Yang-Mills theory, lattice gauge theory and simulations

David Müller

Institute of Analysis
Johannes Kepler University Linz

dmueller@hep.itp.tuwien.ac.at

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Overview

Introduction and physical context

Classical Yang-Mills theory

Lattice gauge theory

Simulating the Glasma in 2+1D

Introduction and physical context

Yang-Mills theory

- Formulated in 1954 by Chen Ning Yang and Robert Mills
- A non-Abelian gauge theory with gauge group $\mathrm{SU}(N_c)$
- A non-linear generalization of electromagnetism, which is a gauge theory based on $\mathrm{U}(1)$
- Gauge theories are a widely used concept in physics: the standard model of particle physics is based on a gauge theory with gauge group $\mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3)$
- All fundamental forces (electromagnetism, weak and strong nuclear force, even gravity) are/can be formulated as gauge theories

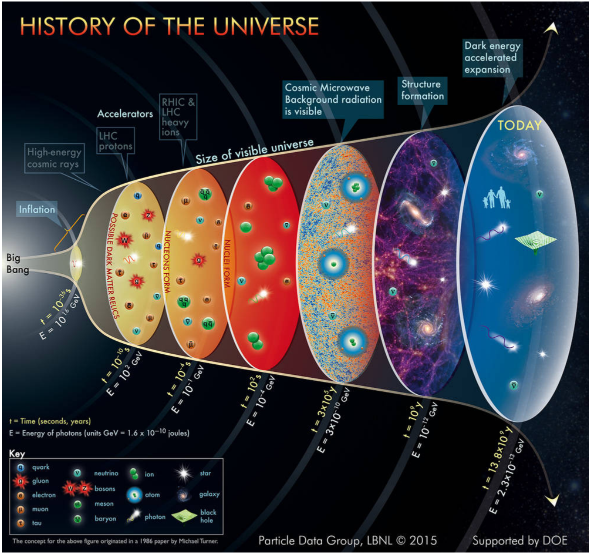
Classical Yang-Mills theory

Classical Yang-Mills theory refers to the study of the classical equations of motion (Euler-Lagrange equations) obtained from the Yang-Mills action

Main topic of this seminar: solving the classical equations of motion of Yang-Mills theory numerically

Not topic of this seminar: quantum field theory, path integrals, lattice quantum chromodynamics (except certain methods), the Millenium problem related to Yang-Mills . . .

Classical Yang-Mills in the early universe



Classical Yang-Mills in the early universe

Electroweak phase transition: the electro-weak force splits into the weak nuclear force and the electromagnetic force

This phase transition can be studied using (extensions of) classical Yang-Mills theory

Literature:

- G. D. Moore and N. Turok, "Classical field dynamics of the electroweak phase transition", PRD 55, 6538 (1997), [arXiv:hep-ph/9608350]
- Y. Akamatsu, A. Rothkopf and N. Yamamoto, "Non-Abelian chiral instabilities at high temperature on the lattice", JHEP 1603, 210 (2016), [arXiv:1512.02374]
- ...

Classical Yang-Mills in heavy-ion collisions

My main application for Yang-Mills theory:

The earliest stages of relativistic heavy-ion collisions

Heavy-ion collisions

- Heavy-ion collision experiments (e.g. LHC at CERN or RHIC at BNL) to investigate the properties of nuclear matter under extreme conditions (high energy)
- Accelerate e.g. gold or lead nuclei to relativistic speeds, perform collisions, detect matter that is created (particle detectors)

Classical Yang-Mills in heavy-ion collisions

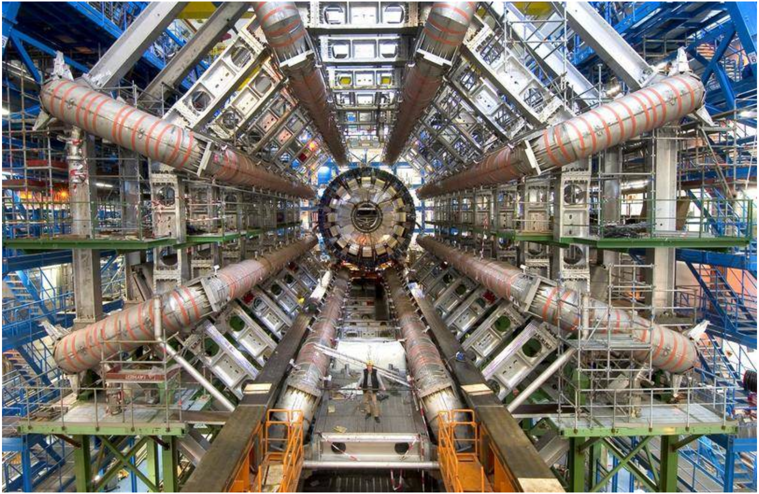


Image from ATLAS @ CERN (2005),
https://home.cern/resources/image/experiments/atlas-images-gallery

Classical Yang-Mills in heavy-ion collisions

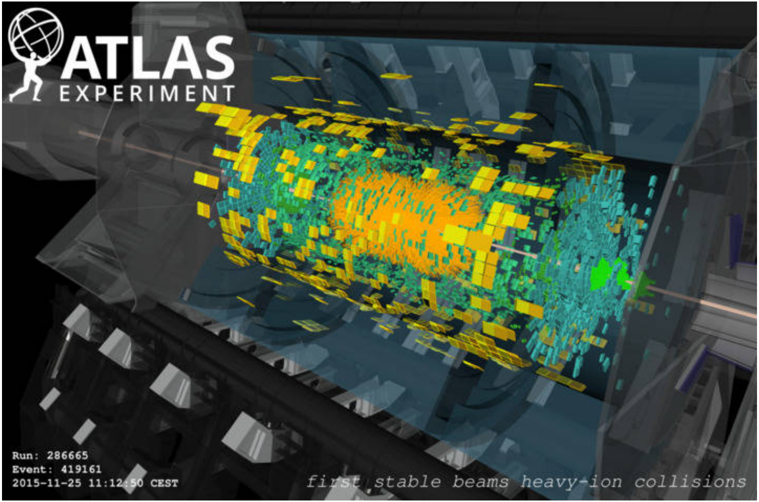


Image from ATLAS @ CERN (2015)
https://atlas.cern/resources/multimedia/physics

Classical Yang-Mills in heavy-ion collisions

- Heavy-ion collision experiments (e.g. LHC at CERN or RHIC at BNL) to investigate the properties of nuclear matter under extreme conditions (high energy)
- Fundamental theory: quantum chromodynamics (gauge group $\mathrm{SU}(3)$) which governs the interactions of quarks and gluons
- At very high energies: nuclei appear as "frozen" thin disks, can be described using classical Yang-Mills theory (color glass condensate)
- Matter created immediately after the collision: "Glasma"
- Dynamics of the Glasma are described by classical Yang-Mills equations

Review:

- F. Gelis, "Color Glass Condensate and Glasma", Int. J. Mod. Phys. A 28, 1330001 (2013) [arXiv:1211.3327]

