Yang-Mills theory, lattice gauge theory and simulations

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Overview

Introduction and physical context

Classical Yang-Mills theory

Lattice gauge theory

Simulating the Glasma in 2+1D

Lattice gauge theory

Motivation

Motivation

Recap: Yang-Mills equations in temporal gauge ($A_0 = 0$)
Equations of motion

$$\partial_0 \pi^i = \partial_j F^{ji} + ig \left[A_j, F^{ji}\right]$$
$$\partial_0 A_i = \pi^i$$

Gauss constraint

$$\partial_i \pi^i + ig \left[A_i, \pi^i\right] = 0$$

Assuming we have consistent initial conditions $A_i(t_0, \vec{x})$, $\pi^i(t_0, \vec{x})$, which satisfy the constraint, can we perform the "time evolution" from t_0 to $t > t_0$ numerically without violating the constraint?

Standard method: finite differences

Discretize Minkowski space $\mathbf{M}$ as a hypercubic lattice Λ with spacings a^μ.

$$\Lambda = \{x \in \mathbf{M} \,|\, x = \sum_{\mu=0}^{3} n_\mu \hat{a}^\mu, \quad n_\mu \in \mathbb{Z}\}, \quad \hat{a}^\mu = a^\mu \hat{e}_\mu \in \mathbf{M} \text{ (no sum)},$$

and unit vectors $\hat{e}_\mu$, e.g. $\hat{e}_0 = (1,0,0,0)^T$, $\hat{e}_1 = (0,1,0,0)^T$, etc. Use finite difference approximations for derivatives, e.g. the forward difference

$$\partial^F_\mu \phi(x) \equiv \frac{\phi(x+\hat{a}^\mu) - \phi(x)}{a^\mu} \simeq \partial_\mu \phi(x) + \mathcal{O}(a^\mu),$$

and the backward difference

$$\partial^B_\mu \phi(x) \equiv \frac{\phi(x) - \phi(x-\hat{a}^\mu)}{a^\mu} \simeq \partial_\mu \phi(x) + \mathcal{O}(a^\mu),$$

Yang-Mills theory on a lattice: first attempt

Naive approach: put Yang-Mills fields on the hypercubic lattice Λ

"Recipe" for the finite difference method:

- At each point $x \in \Lambda$ define a field value $A_\mu(x) \in \mathfrak{su}(N_c)$
- Derivatives of A_μ are approximated using finite differences ∂^F_ν or ∂^B_ν
- Integrals over **M** are approximated as sums over Λ

In principle, this recipe yields a finite difference approximation of the Yang-Mills equations

Problem: what about gauge symmetry?

Yang-Mills theory on a lattice: first try

Naive approach: put Yang-Mills fields on the hypercubic lattice Λ

Gauge field in the continuum:

$$A_\mu : \mathbf{M} \to \mathfrak{su}(N_c)$$

Gauge field on the lattice:

$$A_\mu : \Lambda \to \mathfrak{su}(N_c)$$

Discretized version of gauge transformation?

Consider a “lattice gauge transformation” $\Omega(x) : \Lambda \to \mathrm{SU}(N_c)$ acting on the gauge field A_μ:

$$A'_\mu(x) \equiv \Omega(x) \left(A_\mu(x) + \frac{1}{ig} \partial^F_\mu \right) \Omega^\dagger(x)$$

Yang-Mills theory on a lattice: first attempt

Naive lattice gauge transformation:

$$A'_\mu(x) \equiv \Omega(x)\left(A_\mu(x) + \frac{1}{ig}\partial^F_\mu\right)\Omega^\dagger(x)$$

$\Rightarrow A'_\mu$ is not traceless or hermitian, i.e. not an element of $\mathfrak{su}(N_c)$!

First term $\Omega(x)A_\mu(x)\Omega^\dagger(x)$ is traceless and hermitian.

However, the second term is neither:

$$\begin{aligned}\frac{1}{ig}\Omega(x)\partial^F_\mu\Omega^\dagger(x) &= \frac{1}{iga^\mu}\Omega(x)\left(\Omega^\dagger(x+\hat{a}^\mu) - \Omega^\dagger(x)\right)\\ &= \frac{1}{iga^\mu}\left(\Omega(x)\Omega^\dagger(x+\hat{a}^\mu) - \mathbf{1}\right)\end{aligned}$$

The finite difference approximation of the derivative ∂_μ in the gauge transformation is a problem.

Yang-Mills theory on a lattice: first attempt

As we saw previously, gauge symmetry guarantees us that the equations of motion (here in temporal gauge $A_0 = 0$)

$$\partial_0 \pi^i = \partial_j F^{ji} + ig \left[A_j, F^{ji} \right]$$
$$\partial_0 A_i = \pi^i$$

conserve the Gauss constraint

$$\partial_i \pi^i + ig \left[A_i, \pi^i \right] = 0$$

If we cannot properly formulate gauge symmetry in the discretized version, then there is no guarantee that the discretized Gauss constraint will not be violated.

