{d_{(j,m)} \;:\; j\in\mathbb{Z}, 0\le m\le\lambda\big\}$ with respect to "<" generates a martingale difference sequence. So in view of (2.26) and the UMD property we may introduce the following new Rademacher means in (2.28)

$$\int_0^1 \Big\| \sum_{m=0}^{2^\lambda-1} r_m(t) \; T_{m\cdot e_1} \sum_{Q\in\mathcal{Q}} u_Q \, g_{Q,\lambda} \, |Q|^{-1} \Big\|_{L^p_X} \mathrm{d}t.$$

Hence we have

$$\|u\|_{L^p_X} \approx \int_0^1 \Big\| \sum_{m=0}^{2^\lambda-1} r_m(t) \; T_{m\cdot e_1} \sum_{Q\in\mathcal{Q}} u_Q \, g_{Q,\lambda} \, |Q|^{-1} \Big\|_{L^p_X} \mathrm{d}t. \tag{2.29}$$

Fourth, with $g_{Q,\lambda} = S_\lambda h_Q$ in mind, we apply the cotype inequality (0.3) to (2.29) and see

$$\|u\|_{L^p_X} \gtrsim \Big(\sum_{m=0}^{2^\lambda-1} \big\| T_{m\cdot e_1} \, S_\lambda u \big\|_{L^p_X}^{\mathcal{C}} \Big)^{1/\mathcal{C}}.$$

Finally, utilizing Corollary 2.4 on page 36 concludes the proof

$$\Big(\sum_{m=0}^{2^\lambda-1} \big\| T_{m\cdot e_1} \, S_\lambda u \big\|_{L^p_X}^{\mathcal{C}} \Big)^{1/\mathcal{C}} \approx \Big(\sum_{m=0}^{2^\lambda-1} \big\| S_\lambda u \big\|_{L^p_X}^{\mathcal{C}} \Big)^{1/\mathcal{C}} = 2^{\lambda/\mathcal{C}} \, \|S_\lambda u\|_{L^p_X}.$$

□

Repeating the proof of Theorem 2.5 without Corollary 2.4 and using T. Figiel's bound (0.8) for the shift operator T_m directly, would lead to the weaker result

$$\|S_\lambda u\|_{L^p_X(\mathbb{R}^n)} \le C\cdot\lambda^\alpha \, 2^{-\lambda/\mathcal{C}(L^p_X(\mathbb{R}^n))} \, \|u\|_{L^p_X(\mathbb{R}^n)}, \tag{2.30}$$

where the exponent $0<\alpha<1$ is the exponent occurring in T. Figiel's estimate (0.8).

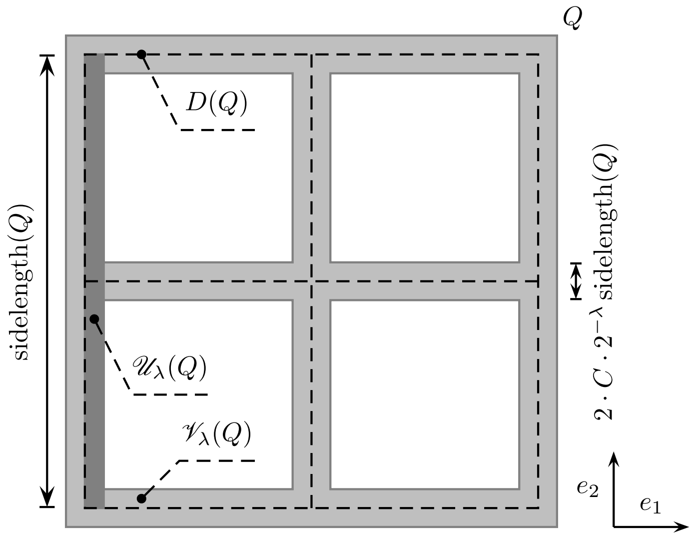

FIGURE 5. The dyadic stripe $\mathscr{U}_\lambda(Q)$ embedded in the ring domain $\mathscr{V}_\lambda(Q)$ in dimension $n = 2$. The picture is drawn as if $C = 1$.

2.4. The Ring Domain Operator.

We define the ring domain operator H_λ, supported in the vicinity of the set of discontinuities of Haar functions. We will show that H_λ can be written as a finite sum of continuous images of stripe operators S_λ. Thus the estimate (2.5) for the stripe operator conveys to the ring domain operator, that is

$$\|H_\lambda u\|_{L^p_X(\mathbb{R}^n)} \leq C \cdot 2^{-\lambda/\mathcal{C}(L^p_X(\mathbb{R}^n))} \|u\|_{L^p_X(\mathbb{R}^n)}. \tag{2.31}$$

For every Q denote by $D(Q)$ the set of discontinuities of the Haar function h_Q and define for all $\lambda \geq 0$.

$$D_\lambda(Q) = \{x \in \mathbb{R}^n : \operatorname{dist}(x, D(Q)) \leq C \cdot 2^{-\lambda} \operatorname{sidelength}(Q)\}.$$

Note that for all $\lambda \geq 0$ and $Q \in \mathscr{Q}$ we have

$$|D_\lambda(Q)| \leq C \cdot 2^{-\lambda} |Q|, \tag{2.32}$$

where C does not depend on λ and Q. Now we cover the set $D_\lambda(Q)$ using dyadic cubes $E(Q)$ with $\operatorname{sidelength}(E(Q)) = 2^{-\lambda} \cdot \operatorname{sidelength}(Q)$, and call the collection of those cubes $\mathscr{V}_\lambda(Q)$. To be more precise

$$\mathscr{V}_\lambda(Q) = \big\{E \in \mathscr{Q} : \operatorname{sidelength}(E) = 2^{-\lambda} \operatorname{sidelength}(Q),\ E \cap D_\lambda(Q) \neq \emptyset\big\}, \tag{2.33}$$

and we define

$$\mathscr{V}_\lambda = \bigcup_{Q \in \mathscr{Q}} \mathscr{V}_\lambda(Q). \tag{2.34}$$

The set covered by $\mathscr{V}_\lambda(Q)$ is illustrated by the shaded region in Figure 5, wherein the dashed lines represent the set of discontinuities $D(Q)$. The cardinality $\# \mathscr{V}_\lambda(Q)$ does not depend on the choice of Q, so we note

$$\# \mathscr{V}_\lambda(Q) \approx 2^{\lambda(n-1)}. \tag{2.35}$$

Finally, define the functions $d_{Q,\lambda}$ associated to the ring domain $\mathscr{V}_\lambda(Q)$ by

$$d_{Q,\lambda} = \sum_{E \in \mathscr{V}_\lambda(Q)} h_E, \tag{2.36}$$

and the ring domain operator H_λ by

$$H_\lambda u = \sum_{Q \in \mathscr{Q}} \langle u, h_Q \rangle \, d_{Q,\lambda} |Q|^{-1}. \tag{2.37}$$

In the subsequent theorem the ring domain operator H_λ is dominated by the stripe operator S_λ. This is done by covering the ring domain function $d_{Q,\lambda}$ with continuous mappings of the dyadic stripe functions $g_{Q,\lambda}$ (see identity (2.40)).

THEOREM 2.6. *Let X be a* UMD *space, $1 < p < \infty$ and $n \in \mathbb{N}$. Let $\lambda \geq 0$ let H_λ denote the ring domain operator given by*

$$H_\lambda u = \sum_{Q \in \mathscr{Q}} \langle u, h_Q \rangle \, d_{Q,\lambda} |Q|^{-1},$$

for all $u \in L^p_X(\mathbb{R}^n)$.

Then we can dominate H_λ by S_λ, that is

$$\big\| H_\lambda u \big\|_{L^p_X} \leq C \cdot \big\| S_\lambda u \big\|_{L^p_X}, \tag{2.38}$$

for all $u \in L^p_X(\mathbb{R}^n)$, where the constant C depends only on n, p and the UMD *constant of X.*

A fortiori, we have the following estimate for H_λ.

COROLLARY 2.7. *Let X be a* UMD *space, $1 < p < \infty$ and $n \in \mathbb{N}$. If $L^p_X(\mathbb{R}^n)$ has cotype $\mathcal{C}(L^p_X(\mathbb{R}^n))$, then there exists a constant $C > 0$ such that for every $u \in L^p_X(\mathbb{R}^n)$ and $\lambda \geq 0$*

$$\| H_\lambda u \|_{L^p_X(\mathbb{R}^n)} \leq C \cdot 2^{-\lambda / \mathcal{C}(L^p_X(\mathbb{R}^n))} \, \| u \|_{L^p_X(\mathbb{R}^n)}, \tag{2.39}$$

where the constant C depends only on n, p, the UMD *constant of X and the cotype $\mathcal{C}(L^p_X(\mathbb{R}^n))$.*

PROOF OF COROLLARY 2.7. Once we have proved Theorem 2.6 we obtain Corollary 2.7 simply by plugging in the estimate for the stripe operator (2.25). □

PROOF OF THEOREM 2.6. Let q denote the lower left corner of Q, that is $q = (q_1, \dots, q_n)$, where

$$q_i = \inf \big\{ x_i \; : \; (x_1, \dots, x_n) \in Q \big\}, \quad \text{for all } 1 \leq i \leq n.$$

Furthermore, denote by M_i the orthogonal transformation swapping e_1 and e_i, that is the linear extension of

$$M_i \, e_1 = e_i, \qquad M_i \, e_i = e_1, \qquad M_i \, e_j = e_j \quad \text{for all } j \notin \{1, i\},$$

and finally define

$$L_i \, g_{Q,\lambda} = g_{Q,\lambda}\big(M_i (x + q) - q \big),$$

for all $Q \in \mathscr{Q}$.

Then we can find a constant $C > 0$ and functions $|c_Q^{(i)}| \leq 1$, $1 \leq i \leq n$ such that

$$d_{Q,\lambda} = \sum_{i=1}^{n} L_i \sum_{m = \lfloor -C \rfloor}^{\lceil C - 1 \rceil} T_{m \cdot e_1} \Big(\operatorname{Id} + T_{(2^{\lambda - 1} - 1) \cdot e_1} + T_{(2^\lambda - 1) \cdot e_1} \Big) \, c_Q^{(i)} \, g_{Q,\lambda}. \tag{2.40}$$

The ring domain $\mathscr{V}_\lambda(Q)$ and the dyadic stripe $\mathscr{U}_\lambda(Q)$ are pictured in Figure 5 on page 40. Since L_i is a bounded operator on L^p_X we can dominate $\|H_\lambda u\|_{L^p_X}$ by

$$\sum_{m=\lfloor -C\rfloor}^{\lceil C-1\rceil} \sum_{i=1}^{n} \Big\| T_{m\cdot e_1}\Big(\operatorname{Id} + T_{(2^{\lambda-1}-1)\cdot e_1} + T_{(2^\lambda-1)\cdot e_1}\Big) S_\lambda \sum_{Q\in\mathscr{Q}} u_Q\, c_Q^{(i)}\, h_Q\, |Q|^{-1}\Big\|_{L^p_X}.$$

If we use inequality (0.8) for $T_{m\cdot e_1}$, $\lfloor -C\rfloor \le m \le \lceil C-1\rceil$, and estimate the operators $T_{(2^{\lambda-1}-1)\cdot e_1} S_\lambda$ and $T_{(2^\lambda-1)\cdot e_1} S_\lambda$ by means of (2.24), we obtain

$$\Big\| T_{m\cdot e_1}\Big(\operatorname{Id} + T_{(2^{\lambda-1}-1)\cdot e_1} + T_{(2^\lambda-1)\cdot e_1}\Big) S_\lambda v\Big\|_{L^p_X} \le C_1 \cdot \|S_\lambda v\|_{L^p_X},$$

for all $v \in L^p_X$, where the constant C_1 is independent of v, m and λ. Combining the last two inequalities we see that

$$\|H_\lambda u\|_{L^p_X} \le 2\cdot C\cdot C_1 \cdot \sum_{i=1}^{n} \Big\| \sum_{Q\in\mathscr{Q}} u_Q\, c_Q^{(i)}\, S_\lambda h_Q\, |Q|^{-1}\Big\|_{L^p_X}.$$

Finally, exploiting the UMD property in order to use Kahane's contraction principle on the functions $|c_Q^{(i)}| \le 1$, $1 \le i \le n$ with respect to the martingale difference sequence $\big\{\sum_{Q\in\mathscr{Q}_j} u_Q\, c_Q^{(i)}\, S_\lambda h_Q\, |Q|^{-1} \;:\; j\in\mathbb{Z}\big\}$ with filtration $\big\{\mathscr{Q}_{j+\lambda} \;:\; j\in\mathbb{Z}\big\}$ concludes the proof. $\square$

3. Decomposition of the Directional Haar Projection $P^{(\varepsilon)}$

Given $1 < p < \infty$ and an integer $n \geq 2$, the directional Haar projection $P^{(\varepsilon)} : L^p_X(\mathbb{R}^n) \longrightarrow L^p_X(\mathbb{R}^n)$ is defined by

$$P^{(\varepsilon)}u = \sum_{Q \in \mathscr{Q}} \langle u, h_Q^{(\varepsilon)} \rangle \, h_Q^{(\varepsilon)} \, |Q|^{-1}, \tag{3.1}$$

for all $u \in L^p_X$.

In order to estimate the directional Haar projection $P^{(\varepsilon)}$, we will decompose $P^{(\varepsilon)}$ in section 3.1 into a series of mollified operators $\sum_l P_l^{(\varepsilon)}$, following [**LMM07**]. Subsequently, wavelet expansions are used in [**LMM07**] to further analyze $P_l^{(\varepsilon)}$.

On the contrary, we will decompose $P_l^{(\varepsilon)}$ into a series of ring domain operators $\sum_{\lambda(l)} c_{\lambda(l)} H_{\lambda(l)}$, using martingale methods feasible in UMD–spaces. In section 3.2 we will use T. Figiel's canonical martingale approach (see [**Fig90**]) to find a suitable representation for $P_l^{(\varepsilon)}$. In the following section 3.3 we define the main cases for the further decomposition of $P_l^{(\varepsilon)}$, which we shall then dominate by weighted series of ring domain operators H_λ in section 3.4. We conclude the first part of this thesis with section 3.5, where we reduce the estimates for $P_l^{(\varepsilon)} R_{i_0}^{-1}$ to inequalities for $P_l^{(\varepsilon)}$.

3.1. Decomposition of $P^{(\varepsilon)}$ into $P_l^{(\varepsilon)}$.

We give a brief overview of the Littlewood–Paley decomposition used in [**LMM07**], and continue with further decompositions in section 3.2 and 3.3, different from the methods in [**LMM07**].

We utilize a compactly supported, smooth approximation of the identity, to obtain a decomposition of the directional projection $P^{(\varepsilon)}$ into a series of mollified operators $P_l^{(\varepsilon)}$,

$$P^{(\varepsilon)} = \sum_{l \in \mathbb{Z}} P_l^{(\varepsilon)}. \tag{3.2}$$

To this end, we fix $b \in C_c^\infty(]0,1[^n)$ such that

$$\int b(x)\,\mathrm{d}x = 1, \quad \text{and} \quad \int x_i \, b(x_1, \dots, x_i, \dots, x_n)\,\mathrm{d}x_i = 0, \tag{3.3}$$

for all $1 \leq i \leq n$. For every integer l define

$$\Delta_l u = u * d_l, \quad \text{where} \quad d_l(x) = 2^{ln} d(2^l x) \quad \text{and} \quad d(x) = 2^n \, b(2\,x) - b(x). \tag{3.4}$$

For all $u \in L^p_X(\mathbb{R}^n)$ holds true that

$$u = \sum_{l \in \mathbb{Z}} \Delta_l u, \tag{3.5}$$

with the series converging in L^p_X. Denoting $\mathscr{Q}_j \subset \mathscr{Q}$ the collection of all dyadic cubes having measure 2^{-jn}, we set

$$P_l^{(\varepsilon)}u = \sum_{j \in \mathbb{Z}} \sum_{Q \in \mathscr{Q}_j} \langle u, \Delta_{j+l}(h_Q^{(\varepsilon)}) \rangle \, h_Q^{(\varepsilon)} \, |Q|^{-1}, \tag{3.6}$$

and observe that by (3.5) for all $u \in L^p_X$

$$P^{(\varepsilon)}u = \sum_{l \in \mathbb{Z}} P_l^{(\varepsilon)}u,$$

where equality holds in the sense of L^p_X. Setting $f_{Q,l}^{(\varepsilon)} = \Delta_{j+l} h_Q^{(\varepsilon)}$, if $Q \in \mathscr{Q}_j$, we rewrite (3.6) as

$$P_l^{(\varepsilon)}u = \sum_{Q \in \mathscr{Q}} \langle u, f_{Q,l}^{(\varepsilon)} \rangle \, h_Q^{(\varepsilon)} \, |Q|^{-1}. \tag{3.7}$$

In contrast to [**LMM07**] we will rather estimate the operator

$$P_- = \sum_{l<0} P_l, \tag{3.8}$$

instead of estimating each P_l, $l < 0$ separately.

3.2. The Integral Kernels $K_l^{(\varepsilon)}$ and $K_-^{(\varepsilon)}$ of $P_l^{(\varepsilon)}$ and $P_-^{(\varepsilon)}$.
From now onwards, we deviate significantly from the methods in [**LMM07**]. In this section we identify the integral kernel $K_l^{(\varepsilon)}$ of the operator $P_l^{(\varepsilon)}$. Then we will expand $K_l^{(\varepsilon)}$ into a Haar series according to T. Figiel's martingale approach (see [**Fig90**]).

Note that

$$\big(P_l^{(\varepsilon)} u\big)(x) = \int K_l^{(\varepsilon)}(x,y)\, u(y)\, \mathrm{d}y, \tag{3.9}$$

where

$$K_l^{(\varepsilon)}(x,y) = \sum_{Q\in\mathscr{Q}} h_Q^{(\varepsilon)}(x)\, f_{Q,l}^{(\varepsilon)}(y)\, |Q|^{-1}. \tag{3.10}$$

Now we expand $K_l^{(\varepsilon)}$ into the series

$$\sum_{\substack{\alpha,\beta\in\{0,1\}^n\\(\alpha,\beta)\neq 0}} \sum_{\substack{K,M,Q\in\mathscr{Q}:\\|K|=|M|}} \langle h_Q^{(\varepsilon)}, h_K^{(\alpha)}\rangle \langle f_{Q,l}^{(\varepsilon)}, h_M^{(\beta)}\rangle |K|^{-1}|M|^{-1}|Q|^{-1} h_K^{(\alpha)}(x)\, h_M^{(\beta)}(y). \tag{3.11}$$

We distinguish the following settings for the parameter β:

(1) $\beta \neq 0$,
(2) $\beta = 0$.

Note that due to the condition $(\alpha,\beta)\neq 0$ in (3.11), case (2) certainly implies $\alpha \neq 0$.

To ease the notation, we will make use of the following convention. We shall write h_Q, denoting one of the functions $h_Q^{(\gamma)}$, $\gamma \in \{0,1\}^n \setminus \{0\}$, and $\mathbb{1}_Q$ for the characteristic function h_Q^0. We may do so since the UMD–property and Kahane's contraction principle enable us to interchange equally supported Haar functions having mean zero.

Using this notation, expansion (3.11) reduces in both cases to

$$K_l(x,y) = \sum_{M,Q\in\mathscr{Q}} \langle f_{Q,l}, h_M\rangle |M|^{-1}|Q|^{-1} h_Q(x)\, h_M(y), \tag{3.12}$$

which is exactly the Haar expansion of K_l in the y–coordinate. Expansion (3.11) breaks up the Haar functions $h_Q^{(\varepsilon)}$ into smaller pieces and reassembles them, subsequently. We might have seen the algebraic form (3.12) simply by plugging the Haar series of u into the operator $P_l^{(\varepsilon)}$. However, after a few purely algebraic manipulations, Figiel's expansion in both coordinates yields identity (3.12).

Now we present an accurate justification for identity (3.12). To this end, we fix $\beta \in \{0,1\}^n \setminus \{0\}$, $\alpha \in \{0,1\}^n$ and rewrite the inner sum of (3.11)

$$\begin{aligned}
&\sum_{\substack{K,M,Q\in\mathscr{Q}:\\|K|=|M|}} \langle h_Q, h_K^{(\alpha)}\rangle\, \langle f_{Q,l}, h_M\rangle\, |K|^{-1}|M|^{-1}|Q|^{-1} h_K^{(\alpha)}(x)\, h_M(y)\\
&= \sum_{M,Q\in\mathscr{Q}} \langle f_{Q,l}, h_M\rangle\, |M|^{-1}|Q|^{-1} h_M(y) \sum_{\substack{K\in\mathscr{Q}:\\|K|=|M|}} \langle h_Q, h_K^{(\alpha)}\rangle |K|^{-1} h_K^{(\alpha)}(x).
\end{aligned}$$

In both cases $\alpha = 0$ and $\alpha \neq 0$ we have

$$\sum_{\substack{K\in\mathscr{Q}:\\|K|=|M|}} \langle h_Q, h_K^{(\alpha)}\rangle\, |K|^{-1} h_K^{(\alpha)}(x) = h_Q(x),$$

for all Q, M and $x \in \mathbb{R}^n$. This is true for the sum being either the conditional expectation of h_Q, or exploiting the orthogonality of the Haar basis, respectively. Hence we obtain (3.12).