Classical Yang-Mills in heavy-ion collisions

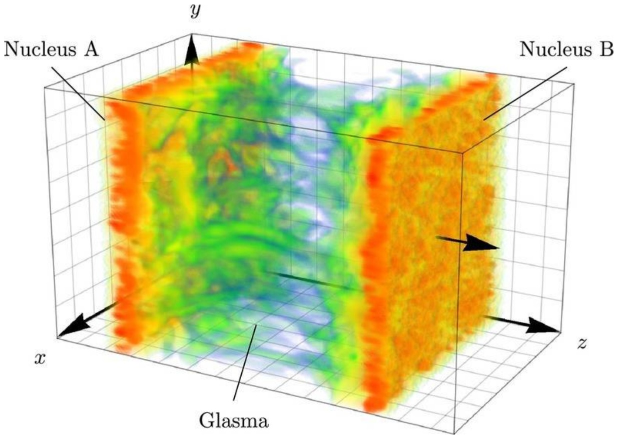


Image from my thesis [arXiv:1904.04267]

Literature

Lattice gauge theory and its application to heavy-ion collisions:

PhD thesis [arXiv:1904.04267] based on:

- D. Gelfand, A. Ipp, DM, "Simulating collisions of thick nuclei in the color glass condensate framework", PRD94, 1, 014020 [arXiv:1605.07184]
- A. Ipp, DM, "Broken boost invariance in the Glasma via finite nuclei thickness", PLB 771, 74 [arXiv:1703.00017]
- A. Ipp, DM, "Implicit schemes for real-time lattice gauge theory", EPJC 78, no. 11, 884 [arXiv:1804.01995]

Literature

General (quantum) field theory, Yang-Mills theory:

- M. E. Peskin, D. V. Schroeder, "An Introduction To Quantum Field Theory" (1995)
- M. Srednicki, "Quantum Field Theory" (2007)
- D. Tong, "Lectures on Quantum Field Theory", lecture notes http://www.damtp.cam.ac.uk/user/tong/qft.html

Lattice gauge theory:

- C. Gattringer, C. B. Lang, "Quantum Chromodynamics on the Lattice: An Introductory Presentation" (2009)

Classical Yang-Mills theory

Classical Yang-Mills theory: overview

- Preliminaries
  - Special relativity
  - Relativistic field theory
- Yang-Mills theory
  - Gauge fields and field strength tensor
  - Yang-Mills action
  - Variation of the action
- Gauge symmetry
  - Gauge fixing
  - Gauss constraint
- Energy-momentum tensor
- Electromagnetism

Preliminaries

Minkowski space

Minkowski space **M** is a four-dimensional real vector space equipped with a metric $g_{\mu\nu}$ with signature $(+1, -1, -1, -1)$ ("mostly minus" convention, particle physics).

- Greek indices $\mu \in \{0, 1, 2, 3\}$ to indicate that a vector v^μ is an element of **M** (a "4-vector") or its tangent space
- Naming convention $v^\mu = (v^0, v^1, v^2, v^3)^T$
  - temporal component v^0
  - spatial components $v^i, \ i \in \{1, 2, 3\}$
- Latin indices for spatial components v^i
- Euclidean coordinate vector

$$x^\mu = (x^0, x^1, x^2, x^3)^T = (ct, x, y, z)^T$$

Speed of light c usually set to $c = 1$ ("natural" or particle physics units)

Minkowski metric

- Covariant metric in Euclidean coordinates

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

- Einstein summation: repeated indices are summed over
- Lowering indices

$$v_\mu \equiv g_{\mu\nu} v^\nu = (v^0, -v^i)^T$$

- Contravariant metric $g^{\mu\nu}$ is inverse of $g_{\mu\nu}$

$$g^{\mu\nu} g_{\nu\rho} = \delta^\mu_\rho$$

- Raising indices

$$v^\mu = g^{\mu\nu} v_\nu$$

Inner product and norm

Inner product of v^μ and w^μ

$$
\begin{aligned}
v \cdot w &\equiv v_\mu w^\mu \\
&= g_{\mu\nu} v^\mu w^\nu \\
&= v^T g\, w \\
&= v^0 w^0 - v^i w^i
\end{aligned}
$$

Norm of v^μ

$$v^2 \equiv v_\mu v^\mu = g_{\mu\nu} v^\mu v^\nu = (v^0)^2 - v^i v^i$$

Note: Minkowski norm is not positive-definite

Inner product and norm

Norm of v^μ using Minkowski metric $g_{\mu\nu}$

$$v^2 \equiv v_\mu v^\mu = g_{\mu\nu} v^\mu v^\nu = (v^0)^2 - v^i v^i$$

Nomenclature

- spacelike vector $v^2 < 0$
- timelike vector $v^2 > 0$
- lightlike vector $v^2 = 0$

Nomenclature depends on signature: different signs in general relativity, string theory

Lorentz group

The inner product of two 4-vectors $v_\mu w^\mu$ is invariant under transformations of the Lorentz group $O(1,3)$.

$$v_\nu w^\nu = g_{\mu\nu} v^\mu w^\nu = v^0 w^0 - v^i w^i$$

1. $SO(3)$: rotations in $\mathbb{R}^3$ subspace
2. Lorentz boosts (change of inertial frame)
3. Time reversal $T : v^0 \to -v^0$
4. Space inversion $P : v^i \to -v^i$

- $O(1,3)$ consists of four connected components
- $SO^+(1,3)$: proper orthochronous Lorentz transformations, component connected to identity (leave out T and P)

Lorentz boosts

- Lorentz boosts correspond to a change of the inertial frame ("Bezugssystem")
- Relativistic generalization of Galilean transformations $x' = x - v\,t$ with velocity v

Example: boost along $x^3 = z$ direction with "rapidity" $\eta \in \mathbb{R}$.

$$\begin{aligned} v'^{\mu} &= \Lambda^{\mu}{}_{\nu} v^{\nu} \\ &= \left(v^0 \cosh\eta + v^3 \sinh\eta,\ v^1,\ v^2,\ v^0 \sinh\eta + v^3 \cosh\eta\right)^T, \end{aligned}$$

where

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \cosh\eta & 0 & 0 & -\sinh\eta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh\eta & 0 & 0 & \cosh\eta \end{pmatrix}.$$

Lorentz boosts

Inner product is invariant under Lorentz boosts

$$v'^{\mu} = (v^0 \cosh\eta - v^3 \sinh\eta,\ \ v^1,\ \ v^2,\ \ v^0 \sinh\eta - v^3 \cosh\eta)^T,$$

$$w'^{\mu} = (w^0 \cosh\eta - w^3 \sinh\eta,\ \ w^1,\ \ w^2,\ \ w^0 \sinh\eta - w^3 \cosh\eta)^T,$$

$$\begin{aligned} v'_{\mu} w'^{\mu} &= g_{\mu\nu} v'^{\mu} w'^{\nu} \\ &= v'^0 w'^0 - w'^i v'^i \\ &= v^0 w^0 - w^i v^i \\ &= v_{\mu} w^{\mu}. \end{aligned}$$

Lorentz boosts

- Velocity v_z from rapidity η

$$v = \tanh\eta$$
$$\cosh\eta = \frac{1}{\sqrt{1-v_z^2}} = \gamma$$
$$\sinh\eta = \frac{v_z}{\sqrt{1-v_z^2}} = v_z\gamma$$

- More familiar form of Lorentz boost

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & 0 & 0 & -v_z\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma v_z & 0 & 0 & \gamma \end{pmatrix}$$

- Lorentz factor $\gamma = 1/\sqrt{1-v_z^2}$

Lorentz boosts

Apply boost to coordinate vector x^μ

$$
\begin{aligned}
x'^\mu &= \Lambda^\mu{}_\nu x^\nu, \\
t' &= \gamma(t - v_z\, z), \\
z' &= \gamma(z - v_z\, t).
\end{aligned}
$$

All standard results of special relativity follow from these transformations, e.g.