Yang-Mills theory on a lattice: first attempt

Second problem with this approach: how exactly should one approximate a term like

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig\,[A_\mu, A_\nu] \quad ?$$

Should we use forward differences ∂_μ^F, backward differences ∂_μ^B or some other higher order finite difference scheme?

$\Rightarrow$ A lot of freedom in choosing the specific discretization. Should we just guess?

Can we construct a "consistent" discretization of Yang-Mills theory that has a conserved Gauss constraint without much guesswork?

Yang-Mills theory on a lattice: first attempt

The naive finite difference approach to solving the Yang-Mills equations on a lattice fails when considering gauge symmetry.

We need two "ingredients" to come up with a numerical method that retains some notion of gauge symmetry:

- Different degrees of freedom (other than A_μ), whose gauge transformation law does not involve derivatives of the gauge transformation matrices $\Omega(x)$: gauge links
- A method for deriving "consistent" discretized equations of motion with a conserved Gauss constraint: method of variational integrators

Variational integrators

Variational integrators: basic idea

Variational integrators are a specific numerical integrators that follow from a variational principle.

Usual finite difference approach:

- Vary action S to obtain equations of motion (EOM)
- Replace derivatives in EOM with finite difference approximations to obtain discrete EOM
- Solve discrete EOM on a computer

Variational integrator approach:

- Discretize action S first (replace derivatives with finite differences, integrals with sums, etc) to obtain discretized action S'
- Vary discrete action S' to obtain discrete EOM
- Solve discrete EOM on a computer

Variational integrators: basic idea

Variational integrators: "discretize first, then vary"

Advantage of a variational integrator: if the discretized action S' has some of the symmetry properties of the continuum action S, then the discrete EOM will also respect these symmetries.

Example: if some symmetry of the action S leads to some conservation law (Noethers theorem), then the discrete analogue of that symmetry for S' leads to a discretized version of that conservation law

In the context of Yang-Mills theory: a discretized version of the Yang-Mills action with gauge symmetry leads to discrete equations of motion that conserve a discrete version of the Gauss constraint

Example: planetary motion

Consider a simple mechanical (i.e. not field theoretical) model: the motion of planets around the sun

Trajectory of a planet (mass)

$$\vec{r}(t) = (x(t), y(t))^T$$

Action (mass $m = 1$)

$$S[\vec{r}(t)] = \int_{-\infty}^{\infty} dt \left(\frac{1}{2} (\partial_0 \vec{r})^2 - V(|\vec{r}(t)|) \right)$$

with potential (all constants set to one)

$$V(r) = -\frac{1}{r}$$

Example: planetary motion

Vary the action to derive the equations of motion

$$\delta S[\vec{r}(t), \delta\vec{r}] = \int_{-\infty}^{\infty} dt \left(-\partial_0^2 \vec{r} - \nabla V(\vec{r}(t))\right) \cdot \delta\vec{r}$$

Introduce momentum

$$\vec{p}(t) \equiv \partial_0 \vec{r}(t)$$

Equations of motion

$$\partial_0 \vec{p}(t) = -\nabla V(\vec{r}(t))$$
$$\partial_0 \vec{r}(t) = \vec{p}(t)$$

Example: planetary motion

Action is invariant under rotations

$$\vec{r}' = R\vec{r}, \qquad R = \begin{pmatrix} \cos\omega & -\sin\omega \\ \sin\omega & \cos\omega \end{pmatrix}$$

Action

$$S[\vec{r}'(t)] = \int_{-\infty}^{\infty} dt \left(\frac{1}{2} \left(\partial_0 \vec{r}'\right)^2 - V(|\vec{r}'(t)|) \right) = S[\vec{r}(t)]$$

Consequence: angular momentum is conserved

Example: planetary motion

Action is invariant under infinitesimal rotations

$$\vec{r}' = R\vec{r}, \qquad R = \begin{pmatrix} \cos\omega & -\sin\omega \\ \sin\omega & \cos\omega \end{pmatrix}$$

Expand for small angles ω

$$\vec{r}' = \vec{r} + \Omega\vec{r} + \mathcal{O}(\omega^2), \qquad \Omega = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}$$

Write $\delta\vec{r} = \Omega\vec{r}$ and vary action

$$\delta S[\vec{r}, \delta\vec{r}] = \int_{-\infty}^{\infty} dt\, [(-\partial_0\vec{p} - \nabla V(\vec{r})) \cdot \delta\vec{r} + \partial_0 (\vec{p} \cdot \delta\vec{r})] = 0$$