Let $\beta = 0$, which implies $\alpha \neq 0$ as noted before, therefore the inner sum of (3.11) reads

$$\begin{aligned}&\sum_{\substack{K,M,Q\in\mathscr{Q}:\\|K|=|M|}} \langle h_Q, h_K\rangle\, \langle f_{Q,l}, \mathbb{1}_M\rangle\, |K|^{-1}|M|^{-1}|Q|^{-1}\, h_K(x)\, \mathbb{1}_M(y)\\ &= \sum_{\substack{M,Q\in\mathscr{Q}:\\|M|=|Q|}} \langle f_{Q,l}, \mathbb{1}_M\rangle\, |M|^{-1}|Q|^{-1} h_Q(x)\, \mathbb{1}_M(y).\end{aligned}$$

Developing the y–component of the last expression into a Haar series yields

$$\begin{aligned}&\sum_{\substack{K,M,Q\in\mathscr{Q}:\\|M|=|Q|}} \langle f_{Q,l}, \mathbb{1}_M\rangle\, \langle h_K, \mathbb{1}_M\rangle\, |K|^{-1}|M|^{-1}|Q|^{-1} h_Q(x)\, h_K(y)\\ &= \sum_{K,Q\in\mathscr{Q}} h_Q(x)\, h_K(y)\, |K|^{-1}|Q|^{-1} \sum_{\substack{M\subsetneq K\\|M|=|Q|}} \langle f_{Q,l}, \mathbb{1}_M\rangle\, \langle h_K, \mathbb{1}_M\rangle\, |M|^{-1}\\ &= \sum_{K,Q\in\mathscr{Q}} h_Q(x)\, h_K(y)\, |K|^{-1}|Q|^{-1} \Big\langle f_{Q,l}, \sum_{\substack{M\subsetneq K\\|M|=|Q|}} \mathbb{1}_M\, \langle h_K, \mathbb{1}_M\rangle\, |M|^{-1}\Big\rangle.\end{aligned}$$

Observe that the inner sum with K and Q fixed is the conditional expectation of h_K at a finer scale. Hence h_K is reproduced, so

$$\sum_{\substack{M\subsetneq K\\|M|=|Q|}} \mathbb{1}_M\, \langle h_K, \mathbb{1}_M\rangle\, |M|^{-1} = h_K,$$

and we gain

$$K_l(x,y) = \sum_{K,Q\in\mathscr{Q}} \langle f_{Q,l}, h_K\rangle |K|^{-1}|Q|^{-1} h_Q(x)\, h_K(y).$$

Note that we may lift the restriction $|Q| < |K|$, since the sum (3.12) will be parameterized according to the ratio of the diameters of Q and M in section 3.3, and split using the triangle inequality.

As a consequence we may assume the generic expansion (3.12) of the integral kernel $K_l(x,y)$ in order to estimate P_l.

We omit the superscripts (ε) and summarize the results of the preceding discussion in

PROPOSITION 3.1. *For each $l \in \mathbb{Z}$ let*

$$P_l u = \sum_{Q\in\mathscr{Q}} \langle u, f_{Q,l}\rangle\, h_Q\, |Q|^{-1}, \tag{3.13}$$

where $f_{Q,l} = \Delta_{j+l}\, h_Q$, for all $Q \in \mathscr{Q}_j$ (see (3.4) for details).

Then integral kernel K_l of P_l is given by

$$\big(P_l u\big)(x) = \int K_l(x,y)\, u(y)\, \mathrm{d}y, \tag{3.14}$$

where

$$K_l(x,y) = \sum_{M,Q\in\mathcal{Q}} \langle f_{Q,l}, h_M\rangle\, |M|^{-1}|Q|^{-1} h_Q(x)\, h_M(y). \tag{3.15}$$

If we define

$$P_- = \sum_{l<0} P_l \quad \textit{and} \quad f_Q = \sum_{l<0} f_{Q,l}, \tag{3.16}$$

then the integral kernel K_- of P_- is given by

$$K_-(x,y) = \sum_{M,Q\in\mathcal{Q}} \langle f_Q, h_M\rangle\, |M|^{-1}|Q|^{-1} h_Q(x)\, h_M(y). \tag{3.17}$$

3.3. Decomposition of P_l – The Main Cases.

We will decompose the operator P_l guided by the different behavior of the coefficients $\langle f_{Q,l}, h_M\rangle$, $l \geq 0$, $M \in \mathcal{Q}$ and $\langle f_{Q,l}, h_M\rangle$, $l < 0$, $M \in \mathcal{Q}$. This is primarily caused by the different shape of the support of the functions $f_{Q,l}$, $l \geq 0$ and $f_{Q,l}$, $l < 0$, (compare the support inclusions in (3.18) and (3.19)) in relation to the size of the cubes M.

3.3.1. *Estimates for the Coefficients.*

First, we want to investigate the mollified Haar functions $f_{Q,l}$, $l \in \mathbb{Z}$. To this end, let $D(Q)$ denote the set of discontinuities of the Haar function h_Q, then

$$D_l(Q) = \{x \in \mathbb{R}^n \,:\, \operatorname{dist}(x, D(Q)) \leq C \cdot 2^{-l} \operatorname{diam}(Q)\}.$$

If $l \geq 0$, note that

$$\begin{aligned} &\int f_{Q,l}(x)\,\mathrm{d}x = 0, &&\operatorname{supp} f_{Q,l} \subset D_l(Q),\\ &|f_{Q,l}| \leq C, &&\operatorname{Lip}(f_{Q,l}) \leq C\, 2^l\, (\operatorname{diam}(Q))^{-1}, \end{aligned} \tag{3.18}$$

and if $l \leq 0$, we have

$$\begin{aligned} &\int f_{Q,l}(x)\,\mathrm{d}x = 0, &&\operatorname{supp} f_{Q,l} \subset C\, 2^{|l|} Q,\\ &|f_{Q,l}| \leq C\, 2^{-|l|(n+1)}, &&\operatorname{Lip}(f_{Q,l}) \leq C\, 2^{-|l|(n+2)}\, (\operatorname{diam}(Q))^{-1}, \end{aligned} \tag{3.19}$$

where the constant C does not depend on l and Q.

Recall that for $Q \in \mathcal{Q}_j$ we defined

$$f_{Q,l} = \Delta_{j+l}\, h_Q = h_Q * d_{j+l} = h_Q * (b_{j+l+1} - b_{j+l}).$$

So taking the sum over $l < 0$ yields

$$\sum_{l<0} f_{Q,l} = h_Q * b_j,$$

hence, the mollified Haar functions f_Q defined in (3.16), are given by

$$f_Q = h_Q * b_j, \qquad \text{for all } Q \in \mathcal{Q}_j,$$

where $b_j(x) = 2^{j\,n}\, b(2^j\, x)$. The functions f_Q have the following properties, which are easily verified. There exists a $C > 0$ independent of Q such that

$$\begin{aligned} &\int f_Q(x)\,\mathrm{d}x = 0, &&\operatorname{supp} f_Q \subset C \cdot Q,\\ &|f_Q| \leq C, &&\operatorname{Lip}(f_Q) \leq C \cdot (\operatorname{diam}(Q))^{-1}, \end{aligned} \tag{3.20}$$

for all $Q \in \mathcal{Q}$.

Proposition 3.2 stated below estimates the coefficients $\langle f_{Q,l}, h_M \rangle$, $l \geq 0$ respectively $\langle f_Q, h_M \rangle$. The different behavior of the inequalities is determined by the ratio of the diameters of the cubes Q and M.

PROPOSITION 3.2. *For all dyadic cubes $Q, M \in \mathscr{Q}$ we have the following estimates for the coefficients $\langle f_{Q,l}, h_M \rangle$, $l \geq 0$.*

(1) If $\operatorname{diam}(Q) \leq \operatorname{diam}(M)$, then

$$|\langle f_{Q,l}, h_M \rangle| \leq C \cdot 2^{-l}\, |Q|, \tag{3.21}$$

(2) if $2^{-l} \operatorname{diam}(Q) \leq \operatorname{diam}(M) < \operatorname{diam}(Q)$, we get

$$|\langle f_{Q,l}, h_M \rangle| \leq C \cdot 2^{-l} \operatorname{diam}(Q)\, (\operatorname{diam}(M))^{n-1}, \tag{3.22}$$

(3) and if $\operatorname{diam}(M) < 2^{-l} \operatorname{diam}(Q)$ we obtain

$$|\langle f_{Q,l}, h_M \rangle| \leq C \cdot 2^{l}\, \frac{\operatorname{diam}(M)}{\operatorname{diam}(Q)}\, |M|. \tag{3.23}$$

The constant C does not depend on l, Q and M.

Moreover, for all dyadic cubes $Q, M \in \mathscr{Q}$ we have the subsequent estimates for the coefficients $\langle f_Q, h_M \rangle$.

(1) If $\operatorname{diam}(M) \leq \operatorname{diam}(Q)$, then

$$|\langle f_Q, h_M \rangle| \leq C \cdot (\operatorname{diam}(Q))^{-1}\, (\operatorname{diam}(M))^{n+1}, \tag{3.24}$$

(2) while if $\operatorname{diam}(M) > \operatorname{diam}(Q)$, we have

$$|\langle f_Q, h_M \rangle| \leq C \cdot |Q|. \tag{3.25}$$

The constant C does not depend on Q and M.

PROOF. First, we want to estimate $\langle f_{Q,l}, h_M \rangle$, so we fix $l \geq 0$ and $Q, M \in \mathscr{Q}$.

If $\operatorname{diam}(Q) \leq \operatorname{diam}(M)$, then using $|D_l(Q)| \lesssim 2^{-l}\, |Q|$ and exploiting the boundedness of $f_{Q,l}$ and h_M implies (3.21).

If $2^{-l} \operatorname{diam}(Q) \leq \operatorname{diam}(M) < \operatorname{diam}(Q)$, then the measure estimate

$$|D_l(Q) \cap M| \lesssim 2^{-l} \operatorname{diam}(Q)\, (\operatorname{diam}(M))^{n-1}$$

together with inequality (3.18) yields (3.22).

If $\operatorname{diam}(M) < 2^{-l} \operatorname{diam}(Q)$, then in view of $\operatorname{Lip}(f_{Q,l}) \lesssim 2^{l}\, (\operatorname{diam}(Q))^{-1}$ and $\int h_M = 0$ in inequality (3.18) we may infer (3.23).

Now we turn to the estimates for $\langle f_Q, h_M \rangle$, $Q, M \in \mathscr{Q}$.

If $\operatorname{diam}(M) \leq \operatorname{diam}(Q)$, we make use of

$$\operatorname{Lip}(f_Q) \leq C\, (\operatorname{diam}(Q))^{-1},$$

according to (3.20) and we gain (3.24).

For $\operatorname{diam}(M) > \operatorname{diam}(Q)$, one can exploit

$$|f_Q| \leq C \qquad \text{and} \qquad \operatorname{supp} f_Q \subset C \cdot Q$$

in (3.20) to obtain (3.25). □

REMARK 3.3. Observe that the coefficients $\langle f_{Q,l}, h_M \rangle$ respectively $\langle f_Q, h_M \rangle$ vanish, if the support of $f_{Q,l}$ respectively f_Q is contained in a set where h_M is constant (see Figure 6 on page 50). More precisely, if we can find a $K \in \mathscr{Q}$ with $\pi(K) = M$ such that

$$\operatorname{supp} f_{Q,l} \subset K \quad \text{respectively} \quad \operatorname{supp} f_Q \subset K,$$

then certainly

$$\langle f_{Q,l}, h_M \rangle = 0 \quad \text{respectively} \quad \langle f_Q, h_M \rangle = 0.$$

So we note that for $\operatorname{diam}(M) > \operatorname{diam}(Q)$ the cubes Q for which $\langle f_{Q,l}, h_M\rangle \neq 0$ respectively $\langle f_Q, h_M\rangle \neq 0$, cluster in the vicinity of $D(M)$, the set of h_M's discontinuities.

3.3.2. *Definition of the Main Cases.*

For each $l \geq 0$ we split the set $\mathscr{Q} \times \mathscr{Q}$ according to the cases in Proposition 3.2 on the preceding page into the three disjoint collections

$$\mathscr{A}_l = \big\{(Q, M) \,:\, \operatorname{diam}(Q) \leq \operatorname{diam}(M)\big\}, \tag{3.26}$$

$$\mathscr{B}_l = \big\{(Q, M) \,:\, 2^{-l} \operatorname{diam}(Q) \leq \operatorname{diam}(M) < \operatorname{diam}(Q)\big\}, \tag{3.27}$$

$$\mathscr{C}_l = \big\{(Q, M) \,:\, \operatorname{diam}(M) < 2^{-l} \operatorname{diam}(Q)\big\}, \tag{3.28}$$

respectively the two disjoint collections

$$\mathscr{A}_- = \big\{(Q, M) \,:\, \operatorname{diam}(M) \leq \operatorname{diam}(Q)\big\}, \tag{3.29}$$

$$\mathscr{B}_- = \big\{(Q, M) \,:\, \operatorname{diam}(M) > \operatorname{diam}(Q)\big\}. \tag{3.30}$$

Then we define the integral kernels

$$A_l(x, y) = \sum_{(Q,M) \in \mathscr{A}_l} \langle f_{Q,l}, h_M\rangle\, h_Q(x)\, h_M(y)\, |Q|^{-1}|M|^{-1}, \tag{3.31}$$

$$B_l(x, y) = \sum_{(Q,M) \in \mathscr{B}_l} \langle f_{Q,l}, h_M\rangle\, h_Q(x)\, h_M(y)\, |Q|^{-1}|M|^{-1}, \tag{3.32}$$

$$C_l(x, y) = \sum_{(Q,M) \in \mathscr{C}_l} \langle f_{Q,l}, h_M\rangle\, h_Q(x)\, h_M(y)\, |Q|^{-1}|M|^{-1}, \tag{3.33}$$

respectively

$$A_-(x, y) = \sum_{(Q,M) \in \mathscr{A}_-} \langle f_Q, h_M\rangle\, h_Q(x)\, h_M(y)\, |Q|^{-1}|M|^{-1}, \tag{3.34}$$

$$B_-(x, y) = \sum_{(Q,M) \in \mathscr{B}_-} \langle f_Q, h_M\rangle\, h_Q(x)\, h_M(y)\, |Q|^{-1}|M|^{-1}, \tag{3.35}$$

accordingly, and associate to each integral kernel the induced operator, precisely

$$\big(A_l\, u\big)(x) = \int A_l(x, y)\, u(y)\, \mathrm{d}y, \tag{3.36}$$

$$\big(B_l\, u\big)(x) = \int B_l(x, y)\, u(y)\, \mathrm{d}y, \tag{3.37}$$

$$\big(C_l\, u\big)(x) = \int C_l(x, y)\, u(y)\, \mathrm{d}y, \tag{3.38}$$

respectively

$$\big(A_-\, u\big)(x) = \int A_-(x, y)\, u(y)\, \mathrm{d}y, \tag{3.39}$$

$$\big(B_-\, u\big)(x) = \int B_-(x, y)\, u(y)\, \mathrm{d}y. \tag{3.40}$$

Finally note that

$$P_l = A_l + B_l + C_l, \quad \text{for all } l \geq 0, \tag{3.41}$$

and

$$P_- = A_- + B_-. \tag{3.42}$$

3.4. Estimates for P_l, $l \geq 0$ and P_-.

In section 3.3.2 on the preceding page we determined our decomposition of P_l, $l \geq 0$ and P_- into

$$P_l = A_l + B_l + C_l, \qquad \text{respectively} \qquad P_- = A_- + B_-.$$

see (3.41) and (3.42).

We will show that each of the operators A_l, B_l^*, C_l^* and A_-^*, B_- can be controlled by certain weighted series of ring domain operators; for details on H_λ we refer the reader to section 2.4.

Combining the results for A_l, B_l and C_l respectively A_-^* and B_- yields Theorem 3.4 below.

THEOREM 3.4. *Let X be a* UMD *space, $1 < p < \infty$ and $n \in \mathbb{N}$. Let $L^p_X(\mathbb{R}^n)$ have type $\mathcal{T}(L^p_X(\mathbb{R}^n))$.*

Then there exists a constant $C > 0$ such that for all $l \geq 0$ and every $u \in L^p_X(\mathbb{R}^n)$ we have

$$\|P_l u\|_{L^p_X(\mathbb{R}^n)} \leq C \cdot 2^{-l(1-\frac{1}{\mathcal{T}(L^p_X(\mathbb{R}^n))})} \|u\|_{L^p_X(\mathbb{R}^n)}, \tag{3.43}$$

where the constant C depends only on n, p, the UMD *constant of X and the type $\mathcal{T}(L^p_X(\mathbb{R}^n))$.*

Moreover, there exists a constant $C > 0$ such that for all $u \in L^p_X(\mathbb{R}^n)$

$$\|P_- u\|_{L^p_X(\mathbb{R}^n)} \leq C \cdot \|u\|_{L^p_X(\mathbb{R}^n)}, \tag{3.44}$$

where the constant C depends only on n, p, the UMD *constant of X and the type $\mathcal{T}(L^p_X(\mathbb{R}^n))$.*

The proof of the theorem is divided into seven parts

Section 3.4.1: Estimates for A_l
Section 3.4.2: Estimates for B_l
Section 3.4.3: Estimates for C_l
Section 3.4.4: Summary for P_l
Section 3.4.5: Estimates for A_-
Section 3.4.6: Estimates for B_-
Section 3.4.7: Summary for P_-

Keeping in mind that

$$P_l = A_l + B_l + C_l, \qquad \text{respectively} \qquad P_- = A_- + B_-,$$

we proved the theorem once we established the inequalities (3.45), (3.46) and (3.47), summarized in section 3.4.4, respectively (3.48) and (3.49), summarized in section 3.4.7.

3.4.1. *Estimates for A_l.*

In view of (3.26), (3.31) and (3.36) note that $\operatorname{diam}(Q) \leq \operatorname{diam}(M)$, and so we may utilize inequality (3.21) in Proposition 3.2 on page 47. Recall that it was mentioned in Remark 3.3 that the coefficients $\langle f_{Q,l}, h_M \rangle$ vanish if h_M is constant on the support of $f_{Q,l}$.

This setting is illustrated in Figure 6 on the next page.