- Time dilation (fast moving clocks appear to run slower)
- Length contraction (fast moving objects appear length contracted)

Nuclei at relativistic speeds: "frozen", thin disks of nuclear matter

Partial derivatives and integrals

Shorthand notation for partial derivatives with respect to coordinates

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu}$$

Partial derivative with raised index

$$\partial^\mu \equiv g^{\mu\nu}\partial_\nu = \frac{\partial}{\partial x_\mu}$$

Example: d'Alembert operator acting on function $\phi : \mathbf{M} \to \mathbb{R}$

$$\partial_\mu \partial^\mu \phi(x) = \frac{\partial^2 \phi(x)}{\partial t^2} - \Delta\phi(x)$$

with $x^0 = t$ as the time coordinate
Integrals over $\mathbf{M}$ denoted as

$$\int_x \phi(x) = \int d^4x\, \phi(x) = \int dt\, dx\, dy\, dz\, \phi(t, x, y, z)$$

Free scalar field

Action functional of a free scalar field $\phi(x) : \mathbf{M} \to \mathbb{R}$, which maps $\phi(x)$ to a real number

$$S[\phi] = \int d^4x \left(\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 \right)$$
$$= \int d^4x \, \mathcal{L}(\partial_\mu \phi(x), \phi(x), x)$$

with mass parameter $m > 0$ and Lagrange density

$$\mathcal{L}(\partial_\mu \phi(x), \phi(x), x) = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2$$

Free scalar field

Action functional of a free scalar field

$$S[\phi] = \int d^4x \left(\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 \right)$$

- Invariant under Lorentz group (rotations, boosts, time reversal, space inversion) and translations

$$x'^\mu = x^\mu + w^\mu$$

  (Lorentz group + translations: Poincaré group)
- Relativistic field theory:
  consistent use of contracted 4-vector index pairs

Free scalar field

Principle of stationary action: a field ϕ which is an extremum of $S[\phi]$ satisfies the equations of motion (EOM) or Euler-Lagrange equations.

Directional functional derivative of $S[\phi]$ in "direction" $\alpha(x)$

$$\delta S[\phi, \alpha] \equiv \lim_{\epsilon \to 0} \frac{S[\phi + \epsilon\alpha] - S[\phi]}{\epsilon} = \int d^4x \frac{\delta S[\phi]}{\delta\phi(x)} \alpha(x)$$

Expression on the right requires integration by parts, $\alpha(x)$ has compact support on $\mathbf{M}$

Principle of stationary action

Different way of writing the same thing:
Variation of the action

$$\delta S[\phi, \delta\phi] = \int d^4x \frac{\delta S[\phi]}{\delta\phi(x)} \delta\phi(x)$$

Compare to total differential of $F(x_1, x_2, \ldots, x_n) : \mathbb{R}^n \to \mathbb{R}$

$$dF = \sum_{i=1}^{n} \frac{\partial F}{\partial x_i} dx_i$$

Equations of motion (Euler-Lagrange eqs.) follow from

$$\delta S[\phi, \delta\phi] = 0 \qquad \Leftrightarrow \qquad \frac{\delta S[\phi]}{\delta\phi(x)} = 0$$

Free scalar field

Action of a free scalar field

$$S[\phi] = \int d^4x \left(\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 \right)$$

Variation of the action

$$\begin{aligned} \delta S[\phi, \delta\phi] &= \int d^4x \left(\partial_\mu \phi(x) \partial^\mu \delta\phi(x) - m^2 \phi(x) \delta\phi(x) \right) \\ &= \int d^4x \left(-\partial_\mu \partial^\mu \phi(x) - m^2 \phi(x) \right) \delta\phi(x) \end{aligned}$$

Note: integration by parts, no boundary terms

Functional derivative

$$\frac{\delta S[\phi]}{\delta\phi(x)} = -\partial_\mu \partial^\mu \phi(x) - m^2 \phi(x)$$

Free scalar field

Action of a free scalar field

$$S[\phi] = \int d^4x \left(\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 \right)$$

Principle of stationary action

$$\frac{\delta S[\phi]}{\delta \phi(x)} = -\partial_\mu \partial^\mu \phi(x) - m^2 \phi(x) = 0$$

Klein-Gordon equation (second order in time derivatives)

$$\frac{\partial^2 \phi(x)}{\partial t^2} - \Delta \phi(x) + m^2 \phi(x) = 0$$

Free scalar field

Lagrangian density for free scalar field

$$\mathcal{L}(\partial_\mu \phi(x), \phi(x), x) = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2$$

Introduce conjugate momentum to $\phi(x)$

$$\pi(x) \equiv \frac{\partial \mathcal{L}}{\partial\left(\partial^0 \phi\right)} = \partial_0 \phi(x)$$

Rewrite Klein-Gordon equation(s) (first order in time derivatives)

$$\begin{aligned}
\partial_0 \pi(x) &= \Delta \phi(x) - m^2 \phi(x) \\
\partial_0 \phi(x) &= \pi(x)
\end{aligned}$$

Initial value problem: specify initial values $\phi(t_0, \vec{x})$ and $\pi(t_0, \vec{x})$ at some time t_0 and solve the equations of motion

Yang-Mills theory

Yang-Mills gauge fields

Degrees of freedom (DOF) in Yang-Mills theory are
gauge fields $A_\mu(x) : \mathbf{M} \to \mathfrak{su}(N_c)$

- Number of colors N_c
  Quantum chromodynamics $N_c = 3$ (strong nuclear force)
  Weak nuclear force $N_c = 2$
- Lie algebra $\mathfrak{su}(N_c)$
  Traceless hermitian matrices in $\mathbb{C}^{N_c \times N_c}$

$$t \in \mathbb{C}^{N_c \times N_c}, \qquad t = t^\dagger, \qquad \mathrm{tr}\,[t] = 0$$

For $t, t' \in \mathfrak{su}(N_c)$ we have

$$\begin{aligned} &\text{Scalar multiplication:} && \alpha t \in \mathfrak{su}(N_c), \quad \alpha \in \mathbb{R} \\ &\text{Addition:} && t + t' \in \mathfrak{su}(N_c), \\ &\text{Commutator:} && \left[t, t'\right]/i \in \mathfrak{su}(N_c), \quad i^2 = -1 \end{aligned}$$

Yang-Mills gauge fields

Degrees of freedom (DOF) in Yang-Mills theory are
gauge fields $A_\mu(x) : \mathbf{M} \to \mathfrak{su}(N_c)$

$$A_\mu(x) = A_\mu^a(x) t^a$$

- Color indices $a \in \{1, 2, \dots, N_c^2 - 1\}$ (Einstein summation)
- Color components $A_\mu^a(x)$ of gauge field
  ($4(N_c^2 - 1)$ functions $\mathbf{M} \to \mathbb{R}$)

Generators $t^a \in \mathfrak{su}(N_c)$

- $N_c = 2$: Pauli matrices $t^a = \frac{1}{2}\sigma^a$
- $N_c = 3$: Gell-Mann matrices $t^a = \frac{1}{2}\lambda^a$

Gauge fields are traceless and hermitian

Pauli-Matrices

$$\sigma_1 = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$\sigma_2 = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$
$$\sigma_3 = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Generators of $\mathfrak{su}(2)$:

$$t^a = \frac{1}{2}\sigma^a$$

Gell-Mann matrices

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}$$

$$\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

Generators of $\mathfrak{su}(3)$:

$$t^a = \frac{1}{2}\lambda^a$$

Properties of generators

- Traceless

$$\mathrm{tr}\,[t^a] = 0$$

- Hermitian

$$t^{a\dagger} = t^a$$

- Normalization

$$\mathrm{tr}\left[t^a t^b\right] = \frac{1}{2}\delta^{ab}$$

- Antisymmetric structure constants f^{abc} (commutator)

$$\left[t^a, t^b\right] = t^a t^b - t^b t^a = i f^{abc} t^c, \quad f^{abc} \in \mathbb{R}$$

- Symmetric structure constants d^{abc} (anti-commutator)

$$\left\{t^a, t^b\right\} = t^a t^b + t^b t^a = \frac{1}{N_c}\delta^{ab}\mathbf{1} + d^{abc} t^c, \quad d^{abc} \in \mathbb{R}$$

Yang-Mills field strength tensor

Definition: field strength tensor

$$F_{\mu\nu}(x) \equiv \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) + ig\left[A_\mu(x), A_\nu(x)\right]$$

with Yang-Mills coupling constant $g > 0$

$F_{\mu\nu}$ is antisymmetric in index pair μ, ν

$F_{\mu\nu}$ is traceless and hermitian

$$F_{\mu\nu}(x) = F^a_{\mu\nu}(x) t^a$$

Using $\left[t^a, t^b\right] = if^{abc} t^c$ we can write

$$F^a_{\mu\nu}(x) = \partial_\mu A^a_\nu(x) - \partial_\nu A^a_\mu(x) - gf^{abc} A^b_\mu(x) A^c_\nu(x)$$

Yang-Mills field strength tensor

Physical interpretation: field strength tensor contains the chromo-electric and -magnetic fields

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}$$

Electric fields

$$F_{0i} = E_i \in \mathfrak{su}(N_c).$$

Magnetic fields

$$F_{ij} = \varepsilon_{ijk} B_k, \quad B_i = -\frac{1}{2}\varepsilon_{ijk} F_{jk} \in \mathfrak{su}(N_c),$$

where ε_{ijk} is the Levi-Civita symbol

$$\varepsilon_{123} = 1, \quad \varepsilon_{ijk} = -\varepsilon_{jik} = -\varepsilon_{ikj}.$$

Yang-Mills action

Using $F_{\mu\nu}$ we can define the Yang-Mills action $S[A_\mu]$

$$S[A_\mu] = \int d^4x \, \mathrm{tr}\left(-\frac{1}{2}F_{\mu\nu}(x)F^{\mu\nu}(x)\right)$$
$$= \int d^4x \, \mathrm{tr}\left(-\frac{1}{2}g^{\mu\rho}g^{\nu\sigma}F_{\mu\nu}(x)F_{\rho\sigma}(x)\right)$$

Consistent use of contracted index pairs: invariant under Lorentz transformations (rotations, boosts, time reversal, spatial inversion) and translations

$\Rightarrow$ Varying this action yields the Yang-Mills equations for $A_\mu(x)$

Varying the Yang-Mills action

Using integration by parts, properties of the commutator and the trace, we find

$$\begin{aligned}\delta S[A_\mu, \delta A_\mu] &= \int d^4x \, \mathrm{tr}\big[-F^{\mu\nu}\delta F_{\mu\nu}\big] \\ &= \int d^4x \, \mathrm{tr}\big[-F^{\mu\nu}(\partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu \\ &\qquad + ig\,[\delta A_\mu, A_\nu] + ig\,[A_\mu, \delta A_\nu]\,)\big] \\ &= -2\int d^4x \, \mathrm{tr}\big[\,(\partial_\nu F^{\mu\nu} + ig\,[A_\nu, F^{\mu\nu}])\,\delta A_\mu\big]\end{aligned}$$

Vanishing variation $\delta S[A_\mu, \delta A_\mu] = 0$:

Yang-Mills equations

$$\partial_\nu F^{\mu\nu} + ig\,[A_\nu, F^{\mu\nu}] = 0$$

Yang-Mills equations

Yang-Mills equations

$$\partial_\nu F^{\mu\nu} + ig\,[A_\nu, F^{\mu\nu}] = 0$$

with field strength

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu + ig\,[A_\mu, A_\nu]$$

Shorthand: (gauge) covariant derivative D_μ
For an algebra-valued field $B(x)$ we define

$$D_\mu B(x) \equiv \partial_\mu B(x) + ig\,[A_\mu(x), B(x)]$$

Write Yang-Mills equations as

$$D_\nu F^{\mu\nu} = 0$$

Yang-Mills equations

Yang-Mills equations in terms of color components A_μ^a

$$A_\mu = A_\mu^a t^a$$

$$\partial_\nu \partial^\mu A^{a,\nu} - \partial_\nu \partial^\nu A^{a,\mu}$$
$$- g\, f^{abc} \left(\partial_\nu A^{b,\mu} A^{c,\nu} + A^{b,\mu} \partial_\nu A^{c,\nu} \right) - g\, f^{abc} A_\nu^b \left(\partial^\mu A^{c,\nu} - \partial^\nu A^{c,\mu} \right)$$
$$+ g^2 f^{abc} f^{cde} A_\nu^b A^{d,\mu} A^{e,\nu} = 0$$

For $N_c = 3$: system of 32 coupled, second order hyperbolic, non-linear PDEs

Yang-Mills equations

Reformulate the second order system into a first order system
Introduce conjugate momenta

$$\pi^{a,\mu} = \frac{\partial \mathcal{L}}{\partial \left(\partial_0 A^a_\mu \right)} = -F^{a,0\mu}$$

$$\pi^0 = 0, \qquad \pi^i = F_{0i} = E_i$$

The momentum π^0 conjugate to A_0 vanishes.

Degrees of freedom

- A^a_0, A^a_i: $4(N_c^2 - 1)$ gauge fields
- π^i: $3(N_c^2 - 1)$ momenta

Yang-Mills equations

Rewrite Yang-Mills equations using canonical momenta π^i

Equations of motion

$$\partial_\mu F^{\mu i} + ig\left[A_\mu, F^{\mu i}\right] = 0$$
$$\pi^i = F_{0i} = \partial_0 A_i - \partial_i A_0 + ig\left[A_0, A_i\right]$$

$$\Rightarrow \partial_0 \pi^i = -ig\left[A_0, \pi^i\right] + \partial_j F^{ji} + ig\left[A_j, F^{ji}\right]$$
$$\Rightarrow \partial_0 A_i = \pi^i + \partial_i A_0 - ig\left[A_0, A_i\right]$$

Gauss constraint (contains no time derivatives of π^i or A_μ)

$$\partial_\mu F^{\mu 0} + ig\left[A_\mu, F^{\mu 0}\right] = 0$$
$$\Rightarrow \partial_i \pi^i + ig\left[A_i, \pi^i\right] = 0$$

Yang-Mills equations

Reformulated system

$$\partial_0 \pi^i = -ig\left[A_0, \pi^i\right] + \partial_j F^{ji} + ig\left[A_j, F^{ji}\right]$$
$$\partial_0 A_i = \pi^i + \partial_i A_0 - ig\left[A_0, A_i\right]$$
$$\partial_i \pi^i + ig\left[A_i, \pi^i\right] = 0$$

There is no term $\partial_0 A_0$. This system is not solvable as a standard initial value problem. Specifying $A_0(t_0, \vec{x})$, $A_i(t_0, \vec{x})$ and $\pi^i(t_0, \vec{x})$ at some initial time t_0 is not enough information to determine the fields at some later time $t_1 > t_0$.