First we split the set $\mathscr{A}_l$ (see (3.26)) into the disjoint collections $\mathscr{A}_{l,\lambda}$, $\lambda \geq 0$, given by

$$\mathscr{A}_{l,\lambda} = \big\{(Q, M) \in \mathscr{A}_l \ : \ \operatorname{diam}(Q) = 2^{-\lambda} \operatorname{diam}(M)\big\},$$

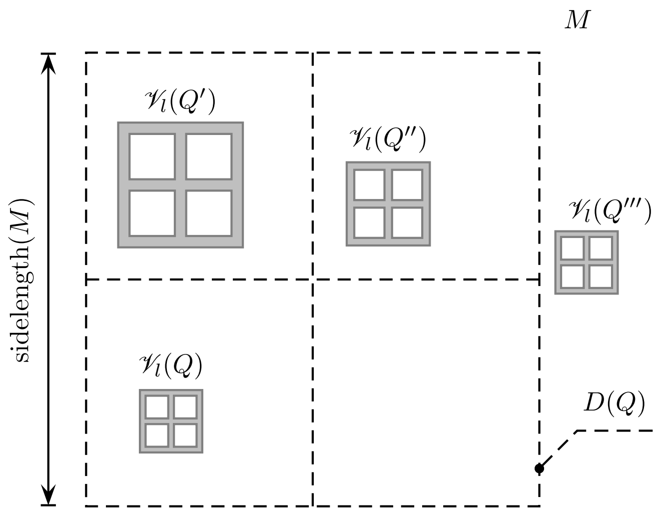

FIGURE 6. The ring domains $\mathscr{V}_l(Q)$, $\mathscr{V}_l(Q')$, $\mathscr{V}_l(Q'')$, $\mathscr{V}_l(Q''')$ are contained in sets where the Haar function h_M is constant.

and define the operator $A_{l,\lambda}$ accordingly, that is

$$A_{l,\lambda} u = \sum_{(Q,M)\in\mathscr{A}_{l,\lambda}} \langle f_{Q,l}, h_M\rangle\, h_Q\, u_M\, |Q|^{-1}\, |M|^{-1},$$

for all $u = \sum_{K\in\mathscr{Q}} u_K\, h_K\, |K|^{-1}$. Obviously, the identity

$$A_l u = \sum_{\lambda=0}^{\infty} A_{l,\lambda}\, u$$

holds true. Recalling that the coefficients $\langle f_{Q,l}, h_M\rangle$ vanish if h_M is constant on the support of $f_{Q,l}$ (see Remark 3.3) and the definition of the ring domain (2.33), we see that

$$\{Q : \langle f_{Q,l}, h_M\rangle \neq 0\} \subset \{Q : Q\cap D_\lambda(M) \neq \emptyset\} = \mathscr{V}_\lambda(M).$$

Using this fact, one has the identity

$$A_{l,\lambda}\, u = \sum_{M\in\mathscr{Q}} u_M\, |M|^{-1} \sum_{Q\in\mathscr{V}_\lambda(M)} \langle f_{Q,l}, h_M\rangle\, |Q|^{-1}\, h_Q,$$

hence glancing at inequality (3.21), utilizing the UMD–property and Kahane's contraction principle, we obtain

$$\begin{aligned}
\|A_{l,\lambda} u\|_{L^p_X(\mathbb{R}^n)} &\lesssim 2^{-l}\, \Big\| \sum_{M\in\mathscr{Q}} u_M\, |M|^{-1} \sum_{Q\in\mathscr{V}_\lambda(M)} h_Q \Big\|_{L^p_X(\mathbb{R}^n)} \\
&= 2^{-l}\, \Big\| \sum_{M\in\mathscr{Q}} u_M\, g_{M,\lambda}\, |M|^{-1} \Big\|_{L^p_X(\mathbb{R}^n)} \\
&= 2^{-l}\, \big\| H_\lambda u \big\|_{L^p_X(\mathbb{R}^n)}.
\end{aligned}$$

The last equality is the definition of the ring domain operator H_λ (see (2.37)). Applying the triangle inequality, using the above estimate for $A_{l,\lambda}$ and invoking Corollary 2.7 yields

$$\|A_l u\|_{L^p_X(\mathbb{R}^n)} \lesssim 2^{-l} \sum_{\lambda=0}^{\infty} 2^{-\lambda/\mathcal{C}(L^p_X(\mathbb{R}^n))}\, \|u\|_{L^p_X(\mathbb{R}^n)}.$$

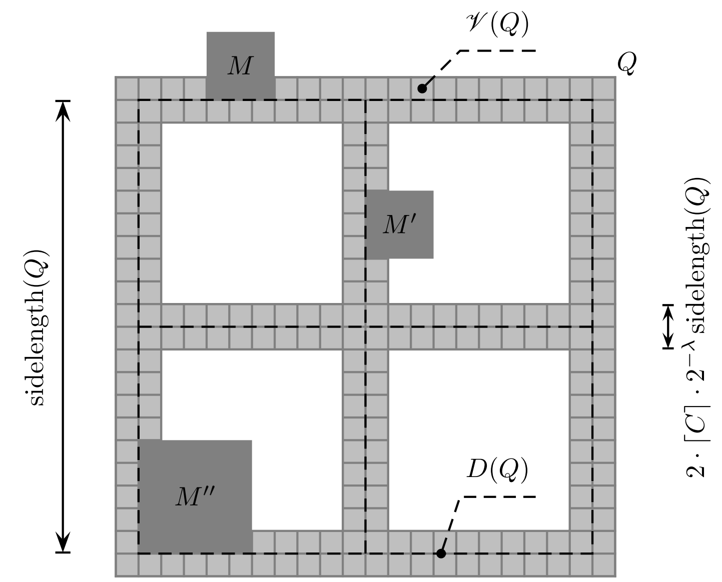

FIGURE 7. The cubes M, M' and M'' intersect the ring domain $\mathscr{V}_l(Q)$.

Evaluating the geometric series we obtain the estimate

$$\|A_l u\|_{L^p_X(\mathbb{R}^n)} \le C \cdot 2^{-l} \, \|u\|_{L^p_X(\mathbb{R}^n)}, \tag{3.45}$$

where the constant C depends on n, p, the UMD constant of X and the cotype $\mathcal{C}(L^p_X(\mathbb{R}^n))$.

REMARK 3.5. Note that with $\lambda \ge 0$ fixed, the collections $\mathscr{V}_\lambda(M)$ are not disjoint as M ranges over $\mathscr{Q}$. But since the number of overlaps is bounded by a constant depending solely on the dimension n and the constant appearing in the definition of $D_\lambda(Q)$, the above proof still applies.

3.4.2. *Estimates for B_l.*
In view of (3.27), (3.32) and (3.37) note that $2^{-l}\operatorname{diam}(Q) \le \operatorname{diam}(M) < \operatorname{diam}(Q)$, and so we may utilize inequality (3.22) in Proposition 3.2 on page 47.
This setting is visualized in Figure 7.

This time we prefer to analyze B_l^*, certainly with respect to the norm $\|\cdot\|_{L^q_Y(\mathbb{R}^n)}$, where $Y = X^*$ and $\frac{1}{p} + \frac{1}{q} = 1$. As before we parameterize the series according to the ratio of the sizes of Q and M. So we split the set $\mathscr{B}_l$ (see (3.27)) into the disjoint collections $\mathscr{B}_{l,\lambda}$, $\lambda \ge 0$, given by

$$\mathscr{B}_{l,\lambda} = \big\{(Q,M) \in \mathscr{B}_l \;:\; \operatorname{diam}(M) = 2^{-\lambda} \operatorname{diam}(Q)\big\},$$

and define the operator $B_{l,\lambda}$ accordingly, that is

$$B_{l,\lambda} u = \sum_{(Q,M)\in\mathscr{B}_{l,\lambda}} \langle f_{Q,l}, h_M\rangle \, h_Q \, u_M \, |Q|^{-1} \, |M|^{-1},$$

for all $u = \sum_{K\in\mathscr{Q}} u_K \, h_K \, |K|^{-1}$.
Note that for $(Q,M) \in \mathscr{B}_{l,\lambda}$ we have

$$\{M \,:\, \langle f_{Q,l}, h_M\rangle \ne 0\} \subset \{M \,:\, M \cap D_l(Q) \ne \emptyset\} = \mathscr{V}_\lambda(Q),$$

hence we can rewrite $B^*_{l,\lambda} u$ as

$$B^*_{l,\lambda} u = \sum_{Q \in \mathscr{Q}} u_Q \, |Q|^{-1} \sum_{M \in \mathscr{V}_\lambda(Q)} \langle f_{Q,l}, h_M \rangle \, |M|^{-1} h_M.$$

Taking the norm, utilizing the UMD–property and applying Kahane's contraction principle to (3.22) yields the estimate

$$\begin{aligned} \|B^*_{l,\lambda} u\|_{L^q_Y(\mathbb{R}^n)} &\lesssim 2^{-l} \, \Big\| \sum_{Q \in \mathscr{Q}} u_Q \, |Q|^{-1} \sum_{M \in \mathscr{V}_\lambda(Q)} h_M \Big\|_{L^q_Y(\mathbb{R}^n)} \\ &= 2^{-l} \, \Big\| \sum_{Q \in \mathscr{Q}} u_Q \, g_{Q,\lambda} |Q|^{-1} \Big\|_{L^q_Y(\mathbb{R}^n)} \\ &= 2^{-l} \, \big\| H_\lambda u \big\|_{L^q_Y(\mathbb{R}^n)}. \end{aligned}$$

The last equality is the definition of the ring domain operator H_λ (see (2.37)). Recall

$$B^*_l u = \sum_{\lambda=0}^{\infty} B^*_{l,\lambda} \, u,$$

so applying the triangle inequality, using the above estimate for $B^*_{l,\lambda}$ and invoking Corollary 2.7 yields

$$\|B^*_l u\|_{L^q_Y(\mathbb{R}^n)} \lesssim 2^{-l} \sum_{\lambda=1}^{l} 2^\lambda \, \big\| H_\lambda u \big\|_{L^q_Y(\mathbb{R}^n)} \lesssim 2^{-l} \sum_{\lambda=1}^{l} 2^{\lambda(1-1/\mathfrak{C}(L^q_Y(\mathbb{R}^n)))} \, \big\| u \big\|_{L^q_Y(\mathbb{R}^n)}.$$

Evaluating the geometric series we obtain the estimate

$$\|B^*_l u\|_{L^q_Y(\mathbb{R}^n)} \leq C \cdot 2^{-l/\mathfrak{C}(L^q_Y(\mathbb{R}^n))} \big\| u \big\|_{L^q_Y(\mathbb{R}^n)}, \tag{3.46}$$

where the constant C depends only on n, q, the UMD constant of Y and the cotype $\mathfrak{C}(L^q_Y(\mathbb{R}^n))$.

3.4.3. *Estimates for C_l.*

In view of (3.28), (3.33) and (3.38) note that now $\operatorname{diam}(M) < 2^{-l} \operatorname{diam}(Q)$, and so we may utilize inequality (3.23) in Proposition 3.2 on page 47.

This setting is visualized in Figure 8 on the next page.

As in the preceding case we aim at estimating the adjoint operator C^*_l; so with $Y = X^*$ and $\frac{1}{p} + \frac{1}{q} = 1$, the usual parameterization splits the collection $\mathscr{C}_l$ (see (3.28)) into the disjoint collections $\mathscr{C}_{l,\lambda}$, $\lambda \geq l+1$, given by

$$\mathscr{C}_{l,\lambda} = \big\{ (Q,M) \in \mathscr{B}_l \; : \; \operatorname{diam}(M) = 2^{-\lambda} \operatorname{diam}(Q) \big\}.$$

We define the operator $C_{l,\lambda}$ accordingly, that is

$$C_{l,\lambda} u = \sum_{(Q,M) \in \mathscr{C}_{l,\lambda}} \langle f_{Q,l}, h_M \rangle \, h_Q \, u_M \, |Q|^{-1} \, |M|^{-1},$$

for all $u = \sum_{K \in \mathscr{Q}} u_K \, h_K \, |K|^{-1}$. The adjoint operators C^*_l and $C^*_{l,\lambda}$ are given by

$$\begin{aligned} C^*_l u &= \sum_{\lambda=l+1}^{\infty} \sum_{Q,M \in \mathscr{C}_{l,\lambda}} \langle f_{Q,l}, h_M \rangle \, |M|^{-1} h_M \, u_Q \, |Q|^{-1} \\ &= \sum_{\lambda=l+1}^{\infty} C^*_{l,\lambda} u. \end{aligned}$$

Observe that for $(Q,M) \in \mathscr{C}_{l,\lambda}$ holds true that

$$\{ M \; : \; \langle f_{Q,l}, h_M \rangle \neq 0 \} \subset \{ M \; : \; M \cap D_l(Q) \neq \emptyset \},$$

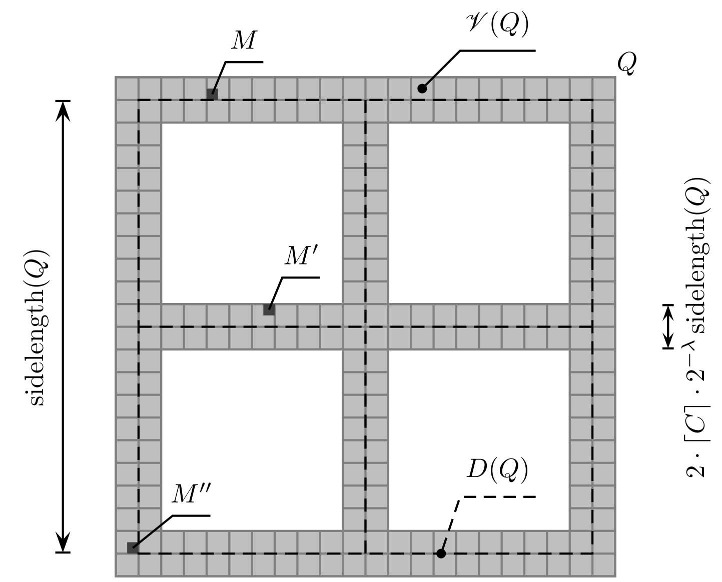

FIGURE 8. The tiny cubes M, M' and M'' are contained in the cover of the ring domain $\mathscr{V}_l(Q)$.

thus we have

$$\Big|\sum_{\substack{(Q,M)\in\mathscr{C}_{l,\lambda}\,:\\ \langle f_{Q,l},h_M\rangle\neq 0}} h_M\Big| \leq \Big|\sum_{M\in\mathscr{V}_l(Q)} h_M\Big| = |g_{Q,l}|.$$

We proceed by applying essentially the same steps as in the preceding cases. Using the UMD–property and subsequently Kahane's contraction principle we obtain

$$\begin{aligned}\|C^*_{l,\lambda}u\|_{L^q_Y(\mathbb{R}^n)} &\lesssim 2^l\, 2^{-\lambda}\Big\|\sum_{Q\in\mathscr{Q}} u_Q\, g_{Q,l}\,|Q|^{-1}\Big\|_{L^q_Y(\mathbb{R}^n)}\\ &= 2^l\, 2^{-\lambda}\big\|H_l u\big\|_{L^q_Y(\mathbb{R}^n)}.\end{aligned}$$

Hence, applying the triangle inequality and using the above estimate for $C^*_{l,\lambda}$ gives us

$$\|C^*_l u\|_{L^q_Y(\mathbb{R}^n)} \lesssim \big\|H_l u\big\|_{L^q_Y(\mathbb{R}^n)}.$$

Finally, Corollary 2.7 yields

$$\|C^*_l u\|_{L^q_Y(\mathbb{R}^n)} \leq C\cdot 2^{-l/\mathcal{C}(L^q_Y(\mathbb{R}^n))}, \tag{3.47}$$

where the constant C depends only on n, q, the UMD constant of Y and the cotype $\mathcal{C}(L^q_Y(\mathbb{R}^n))$.

3.4.4. *Summary for P_l.*

First, note that for $Y = X^*$ and $\frac{1}{p}+\frac{1}{q}=1$ holds true that

$$(L^p_X(\mathbb{R}^n))^* = L^q_Y(\mathbb{R}^n) \qquad \text{and} \qquad \frac{1}{\mathcal{T}(L^p_X(\mathbb{R}^n))}+\frac{1}{\mathcal{C}(L^q_Y(\mathbb{R}^n))}=1.$$

Second, we use that

$$\big\|B^*_l\ :\ L^q_Y(\mathbb{R}^n)\to L^q_Y(\mathbb{R}^n)\big\| \lesssim \big\|B_l\ :\ L^p_X(\mathbb{R}^n)\to L^p_X(\mathbb{R}^n)\big\|,$$

and

$$\big\|C^*_l\ :\ L^q_Y(\mathbb{R}^n)\to L^q_Y(\mathbb{R}^n)\big\| \lesssim \big\|C_l\ :\ L^p_X(\mathbb{R}^n)\to L^p_X(\mathbb{R}^n)\big\|,$$

to combine the inequalities (3.45), (3.46), (3.47) via the identity

$$P_l = A_l + B_l + C_l.$$

Thereby we obtain

$$\|P_l : L^p_X(\mathbb{R}^n) \to L^p_X(\mathbb{R}^n)\| \le C \cdot 2^{-l(1-\frac{1}{\mathcal{T}(L^p_X(\mathbb{R}^n))})},$$

where $L^p_X(\mathbb{R}^n)$ has type $\mathcal{T}(L^p_X(\mathbb{R}^n))$ and the constant C depends only on n, p, the UMD constant of X and the type $\mathcal{T}(L^p_X(\mathbb{R}^n))$.

3.4.5. *Estimates for A_-.*

In view of (3.29), (3.34) and (3.39) note that $\operatorname{diam}(M) \le \operatorname{diam}(Q)$, and so we may utilize inequality (3.24) in Proposition 3.2 on page 47. In this case the size of the cube M cannot exceed the size of Q, so we may use inequality (3.24). We rather want to estimate A_-^* than A_-, therefore set $Y = X^*$ and q such that $\frac{1}{p} + \frac{1}{q} = 1$.

First, we split the set $\mathscr{A}_-$ (see (3.29)) into the disjoint collections $\mathscr{A}_{-,\lambda}$, $\lambda \ge 0$, given by

$$\mathscr{A}_{-,\lambda} = \big\{(Q, M) \in \mathscr{A}_- \ : \ \operatorname{diam}(M) = 2^{-\lambda} \operatorname{diam}(Q)\big\},$$

and define the operator $A_{-,\lambda}$ accordingly, that is

$$A_{-,\lambda} u = \sum_{(Q,M)\in\mathscr{A}_{-,\lambda}} \langle f_Q, h_M\rangle\, h_Q\, u_M\, |Q|^{-1}\, |M|^{-1},$$

for all $u = \sum_{K\in\mathscr{Q}} u_K\, h_K\, |K|^{-1}$. The adjoint operators A_-^* and $A_{-,\lambda}^*$ are given by

$$\begin{aligned} A_-^* u &= \sum_{\lambda=0}^{\infty} \sum_{Q,M\in\mathscr{A}_{-,\lambda}} \langle f_Q, h_M\rangle\, u_Q\, h_M\, |Q|^{-1}\, |M|^{-1} \\ &= \sum_{\lambda=0}^{\infty} A_{-,\lambda}^* u. \end{aligned}$$

Utilizing the UMD–property and subsequently Kahane's contraction principle (0.4) with respect to (3.24), we infer

$$\big\|A_{-,\lambda}^* u\big\|_{L^q_Y(\mathbb{R}^n)} \lesssim 2^{-\lambda} \Big\| \sum_{Q\in\mathscr{Q}} \sum_{\substack{(Q,M)\in\mathscr{A}_{-,\lambda}\\ M\cap(C\cdot Q)\neq\emptyset}} u_Q\, |Q|^{-1} h_M \Big\|_{L^q_Y(\mathbb{R}^n)}.$$

For every $Q \in \mathscr{Q}$ we apply Kahane's contraction principle to

$$\Big| \sum_{\substack{(Q,M)\in\mathscr{A}_{-,\lambda}\\ M\cap(C\cdot Q)\neq\emptyset}} h_M \Big| \le |h_Q|,$$

and note that we would actually need a constant number of shifts T_m, $|m| \le C_1$ of h_Q to cover the whole support of the sum on the left-hand side. In view of estimate (0.8) we omit this detail and continue the proof with the estimate

$$\begin{aligned} \big\|A_{-,\lambda}^* u\big\|_{L^q_Y(\mathbb{R}^n)} &\lesssim 2^{-\lambda} \Big\| \sum_{Q\in\mathscr{Q}} u_Q\, |Q|^{-1} \sum_{\substack{(Q,M)\in\mathscr{A}_{-,\lambda}\\ M\cap(C\cdot Q)\neq\emptyset}} h_M \Big\|_{L^q_Y(\mathbb{R}^n)} \\ &\lesssim 2^{-\lambda} \big\|u\big\|_{L^q_Y(\mathbb{R}^n)}. \end{aligned}$$

Summing over $\lambda \ge 0$ yields

$$\|A_-^* u\|_{L^q_Y(\mathbb{R}^n)} \le C \cdot \|u\|_{L^q_Y(\mathbb{R}^n)}, \tag{3.48}$$

where the constant C depends only on n, q, the UMD constant of Y and the cotype $\mathcal{C}(L^q_Y(\mathbb{R}^n))$.

3.4.6. *Estimates for B_-.*

In view of (3.30), (3.35) and (3.40) note that $\operatorname{diam}(M) > \operatorname{diam}(Q)$, and so we may utilize inequality (3.25) in Proposition 3.2 on page 47.

As usual we split the set $\mathscr{B}_-$ (see (3.30)) into the disjoint collections $\mathscr{B}_{-,\lambda}$, $\lambda \geq 1$, given by

$$\mathscr{B}_{-,\lambda} = \big\{(Q, M) \in \mathscr{B}_- \ : \ \operatorname{diam}(Q) = 2^{-\lambda} \operatorname{diam}(M)\big\},$$

and define the operator $B_{-,\lambda}$ accordingly, that is

$$B_{-,\lambda} u = \sum_{(Q,M) \in \mathscr{B}_{-,\lambda}} \langle f_Q, h_M \rangle \, h_Q \, u_M \, |Q|^{-1} \, |M|^{-1},$$

for all $u = \sum_{K \in \mathscr{Q}} u_K \, h_K \, |K|^{-1}$. Obviously, the identity

$$B_- u = \sum_{\lambda=1}^{\infty} B_{-,\lambda} \, u$$

holds true. For all $(Q, M) \in \mathscr{B}_{-,\lambda}$ we have the inclusions

$$\{Q \, : \, \langle f_Q, h_M \rangle \neq 0\} \subset \{Q \, : \, (C \cdot Q) \cap D(Q) \neq \emptyset\} \subset \mathscr{V}_\lambda(M).$$

Successively using the UMD–property, Kahane's contraction principle applied to (3.25) and the inclusion above we obtain

$$\begin{aligned}
\big\|B_{-,\lambda} u\big\|_{L^p_X(\mathbb{R}^n)} &\lesssim \big\| \sum_{M \in \mathscr{Q}} u_M \, |M|^{-1} \sum_{Q \in \mathscr{V}_\lambda(M)} h_Q \big\|_{L^p_X(\mathbb{R}^n)} \\
&= \big\| \sum_{M \in \mathscr{Q}} u_M \, g_{M,\lambda} \, |M|^{-1} \big\|_{L^p_X(\mathbb{R}^n)} \\
&= \big\|H_\lambda u\big\|_{L^p_X(\mathbb{R}^n)}.
\end{aligned}$$

The last equality is the definition of the ring domain operator H_λ (see (2.37)). The main result on ring domain operators Corollary 2.7 yields

$$\|B_{-,\lambda} u\|_{L^p_X(\mathbb{R}^n)} \lesssim \|H_\lambda u\|_{L^p_X(\mathbb{R}^n)} \lesssim 2^{-\lambda/\mathcal{C}(L^p_X(\mathbb{R}^n))} \, \|u\|_{L^p_X(\mathbb{R}^n)}.$$

Hence, summation over $\lambda \geq 1$ gives us

$$\|B_- u\|_{L^p_X(\mathbb{R}^n)} \leq C \, \|u\|_{L^p_X(\mathbb{R}^n)}, \tag{3.49}$$

where the constant C depends only on n, p, the UMD constant of X and the cotype $\mathcal{C}(L^p_X(\mathbb{R}^n))$.