The Yang-Mills equations (as stated above) are under-determined: gauge symmetry

Yang-Mills in 1+1D, single color component

Reduce dimensions to $1+1$ (t and x) $\Rightarrow A_0(t,x)$ and $A_1(t,x)$

Yang-Mills equations:

$$\begin{aligned}
\partial_0 \pi^1 &= -ig\left[A_0, \pi^1\right] \\
\partial_0 A_1 &= \pi^1 + \partial_1 A_0 - ig\left[A_0, A_1\right] \\
\partial_1 \pi^1 + ig\left[A_1, \pi^1\right] &= 0
\end{aligned}$$

Reduce to a single color component:
$A_0 = A_0^1 t^1$, $A_1 = A_1^1 t^1 \Rightarrow$ drop all commutator terms

Yang-Mills in 1+1D, single color component

Remove commutator terms:

$$\partial_0 \pi^1 = 0$$
$$\partial_0 A_1 = \pi^1 + \partial_1 A_0$$
$$\partial_1 \pi^1 = 0$$

From first and third equation: π^1 must be constant w.r.t t and x

$$\pi^1 = C$$

Solution of second equation with initial data at t_0:

$$A_1(t,x) = A_1(t_0,x) + (t - t_0)C + \int_{t_0}^{t} dt' \partial_1 A_0(t',x)$$

Cannot compute $A_1(t,x)$ without specifying $A_0(t,x)$.

Gauge symmetry

Gauge symmetry

Special unitary group $\mathrm{SU}(N_c)$

Special unitary matrices acting on $\mathbb{C}^{N_c}$

$$U \in \mathbb{C}^{N_c \times N_c}, \qquad UU^\dagger = U^\dagger U = \mathbf{1}, \qquad \det U = 1$$

For $U, U', U'' \in \mathrm{SU}(N_c)$ we have

$$\begin{aligned} &\text{Multiplication:} && UU' \in \mathrm{SU}(N_c) \\ &\text{Associativity:} && (UU')\, U'' = U\,(U'U'') \\ &\text{Identity:} && \mathbf{1}U = U\mathbf{1} \\ &\text{Inverse:} && U^\dagger U = \mathbf{1} \end{aligned}$$

$\mathrm{SU}(N_c)$ is a finite-dimensional real smooth manifold. Inverse and multiplication are smooth maps. $\mathrm{SU}(N_c)$ is a Lie group.

Gauge symmetry

Connection between Lie algebra $\mathfrak{su}(N_c)$ and Lie group $\mathrm{SU}(N_c)$:
the exponential map $\exp : \mathfrak{su}(N_c) \to \mathrm{SU}(N_c)$

Elements of the Lie algebra "generate" elements of the Lie group via

$$U = \exp(it), \qquad t \in \mathfrak{su}(N_c), \quad i^2 = -1$$

Definition as a series:

$$U = \exp(it) = \sum_{n=0}^{\infty} \frac{1}{n!} (it)^n$$

Some useful properties:

$$\exp(it)\exp(it') = \exp(i(t+t')), \qquad [t, t'] = 0, \qquad t, t' \in \mathfrak{su}(N_c)$$

$$(\exp(it))^{-1} = \exp(-it), \qquad t \in \mathfrak{su}(N_c)$$

Gauge symmetry

The Yang-Mills action for N_c colors exhibits a particular local symmetry: SU(N_c) gauge symmetry

Consider a "local gauge transformation", a smooth function $\Omega(x) : \mathbf{M} \to \mathrm{SU}(N_c)$ acting on a gauge field $A_\mu(x)$

$$
\begin{aligned}
A'_\mu(x) &= \Omega(x) \left(A_\mu(x) + \frac{1}{ig} \partial_\mu \right) \Omega^\dagger(x) \\
&= \Omega(x) A_\mu(x) \Omega^\dagger(x) + \frac{1}{ig} \Omega(x) \partial_\mu \Omega^\dagger(x)
\end{aligned}
$$

$\Rightarrow$ Gauge transformation of the field strength tensor

$$
F'_{\mu\nu}(x) = \Omega(x) F_{\mu\nu}(x) \Omega^\dagger(x)
$$

How does $S[A_\mu]$ change under this transformation?

Gauge symmetry

Gauge transformation of the field strength tensor

$$F'_{\mu\nu}(x) = \Omega(x)F_{\mu\nu}(x)\Omega^\dagger(x)$$

Transformation of $S[A_\mu]$

$$\begin{aligned} S[A'_\mu] &= \int d^4x \, \mathrm{tr}\left(-\frac{1}{2}F'_{\mu\nu}(x)F'^{\mu\nu}(x)\right) \\ &= \int d^4x \, \mathrm{tr}\left(-\frac{1}{2}\Omega(x)F_{\mu\nu}(x)\Omega^\dagger(x)\Omega(x)F^{\mu\nu}(x)\Omega^\dagger(x)\right) \\ &= S[A_\mu] \end{aligned}$$

The Yang-Mills action is invariant under local gauge transformations $\Omega(x)$.

Gauge symmetry: implications

- If A_μ solves the Yang-Mills (YM) equations, then A'_μ does too
- Gauge symmetry reflects the degree of redundancy in the gauge field description of gauge field theories
- Physical observables must be gauge invariant
  - Gauge field A_μ is not an observable

$$A'_\mu = \Omega \left(A_\mu + \frac{1}{ig} \partial_\mu \right) \Omega^\dagger$$

  - Field strength tensor $F_{\mu\nu}$ is not an observable

$$F'_{\mu\nu} = \Omega F_{\mu\nu} \Omega^\dagger$$

  - Physical observables like the energy-momentum tensor $T_{\mu\nu}$ are gauge invariant

$$T'_{\mu\nu} = T_{\mu\nu}$$

Gauge symmetry

Proof: if A_μ solves the YM equations, then A'_μ does too.

1) Assume that A_μ solves the YM equations

$$\partial_\nu F^{\mu\nu} + ig\left[A_\nu, F^{\mu\nu}\right] = 0$$

2) Check if A'_μ solves them too:

$$A'_\mu = \Omega\left(A_\mu + \frac{1}{ig}\partial_\mu\right)\Omega^\dagger, \qquad F'_{\mu\nu} = \Omega F_{\mu\nu}\Omega^\dagger$$

$$\partial_\nu F'^{\mu\nu} = \Omega\left(\partial_\nu F^{\mu\nu} - \left[\partial_\nu\Omega^\dagger\Omega, F^{\mu\nu}\right]\right)\Omega^\dagger$$

$$ig\left[A'_\nu, F'^{\mu\nu}\right] = \Omega\left(ig\left[A_\nu, F^{\mu\nu}\right] + \left[\partial_\nu\Omega^\dagger\Omega, F^{\mu\nu}\right]\right)\Omega^\dagger$$

$$\Rightarrow \partial_\nu F'^{\mu\nu} + ig\left[A'_\nu, F'^{\mu\nu}\right] = \Omega\left(\partial_\nu F^{\mu\nu} + ig\left[A_\nu, F^{\mu\nu}\right]\right)\Omega^\dagger = 0$$

Gauge symmetry

Proof: if A_μ solves the YM equations, then A'_μ does too.