3.4.7. *Summary for P_-.*

First note that for $Y = X^*$ and $\frac{1}{p} + \frac{1}{q} = 1$ holds true

$$(L^p_X(\mathbb{R}^n))^* = L^q_Y(\mathbb{R}^n) \qquad \text{and} \qquad \frac{1}{\mathcal{T}(L^p_X(\mathbb{R}^n))} + \frac{1}{\mathcal{C}(L^q_Y(\mathbb{R}^n))} = 1.$$

Second, we use that

$$\big\|A_-^* \ : \ L^q_Y(\mathbb{R}^n) \to L^q_Y(\mathbb{R}^n)\big\| \lesssim \big\|A_- \ : \ L^p_X(\mathbb{R}^n) \to L^p_X(\mathbb{R}^n)\big\|,$$

to combine the inequalities (3.48) and (3.49) via the identity

$$P_- = A_- + B_-$$

so that we obtain

$$\|P_- \, : \, L^p_X(\mathbb{R}^n) \to L^p_X(\mathbb{R}^n)\| \leq C,$$

where $L^p_X(\mathbb{R}^n)$ has type $\mathcal{T}(L^p_X(\mathbb{R}^n))$ and the constant C depends only on n, p, the UMD constant of X and the type $\mathcal{T}(L^p_X(\mathbb{R}^n))$.

3.5. Estimates for $P_l^{(\varepsilon)} R_{i_0}^{-1}$.

Following [**LMM07**] we will establish estimates for $P_l^{(\varepsilon)} R_{i_0}^{-1}$, $l \in \mathbb{Z}$ by reducing them to estimates for $P_l^{(\varepsilon)}$. We exploit that $\big(R_{i_0}^{-1}\big)^*$ maps the mollified Haar functions $f_{Q,l}^{(\varepsilon)}$ to functions $k_{Q,l}^{(\varepsilon)}$ having similar properties. Due to the algebraic identity (3.50), this amounts to controlling the support of the $k_{Q,l}$, besides factors depending on l. Assuming $\varepsilon_{i_0} = 1$, we have

$$\operatorname{supp}\big(\mathbb{E}_{i_0} h_Q^{(\varepsilon)}\big) \subset Q,$$

restricting the support of the functions $k_{Q,l,i}$ defined in (3.51), and exhibiting the conditions asserted in (3.54) and (3.55).

We do not omit the superscripts (ε) at this time.

It is a well known fact that one can write the inverse of the Riesz transform $R_{i_0}^{-1}$ as

$$R_{i_0}^{-1} = R_{i_0} + \sum_{\substack{1 \le i \le n \\ i \ne i_0}} \mathbb{E}_{i_0} \partial_i R_i, \tag{3.50}$$

where $\mathbb{E}_{i_0}$ is given by

$$\mathbb{E}_{i_0} f(x) = \int_{-\infty}^{x_{i_0}} f(x_1, \dots, x_{i_0-1}, s, x_{i_0+1}, \dots, x_n)\, \mathrm{d}s, \qquad x = (x_1, \dots, x_n).$$

Now we introduce the family of functions

$$k_{Q,l,i}^{(\varepsilon)} = \Delta_{j+l}\big(\mathbb{E}_{i_0} \partial_i h_Q^{(\varepsilon)}\big), \qquad \text{if } Q \in \mathscr{Q}_j, \tag{3.51}$$

and consider

$$\begin{aligned} P_l^{(\varepsilon)} R_{i_0}^{-1} u = & \sum_{j \in \mathbb{Z}} \sum_{Q \in \mathscr{Q}_j} \big\langle R_{i_0} u, \Delta_{j+l}(h_Q^{(\varepsilon)})\big\rangle\, h_Q^{(\varepsilon)}\, |Q|^{-1} \\ & + \sum_{\substack{1 \le i \le n \\ i \ne i_0}} \sum_{j \in \mathbb{Z}} \sum_{Q \in \mathscr{Q}_j} \big\langle \mathbb{E}_{i_0} \partial_i R_i u, \Delta_{j+l}(h_Q^{(\varepsilon)})\big\rangle\, h_Q^{(\varepsilon)}\, |Q|^{-1}. \end{aligned} \tag{3.52}$$

Since the Riesz transforms R_i, $1 \le i \le n$ are continuous on $L_X^p(\mathbb{R}^n)$, it is obvious that the first sum of (3.52) can be treated as if it were P_l (also see (3.6)).

For the second sum of (3.52), we fix a coordinate $i \ne i_0$, rearrange the operators in the scalar product and use the functions defined in (3.51), hence

$$\sum_{j \in \mathbb{Z}} \sum_{Q \in \mathscr{Q}_j} \langle \mathbb{E}_{i_0} \partial_i R_i u, \Delta_{j+l}(h_Q^{(\varepsilon)})\rangle\, h_Q^{(\varepsilon)}\, |Q|^{-1} = \sum_{Q \in \mathscr{Q}} \langle R_i u, k_{Q,l,i}^{(\varepsilon)}\rangle\, h_Q^{(\varepsilon)}\, |Q|^{-1}.$$

Due to the continuity of the Riesz transforms $R_i : L_X^p(\mathbb{R}^n) \to L_X^p(\mathbb{R}^n)$ we may estimate the following type of operator

$$K_{l,i}^{(\varepsilon)} u = \sum_{Q \in \mathscr{Q}} \langle u, k_{Q,l,i}^{(\varepsilon)}\rangle\, h_Q^{(\varepsilon)}\, |Q|^{-1}, \tag{3.53}$$

instead of the second sum in (3.52).

In order to estimate $K_{l,i}^{(\varepsilon)}$ we need to analyze the analytic properties of the functions $k_{Q,l,i}^{(\varepsilon)}$. If $l \ge 0$, then

$$\begin{gathered} \int k_{Q,l,i}^{(\varepsilon)}(x)\, \mathrm{d}x = 0, \qquad \operatorname{supp} k_{Q,l,i}^{(\varepsilon)} \subset D_l^{(\varepsilon)}(Q), \\ |k_{Q,l,i}^{(\varepsilon)}| \le C\, 2^l, \quad \operatorname{Lip}(k_{Q,l,i}^{(\varepsilon)}) \le C\, 2^{2l}\, (\operatorname{diam}(Q))^{-1}, \end{gathered} \tag{3.54}$$

and for $l \leq 0$

$$\int k^{(\varepsilon)}_{Q,l,i}(x)\,\mathrm{d}x = 0, \qquad \operatorname{supp} k^{(\varepsilon)}_{Q,l,i} \subset C\,2^{|l|}Q,$$
$$|k^{(\varepsilon)}_{Q,l,i}| \leq C\,2^{-|l|(n+1)}, \quad \operatorname{Lip}(k^{(\varepsilon)}_{Q,l,i}) \leq C\,2^{-|l|(n+2)}\,(\operatorname{diam}(Q))^{-1}. \tag{3.55}$$

Note that the above properties of $k^{(\varepsilon)}_{Q,l,i}$ depend particularly on the coordinate-wise vanishing moments of b (3.3), introduced by Δ_l in equations (3.4) and (3.6). Furthermore observe the definition of $k^{(\varepsilon)}_{Q,l,i}$ involves an integration of $h^{(\varepsilon)}_Q$ with respect to the variable x_{i_0}. We specifically want to stress that if $\varepsilon_{i_0} = 1$, then $\mathbb{E}_{i_0} h^{(\varepsilon)}_Q$ is compactly supported in Q, but if $\varepsilon_{i_0} = 0$, then $\operatorname{supp}\big(\mathbb{E}_{i_0} h^{(\varepsilon)}_Q\big)$ is unbounded.

If we compare this with the properties (3.18) and (3.19) regarding the functions $f^{(\varepsilon)}_{Q,l}$, it turns out that the properties coincide if $l \leq 0$, and that $2^{-l}\,k^{(\varepsilon)}_{Q,l,i}$, satisfies the same conditions as $f^{(\varepsilon)}_{Q,l}$, if $l \geq 0$. Reconsidering the proof of Theorem 3.4 on page 49, we note that those arguments where solely depending on the analytic properties (3.18) and (3.19) of the functions $f^{(\varepsilon)}_{Q,l}$. With regard to (3.54) respectively (3.55), the same proofs are feasible with the functions $k^{(\varepsilon)}_{Q,l,i}$ replacing $f_{Q,l}$ if $l \leq 0$, respectively $2^{-l}\,k^{(\varepsilon)}_{Q,l,i}$ replacing $f_{Q,l}$ if $l \geq 0$. Furthermore we have to replace P_l by $K_{l,i}$, for every $1 \leq i \leq n$.

Stringing this all together we obtain the following result from the estimates of Theorem 3.4 on page 49.

Theorem 3.6. *Let X be a* UMD *space, $1 < p < \infty$ and $n \in \mathbb{N}$, and let $L^p_X(\mathbb{R}^n)$ have type $\mathcal{T}(L^p_X(\mathbb{R}^n))$. Furthermore denote by R_{i_0} the Riesz transform acting in direction i_0 and let $\varepsilon_{i_0} = 1$.*

Then there exists a constant $C > 0$ such that for every $l \geq 0$ and all $u \in L^p_X(\mathbb{R}^n)$ we have

$$\|P^{(\varepsilon)}_l R^{-1}_{i_0} u\|_{L^p_X(\mathbb{R}^n)} \leq C \cdot 2^{\overline{\frac{l}{\mathcal{T}(L^p_X(\mathbb{R}^n))}}}\,\|u\|_{L^p_X(\mathbb{R}^n)}, \tag{3.56}$$

where the constant C depends only on n, p, the UMD *constant of X and the type $\mathcal{T}(L^p_X(\mathbb{R}^n))$.*

Moreover, there exists a constant $C > 0$ such that for all $u \in L^p_X(\mathbb{R}^n)$

$$\|P^{(\varepsilon)}_- R^{-1}_{i_0} u\|_{L^p_X(\mathbb{R}^n)} \leq C \cdot \|u\|_{L^p_X(\mathbb{R}^n)}, \tag{3.57}$$

where the constant C depends only on n, p, the UMD *constant of X and the type $\mathcal{T}(L^p_X(\mathbb{R}^n))$.*

4. Auxiliary Results

In order to keep this thesis self–contained we include here several auxiliary results used in Chapter 2.

Lipschitz Estimate for Separately Convex Functions.

We record here a Lipschitz estimate for separately convex functions satisfying convenient growth estimates on the Banach space X. The resulting inequality holds true without any assumptions on the underlying normed vector space X.

THEOREM 4.1. *Let X be a normed vector space, $n \geq 1$, $f : X^n \to \mathbb{R}$ separately convex and $g : X^n \to \mathbb{R}$, where $g(x) = 1 + \sum_{i=1}^n \|x_i\|_X^p$. If $0 \leq f(x) \leq g(x)$, $x \in X$, then*

$$\big|f(x) - f(y)\big| \leq C \cdot \big(1 + \|x\|_{X^n} + \|y\|_{X^n}\big)^{p-1} \cdot \|x - y\|_{X^n}, \tag{4.1}$$

for all $x, y \in X^n$. The constant $C > 0$ depends only on n and p.

PROOF. Let $x \neq y \in X^n$, and with $1 \leq k \leq n$ fixed define

$$\begin{aligned} f_k(t) &= f(x_1, \ldots, x_{k-1}, x_k + t(y_k - x_k), x_{k+1}, \ldots, x_n), \\ g_k(t) &= g(x_1, \ldots, x_{k-1}, x_k + t(y_k - x_k), x_{k+1}, \ldots, x_n), \\ n_k(t) &= \big\|x_k + t(y_k - x_k)\big\|_X, \end{aligned}$$

for all $t \in \mathbb{R}$. We may assume that $f_k(0) \leq f_k(1)$, otherwise we would switch x_k and y_k. Observe that $n_k(t)$ is increasing if $t \geq \frac{2\|x_k\|}{\|y_k - x_k\|}$, hence $g_k(t)$ is increasing if $t \geq \frac{2\|x_k\|}{\|y_k - x_k\|}$.

Given $t_0 < t_1$ which will be specified later, we define the affine functions

$$\begin{aligned} \ell_1(t) &= f_k(0) + t \cdot (f_k(1) - f_k(0)), \\ \ell_2(t) &= g_k(0) + \frac{g_k(t_1) - g_k(t_0)}{t_1 - t_0} \cdot (t - t_0), \end{aligned}$$

and let $\bar{t}$ denote the point where $\ell_2(\bar{t}) = 0$, that is

$$\bar{t} = t_0 - \frac{g_k(t_0)}{g_k(t_1) - g_k(t_0)} \cdot (t_1 - t_0). \tag{4.2}$$

Now we prove that if $1 \leq \bar{t} < t_0 < t_1$ and $t_0 \geq 2\|x_k\|/\|y_k - x_k\|$, then

$$f_k(1) - f_k(0) \leq \frac{g_k(t_1) - g_k(t_0)}{t_1 - t_0}. \tag{4.3}$$

Assume (4.3) does not hold true, then since $f_k(0) \geq 0$ and $\bar{t} \geq 1$ we have $\ell_1(t) > \ell_2(t)$, for all $t > \bar{t}$. Since $f_k(t)$ is convex, we know that $f_k(t) \geq \ell_1(t)$, $t \geq \bar{t}$, hence $f_k(t_1) \geq \ell_1(t_1) > \ell_2(t_1) = g_k(t_1)$, which contradicts $f_k(t) \leq g_k(t)$, $t \in \mathbb{R}$.

Now we want to impose conditions on $t_0 < t_1$, such that $\bar{t} \geq 1$. Observe,

$$\begin{aligned} \frac{g_k(t_1) - g_k(t_0)}{t_1 - t_0} &\geq p \cdot n_k(t_0)^{p-1} \cdot \big(n_k(t_1) - n_k(t_0)\big)/(t_1 - t_0) \\ &\geq p \cdot n_k(t_0)^{p-1} \cdot \big(\|y_k - x_k\| - \frac{2\|x_k\|}{t_1 - t_0}\big), \end{aligned}$$

and plugging this estimate into (4.2) we gain

$$\bar{t} \geq t_0 - \frac{g_k(t_0)}{p \cdot \|x_k + t_0(y_k - x_k)\|^{p-1} \cdot \big(\|y_k - x_k\| - \frac{2\|x_k\|}{t_1 - t_0}\big)} \tag{4.4}$$

If we impose the following constraints

- $(t_1 - t_0) \cdot \|y_k - x_k\| \geq 2C \cdot \|x_k\|$,
- $t_0 \cdot \|y_k - x_k\| \geq 2C \cdot \|x_i\|$, $1 \leq i \leq n$,

- $t_0 \cdot \|y_k - x_k\| \geq C$,
- $t_0 \cdot \|y_k - x_k\| \geq 2 \cdot \|x_k\|$,

in order to estimate (4.4), we get

$$\bar{t} \geq t_0 - A_1 - A_2 - A_3,$$

where

$$A_1 = \frac{1}{p \cdot (1 - \frac{1}{C}) \cdot \|x_k + t_0(y_k - x_k)\|^{p-1} \cdot \|y_k - x_k\|} \leq \frac{t_0}{p \cdot (C-1)^p},$$

$$A_2 = \sum_{i \neq k} \frac{\|x_i\|^p}{p \cdot \|x_k + t_0(y_k - x_k)\|^{p-1} \cdot \|y_k - x_k\| \cdot \left(1 - \frac{1}{C}\right)} \leq \frac{t_0 \cdot (n-1)}{p \cdot (C-1)^p},$$

$$A_3 = \frac{\|x_k + t_0(y_k - x_k)\|}{p \cdot (1 - \frac{1}{C}) \cdot \|y_k - x_k\|} \leq \frac{t_0 \cdot (1 + C)}{p \cdot (C-1)}.$$

Using these estimates we gain

$$\bar{t} \geq t_0 \cdot \left(1 - \frac{1}{p \cdot (C-1)^p} - \frac{n-1}{p \cdot (C-1)^p} - \frac{1+C}{p \cdot (C-1)}\right) = t_0 \cdot \alpha. \tag{4.5}$$

If we choose C large enough, so that $\alpha \geq \frac{p-1}{2p}$, and define

$$\begin{aligned} t_0 &= \sum_{i=1}^{n} \frac{C \cdot \|x_i\|}{\|y_k - x_k\|} + \frac{C}{\|y_k - x_k\|} + \frac{1}{\alpha} \\ t_1 &= 3t_0, \end{aligned} \tag{4.6}$$

so that $t_0 < t_1$ satisfies our constraints. Hence we can infer (4.5), and get $1 \leq \bar{t} < t_0 < t_1$, $t_0 \geq 2\|x_k\|/\|y_k - x_k\|$. Thus (4.3) yields

$$f_k(1) - f_k(0) \leq \frac{g_k(t_1) - g_k(t_0)}{t_1 - t_0}, \tag{4.7}$$

where t_0, t_1 are defined in (4.6). A straightforward computation shows

$$\frac{g_k(t_1) - g_k(t_0)}{t_1 - t_0} \lesssim 1 + \|y_k - x_k\|_X + \|x\|_{X^n},$$

and so we have

$$\begin{aligned} &\big|f(x_1, \ldots, x_{k-1}, x_k, x_{k+1}, \ldots, x_n) - f(x_1, \ldots, x_{k-1}, y_k, x_{k+1}, \ldots, x_n)\big| \\ &\qquad \lesssim \big(1 + \|y_k - x_k\|_X + \|x\|_{X^n}\big)^{p-1} \|y_k - x_k\|_X. \end{aligned}$$

By induction one can verify

$$\big|f(x) - f(y)\big| \leq C \cdot \big(1 + \|x\|_{X^n} + \|y\|_{X^n}\big)^{p-1} \cdot \|x - y\|_{X^n},$$

where C depends only on n and p. □

Convolution Operators on L^p_X.

So we recall a well–known criterion for compactness in vector–valued L^p_X. The result is applied to vector–valued convolution operators with integrable kernel. The conclusion holds without any assumption on the underlying Banach space X.

Theorem 4.2 (Kolmogorov–Riesz). *Let X be a Banach space, $1 \leq p < \infty$ and $\mathcal{F} \subset L^p_X(\mathbb{R}^n)$. Then $\mathcal{F}$ is precompact in $L^p_X(\mathbb{R}^n)$ if and only if the following conditions are satisfied*

(i) $\displaystyle\sup_{f \in \mathcal{F}} \int_{\mathbb{R}^n} \big\|f(x)\big\|_X^p \,\mathrm{d}x < \infty,$

(ii) $\displaystyle\sup_{f \in \mathcal{F}} \int_{\mathbb{R}^n} \big\|f(x+h) - f(x)\big\|_X^p \,\mathrm{d}x \xrightarrow{h \to 0} 0,$

(iii) $\sup\limits_{f\in\mathcal{F}} \int\limits_{\{|x|>R\}} \big\|f(x)\big\|_X^p \,\mathrm{d}x \xrightarrow{R\to\infty} 0.$

For more details see [**Alt06**, Theorem 2.5].

THEOREM 4.3. *Let $K \in L^1_{\mathbb{C}}(\mathbb{R}^n)$, $1 \le p < \infty$ and X be a Banach space. Then the convolution operator T given by*

$$Tf = K * f$$

maps $L^p_X(\mathbb{R}^n)$, $1 \le p < \infty$ compactly into itself.

PROOF. Let $1 \le p < \infty$. Given $m \ge 1$, we define the operator $R_m f = f \cdot \mathbb{1}_{\{|x|\le m\}}$. Note that R_m, $m \ge 1$ are contractions from L^p_X to itself, and $R_m f \to f$ in L^p_X, as $m \to \infty$. Thus, by the uniform boundedness principle we have $\big\|R_m - \mathrm{Id} : L^p_X \to L^p_X\big\| \to 0$, as $m \to \infty$.

Let $m \ge 1$ be fixed and $\mathcal{B} \subset L^p_X$ be a bounded set. Denoting $T_m = R_m \circ T$ and $\mathcal{F} = T_m(\mathcal{B})$, we will verify the conditions (i)–(iii) of the Kolmogorov–Riesz compactness criterion, see Theorem 4.2 on the preceding page, for the set $\mathcal{F}$.