Short version: if A_μ is an extremum of the action ($\delta S = 0$), then A'_μ also satisfies $\delta S = 0$ due to gauge symmetry $S[A_\mu] = S[A'_\mu]$.

$$A'_\mu + \delta A'_\mu = \Omega \left(A_\mu + \delta A_\mu + \frac{1}{ig}\partial_\mu \right) \Omega^\dagger$$

$$\Rightarrow \delta A'_\mu = \Omega \delta A_\mu \Omega^\dagger$$

Variation is invariant:

$$\begin{aligned} \delta S[A'_\mu, \delta A'_\mu] &= -2 \int d^4x \, \mathrm{tr} \left[\left(\partial_\nu F'^{\mu\nu} + ig \left[A'_\nu, F'^{\mu\nu} \right] \right) \delta A'_\mu \right] \\ &= -2 \int d^4x \, \mathrm{tr} \left[\Omega \left(\partial_\nu F^{\mu\nu} + ig \left[A_\nu, F^{\mu\nu} \right] \right) \Omega^\dagger \Omega \delta A_\mu \Omega^\dagger \right] \\ &= \delta S[A_\mu, \delta A_\mu] = 0. \end{aligned}$$

Gauge fixing

A_μ and A'_μ are said to be gauge equivalent (belong to the same equivalence class) if there exists a gauge transformation Ω which satisfies

$$A'_\mu = \Omega(A_\mu + \frac{1}{ig}\partial_\mu)\Omega^\dagger$$

Equivalence classes are also known as "gauge orbits".

Lots of freedom to choose how a particular solution to the YM equations looks. Is there a solution within an equivalence class that is particularly simple? Is there a way to make the YM equations easier to solve by restricting the "gauge freedom"?

Idea: reduce the gauge freedom by "fixing" the gauge symmetry. Supplement YM equations with a gauge fixing condition $G[A_\mu] = 0$.

Gauge fixing

Supplement YM equations with a gauge fixing condition

$$G[A_\mu] = 0$$

What can $G[A_\mu]$ be? Gauge fixing condition must be realizable:

Suppose A_μ does not satisfy the gauge condition $G[A_\mu] \neq 0$. If G is realizable, then there must exist a gauge transformation Ω such that $A'_\mu = \Omega(A_\mu + \frac{1}{ig}\partial_\mu)\Omega^\dagger$ satisfies $G[A'_\mu] = 0$.

Gauge fixing

Some popular, commonly used gauge fixing conditions:

- Temporal (axial) gauge

$$A^0(x) = A_0(x) = 0, \qquad \forall x \in \mathbf{M}$$

  Similar: spatial axial gauges $A_i(x) = 0$

- Coulomb gauge

$$\partial_i A^i(x) = 0, \qquad \forall x \in \mathbf{M}$$

  Note: sum only over spatial indices $i \in \{1, 2, 3\}$

- Covariant (Lorenz) gauge

$$\partial_\mu A^\mu(x) = 0, \qquad \forall x \in \mathbf{M}$$

  Note: use of contracted 4-vector indices, invariant under Lorentz group

Temporal gauge

Temporal gauge $A_0(x) = 0$ is very useful for numerical simulations

Is temporal gauge realizable?
Consider A_μ with $A_0 \neq 0$. Can we find a gauge transformation Ω such that

$$A_0' = \Omega \left(A_0 + \frac{1}{ig} \partial_0 \right) \Omega^\dagger = 0$$

$\Rightarrow \Omega^\dagger(t, \vec{x})$ must satisfy

$$\partial_0 \Omega^\dagger(t, \vec{x}) = -igA_0(t, \vec{x}) \Omega^\dagger(t, \vec{x})$$

where $x^0 = t$ and $\vec{x} = (x^1, x^2, x^3)^T$

Temporal gauge

The equation

$$\partial_t \Omega^\dagger(t, \vec{x}) = -igA_0(t, \vec{x})\Omega^\dagger(t, \vec{x})$$

is solved by the path-ordered exponential

$$\Omega^\dagger(t, \vec{x}) = \mathcal{P} \exp\left(-ig \int_{-\infty}^{t} dt' A_0(t', \vec{x})\right)$$

with $\lim_{t \to -\infty} \Omega(t, \vec{x}) = \mathbf{1}$ and $\mathcal{P}$ denotes path ordering.

Path-ordering

Consider a smooth path $x(s) : \mathbb{R} \to \mathbf{M}$ parameterized by $s \in [0, 1]$ and the gauge field along the path $A(s) = \frac{dx^\mu(s)}{ds} A_\mu(x(s))$. The path ordering symbol $\mathcal{P}$ orders products according to the parameter s

$$\mathcal{P}\left[A(s)A(s')\right] = \begin{cases} A(s)A(s'), & \text{for } s \geq s' \\ A(s')A(s), & \text{for } s < s' \end{cases}$$

Convention: "left means later"

Alternative expression using the Heaviside step function θ

$$\mathcal{P}\left[A(s)A(s')\right] = \theta(s - s')A(s)A(s') + \theta(s' - s)A(s')A(s)$$

Path-ordered exponential

Definition as series

For $t_A < t_B$:

$$\mathcal{P}\exp\left(-ig\int_{t_A}^{t_B} dt' A_0(t',\vec{x})\right) = \mathbf{1} + \sum_{n=1}^{\infty}\frac{1}{n!}\mathcal{P}\left[-ig\int_{t_A}^{t_B} dt' A_0(t')\right]^n$$

$$= \mathbf{1} + \sum_{n=1}^{\infty}\frac{1}{n!}(-ig)^n\int_{t_A}^{t_B} dt'_1 \int_{t_A}^{t_B} dt'_2 \cdots \int_{t_A}^{t_B} dt'_n \mathcal{P}\left[A_0(t'_1)A_0(t'_2)\dots A_0(t'_n)\right]$$

$$= \mathbf{1} + \sum_{n=1}^{\infty}(-ig)^n\int_{t_A}^{t_B} dt'_1 \int_{t_A}^{t'_1} dt'_2 \cdots \int_{t_A}^{t'_{n-1}} dt'_n A_0(t'_1)A_0(t'_2)\dots A_0(t'_n)$$

Definition using products

Discretize interval $t \in [t_A, t_B]$ as set: $t \in \{t_0, t_1, \cdots, t_n\}$ with $t_0 = t_A$, $t_n = t_B$ and $\Delta t = (t_B - t_A)/n$.

$$\mathcal{P} \exp\left(-ig \int_{t_A}^{t_B} dt' A_0(t', \vec{x})\right) = \lim_{n\to\infty} \mathcal{P} \prod_{i=0}^{n} (\mathbf{1} - ig\Delta t A_0(t_i))$$

$$= \lim_{n\to\infty} (\mathbf{1} - ig\Delta t A_0(t_n))(\mathbf{1} - ig\Delta t A_0(t_{n-1}))\cdots(\mathbf{1} - ig\Delta t A_0(t_0))$$

Derivative of path ordered exponential

What is $\partial_t \Omega^\dagger(t)$?

$$\partial_t \Omega^\dagger(t, \vec{x}) = \lim_{\epsilon \to 0} \frac{\Omega^\dagger(t+\epsilon) - \Omega^\dagger(t)}{\epsilon}$$

From product definition of the path ordered exponential we know

$$\Omega^\dagger(t+\epsilon) \approx (\mathbf{1} - ig\epsilon A_0(t))\,\Omega^\dagger(t) + \mathcal{O}(\epsilon^2)$$

Inserting this into the differential quotient yields

$$\partial_t \Omega^\dagger(t) = -igA_0(t)\Omega^\dagger(t)$$

Temporal gauge: summary

Temporal gauge is defined as the condition

$$A_0 = 0$$

Temporal gauge is realizable: for any $A_0 \neq 0$ we can find a gauge transformation such that $A'_0 = 0$.