Due to Young's Inequality we have

$$\sup_{f\in\mathcal{B}} \big\|T\,f\big\|_{L^p_X} \le \big\|K\big\|_{L^1_{\mathbb{C}}} \cdot \sup_{f\in\mathcal{B}} \big\|f\big\|_{L^p_X},$$

thus $\mathcal{F}$ is bounded, and condition (i) is satisfied.

Now we verify (ii). For all $h \in \mathbb{R}^n$ we have

$$\begin{aligned}&\sup_{f\in\mathcal{B}} \int\limits_{\mathbb{R}^n} \big\|T_m f(x+h) - T_m f(x)\big\|_X^p \,\mathrm{d}x \\ &\qquad\le \sup_{f\in\mathcal{B}} \int\limits_{\mathbb{R}^n} \Big\| \int\limits_{\mathbb{R}^n} \big(K(x+h-y) - K(x-y)\big) \cdot f(y)\,\mathrm{d}y \Big\|_X^p \,\mathrm{d}x\end{aligned}$$

Noting that the inner integral is a convolution, we can apply Young's Inequality and gain

$$\sup_{f\in\mathcal{B}} \int\limits_{\mathbb{R}^n} \big\|T_m f(x+h) - T_m f(x)\big\|_X^p \,\mathrm{d}x \le \sup_{f\in\mathcal{B}} \|f\|^p_{L^p_X} \cdot \Big(\int\limits_{\mathbb{R}^n} \big|\big(K(x+h) - K(x)\big|\,\mathrm{d}x\Big)^p.$$

Since $\mathcal{B}$ is bounded in L^p_X and the latter expression tends to zero as $h \to 0$, we have established condition (ii).

The last condition is satisfied since $T_m f$ is supported in $B(0,m)$.

Now we know that $T_m : L^p_X \longrightarrow L^p_X$ is compact, and

$$\big\|T_m - T : L^p_X \to L^p_X\big\| \le \big\|R_m - \mathrm{Id} : L^p_X \to L^p_X\big\| \cdot \big\|T : L^p_X \to L^p_X\big\|.$$

Due to Young's Inequality T is bounded, and $\big\|R_m - \mathrm{Id}\,\big\| \to 0$, as $m \to \infty$, thus we have established the Theorem. □

Fourier Multipliers on L^p_X.

The following is a well–known criterion for obtaining Fourier multiplier with integrable kernel.

THEOREM 4.4. *Given a real number $\mu > 0$ and a positive integer n let the function $m : \mathbb{R}^n \setminus \{0\} \longrightarrow \mathbb{C}$, be such that*

$$\big|\partial_\xi^\alpha m(\xi)\big| \le A \cdot \langle\xi\rangle^{-\mu-|\alpha|}, \quad \textit{for all multi-indices } |\alpha| \le n+1. \tag{4.8}$$

If we define the kernel K formally by $K = \mathcal{F}^{-1}m$, then

$$\int\limits_{\mathbb{R}^n} |K(x)|\,\mathrm{d}x \le C, \tag{4.9}$$

where the constant C depends only on A, n and μ.

PROOF. First we let $\delta \in C_0^\infty(-1,1)^n$ be such that $0 \leq \delta(\xi) \leq 1$ for all ξ, and $\delta(\xi) = 1$, if $|\xi| \leq 1/2$. Then define $\delta_j(\xi) = \delta(2^{-j-1}\xi) - \delta(2^{-j}\xi)$, for all $j \geq 1$, and $\delta_0(\xi) = \delta(\xi)$. Then $1 = \sum_{j\geq 0} \delta_j(\xi)$, for all $\xi \in \mathbb{R}^n$. If we set

$$\begin{aligned} m_j(\xi) &= \delta_j(\xi)\, m(\xi), \\ K_j(x) &= (\mathcal{F}^{-1} m_j)(x), \end{aligned}$$

then $K_j = \sum_{j\geq 0} K_j$. Now let α be an arbitrary multi-index such that $|\alpha| \leq n+1$, then integrating by parts and using (4.8) yields

$$\begin{aligned} |K_j(x)| &\leq |x^{-\alpha}| \cdot \int_{\mathbb{R}^n} \left|\partial_\xi^\alpha m_j(\xi)\right| \mathrm{d}\xi \\ &\leq A\,|x^{-\alpha}| \cdot \int_{\mathbb{R}^n} \langle\xi\rangle^{-\mu-|\alpha|}\,\mathrm{d}\xi \\ &\leq A\,|x^{-\alpha}|\, 2^{j(n-\mu-|\alpha|)}. \end{aligned}$$

Thus we gain

$$|K_j(x)| \leq A\,|x|^{-N}\, 2^{j(n-\mu-N)},$$

for all $0 \leq N \leq n+1$. Using this estimate for $N = n-1$ and $N = n$ we obtain

$$|K_j(x)| \leq A\,|x|^{-n+\mu/2}\, 2^{-j\mu/2},$$

Using the above estimates with $N = n+1$ if $|x| \geq 1$, and the latter one if $|x| \leq 1$, then

$$\int_{\mathbb{R}^n} |K_j(x)|\,\mathrm{d}x \leq C \cdot 2^{-j(-1-\mu)} + 2^{-j\,\mu/2}.$$

Since $\mu > 0$ summing over $j \geq 0$ yields estimate (4.9). □

Facts on the Sobolev Space $W_X^{-1,p}$.

From now onwards the Banach space X has the UMD–property. We gather some facts contributing to the proof of Theorem 1.2.

THEOREM 4.5. *Let X be a* UMD *space, $n \geq 1$ and $1 < p < \infty$. If $\alpha \in \mathcal{S}(\mathbb{R}^n;\mathbb{C})$, then there exists a constant $C > 0$ such that*

$$\left\|\alpha \cdot u\right\|_{W^{-1,p}(\mathbb{R}^n;X)} \leq C \cdot \left\|u\right\|_{W^{-1,p}(\mathbb{R}^n;X)} \tag{4.10}$$

for all $u \in W^{-1,p}(\mathbb{R}^n;X)$.

PROOF. Note that in UMD spaces

$$\left\|u\right\|_{W^{-1,p}(\mathbb{R}^n;X)} = \left\|\mathcal{F}^{-1}\big(\langle\xi\rangle^{-1}\mathcal{F}u\big)\right\|_{L^p(\mathbb{R}^n;X)},$$

where $\langle\xi\rangle = (1+|\xi|^2)^{1/2}$, and $\mathcal{F}$ denotes the Fourier transform. Since

$$\mathcal{F}^{-1}\big(\langle\xi\rangle^{-1}\mathcal{F}(\alpha\cdot u)\big)(x) = \int_{\mathbb{R}^n} e^{ix\cdot\eta}\,\mathcal{F}\alpha(\eta)\,\langle\eta\rangle^N T_{m_\eta}\big(\mathcal{F}^{-1}(\langle\xi\rangle^{-1}\mathcal{F}u)\big)\,\mathrm{d}\eta,$$

where

$$T_{m_\eta} f = \mathcal{F}^{-1}(m_\eta(\xi)\,\mathcal{F}f(\xi)) \qquad\qquad m_\eta(\xi) = \langle\xi\rangle\,\langle\xi+\eta\rangle^{-1}\langle\eta\rangle^{-N},$$

we obtain

$$\left\|\alpha\cdot u\right\|_{W^{-1,p}(\mathbb{R}^n;X)} \leq \left\|\mathcal{F}\alpha(\eta)\,\langle\eta\rangle^N\right\|_{L^1(\mathbb{R}^n;\mathbb{R})} \cdot \sup_{\eta\in\mathbb{R}^n} \left\|T_{m_\eta}\big(\mathcal{F}^{-1}(\langle\xi\rangle^{-1}\mathcal{F}u(\xi))\big)\right\|_{L^p(\mathbb{R}^n;X)}.$$

Observe $\langle\xi+\eta\rangle\cdot\langle\eta\rangle \geq c\cdot\langle\xi\rangle$, for a constant $c > 0$, hence

$$\left|\partial_\xi^\beta m_\eta(\xi)\right| \leq A\cdot\langle\xi\rangle^{-|\beta|},$$

for all multi-indices β. Note that the constant A does not depend on η, if $N = N(\beta)$ is chosen sufficiently large. Setting $N = n+2$ will be good enough for our purposes. Thus we know by [**McC84**, Theorem 1.1] that

$$\big\|T_{m_\eta} : L^p(\mathbb{R}^n; X) \to L^p(\mathbb{R}^n; X)\big\| \le C,$$

where C does not depend on η. Hence

$$\|\alpha \cdot u\|_{W^{-1,p}(\mathbb{R}^n;X)} \le C \cdot \big\|\mathcal{F}\alpha(\eta)\,\langle\eta\rangle^{n+2}\big\|_{L^1(\mathbb{R}^n;\mathbb{R})} \cdot \big\|\mathcal{F}^{-1}(\langle\xi\rangle^{-1}\mathcal{F}u(\xi))\big\|_{L^p(\mathbb{R}^n;X)},$$

with α being in $\mathcal{S}(\mathbb{R}^n;\mathbb{C})$, and we proved the assertion. □

THEOREM 4.6. *Let X be a* UMD *space, $n \ge 1$, $1 \le i \le n$ and $1 < p < \infty$. Then there exist operators T_1 mapping $L^p(\mathbb{R}^n; X)$ compactly into itself, and T_2 mapping $L^p(\mathbb{R}^n; X)$ boundedly into itself, such that*

$$\|u\|_{W^{-1,p}(\mathbb{R}^n;X)} \le \|T_1 u\|_{L^p(\mathbb{R}^n;X)} + \big\|T_2\big(\mathcal{F}^{-1}\langle\xi\rangle^{-1}\xi^\alpha\mathcal{F}u\big)\big\|_{L^p(\mathbb{R}^n;X)} \tag{4.11}$$

for all multi-indices $|\alpha| = 1$ and $u \in W^{-1,p}(\mathbb{R}^n; X)$.

PROOF. As usual, we shall abbreviate $L^p(\mathbb{R}^n; X)$ by L^p_X and $W^{-1,p}(\mathbb{R}^n; X)$ by $W^{-1,p}_X$. We may assume that $\alpha = (1, 0, \dots, 0)$ throughout the proof. Choose a $\psi \in C_0^\infty(-1, 1)$ such that $0 \le \psi(t) \le 1$, for all t, and $\psi(t) = 1$, if $|t| \le 1/2$. Using the definition of the $W^{-1,p}_X$ for UMD spaces and splitting the Fourier spectrum according to ψ and $1 - \psi$ yields

$$\|u\|_{W^{-1,p}_X} \le \big\|\mathcal{F}^{-1}\big(\psi(\xi_1)\langle\xi\rangle^{-1}\mathcal{F}u\big)\big\|_{L^p_X} + \big\|\mathcal{F}^{-1}\big(\xi_1^{-1}(1-\psi(\xi_1))\,\langle\xi\rangle^{-1}\xi_1\mathcal{F}u\big)\big\|_{L^p_X}.$$

Let us define

$$\begin{aligned} T_1 f &= \mathcal{F}^{-1}\big(m_1\,\mathcal{F}f\big), & m_1(\xi) &= \psi(\xi_1)\,\langle\xi\rangle^{-1}, & \xi &\in \mathbb{R}^n,\\ \widetilde{T}_2 f &= \mathcal{F}^{-1}\big(m_2(t)\,\mathcal{F}f\big), & m_2(t) &= t^{-1}(1-\psi(t)), & t &\in \mathbb{R}, \end{aligned}$$

If we can establish that T_1 is compact, and $\widetilde{T}_2$ is bounded, we proved (4.10).

Observe that since

$$\begin{aligned} |\partial_\xi^\alpha m_1(\xi)| &\le C\,\langle\xi\rangle^{-|\alpha|-1}\\ |\partial_t^\alpha m_2(t)| &\le C\,\langle t\rangle^{-|\alpha|-1} \end{aligned}$$

we can make use of Theorem 4.4 on page 60, thus the associated kernels K_1 and K_2 are in $L^1(\mathbb{R}^n, \mathbb{C})$ and $L^1(\mathbb{R}^1, \mathbb{C})$, respectively. A glance at Theorem 4.3 yields that T_1 maps $L^p(\mathbb{R}^n; X)$ compactly into itself, and $\widetilde{T}_2$ maps $L^p(\mathbb{R}^1; X)$ compactly into itself. A fortiori, the operator T_2 given by

$$(T_2 f)(x_1, x_2 \dots, x_n) = \big(\widetilde{T}_2(t \mapsto f(t, x_2, \dots, x_n))\big)(x_1)$$

maps $L^p(\mathbb{R}^n; X)$ boundedly into itself. □

THEOREM 4.7. *Let X be a* UMD *space, $n \ge 1$ and $1 < p < \infty$. If $\alpha \in \mathcal{S}(\mathbb{R}^n)$ and*

$$u_r \longrightarrow u, \qquad \textit{weakly in } L^p(\mathbb{R}^n; X), \tag{4.12}$$

$$\partial_i u_r \longrightarrow \partial_i u, \qquad \textit{strongly in } W^{-1,p}(\mathbb{R}^n; X), \tag{4.13}$$

then

$$\partial_i\big(\alpha\,u_r\big) \longrightarrow \partial_i\big(\alpha\,u\big), \qquad \textit{strongly in } W^{-1,p}(\mathbb{R}^n; X). \tag{4.14}$$

PROOF. First, define $v_r = u_r - u$, then

$$\big\|\partial_i\big(\alpha\, v_r\big)\big\|_{W_X^{-1,p}} \leq \big\|\partial_i(\alpha)\, v_r\big\|_{W_X^{-1,p}} + \big\|\alpha\, \partial_i(v_r)\big\|_{W_X^{-1,p}}$$

Due to Theorem 4.5, we may dominate the right hand-side by a constant multiple of

$$\big\|v_r\big\|_{W_X^{-1,p}} + \big\|\partial_i v_r\big\|_{W_X^{-1,p}}.$$

Now we apply Theorem 4.6 on the facing page to the first term and obtain

$$\big\|v_r\big\|_{W_X^{-1,p}} \leq \big\|T_1 v_r\big\|_{L_X^p} + \big\|T_2\big(\mathcal{F}^{-1}(\langle\xi\rangle^{-1}\xi_i\, \mathcal{F} v_r)\big)\big\|_{L_X^p},$$

where $T_1 : L_X^p \longrightarrow L_X^p$ is compact and $T_2 : L_X^p \longrightarrow L_X^p$ is bounded. To summarize, we have

$$\big\|\partial_i\big(\alpha\, v_r\big)\big\|_{W_X^{-1,p}} \leq \big\|T_1 v_r\big\|_{L_X^p} + \Big(\big\|T_2 : L_X^p \to L_X^p\big\| + 1\Big) \cdot \big\|\partial_i v_r\big\|_{W_X^{-1,p}}$$

The first term converges to zero as $r \to \infty$, since T_1 is compact, and $v_r \to 0$ weakly in L_X^p. A glance at (4.13) shows that the latter term vanishes as $r \to \infty$, as well. □

CHAPTER 3

Shift Operators and the One–Third–Trick

We will present a new proof for the estimates on the shift operators T_m and U_m, first established by T. Figiel. For a dyadic interval I let $\tau_m(I) = I + m\,|I|$, and define the operators T_m and U_m as the linear extension of

$$\begin{aligned} T_m h_I &= h_{\tau_m(I)}, \\ U_m h_I &= \mathbb{1}_{\tau_m(I)} - \mathbb{1}_I, \end{aligned}$$

where h_I denotes the standard mean zero Haar function supported on I, and $\mathbb{1}_I$ the characteristic function of I. The result of T. Figiel in **[Fig88]** was

$$\begin{aligned} \|T_m \,:\, L^p_X \to L^p_X\| &\leq C\,\big(\log_2(2 + |m|)\big)^\alpha, \\ \|U_m \,:\, L^p_X \to L^p_X\| &\leq C\,\big(\log_2(2 + |m|)\big)^\beta, \end{aligned}$$

where the constant $C > 0$ depends only on p, X and $\alpha, \beta < 1$. The Banach space X has to be a UMD-space. The proof of T. Figiel involves hard combinatorics and has many cases to be considered, especially for the structurally more complicated operator U_m.

In Section 1 we will use the well-known one-third-trick (see **[Wol82]** and **[CWW85]**), to define the bilateral alternating one-third-trick operator S and its unilateral variants S_0 and S_1, each mapping L^p_X isomorphic into itself.

In Section 2 we will use the operator S to avoid the hard combinatorics of T. Figiel and reduce the estimates for T_m to the simplest case. The key feature of the isomorphism S will be that it commutes with T_m, that is the identity

$$(S \circ T_m)(u) = (T_m \circ S)(u),$$

for all $u \in L^p_X$.

In Section 3 we will decompose the more complex operator U_m into the five parts

$$U_m = U_m \circ Q^{(0)} + \sum_{\varepsilon \in \{0,1\}} \big(A^{(\varepsilon)}_m + B^{(\varepsilon)}_m\big) \circ Q^{(1,0)},$$

each behaving like the simpler operator T_m, for which we can assume the simplest case. So the one-third-trick operators S, S_0, S_1 in conjunction with this decomposition of U_m allow us not only to treat the operators T_m and U_m equally, but also to consider solely the simplest case for both operators T_m and U_m.

1. The One–Third–Trick

In this section we will first introduce the bilateral alternating one-third-shift operator S given by $S(h_I) = h_{\sigma(I)}$, see (1.4). In Theorem 1.2 we establish that $S : L^p_X \longrightarrow L^p_X$ is an isomorphism by means of Bourgain's version of Stein's martingale inequality. Finally, we will consider the unilateral variants S_0 and S_1 of the one-third-shift operator, and establish in Theorem 1.3 that both are isomorphic maps from L^p_X to itself, as well.

The one-third-trick may be found in [**Wol82**] and [**CWW85**].

1.1. Bilateral Alternating One-Third-Shift.

For every $j \in \mathbb{Z}$ let

$$s_j = (-1)^j \, 2^{-j}/3, \tag{1.1}$$

and define

$$s(I) = s_j, \tag{1.2}$$

for all intervals I having measure $|I| = 2^{-j}$. Then define the one–third–shift map

$$\sigma(I) = I + s(I), \tag{1.3}$$

and the one–third–shift operator

$$S(h_I) = h_{\sigma(I)}, \tag{1.4}$$

where by $h_{\sigma(I)}$ we denote the function $h_{\sigma(I)}(x) = h_I(x - s(I))$. The one–third–shift of dyadic intervals for two consecutive levels is illustrated in Figure 1.

From this picture one can see that the collection of one–third–shifted dyadic intervals $\sigma(\mathscr{D})$ is nested, and $\mathscr{D} \cap \sigma(\mathscr{D}) = \emptyset$. Note that if a one–third–shifted dyadic interval $J \in \sigma(\mathscr{D})$ is contained in a non–shifted interval $I \in \mathscr{D}$, then $\operatorname{dist}(J, I^c) \geq |J|/3$. For every given an interval $I \in \mathscr{D}$ exists a unique one–third–shifted interval $J \in \sigma(\mathscr{D})$, $|J| = |I|/2$ being contained in I. First observe that for every $j \in \mathbb{Z}$ and $I \in \mathscr{D}_j$ we have

$$\begin{aligned} \#\big\{J \in \sigma(\mathscr{D}_{j+1}) \,:\, J \cap I \neq \emptyset\big\} &= 3, \\ \#\big\{J \in \sigma(\mathscr{D}_{j+1}) \,:\, J \subset I\big\} &= 1. \end{aligned}$$

So we can define $\omega(I)$ by

$$\omega(I) = J, \quad \text{where } J \in \sigma(\mathscr{D}),\ |J| = |I|/2 \text{ and } J \subset I, \tag{1.5}$$

see Figure 2 on the facing page.

Note the basic properties summarized in

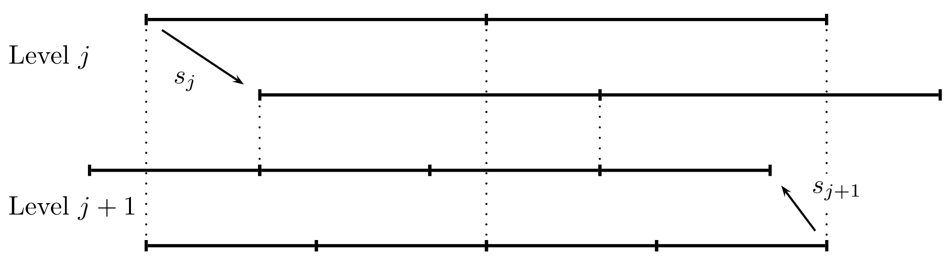

FIGURE 1. One–third–shift of two consecutive levels of intervals. In this illustration j is even.

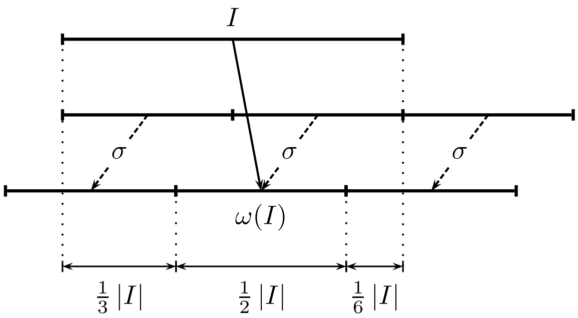

FIGURE 2. The interval I has measure $|I| = 2^{-j}$ with j being even.

LEMMA 1.1. *The following statements are true.*

(i) $\sigma(\mathscr{D})$ *is a nested collection of dyadic intervals, and* $\mathscr{D} \cap \sigma(\mathscr{D}) = \emptyset$.
(ii) $\omega : \mathscr{D} \longrightarrow \sigma(\mathscr{D})$ *is well defined and injective.*
(iii) *Let* $I \in \mathscr{D}$, *then* $\omega(I) \subset I$.
(iv) *For every* $I \in \mathscr{D}$ *we have* $\operatorname{dist}(\omega(I), I^c) = |I|/6$.
(v) *Let* $I, J \in \mathscr{D}$, $|I| = |J|$, *then* $\operatorname{dist}(\omega(I), \omega(J)) < |\omega(I)|$ *if and only if* $I = J$.
(vi) *For all* $I \in \mathscr{D}$ *we have the identity* $\sigma(I) = \omega(I) \cup \big(\omega(I) + \operatorname{sign}(s(I)) \cdot |\omega(I)|\big)$.

PROOF. The assertions are easily verified. □

We need to build up some more notation. For all $j \in \mathbb{Z}$ and

$$u = \sum_{I \in \mathscr{D}} u_I \, h_I \, |I|^{-1}$$

let $(u)_j$ restrict the function u to level j, precisely

$$(u)_j = \sum_{I \in \mathscr{D}_j} u_I \, h_I \, |I|^{-1}. \tag{1.6}$$

Eventually, we define

$$\mathbb{I}(u)_j = \sum_{I \in \mathscr{D}_j} u_I \, \mathtt{1}_I \, |I|^{-1}, \tag{1.7}$$

and find due to Kahane's contraction principle (0.4) that

$$\int_0^1 \Big\| \sum_{j \in \mathbb{Z}} r_j(t) \, (u)_j \Big\|_{L^p_X} \mathrm{d}t = \int_0^1 \Big\| \sum_{j \in \mathbb{Z}} r_j(t) \; \mathbb{I}(u)_j \Big\|_{L^p_X} \mathrm{d}t. \tag{1.8}$$

The following theorem establishes that the one–third–shift operator $S : L^p_X \longrightarrow L^p_X$ is an isomorphism.

THEOREM 1.2. *Let* $1 < p < \infty$ *and* X *a Banach space with the* UMD*–property, then there exists a constant* $C > 0$ *such that*

$$\frac{1}{C} \, \|u\|_{L^p_X} \le \|Su\|_{L^p_X} \le C \, \|u\|_{L^p_X},$$

for all $u \in L^p_X$.

PROOF. Let $u = \sum_{I \in \mathscr{D}} u_I \, h_I \, |I|^{-1} \in L^p_X$ be fixed throughout this proof and set

$$v = \sum_{I \in \mathscr{D}} u_I \, h_{\omega(I)} \, |\omega(I)|^{-1}.$$

Note that $\{\omega(I) : I \in \mathscr{D}\}$ is nested, see Lemma 1.1, assertion (i) and (ii). Observe we have due to Lemma 1.1, assertion (iii) that $\mathbb{I}(u)_j = \mathbb{E}(\mathbb{I}(v)_j|\mathscr{D}_j)$, so the UMD–property and Kahane's contraction principle (0.4) yield

$$\begin{aligned}\|u\|_{L^p_X} &\lesssim \int_0^1 \|\sum_{j\in\mathbb{Z}} r_j(t)\,(u)_j\|_{L^p_X}\,\mathrm{d}t\\ &= \int_0^1 \|\sum_{j\in\mathbb{Z}} r_j(t)\;\mathbb{I}(u)_j\|_{L^p_X}\,\mathrm{d}t\\ &= \int_0^1 \|\sum_{j\in\mathbb{Z}} r_j(t)\;\mathbb{E}(\mathbb{I}(v)_j|\mathscr{D}_j)\|_{L^p_X}\,\mathrm{d}t.\end{aligned}$$

Now we apply Stein's martingale inequality (0.5) followed by identity (1.8) to pass from $\mathbb{I}(v)_j$ to $(v)_j$, so

$$\begin{aligned}\int_0^1 \|\sum_{j\in\mathbb{Z}} r_j(t)\;\mathbb{E}(\mathbb{I}(v)_j|\mathscr{D}_j)\|_{L^p_X}\,\mathrm{d}t &\lesssim \int_0^1 \|\sum_{j\in\mathbb{Z}} r_j(t)\;\mathbb{I}(v)_j\|_{L^p_X}\,\mathrm{d}t\\ &= \int_0^1 \|\sum_{j\in\mathbb{Z}} r_j(t)\,(v)_j\|_{L^p_X}\,\mathrm{d}t.\end{aligned}$$

Recalling definition (1.4) and applying Kahane's contraction principle in consideration of $\omega(I) \subset \sigma(I)$ (see identity (vi) in Lemma 1.1), we estimate

$$\int_0^1 \|\sum_{j\in\mathbb{Z}} r_j(t)\,(v)_j\|_{L^p_X}\,\mathrm{d}t \le 2\cdot \int_0^1 \|\sum_{j\in\mathbb{Z}} r_j(t)\,(Su)_j\|_{L^p_X}\,\mathrm{d}t,$$

so the UMD–property implies

$$\int_0^1 \|\sum_{j\in\mathbb{Z}} r_j(t)\,(v)_j\|_{L^p_X}\,\mathrm{d}t \lesssim \|Su\|_{L^p_X}.$$

Thus, collecting the inequalities yields

$$\|u\|_{L^p_X} \lesssim \|Su\|_{L^p_X}.$$

One can repeat the preceding argument with the roles of u and Su interchanged and obtain the converse inequality

$$\|Su\|_{L^p_X} \lesssim \|u\|_{L^p_X}.$$

□

1.2. Unilateral One-Third-Shift.

Now we want analyze modified versions σ_0 and σ_1 of the one-third-shift map σ. To this end we define $\sigma_0, \sigma_1 : \mathscr{D} \longrightarrow \sigma(\mathscr{D})$,

$$\sigma_0(I) = J, \quad \text{where } J \in \sigma(\mathscr{D}),\ |J| = |I| \text{ and } \sup J \in I, \tag{1.9}$$

$$\sigma_1(I) = J, \quad \text{where } J \in \sigma(\mathscr{D}),\ |J| = |I| \text{ and } \inf J \in I, \tag{1.10}$$

see Figure 3 on the next page. This induces the one-third-shift operators S_0 and S_1 given by the linear extension of

$$S_0(h_I) = h_{\sigma_0(I)}, \qquad I \in \mathscr{D}, \tag{1.11}$$

$$S_1(h_I) = h_{\sigma_1(I)}, \qquad I \in \mathscr{D}. \tag{1.12}$$

Observe that we have either

$$\sigma(I) = \sigma_0(I) \qquad \text{or} \qquad \sigma(I) = \sigma_1(I),$$

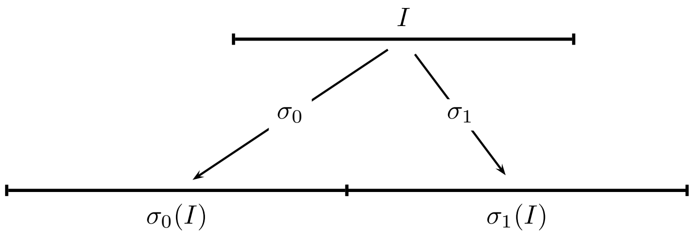

FIGURE 3. Unilateral One-Third-Shifts σ_0 and σ_1 applied to $I \in \mathscr{D}$. In this picture the one-third-shift map σ shifts to the right, so $\sigma_1(I) = \sigma(I)$.

depending on the direction in which σ one-third-shifts the interval I. Anyhow we can see that

$$|I \cap \sigma_0(I)| \geq \frac{1}{3}|I|, \qquad\qquad |I \cap \sigma_1(I)| \geq \frac{1}{3}|I|,$$

for all $I \in \mathscr{D}$. This is what enables us to apply the proof of Theorem 1.2 on page 67 with some small tweaks to obtain Theorem 1.3 below.

THEOREM 1.3. *Let $1 < p < \infty$ and X a Banach space with the* UMD*–property, then there exists a constant $C > 0$ such that*

$$\frac{1}{C}\left\|u\right\|_{L^p_X} \leq \left\|S_0 u\right\|_{L^p_X} \leq C\left\|u\right\|_{L^p_X},$$
$$\frac{1}{C}\left\|u\right\|_{L^p_X} \leq \left\|S_1 u\right\|_{L^p_X} \leq C\left\|u\right\|_{L^p_X},$$

for all $u \in L^p_X$.

PROOF. Define ω_0 and ω_1 by

$$\omega_0(I) = J, \quad \text{where } J \in \sigma(\mathscr{D}),\ |J| = |I|/4 \text{ and } \sup J = \sup \sigma_0(I),$$
$$\omega_1(I) = J, \quad \text{where } J \in \sigma(\mathscr{D}),\ |J| = |I|/4 \text{ and } \inf J = \inf \sigma_1(I),$$

for all $I \in \mathscr{D}$. Now all we need to do is repeat the proof of Theorem 1.3 with ω replaced by ω_δ to estimate S_δ, for each $\delta \in \{0, 1\}$. $\square$

2. The Shift Operator T_m

Here we will define and analyze the shift map τ_m and the shift operator T_m. We will give an alternative proof for the estimate

$$\|T_m : L^p_X \to L^p_X\| \leq C\left(\log_2(2+|m|)\right)^{\alpha},$$

first established by T. Figiel in [**Fig88**]. The proof of T. Figiel involves hard combinatorics and considering a variety of different cases. We will use use the one–third–shift operator S introduced in Section 1 to circumvent the hard combinatorics of T. Figiel and thereby reduce the estimates for T_m to the simplest case.

For $m \in \mathbb{Z}$ define the shift map τ_m by

$$\tau_m(I) = I + m\,|I|, \tag{2.1}$$

for all $I \in \mathscr{D} \cup \sigma(\mathscr{D})$. This induces the shift operator T_m, given by

$$T_m h_I = h_{\tau_m(I)}, \tag{2.2}$$

for all $I \in \mathscr{D} \cup \sigma(\mathscr{D})$. It is crucial that the one–third–shift operator S defined in (1.4) and the shift operator T_m commute, that is the identity

$$(S \circ T_m)(u) = (T_m \circ S)(u), \tag{2.3}$$

for all $u \in L^p_X$. Analogously, we have that

$$(S_0 \circ T_m)(u) = (T_m \circ S_0)(u), \tag{2.4}$$

$$(S_1 \circ T_m)(u) = (T_m \circ S_1)(u), \tag{2.5}$$

for all $u \in L^p_X$, see (1.9), (1.10), (1.11) and (1.12).

We aim at splitting the dyadic intervals $\mathscr{D}$ into collections $\mathscr{B}_i^{(\delta)}$, such that we may bound $T_m \circ S^\delta$ on functions supported on $\sigma^\delta\big(\mathscr{B}_i^{(\delta)}\big)$, $\delta \in \{0,1\}$. Note that if $\delta = 0$, then $S^\delta = \mathrm{Id}$ and $\sigma^\delta = \mathrm{Id}$.

For a given a shift width $m \in \mathbb{Z}$, $m \neq 0$, the following lemma splits the dyadic intervals $\mathscr{D}$ into $16 + 4 \cdot \log_2(|m|)$ disjoint collections $\mathscr{B}_i^{(\delta)}$. The collections are constructed such that for all $\delta \in \{0,1\}$ and $I \in \mathscr{B}_i^{(\delta)}$ the intervals $\sigma^\delta(I)$ and $\big(\tau_m \circ \sigma^\delta\big)(I)$ have the same dyadic predecessor with respect to $\sigma^\delta\big(\mathscr{B}_i^{(\delta)}\big)$.

LEMMA 2.1. *For every integer $m \in \mathbb{Z}$, $m \neq 0$ let τ_m denote the map given by*

$$\tau_m(I) = I + m\,|I|,$$

for all $I \in \mathscr{D} \cup \sigma(\mathscr{D})$, see (2.1).

Then there exist a constant $K(m) \leq 7 + 2 \cdot \log_2(|m|)$ and disjoint collections of dyadic intervals $\mathscr{B}_i^{(\delta)}$, $0 \leq i \leq K(m)$, $\delta \in \{0,1\}$ with

$$\mathscr{D} = \bigcup_{\delta \in \{0,1\}} \bigcup_{i=0}^{K(m)} \mathscr{B}_i^{(\delta)},$$

such that

$$\big\{I,\, \tau_m(I),\, I \cup \tau_m(I) \,:\, I \in \sigma^\delta(\mathscr{B}_i^{(\delta)})\big\} \tag{2.6}$$

is a nested collection of sets, for all $0 \leq i \leq K(m)$ and $\delta \in \{0,1\}$.

PROOF. Due to symmetry we may assume that $m \geq 1$, and we set $K(m) = K(-m)$, if $m \leq -1$. So fix a shift width $m \geq 2$ and a $\lambda \geq 4$ such that

$$2^{\lambda-3} \leq m < 2^{\lambda-2}, \tag{2.7}$$

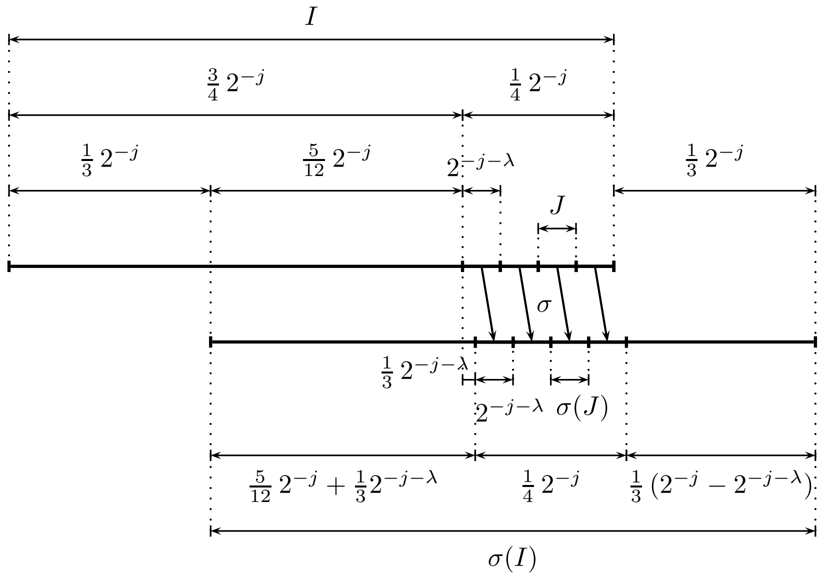

FIGURE 4. The one–third–shift map σ acting on $I \in \mathscr{D}$, $|I| = 2^{-j}$ and $J \in \mathscr{D}$, $|J| = 2^{-j-\lambda}$, where $J \subset I$ and $\tau_m(J) \cap I = \emptyset$. In this picture λ is even.

and set $L(m) = \lambda - 1$. If $m = 1$, then let $\lambda = 4$ and set $L(1) = 3$. Now we split $\mathscr{D}$ into disjoint collections $\mathscr{A}_i$, $0 \le i \le L(m)$, by omitting $L(m)$ consecutive levels of $\mathscr{D}$. More precisely, for every $0 \le i \le L(m)$ we define

$$\mathscr{A}_i = \bigcup_{j\in\mathbb{Z}} \big\{ I \in \mathscr{D} \,:\, |I| = 2^{-(\lambda\cdot j + i)} \big\}. \tag{2.8}$$

Next we want to divide each of the $\mathscr{A}_i$ into two collections $\mathscr{A}_i^{(0)}$ and $\mathscr{A}_i^{(1)}$, such that every $I \in \mathscr{A}_i^{(0)}$ has the same predecessor in $\mathscr{A}_i^{(0)}$ as $\tau_m(I)$, and $\mathscr{A}_i^{(0)}$ is maximal. As a consequence, the collection $\mathscr{A}_i^{(1)}$ consists all intervals I such that I and $\tau_m(I)$ do not share the same predecessor. But, if we apply the one–third–shift map σ to the collection $\mathscr{A}_i^{(1)}$, then every $I \in \sigma\big(\mathscr{A}_i^{(1)}\big)$ has the same predecessor in $\sigma\big(\mathscr{A}_i^{(1)}\big)$ as $\tau_m(I)$. We will now construct these two collections. To this end let $\mathscr{G}$ denote one of the collections $\mathscr{A}_i$, $\sigma\big(\mathscr{A}_i\big)$, $0 \le i \le L(m)$ and define

$$\begin{aligned} \mathscr{C}_0(\mathscr{G}, I) &= \big\{ J \in \mathscr{G} \,:\, |J| = 2^{-\lambda}\,|I|,\ J \subset I \text{ and } \tau_m(J) \subset I \big\}, \\ \mathscr{C}_1(\mathscr{G}, I) &= \big\{ J \in \mathscr{G} \,:\, |J| = 2^{-\lambda}\,|I|,\ J \subset I \text{ and } \tau_m(J) \cap I = \emptyset \big\}. \end{aligned} \tag{2.9}$$

Revisiting the definition of the one–third–shift map (1.3) and considering the restriction (2.7) one can see that

$$\sigma\Big(\mathscr{C}_1(\mathscr{A}_i, I)\Big) \subset \mathscr{C}_0\Big(\sigma(\mathscr{A}_i), \sigma(I)\Big), \tag{2.10}$$

for all $I \in \mathscr{A}_i$, $0 \le i \le L(m)$. This means that all intervals $J \in \sigma\big(\mathscr{C}_1(\mathscr{A}_i, I)\big)$ are such that J and $\tau_m(J)$ share $\sigma(I)$ as common predecessor with respect to the collection $\sigma\big(\mathscr{A}_i^{(1)}\big)$. In Figure 4 one can see the action of the one–third–shift map σ on the collection $\mathscr{A}_i$. Now define for every $0 \le i \le L(m)$ the following collections of dyadic intervals

$$\begin{aligned} \mathscr{A}_i^{(0)} &= \bigcup \big\{ \mathscr{C}_0(\mathscr{A}_i, I) \,:\, I \in \mathscr{A}_i \big\}, \\ \mathscr{A}_i^{(1)} &= \mathscr{A}_i \setminus \mathscr{A}_i^{(0)}. \end{aligned} \tag{2.11}$$

Finally, for all $0 \leq i \leq L(m)$ and $\delta \in \{0,1\}$ we split $\mathscr{A}_i^{(\delta)}$ into two disjoint collections

$$\mathscr{B}_i^{(\delta)} \quad \text{and} \quad \mathscr{B}_{i+L(m)+1}^{(\delta)}, \tag{2.12}$$

such that

$$\mathscr{B}_i^{(\delta)} \cap \tau_m\big(\mathscr{B}_i^{(\delta)}\big) = \emptyset, \tag{2.13}$$

for all $0 \leq i \leq K(m)$ and $\delta \in \{0,1\}$, where we set $K(m) = 2 \cdot L(m) + 1$. Considering (2.7) and $L(m) = \lambda - 1$ we find that $K(m) \leq 7 + 2 \cdot \log_2(m)$. For this purpose consider the collection

$$\mathscr{E} = \big\{ \tau_k(I) \,:\, I \in \mathscr{D},\, \inf I = 0,\, 0 \leq k \leq m-1 \big\},$$

and observe that

$$\mathscr{D} = \bigcup_{\substack{j \in \mathbb{Z} \\ j \text{ even}}} \tau_{j\cdot m}\big(\mathscr{E}\big) \cup \bigcup_{\substack{j \in \mathbb{Z} \\ j \text{ odd}}} \tau_{j\cdot m}\big(\mathscr{E}\big) = \mathscr{D}_{\text{even}} \cup \mathscr{D}_{\text{odd}}.$$

Now define the collections

$$\begin{aligned} \mathscr{B}_i^{(\delta)} &= \mathscr{A}_i^{(\delta)} \cap \mathscr{D}_{\text{even}}, \\ \mathscr{B}_{i+L(m)+1}^{(\delta)} &= \mathscr{A}_i^{(\delta)} \cap \mathscr{D}_{\text{odd}}, \end{aligned} \tag{2.14}$$

for all $0 \leq i \leq L(m)$ and $\delta \in \{0,1\}$.