The gauge transformed fields are given by

$$A'_i = \Omega(A_i + \frac{1}{ig}\partial_i)\Omega^\dagger, \qquad A'_0 = 0$$

with the path-ordered exponential

$$\Omega^\dagger(t, \vec{x}) = \mathcal{P}\exp\left(-ig\int_{-\infty}^{t} dt' A_0(t', \vec{x})\right)$$

Yang-Mills equations in temporal gauge

Back to the Yang-Mills equations ...

Recall conjugate momentum π^i

$$\begin{aligned}\pi^i &= \frac{\partial \mathcal{L}}{\partial\left(\partial_0 A_i\right)} \\ &= F_{0i} \\ &= \partial_0 A_i - \partial_i A_0 + ig\left[A_0, A_i\right] \\ &= \partial_0 A_i\end{aligned}$$

Much simpler expression in temporal gauge.

Yang-Mills equations in temporal gauge

By eliminating A_0, the Yang-Mills equations can be solved as an initial value problem.

$$\partial_0 \pi^i = \partial_j F^{ji} + ig \left[A_j, F^{ji} \right]$$
$$\partial_0 A_i = \pi^i$$
$$\partial_i \pi^i + ig \left[A_i, \pi^i \right] = 0$$

It is sufficient to specify $A_i(t_0, \vec{x})$, $\pi^i(t_0, \vec{x})$ (assuming they satisfy the Gauss constraint) to find A_i and π^i at some later time $t_1 > t_0$.

Gauss constraint

The Yang-Mills equations (in temporal gauge) are

1) the equations of motion which follow from $\frac{\delta S[A_\mu]}{\delta A_i} = 0$ for $i \in \{1, 2, 3\}$

$$\partial_0 \pi^i = \partial_j F^{ji} + ig \left[A_j, F^{ji} \right]$$
$$\partial_0 A_i = \pi^i$$

2) and the Gauss constraint which follows from $\frac{\delta S[A_\mu]}{\delta A_0} = 0$

$$\partial_i \pi^i + ig \left[A_i, \pi^i \right] = 0$$

The Gauss constraint does not tell us about the "dynamics" of the fields (no time derivatives), but constrains the possible solutions for π^i and A_i.

Gauss constraint

If we choose initial values $\pi^i(t_0, \vec{x})$ and $A_i(t_0, \vec{x})$ at some initial time t_0, which satisfy the Gauss constraint

$$\partial_i \pi^i(t_0, \vec{x}) + ig \left[A_i(t_0, \vec{x}), \pi^i(t_0, \vec{x}) \right] = 0,$$

the solutions of the equations of motion (EOM) $\pi^i(t, \vec{x})$ and $A_i(t, \vec{x})$ with $t > t_0$ will also satisfy the constraint.
More generally: if $\pi^i(t_0, \vec{x})$ and $A_i(t_0, \vec{x})$ satisfy

$$\partial_i \pi^i(t_0, \vec{x}) + ig \left[A_i(t_0, \vec{x}), \pi^i(t_0, \vec{x}) \right] = C(\vec{x}) \in \mathfrak{su}(N_c),$$

then the solutions of the EOM will conserve the quantity C, i.e.

$$\partial_i \pi^i(t, \vec{x}) + ig \left[A_i(t, \vec{x}), \pi^i(t, \vec{x}) \right] = C(\vec{x}),$$

for $t > t_0$. The EOM conserve the constraint.

Gauss constraint

Gauss constraint with non-zero right hand side:

$$\partial_i \pi^i(t,\vec{x}) + ig\left[A_i(t,\vec{x}), \pi^i(t,\vec{x})\right] = C(\vec{x}),$$

Explicit proof: consider $C(t,\vec{x})$ as a function of time t. Then compute

$$\frac{dC}{dt} = \partial_i \partial_0 \pi^i(t,\vec{x}) + ig\left[\partial_0 A_i(t,\vec{x}), \pi^i(t,\vec{x})\right] + ig\left[A_i(t,\vec{x}), \partial_0 \pi^i(t,\vec{x})\right]$$

and insert EOM

$$\partial_0 \pi^i = \partial_j F^{ji} + ig\left[A_j, F^{ji}\right],$$
$$\partial_0 A_i = \pi^i,$$

to find $dC/dt = 0$. Also works without gauge fixing.

Gauss constraint

More general: the conservation of the Gauss constraint is a consequence of gauge symmetry.
The action $S[A_\mu]$ is invariant under gauge transformations

$$A'_\mu = \Omega\left(A_\mu + \frac{1}{ig}\partial_\mu\right)\Omega^\dagger$$

Consider a "small" gauge transformation

$$\Omega = \exp(ig\alpha) \simeq \mathbf{1} + ig\alpha + \mathcal{O}(\alpha^2)$$

We then have

$$\begin{aligned} A'_\mu &\simeq A_\mu + \partial_\mu\alpha + ig\left[A_\mu, \alpha\right] + \mathcal{O}(\alpha^2) \\ &\simeq A_\mu + D_\mu\alpha + \mathcal{O}(\alpha^2) \end{aligned}$$

Gauss constraint

Gauge symmetry: $S[A'_\mu] = S[A_\mu]$

Since $S[A_\mu + D_\mu\alpha] = S[A_\mu]$ (gauge invariance) we can expand $S[A'_\mu]$ up to linear order in α and set the coefficient to zero.

$$\delta S[A_\mu, D_\mu\alpha] = \int d^4x \frac{\delta S}{\delta A^a_\mu}(D_\mu\alpha)^a,$$

where

$$(D_\mu\alpha)^a = \partial_\mu\alpha^a - gf^{abc}A^b_\mu\alpha^c$$

Use integration by parts and anti-symmetry of f^{abc}

$$\delta S[A_\mu, D_\mu\alpha] = -\int d^4x\, \alpha^a \left(\left(\delta^{ac}\partial_\mu - gf^{abc}A^b_\mu\right)\frac{\delta S}{\delta A^c_\mu}\right)$$

Gauss constraint

Since S is gauge invariant, this expression must be identically zero

$$\delta S[A_\mu, D_\mu\alpha] = -\int d^4x\, \alpha^a \left(\left(\delta^{ac}\partial_\mu - gf^{abc}A^b_\mu \right) \frac{\delta S}{\delta A^c_\mu} \right) = 0,$$

which implies

$$\left(\delta^{ac}\partial_\mu - gf^{abc}A^b_\mu \right) \frac{\delta S}{\delta A^c_\mu} = 0$$

or simply

$$D_\mu \frac{\delta S}{\delta A_\mu} = 0, \qquad \frac{\delta S}{\delta A_\mu} = \frac{\delta S}{\delta A^a_\mu} t^a.$$

Gauss constraint

In temporal gauge ($A_0 = 0$) this leads to

$$\partial_0 \frac{\delta S}{\delta A_0} = D_i \frac{\delta S}{\delta A_i} = 0$$

if the EOM are satisfied $\frac{\delta S}{\delta A_i^c} = 0$. The constraint is conserved.

Without gauge fixing we find $D_0 \frac{\delta S}{\delta A_0} = 0$. If the constraint is satisfied at some time t_0, then it will also be satisfied at $t \neq t_0$.