With regard to (2.10), (2.9) and noting that $\tau_m(I) \in \mathscr{D}_{\text{odd}}$ if and only if $I \in \mathscr{D}_{\text{even}}$, we verified (2.6), finishing this proof. □

Remark 2.2. Note that we actually proved the slightly stronger result

$$I \cup \tau_m(I) \subset \pi^\lambda(I), \tag{2.15}$$

for all $I \in \sigma^\delta\big(\mathscr{B}_i^{(\delta)}\big)$, $0 \leq i \leq K(m)$, $\delta \in \{0,1\}$. Conceive the predecessor map π with respect to $\sigma^\delta\big(\mathscr{D}\big)$. To be more precise let $I \in \sigma^\delta(\mathscr{D})$. Then $\pi(I)$ is the unique interval $J \in \sigma^\delta(\mathscr{D})$ such that $J \supset I$, and $\pi^\lambda = \pi \circ \cdots \circ \pi$.

Given $1 < p < \infty$, a Banach space X with the UMD-property and $m \in \mathbb{Z}$, we define the projections

$$P_i^{(\delta)} u = \sum_{I \in \mathscr{B}_i^{(\delta)}} \langle u, h_I \rangle \, h_I \, |I|^{-1}, \tag{2.16}$$

for all $0 \leq i \leq K(m)$, $\delta \in \{0,1\}$ and $u \in L_X^p$. Note the identity

$$u = \sum_{\delta \in \{0,1\}} \sum_{i=0}^{K(m)} P_i^{(\delta)} u \tag{2.17}$$

holds true for all $u \in L_X^p$, since the collections $\mathscr{B}_i^{(\delta)}$, $0 \leq i \leq K(m)$, $\delta \in \{0,1\}$ form a partition of $\mathscr{D}$, see Lemma 2.1.

Exploiting that the one–third–shift operator S is an isomorphism on L_X^p (see Theorem 1.2), we will now estimate the shift operator T_m on the range of each $P_i^{(\delta)}$ in the subsequent theorem.

THEOREM 2.3. *Let $1 < p < \infty$ and X be a Banach space with the* UMD–*property. Then for every $m \in \mathbb{Z}$, $0 \le i \le K(m)$ and $\delta \in \{0,1\}$ the inequality*

$$\big\|T_m \circ P_i^{(\delta)} u\big\|_{L_X^p} \le C \cdot \big\|P_i^{(\delta)} u\big\|_{L_X^p}, \tag{2.18}$$

holds true for all $u \in L_X^p$, where the constant C depends only on p and X. The projections $P_i^{(\delta)}$, $0 \le i \le K(m)$, $\delta \in \{0,1\}$ are defined according to (2.16)*, and* $K(m) \le 7 + 2 \cdot \log_2(1 + |m|)$.

PROOF. Note that due to symmetry once we established (2.18) for $m \ge 1$, the theorem is proved.

Recalling the properties of the partition $\mathscr{B}_i^{(\delta)}$, $0 \le i \le K(m)$, $\delta \in \{0,1\}$ of $\mathscr{D}$, see Lemma 2.1 on page 70, and we know that the collection

$$\big\{I,\, \tau_m(I),\, I \cup \tau_m(I) \,:\, I \in \sigma^\delta(\mathscr{B}_i^{(\delta)})\big\} \tag{2.19}$$

is nested, for all $0 \le i \le K(m)$ and $\delta \in \{0,1\}$. Throughout this proof let $m \in \mathbb{Z}$, $0 \le i \le K(m)$, $\delta \in \{0,1\}$ and $u \in P_i^{(\delta)}(L_X^p)$ be fixed. According to (2.16) we may assume that u has the representation

$$u = \sum_{I \in \mathscr{B}_i^{(\delta)}} u_I\, h_I\, |I|^{-1}.$$

For every $J \in \sigma^\delta(\mathscr{D})$ let

$$A^{(\delta)}(J) = J \cup \tau_m(J), \tag{2.20}$$

and for all $j \in \mathbb{Z}$ define the collection

$$\mathscr{A}_j^{(\delta)} = \big\{A^{(\delta)}(J) \,:\, J \in \sigma^\delta(\mathscr{D}_j)\big\}. \tag{2.21}$$

Then specify the filtration $\{\mathscr{F}_j^{(\delta)}\}_j$ by

$$\mathscr{F}_j^{(\delta)} = \sigma\text{-algebra}\Big(\bigcup_{i \le j} \mathscr{A}_i^{(\delta)}\Big), \tag{2.22}$$

and observe that due to (2.19) every $A^{(\delta)}(J)$, $J \in \sigma^\delta(\mathscr{D}_j)$ is an atom for $\mathscr{F}_j^{(\delta)}$. The one–third–shift operator is given by

$$S^\delta u = \sum_{I \in \mathscr{B}_i^{(\delta)}} u_I\, h_{\sigma^\delta(I)}\, |I|^{-1} = \sum_{J \in \sigma^\delta(\mathscr{B}_i^{(\delta)})} u_{\sigma^{-\delta}(J)}\, h_J\, |J|^{-1}, \tag{2.23}$$

see (1.4) for details. We recall the notation

$$(u)_j = \sum_{|I| = 2^{-j}} u_I\, h_I\, |I|^{-1} \quad \text{and} \quad \mathbb{I}(u)_j = \sum_{|I| = 2^{-j}} u_I\, 1_I\, |I|^{-1},$$

and note that

$$\big\|T_m\, S^\delta u\big\|_{L_X^p} \approx \int_0^1 \big\|\sum_{j \in \mathbb{Z}} r_j(t)\ \mathbb{I}\left(T_m\, S^\delta u\right)_j\big\|_{L_X^p}\, \mathrm{d}t,$$

see (1.6), (1.7) and (1.8). Obviously, $\mathbb{I}\left(T_m\, S^\delta u\right)_j \le 2 \cdot \mathbb{E}\left(\mathbb{I}(S^\delta u)_j | \mathscr{F}_j^{(\delta)}\right)$, hence Kahane's contraction principle and Bourgain's version of Stein's martingale inequality yield

$$\begin{aligned}\int_0^1 \big\|\sum_{j\in\mathbb{Z}} r_j(t)\; \mathbb{I}\big(T_m\, S^\delta u\big)_j\big\|_{L^p_X}\, \mathrm{d}t &\le \int_0^1 \big\|\sum_{j\in\mathbb{Z}} r_j(t)\, 2\cdot \mathbb{E}\,\big(\mathbb{I}(S^\delta u)_j|\mathscr{F}_j^{(\delta)}\big)\big\|_{L^p_X}\, \mathrm{d}t\\ &\lesssim \int_0^1 \big\|\sum_{j\in\mathbb{Z}} r_j(t)\; \mathbb{I}(S^\delta u)_j\big\|_{L^p_X}\, \mathrm{d}t\\ &\approx \big\|S^\delta u\big\|_{L^p_X}.\end{aligned}$$

Combining the latter two estimates with Theorem 1.2 on page 67 proves

$$\big\|T_m\, S^\delta u\big\|_{L^p_X} \lesssim \big\|u\big\|_{L^p_X}. \tag{2.24}$$

According to (2.3) the shift operator T_m and the one–third–shift operator S commute, so we have the identity

$$T_m u = \big(S^{-\delta}\circ T_m \circ S^\delta\big)(u),$$

and we obtain by an application of Theorem 1.2 on page 67

$$\big\|T_m u\big\|_{L^p_X} \lesssim \big\|\big(T_m\circ S^\delta\big)(u)\big\|_{L^p_X}. \tag{2.25}$$

We conclude the proof by joining (2.25) and (2.24).

□

Remark 2.4. By slightly adjusting the construction of $\mathscr{B}_i^{(\delta)}$ we could replace Bourgain's version of Stein's martingale inequality by the martingale transforms in [**Fig88**, Proposition 2, Step 0] in order to gain (2.24). To this end we will basically have to replace λ by $\lambda+1$ and redefine $\mathscr{C}_0$ and $\mathscr{C}_1$ as follows

$$\begin{aligned}\mathscr{C}_0(I,\mathscr{A}_i) &= \big\{J\in\mathscr{A}_i\,:\,|J| = 2^{-\lambda}\,|I|,\ J\subset I_0 \text{ and } \tau_m(J)\subset I_0\big\}\\ &\qquad\cup\big\{J\in\mathscr{A}_i\,:\,|J| = 2^{-\lambda}\,|I|,\ J\subset I_1 \text{ and } \tau_m(J)\subset I_1\big\},\\ \mathscr{C}_1(I,\mathscr{A}_i) &= \big\{J\in\mathscr{A}_i\,:\,|J| = 2^{-\lambda}\,|I|,\ J\subset I_0 \text{ and } \tau_m(J)\cap I_0=\emptyset\big\}\\ &\qquad\cup\big\{J\in\mathscr{A}_i\,:\,|J| = 2^{-\lambda}\,|I|,\ J\subset I_1 \text{ and } \tau_m(J)\cap I_1=\emptyset\big\},\end{aligned}$$

confer (2.8) and (2.9). This results in the collection

$$\big\{J_0,\,\tau_m(J)_0,\,J_1,\,\tau_m(J)_1,\,J\cup\tau_m(J)\,:\,J\in\sigma^\delta(\mathscr{B}_i^{(\delta)})\big\} \tag{2.26}$$

being nested for all $0\le i\le K(m)$ and $\delta\in\{0,1\}$. With this modifications let us define

$$d_{J,1}^{(\delta)} = \frac{1}{2}\big(h_J + h_{\tau_m(J)}\big) \quad\text{and}\quad d_{J,2}^{(\delta)} = \frac{1}{2}\big(h_J - h_{\tau_m(J)}\big),$$

for all $J\in\sigma^\delta\big(\mathscr{B}_i^{(\delta)}\big)$. Since (2.26) is nested, $\big\{d_{J,1}^{(\delta)}, d_{J,2}^{(\delta)}\,:\,J\in\sigma\big(\mathscr{B}_i^{(\delta)}\big)\big\}$ forms a martingale difference sequence. Observe $h_J = d_{J,1}^{(\delta)} + d_{J,2}^{(\delta)}$ and $h_{\tau_m(J)} = d_{J,1}^{(\delta)} - d_{J,2}^{(\delta)}$, hence we may swap h_J and $h_{\tau_m(J)}$ without using Bourgain's version of Stein's martingale inequality.

3. A Martingale Decomposition for U_m

In this section we will study the shift operator U_m, and prove the estimate

$$\|U_m : L^p_X \to L^p_X\| \le C\left(\log_2(2+|m|)\right)^\beta,$$

due to T. Figiel, see **[Fig88]**. The combinatorics of T. Figiel in order to estimate U_m are even harder than for the operator T_m. This is mainly due to the observation that $\{T_m h_I\}_{I\in\mathscr{A}}$ is a martingale difference sequence for any choice of $\mathscr{A}\subset\mathscr{D}$, whereas whether $\{U_m h_I\}_{I\in\mathscr{B}}$ forms a martingale difference sequence strongly depends on the choice of $\mathscr{B}\subset\mathscr{D}$. Making use the one–third–shift operators introduced in Section 1, we will decompose the operator U_m into the five parts

$$U_m = U_m\circ Q^{(0)} + \sum_{\varepsilon\in\{0,1\}}\left(A_m^{(\varepsilon)} + B_m^{(\varepsilon)}\right)\circ Q^{(1,0)}$$

each behaving like T_m. Thus, the one–third–trick allows us to reduce the estimates for U_m to the simplest case, as well.

For every $m\in\mathbb{Z}$ we defined in the (2.1) the shift map τ_m by

$$\tau_m(I) = I + m\,|I|,$$

for all $I\in\mathscr{D}\cup\sigma(\mathscr{D})$. Now we introduce the shift operator U_m by setting

$$U_m h_I = \mathbb{1}_{\tau_m(I)} - \mathbb{1}_I, \tag{3.1}$$

for all $I\in\mathscr{D}\cup\sigma(\mathscr{D})$. Essentially the same method we used to bound T_m for functions supported on the collections $\mathscr{B}_i^{(0)}$, $0\le i\le K(m)$ qualifies for estimating U_m. This is primarily due to the fact that $\{U_m h_I : I\in\mathscr{B}_i^{(0)}\}$ forms a martingale difference sequence, which is ensured by Lemma 2.1. The main obstacle is to estimate U_m on $\mathscr{B}_i^{(1)}$, since $\{U_m h_I : I\in\mathscr{B}_i^{(1)}\}$ is *not* a martingale difference sequence. The remedy to this problem is the martingale difference sequence decomposition of U_m into

$$U_m h_I = a_I^{(\varepsilon)} + b_I^{(\varepsilon)} - b_{\tau_m(I)}^{(\varepsilon)}, \qquad I\in\mathscr{B}_i^{(1,\varepsilon)}$$

where

$$\mathscr{B}_i^{(1,0)} = \left\{I\in\mathscr{B}_i^{(1)} : \inf\tau_m(I)\ne\inf\pi^\lambda(\tau_m(I))\right\},$$
$$\mathscr{B}_i^{(1,1)} = \left\{I\in\mathscr{B}_i^{(1)} : \inf\tau_m(I)=\inf\pi^\lambda(\tau_m(I))\right\}.$$

Recall that given $\delta\in\{0,1\}$ and an interval $I\in\sigma^\delta(\mathscr{D})$, the interval $\pi(I)$ is the unique $J\in\sigma^\delta(\mathscr{D})$ such that $J\supset I$, and $\pi^\lambda=\pi\circ\cdots\circ\pi$. The collections $\{a_I^{(\varepsilon)} : I\in\mathscr{B}_i^{(1,\varepsilon)}\}$ and $\{b_I^{(\varepsilon)}, b_{\tau_m(I)}^{(\varepsilon)} : I\in\mathscr{B}_i^{(1,\varepsilon)}\}$ are martingale difference sequences, each, see Theorem 3.1. This is what enables us to treat U_m like T_m, which is elaborated in Theorem 3.3.

First, we define $\alpha_0,\alpha_1 : \mathscr{D}\longrightarrow\sigma(\mathscr{D})$,

$$\alpha_0(I) = J, \qquad \text{where } J\in\sigma(\mathscr{D}),\ |J|=|I| \text{ and } \sup J\in I, \tag{3.2}$$
$$\alpha_1(I) = J, \qquad \text{where } J\in\sigma(\mathscr{D}),\ |J|=|I| \text{ and } \inf J\in I. \tag{3.3}$$

Note that $\alpha_\delta=\sigma_\delta$, $\delta\in\{0,1\}$ where σ_δ was defined in Subsection 1.2. Second, define the maps $\beta_0,\beta_1 : \mathscr{D}\longrightarrow\beta_0(\mathscr{D})$,

$$\beta_0(I) = \alpha_0(I)\setminus I, \tag{3.4}$$
$$\beta_1(I) = \alpha_1(I)\cap I, \tag{3.5}$$

and set

$$\beta(I) = \beta_0(I)\cup\beta_1(I). \tag{3.6}$$

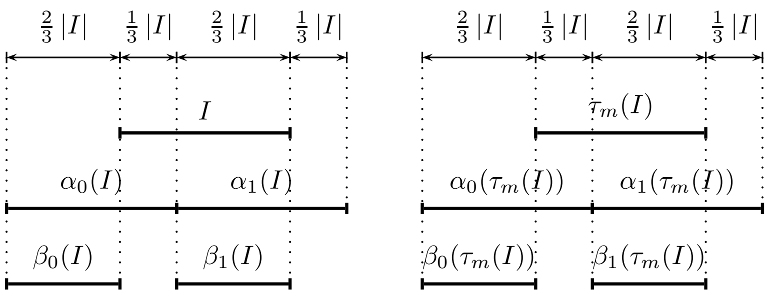

FIGURE 5. The support functions α_0, α_1, β_0, β_1 for I and $\tau_m(I)$.

Finally, let $\gamma_0, \gamma_1 : \mathscr{D} \longrightarrow \mathscr{D}$,

$$\gamma_0(I) = \tau_{-1}(I), \tag{3.7}$$

$$\gamma_1(I) = I, \tag{3.8}$$

and define

$$\gamma(I) = \gamma_0(I) \cup \gamma_1(I). \tag{3.9}$$

The functions α_0, α_1, β_0 and β_1 are visualized in Figure 5 for an arbitrary $I \in \mathscr{D}$.

With $m \in \mathbb{Z}$, $m \neq 0$ fixed, we introduce the functions

$$a_I^{(0)} = \mathbb{1}_{\alpha_0(\tau_m(I))} - \mathbb{1}_{\alpha_0(I)}, \qquad I \in \mathscr{D}, \tag{3.10}$$

$$b_I^{(0)} = \mathbb{1}_{\beta_0(I)} - \mathbb{1}_{\beta_1(I)}, \qquad I \in \mathscr{D}, \tag{3.11}$$

and

$$a_I^{(1)} = \mathbb{1}_{\alpha_1(\tau_m(I))} - \mathbb{1}_{\alpha_1(I)}, \qquad I \in \mathscr{D}, \tag{3.12}$$

$$b_I^{(1)} = \mathbb{1}_{I\setminus\beta_1(I)} - \mathbb{1}_{I\setminus\beta_0(I)}, \qquad I \in \mathscr{D}. \tag{3.13}$$

see Figures 6 and 7. We define the operators $A_m^{(\varepsilon)}$, $B^{(\varepsilon)}$ and $B_m^{(\varepsilon)}$ as the linear extension of

$$A_m^{(\varepsilon)} h_I = a_I^{(\varepsilon)}, \qquad I \in \mathscr{D}, \tag{3.14}$$

$$B^{(\varepsilon)} h_I = b_I^{(\varepsilon)}, \qquad I \in \mathscr{D}, \tag{3.15}$$

$$B_m^{(\varepsilon)} h_I = b_I^{(\varepsilon)} - b_{\tau_m(I)}^{(\varepsilon)}, \qquad I \in \mathscr{D}, \tag{3.16}$$

for $\varepsilon \in \{0, 1\}$. Note the identities

$$U_m = A_m^{(\varepsilon)} + B_m^{(\varepsilon)} = A_m^{(\varepsilon)} + B^{(\varepsilon)} - B^{(\varepsilon)} \circ T_m, \tag{3.17}$$

hold true for $\varepsilon \in \{0, 1\}$, see (3.10), (3.11), (3.12), (3.13) and Figures 6 and 7.

Now we split the collections $\mathscr{B}_i^{(1)}$ into

$$\mathscr{B}_i^{(1)} = \mathscr{B}_i^{(1,0)} \cup \mathscr{B}_i^{(1,1)}, \tag{3.18}$$

where

$$\mathscr{B}_i^{(1,0)} = \big\{ I \in \mathscr{B}_i^{(1)} \,:\, \inf \operatorname{sign}(m)\, \tau_m(I) \neq \inf \operatorname{sign}(m)\ \pi^\lambda(\tau_m(I)) \big\}, \tag{3.19}$$

$$\mathscr{B}_i^{(1,1)} = \big\{ I \in \mathscr{B}_i^{(1)} \,:\, \inf \operatorname{sign}(m)\, \tau_m(I) = \inf \operatorname{sign}(m)\ \pi^\lambda(\tau_m(I)) \big\}, \tag{3.20}$$

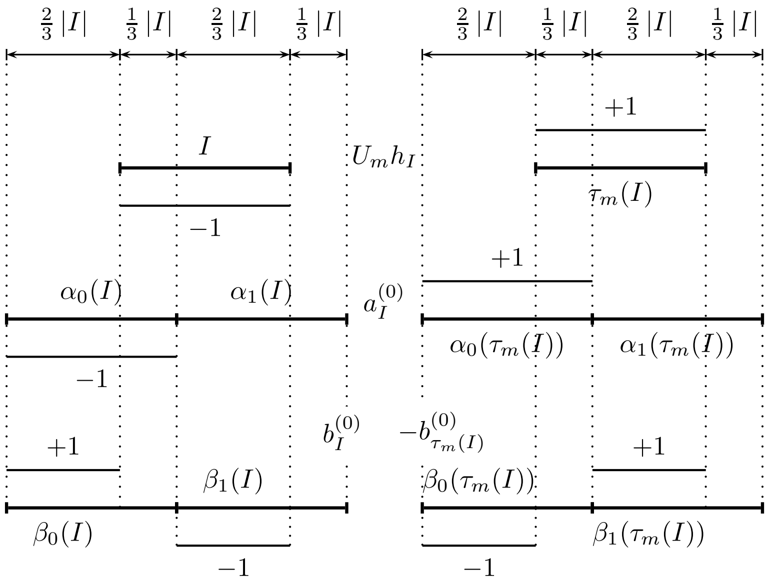

FIGURE 6. Martingale decomposition of U_m to the left.

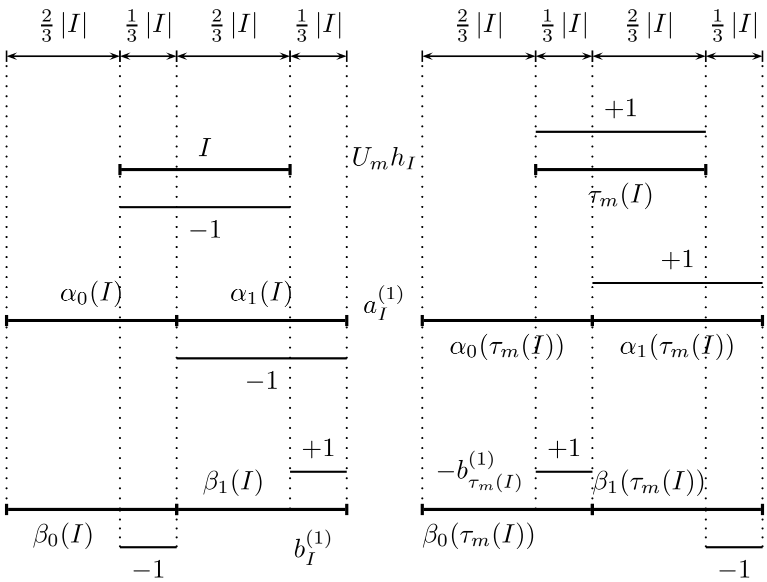

FIGURE 7. Martingale decomposition of U_m to the right.

for all $0 \leq i \leq K(m)$, $m \neq 0$. Next we define

$$\begin{aligned} Q_i^{(0)} u &= \sum_{I \in \mathscr{B}_i^{(0)}} \langle u, h_I \rangle \, h_I \, |I|^{-1}, \\ Q_i^{(1,\varepsilon)} u &= \sum_{I \in \mathscr{B}_i^{(1,\varepsilon)}} \langle u, h_I \rangle \, h_I \, |I|^{-1}, \end{aligned} \tag{3.21}$$

for all $0 \leq i \leq K(m)$ and $\varepsilon \in \{0,1\}$. The collections $\mathscr{B}_i^{(\delta)}$ are specified in Lemma 2.1, and $\mathscr{B}_i^{(1,\varepsilon)}$ is defined in (3.19) and (3.20). Now, if we set

$$\begin{aligned} Q^{(0)} &= \sum_{i=0}^{K(m)} Q_i^{(0)}, \\ Q^{(1,\varepsilon)} &= \sum_{i=0}^{K(m)} Q_i^{(0,\varepsilon)}, \end{aligned} \tag{3.22}$$

for all $\varepsilon \in \{0,1\}$, then certainly

$$u = Q^{(0)}\, u + Q^{(1,0)}\, u + Q^{(1,1)}\, u \tag{3.23}$$

for all $u \in L^p_X$.

Note that $Q_i^{(0)} = P_i^{(0)}$ and $P_i^{(1)} = Q_i^{(1,0)} + Q_i^{(1,1)}$, where $P_i^{(\delta)}$, $\delta \in \{0,1\}$ was defined in (2.16).

The following theorem decomposes the operator U_m into five parts, which form martingale difference sequences, each.

THEOREM 3.1. *Let $m \in \mathbb{Z}$ and fix $0 \leq i \leq K(m)$. The collection $\mathscr{B}_i^{(0)}$ is defined in Lemma 2.1 and $\mathscr{B}_i^{(1,\varepsilon)}$, $\varepsilon \in \{0,1\}$ is given by* (3.19) *and* (3.20). *Then the identity*

$$U_m u = U_m \circ Q^{(0)}\, u + \sum_{\varepsilon \in \{0,1\}} \big(A_m^{(\varepsilon)} + B_m^{(\varepsilon)}\big) \circ Q^{(1,\varepsilon)}\, u \tag{3.24}$$

holds true for all $u \in L^p_X$. Furthermore, for every $0 \leq i \leq K(m)$, each of the collections

$$\big\{U_m h_I \,:\, I \in \mathscr{B}_i^{(0)}\big\}, \tag{3.25}$$

as well as

$$\big\{a_I^{(\varepsilon)} \,:\, I \in \mathscr{B}_i^{(1,\varepsilon)}\big\} \tag{3.26}$$

and

$$\big\{b_I^{(\varepsilon)}, b_{\tau_m(I)}^{(\varepsilon)} \,:\, I \in \mathscr{B}_i^{(1,\varepsilon)}\big\} \tag{3.27}$$

constitute martingale difference sequences, for every $\varepsilon \in \{0,1\}$.

PROOF. Within the proof we may assume that m is non-negative. So let $m \in \mathbb{Z}$ be non-negative and $0 \leq i \leq K(m)$ be fixed throughout the rest of this proof. Whenever we apply the predecessor map π to an interval $I \in \sigma^\delta\big(\mathscr{D}\big)$, we understand it with respect to $\sigma^\delta\big(\mathscr{D}\big)$, where $\delta \in \{0,1\}$.

Observe, identity (3.24) follows immediately from (3.23) and (3.17).

First, note that Lemma 2.1 implies that

$$\big\{I,\, \tau_m(I),\, I \cup \tau_m(I) \,:\, I \in \mathscr{B}_i^{(0)}\big\}$$

is a nested collection of sets, hence

$$\big\{U_m h_I \,:\, I \in \mathscr{B}_i^{(0)}\big\}$$

is a martingale difference sequence.

Second, we will show that $\big\{a_I^{(0)} \,:\, I \in \mathscr{B}_i^{(1,0)}\big\}$ forms a martingale difference sequence. Henceforth, we shall abbreviate $\mathscr{B}_i^{(1,0)}$ by $\mathscr{B}$. Now, fix $I, J \in \mathscr{B}$, $|J| < |I|$ such that $\operatorname{supp} a_J^{(0)} \cap \operatorname{supp} a_I^{(0)} \neq \emptyset$. Note that $J \subset \big(\pi^\lambda(J)\big)_{11}$, for all $J \in \mathscr{B}$, where K_{11}, $K \in \mathscr{D}$ denotes the unique $M \subset K$, $M \in \mathscr{D}$, $|M| = |K|/4$ such that $\sup M = \sup K$. From this and the definition of $\mathscr{B}$ it is clear that $\operatorname{supp} a_J^{(0)} \subset \alpha_1(\pi^\lambda(J))$ (see also Remark 2.2), hence

$$\emptyset \neq \alpha_1(\pi^\lambda(J)) \cap \operatorname{supp} a_I^{(0)} = \big(\alpha_1(\pi^\lambda(J)) \cap \alpha_0(I)\big) \cup \big(\alpha_1(\pi^\lambda(J)) \cap \alpha_0(\tau_m(I))\big).$$

Since $|J| < |I|$, $I, J \in \mathscr{B}$, we know that $|\alpha_1(\pi^\lambda(J))| \le |I|$, thus

$$\text{either} \quad \alpha_1(\pi^\lambda(J)) \subset \alpha_0(I) \quad \text{or} \quad \alpha_1(\pi^\lambda(J)) \subset \alpha_0(\tau_m(I)),$$

which finishes the second part of this proof.

The proof that $\{a_I^{(1)} : I \in \mathscr{B}_i^{(1,1)}\}$ forms a martingale difference sequence is essentially the same, and we omit the details.

Third, we will show that $\{b_I^{(0)}, b_{\tau_m(I)}^{(0)} : I \in \mathscr{B}^{(1,0)}\}$ constitutes a martingale difference sequence. Again, we shall abbreviate $\mathscr{B}_i^{(1,0)}$ by $\mathscr{B}$. To this end, we assume there exist $I, J \in \mathscr{B} \cup \tau_m(\mathscr{B})$, $|J| < |I|$ such that

$$\beta(J) \cap \beta(I) \neq \emptyset \qquad \text{and} \qquad \beta(J) \cap \beta(I)^c \neq \emptyset. \tag{$\mathcal{A}$}$$

Since $\beta(J) \subset \gamma(J)$, assumption $(\mathcal{A})$ is covered by the following four cases.

(1) $\gamma(J) \cap I \neq \emptyset$ and $\gamma(J) \cap I^c \neq \emptyset$,
(2) $\gamma(J) \cap \gamma_0(I) \neq \emptyset$ and $\gamma(J) \cap \gamma_0(I)^c \neq \emptyset$,
(3) $\gamma(J) \subset I$ and $\inf \beta_1(I) \in \gamma(J)$,
(4) $\gamma(J) \subset \gamma_0(I)$ and $\inf \beta_0(I) \in \gamma(J)$.

If we assume case (1), then $\inf J = \inf I$ or $\inf J = \sup I$. Anyhow, we have that $\inf J = \inf \pi^\lambda(J)$, so we know $J \notin (\mathscr{B} \cup \tau_m(\mathscr{B}))$, contradicting our assumption. Case (2) is analogous to case (1). Note that we abbreviated $\mathscr{B}_i^{(1,0)}$ by $\mathscr{B}$, so consider the definition of $\mathscr{B}_i^{(1)}$ to see that $J \notin \mathscr{B}_i^{(1,0)}$, and consider (3.19) to determine that also $J \notin \tau_m(\mathscr{B}_i^{(1,0)})$.

Let us now assume case (3) is true. This means that either $\inf I + \frac{1}{3}|I| \in \gamma(J)$ or $\inf I + \frac{2}{3}|I| \in \gamma(J)$, depending on the sign of the one–third–shift for I. We fix $z \in \{1, 2\}$ and assume that

$$\inf I + \frac{z}{3}|I| \in \gamma(J). \tag{3.28}$$

Due to (3.19) we see that $\pi^\lambda(\gamma_0(J)) = \pi^\lambda(J)$, so if we set $K = \pi^\lambda(J)$, then

$$\inf I + \frac{z}{3}|I| \in K.$$

This corresponds to either one of the following being true

$$\inf I + \frac{z}{3}|I| = \inf K + \frac{1}{3}|K| \qquad \text{or} \qquad \inf I + \frac{z}{3}|I| = \inf K + \frac{2}{3}|K|. \tag{3.29}$$

If $J \in \mathscr{B}$ we know $J \subset K_{11}$, thus

$$\begin{aligned} \inf \gamma(J) &\ge \inf K + \frac{3}{4}|K| - 2^{-\lambda}|K| \\ &> \inf K + \frac{2}{3}|K|. \end{aligned} \tag{3.30}$$

Recall that K_{11} denotes the unique $M \subset K$, $M \in \mathscr{D}$, $|M| = |K|/4$ such that $\sup M = \sup K$. The last strict inequality holds true since $\lambda \ge 4$ per construction of $\mathscr{B}$, see (2.7) if $|m| \ge 2$ and note the exception for $|m| = 1$ beneath. Combining (3.28) and (3.30) yields

$$\inf I + \frac{z}{3}|I| > \inf K + \frac{2}{3}|K|,$$

which contradicts (3.29) in both cases.

If $J \in \tau_m(\mathscr{B})$ we know $J \subset K_{00}$, where K_{00} denotes the unique $M \subset K$, $M \in \mathscr{D}$, $|M| = |K|/4$ such that $\inf M = \inf K$. So we note

$$\begin{aligned} \sup \gamma(J) &\le \inf K + \frac{1}{4}|K| + 2^{-\lambda}|K| \\ &< \inf K + \frac{1}{3}|K|. \end{aligned} \tag{3.31}$$

The last strict inequality holds true since $\lambda \geq 4$ per construction of $\mathscr{B}$, see (2.7) if $|m| \geq 2$ and note the exception for $|m| = 1$ beneath. Combining (3.28) and (3.31) yields

$$\inf I + \frac{z}{3}|I| < \inf K + \frac{1}{3}|K|,$$

which contradicts (3.29) in both cases.

Case (4) is analogous to case (3).

Altogether we proved that our assumption ($\mathcal{A}$) was false, therefore

$$\beta(J) \subset \beta_0(I) \qquad \text{or} \qquad \beta(J) \subset \beta_1(I)$$

for all $I, J \in \mathscr{B}$, $|J| < |I|$ such that $\beta(J) \cap \beta(I) \neq \emptyset$. In other words, the support of b_J is contained in a set where $b_I^{(0)}$ is constant, hence

$$\{b_I^{(0)}, b_{\tau_m(I)}^{(0)} \,:\, I \in \mathscr{B}_i^{(1,0)}\}$$

constitutes a martingale difference sequence.

The proof that $\{b_I^{(1)}, b_{\tau_m(I)}^{(1)} \,:\, I \in \mathscr{B}^{(1,1)}\}$ constitutes a martingale difference sequence is essentially the same argument, so we omit it. □

Remark 3.2. Specifically we want to emphasize that (3.25), (3.26) and (3.27) imply that

$$\{U_m \circ Q_i^{(0)} h_I \,:\, I \in \mathscr{D}\},$$

as well as

$$\{A_m^{(\varepsilon)} \circ Q_i^{(1,\varepsilon)} h_I \,:\, I \in \mathscr{D}\},$$

and

$$\{B_m^{(\varepsilon)} \circ Q_i^{(1,\varepsilon)} h_I \,:\, I \in \mathscr{D}\},$$

constitute martingale difference sequences, for each $0 \leq i \leq K(m)$, $\varepsilon \in \{0,1\}$.

Consider the splitting of $\mathscr{D}$ into the sets $\mathscr{B}_i^{(\delta)}$, $0 \leq i \leq K(m)$, $\delta \in \{0,1\}$, see Lemma 2.1 on page 70 for details, which we used in Theorem 2.3 on page 73 to treat the shift operator T_m. Retracing our steps in the proof of Theorem 2.3 we find that we could actually repeat this proof with the operator T_m replaced by any of the operators $U_m \circ Q^{(0)}$, $A_m^{(\varepsilon)} \circ Q^{(1,\varepsilon)}$, $B_m^{(\varepsilon)} \circ Q^{(1,\varepsilon)}$, $\varepsilon \in \{0,1\}$. The details are elaborated in Theorem 3.3 below.

Theorem 3.3. *For all $m \in \mathbb{Z}$, $0 \leq i \leq K(m)$ and $\delta \in \{0,1\}$ let $\mathscr{B}_i^{(\delta)}$ denote the collection specified in Lemma 2.1. Then for every $0 \leq i \leq K(m)$ and $\varepsilon \in \{0,1\}$ we have the estimates*

$$\begin{aligned} \|U_m \circ Q_i^{(0)} u\|_{L_X^p} &\leq C \cdot \|Q_i^{(0)} u\|_{L_X^p}, \\ \|U_m \circ Q_i^{(1,\varepsilon)} u\|_{L_X^p} &\leq C \cdot \|Q_i^{(1,\varepsilon)} u\|_{L_X^p} \end{aligned} \tag{3.32}$$

for all $u \in L_X^p$, where C depends only on p and X. Furthermore, we have the bound $K(m) \leq 7 + 2 \cdot \log_2(1 + |m|)$.

Proof. Let $m \in \mathbb{Z}$ and $0 \leq i \leq K(m)$ be fixed throughout the rest of the proof.

First, we will estimate $U_m \circ Q_i^{(0)}$. Due to Theorem 3.1 respectively Remark 3.2 we know that $\{U_m \circ Q_i^{(0)} h_I \,:\, I \in \mathscr{D}\}$ forms a martingale difference sequence, which enables us to introduce Rademacher functions via the UMD–property. Hence

$$\begin{aligned}\big\|U_m \circ Q_i^{(0)}\, u\big\|_{L^p_X} &\approx \int_0^1 \Big\| \sum_{I \in \mathscr{B}_i^{(0)}} r_I(t)\, \langle u, h_I\rangle\, U_m\, h_I\, |I|^{-1} \Big\|_{L^p_X} \,\mathrm{d}t \\ &= \int_0^1 \Big\| \sum_{I \in \mathscr{B}_i^{(0)}} r_I(t)\, \langle u, h_I\rangle\, (\mathrm{Id} + T_m)\, h_I\, |I|^{-1} \Big\|_{L^p_X} \,\mathrm{d}t\end{aligned}$$

for all $u \in L^p_X$. This is all we need to repeat the proof of Theorem 2.3 in Section 2 with T_m replaced by $\mathrm{Id} + T_m$.

Now we turn to the estimate for $U_m \circ Q_i^{(1,\varepsilon)}$, with $\varepsilon \in \{0,1\}$ fixed throughout the rest of the proof.

First, observe that

$$U_m \circ Q_i^{(1,\varepsilon)}\, u = A_m^{(\varepsilon)} \circ Q_i^{(1,\varepsilon)}\, u + B_m^{(\varepsilon)} \circ Q_i^{(1,\varepsilon)}\, u,$$

for all $u \in L^p_X$, see (3.17). Theorem 3.1 on page 78 ensures that both

$$\big\{A_m^{(\varepsilon)} \circ Q_i^{(1,\varepsilon)}\, h_I \,:\, I \in \mathscr{D}\big\} \quad \text{and} \quad \big\{B_m^{(\varepsilon)} \circ Q_i^{(1,\varepsilon)}\, h_I \,:\, I \in \mathscr{D}\big\}$$

form martingale difference sequences, which allows us to introduce Rademacher means via the UMD–property, hence

$$\big\|A_m^{(\varepsilon)} \circ Q_i^{(1,\varepsilon)}\, u\big\|_{L^p_X} \lesssim \int_0^1 \Big\| \sum_{I \in \mathscr{B}_i^{(1,\varepsilon)}} r_I(t)\, \langle u, h_I\rangle\, a_I^{(\varepsilon)}\, |I|^{-1} \Big\|_{L^p_X} \,\mathrm{d}t$$

and

$$\big\|B_m^{(\varepsilon)} \circ Q_i^{(1,\varepsilon)}\, u\big\|_{L^p_X} \lesssim \int_0^1 \Big\| \sum_{I \in \mathscr{B}_i^{(1,\varepsilon)}} r_I(t)\, \langle u, h_I\rangle\, \big(b_I^{(\varepsilon)} - b_{\tau_m(I)}^{(\varepsilon)}\big)\, h_I\, |I|^{-1} \Big\|_{L^p_X} \,\mathrm{d}t,$$

for all $u \in L^p_X$. Now we can essentially repeat the proof of Theorem 2.3 in Section 2, for $\delta = 1$ and with T_m replaced by $A_m^{(\varepsilon)}$ and $B_m^{(\varepsilon)}$, respectively. We have to utilize the unilateral operators S_0 and S_1 instead of S as well, see Subsection 1.2 on page 68. If we do so, we end up with the estimates

$$\big\|A_m^{(\varepsilon)} \circ Q_i^{(1,\varepsilon)}\, u\big\|_{L^p_X} \lesssim \int_0^1 \Big\| \sum_{I \in \mathscr{B}_i^{(1,\varepsilon)}} r_I(t)\, \langle u, h_I\rangle\, h_{\alpha_\varepsilon(I)}\, |I|^{-1} \Big\|_{L^p_X} \,\mathrm{d}t$$

and

$$\big\|B_m^{(\varepsilon)} \circ Q_i^{(1,\varepsilon)}\, u\big\|_{L^p_X} \lesssim \int_0^1 \Big\| \sum_{I \in \mathscr{B}_i^{(1,\varepsilon)}} r_I(t)\, \langle u, h_I\rangle\, b_I^{(\varepsilon)}\, |I|^{-1} \Big\|_{L^p_X} \,\mathrm{d}t,$$

for all $u \in L^p_X$. Thus, considering $h_{\alpha_\varepsilon(I)} = S_\varepsilon h_I$ and $|b_I^{(\varepsilon)}| \leq |S_0 h_I| + |S_1 h_I|$, see (1.9), (1.10), (1.11), (1.12) and combining our estimates for $A_m^{(\varepsilon)}$ and $B_m^{(\varepsilon)}$ with the inequalities for the unilateral one-third-shift operators S_0 and S_1 in Theorem 1.3 on page 69 yields

$$\begin{aligned}\big\|U_m \circ Q_i^{(1,\varepsilon)}\, u\big\|_{L^p_X} &\lesssim \big\|S_0 \circ Q_i^{(1,\varepsilon)}\, u\big\|_{L^p_X} + \big\|S_1 \circ Q_i^{(1,\varepsilon)}\, u\big\|_{L^p_X} \\ &\lesssim \big\|Q_i^{(1,\varepsilon)}\, u\big\|_{L^p_X},\end{aligned}$$

for all $u \in L^p_X$, concluding the proof. □

Inserting Theorem 2.3 on page 73 and Theorem 3.3 on the facing page into [**Fig88**, Lemma 1] one can obtain [**Fig88**, Theorem 1] stated below for sake of completeness.

THEOREM 3.4. *Let* $1 < p < \infty$*, and* X *be a Banach space with the* UMD–*property. For* $m \in \mathbb{Z}$ *let the map* τ_m *denote the shift map defined by*

$$I \mapsto I + m\,|I|.$$

Let T_m*,* U_m *denote the linear extensions of the maps*

$$T_m h_I = h_{\tau_m(I)},$$

and

$$U_m h_I = \mathtt{1}_{\tau_m(I)} - \mathtt{1}_I,$$

respectively, then

$$\|T_m : L^p_X \to L^p_X\| \leq C \left(\log_2(2 + |m|)\right)^{\alpha},$$
$$\|U_m : L^p_X \to L^p_X\| \leq C \left(\log_2(2 + |m|)\right)^{\beta},$$

where the constant $C > 0$ *depends only on* p*,* X *and* $0 < \alpha, \beta < 1$*. More precisely, if* L^p_X *has type* $\mathcal{T}$ *and cotype* $\mathcal{C}$*, then one can take* $\alpha = \frac{1}{\mathcal{T}} - \frac{1}{\mathcal{C}}$ *and* $\beta = 1 - \frac{1}{\mathcal{C}}$*.*

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