Gauss constraint

Write $C(t, \vec{x}) = \frac{\delta S}{\delta A_0}$. The equation $D_0 C(t, \vec{x}) = 0$ is solved by

$$C(t_1, \vec{x}) = \Omega(t_1, t_0; \vec{x}) C(t_0, \vec{x}) \Omega^\dagger(t_1, t_0; \vec{x}),$$

with

$$\partial_t \Omega(t, t_0; \vec{x}) = -igA_0(t, \vec{x}) \Omega(t, t_0; \vec{x}), \qquad \Omega(t_0, t_0; \vec{x}) = \mathbf{1}.$$

This is solved by the path-ordered exponential

$$\Omega(t_1, t_0; \vec{x}) = \mathcal{P} \exp \left(-ig \int_{t_0}^{t_1} dt' A_0(t', \vec{x}) \right).$$

If $C(t_0, \vec{x}) = 0$ then also $C(t, \vec{x}) = 0$ for $t \neq t_0$.

Gauss constraint: summary

- The Gauss constraint follows from the variation of $S[A_\mu]$ with respect to A_0
- The equations of motion conserve the Gauss constraint
- The conservation of the Gauss constraint does not depend on the exact form of the EOM or the constraint, but is a consequence of gauge invariance

The more general theorem all this follows from is known as Noether's second theorem, which is valid for local (gauge), continuous symmetries of the action. Noether's first theorem applies to global continuous symmetries. See e.g. [arXiv:1601.03616] for a review on (gauge) symmetries and Noether's theorems.

Energy-momentum tensor

Energy-momentum tensor

Many quantities such as A_μ or $F_{\mu\nu}$ change under gauge transformations and can therefore not be physically observable. Physical observables must be gauge invariant.

One particular example: the energy-momentum tensor

$$T^{\mu\nu} = F^{a,\mu\rho} F^{a,\nu}_{\rho} - \frac{1}{4} g^{\mu\nu} F^{a,\rho\sigma} F^{a}_{\rho\sigma}$$

Invariance is easy to show: rewrite

$$T^{\mu\nu} = 2\,\mathrm{tr}\left(F^{\mu\rho} F^{\nu}_{\rho} - \frac{1}{4} g^{\mu\nu} F^{\rho\sigma} F_{\rho\sigma} \right)$$

and use $F'_{\mu\nu} = \Omega F_{\mu\nu} \Omega^\dagger$.

Energy-momentum tensor

Energy-momentum tensor (or stress-energy tensor)

$$T^{\mu\nu} = F^{a,\mu\rho} F^{a,\nu}_{\rho} - \frac{1}{4} g^{\mu\nu} F^{a,\rho\sigma} F^{a}_{\rho\sigma}$$

$T_{\mu\nu}$ is a main object of interest in the earliest stages of heavy-ion collisions. Many experimental observations (properties of particles measured in detectors) depend on $T_{\mu\nu}$ shortly after the collision.

- T^{00}: energy density
- T^{i0}: energy flux in along x^i axis
- T^{ij} for $i = j$: pressure density components
- T^{ij} for $i \neq j$: shear stress

Energy-momentum tensor

Energy-momentum tensor (or stress-energy tensor)

$$T^{\mu\nu} = F^{a,\mu\rho} F^{a,\nu}_{\rho} - \frac{1}{4} g^{\mu\nu} F^{a,\rho\sigma} F^{a}_{\rho\sigma}$$

and its conservation law

$$\partial_\mu T^{\mu\nu} = 0$$

can be derived from the invariance of $S[A_\mu]$ under space-time translations

$$x'^\mu = x^\mu + w^\mu$$

for arbitrary, constant translation vectors w^μ.
This follows from Noether's first theorem, which applies to global (x independent) continuous symmetries.

Electromagnetism

Electromagnetism as a $U(1)$ gauge theory

Electromagnetism is an Abelian gauge theory with a $U(1)$ gauge symmetry

The Lie group $U(1)$

$U(1)$ consists of complex numbers $u \in \mathbb{C}$ with $|u| = 1$

$$\begin{aligned} u &= \exp(i\theta) \in U(1), & \theta \in \mathbb{R} \\ uu' &= \exp(i\theta)\exp(i\theta') = \exp(i(\theta+\theta')) \in U(1), & \theta, \theta' \in \mathbb{R} \\ u^{-1} &= \exp(-i\theta) = u^* \in U(1) \end{aligned}$$

The Lie algebra of $U(1)$ is simply $\mathbb{R}$

Electromagnetism as a $U(1)$ gauge theory

Degrees of freedom: Abelian gauge fields $A_\mu : \mathbf{M} \to \mathbb{R}$

(Abelian) Field strength tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

Note: $U(1)$ is Abelian: no commutator term
Just a "single color component": no need for an index

Action

$$S[A_\mu] = \int d^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right)$$

Note: no trace

Action is invariant under $U(1)$ gauge symmetry

Electromagnetism as a $U(1)$ gauge theory

Gauge symmetry

Gauge transformations in (non-Abelian) Yang-Mills theory

$$A'_\mu(x) = \Omega(x)\left(A_\mu(x) + \frac{1}{ig}\partial_\mu\right)\Omega^\dagger(x)$$

with $\Omega : \mathbf{M} \to \mathrm{SU}(N_c)$, $g > 0$ Yang-Mills coupling constant

Gauge transformations in $U(1)$ gauge theory

$$A'_\mu(x) = \Omega(x)\left(A_\mu(x) + \frac{1}{ie}\partial_\mu\right)\Omega^*(x)$$

with $\Omega(x) : \mathbf{M} \to U(1)$, $e > 0$ elementary electric charge

Electromagnetism as a $U(1)$ gauge theory

Gauge symmetry

Gauge transformations in $U(1)$ gauge theory

$$A'_\mu(x) = \Omega(x)\left(A_\mu(x) + \frac{1}{ie}\right)\Omega^*(x)$$

with $\Omega : \mathbf{M} \to U(1)$, $e > 0$ elementary electric charge

Write $\Omega(x) = \exp(ie\alpha(x))$ with $\alpha : \mathbf{M} \to \mathbb{R}$.

$$\begin{aligned} A'_\mu(x) &= \Omega(x)\left(A_\mu(x) + \frac{1}{ie}\partial_\mu\right)\Omega^*(x) \\ &= \Omega(x)A_\mu(x)\Omega^*(x) + \frac{1}{ie}\Omega(x)\partial_\mu\Omega^*(x) \\ &= A_\mu(x) - \partial_\mu\alpha(x) \end{aligned}$$

Adding a gradient term to A_μ leaves $S[A_\mu]$ invariant

Electromagnetism as a $U(1)$ gauge theory

Gauge symmetry

Field strength tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

Gauge transformations:

$$F'_{\mu\nu} = \Omega F_{\mu\nu} \Omega^* = F_{\mu\nu}$$

In $U(1)$ gauge theory, the field strength tensor is gauge invariant and therefore a physical observable

$\Rightarrow$ Electric and magnetic fields are observables

Maxwell's equations

Vary $S[A_\mu]$ to obtain the classical equations of motion

$$\partial_\mu F^{\mu\nu} = 0$$

Use $F_{0i} = E_i$ and $F_{ij} = \varepsilon_{ijk} B_k$ to find

$$\nabla \cdot \vec{E} = 0, \qquad \frac{\partial \vec{E}}{\partial t} = \nabla \times \vec{B}$$

The other two Maxwell equations

$$\nabla \cdot \vec{B} = 0, \qquad \frac{\partial \vec{B}}{\partial t} = -\nabla \times \vec{E}$$

follow from the definition of the magnetic field $B_i = -\frac{1}{2}\varepsilon_{ijk} F_{jk}$ and the other two equations.