Note: $\delta\vec{r}(t)$ does not have compact support

Example: planetary motion

Action is invariant under infinitesimal rotations

$$\delta S[\vec{r}, \delta\vec{r}] = \int_{-\infty}^{\infty} dt\, [(-\partial_0 \vec{p} - \nabla V(\vec{r})) \cdot \delta\vec{r} + \partial_0 (\vec{p} \cdot \delta\vec{r})] = 0$$

Left term vanishes: equations of motion
Right term: yields conservation law (Noether's first theorem)

$$\partial_0 (\vec{p} \cdot \delta\vec{r}) = 0$$

Use $\delta r = \Omega\vec{r} = (-\omega y(t), \omega x(t))^T$ and find

$$\partial_0 L = \partial_0 (-p_x(t) y(t) + p_y(t) x(t)) = 0.$$

Angular momentum $L = -p_x y + p_y x$ is conserved.

Example: planetary motion

Let's simulate this system numerically!
Naive approach using forward differences: Forward Euler scheme

$$\partial_0 \vec{p}(t) = -\nabla V(\vec{r}) \qquad \Rightarrow \partial_0^F \vec{p}(t) = -\nabla V(\vec{r}(t))$$
$$\partial_0 \vec{r}(t) = \vec{p}(t) \qquad \Rightarrow \partial_0^F \vec{q}(t) = \vec{p}(t)$$

Discrete "time evolution": time step $a^0 = \Delta t$

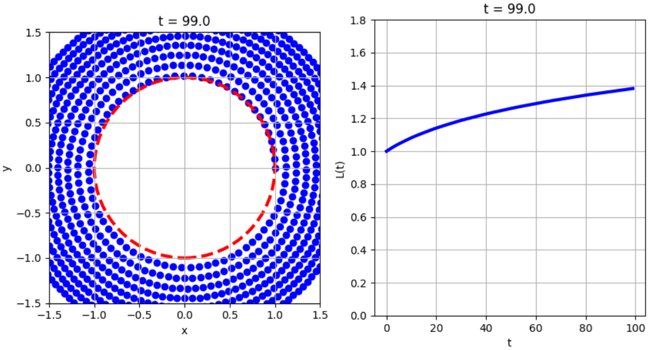
$$\vec{p}(t + \Delta t) = \vec{p}(t) - \Delta t \nabla V(\vec{r}(t))$$
$$\vec{q}(t + \Delta t) = \vec{q}(t) + \Delta t \vec{p}(t)$$

Conserved angular momentum?

$$L(t) = -p_x(t) y(t) + p_y(t) x(t)$$

Example: planetary motion

Animation of simulation data: trajectory $\vec{r}(t)$ and angular momentum $L(t)$ as a function of time t from Forward Euler scheme



Trajectory unstable, no conserved angular momentum

Example: planetary motion

Variational integrator approach: formulate discretized action with rotational symmetry built in

$$S[\vec{r}(t)] = \Delta t \sum_t \left(\frac{1}{2} \left(\partial_0^F \vec{r}(t) \right)^2 - V(|\vec{r}(t)|) \right)$$

Invariance:

$$V(|\vec{r}'(t)|) = V(|R\vec{r}(t)|) = V(|\vec{r}(t)|)$$

$$\partial_0^F \vec{r}'(t) = R \partial_0^F \vec{r}(t), \quad \Rightarrow \quad \left(\partial_0^F \vec{r}'(t) \right)^2 = \left(\partial_0^F \vec{r}(t) \right)^2$$

Example: planetary motion

Discrete action to be "varied":

$$S[\vec{r}(t)] = \Delta t \sum_t \left(\frac{1}{2} \left(\partial_0^F \vec{r}(t) \right)^2 - V(|\vec{r}(t)|) \right)$$

The action is now a function of the positions $\vec{r}(t)$ at the discrete times $t_0, t_1, t_2, \ldots$

The "variation" $\delta S[\vec{r}, \delta r]$ is now just the total differential dS.

I will keep using the $\delta S[\vec{r}, \delta r]$ notation anyways, even though I'm not using functional derivatives.

Example: planetary motion

Useful formulae for finite differences

The product rule(s)

$$\begin{aligned}\partial_0^B(f(t)g(t)) &= (f(t)g(t) - f(t-\Delta t)g(t-\Delta t))\,/\Delta t \\ &\quad + f(t-\Delta t)g(t)/\Delta t - f(t-\Delta t)g(t)/\Delta t \\ &= \partial_0^B f(t)g(t) + f(t-\Delta t)\partial_0^B g(t)\end{aligned}$$

and

$$\partial_0^F(f(t)g(t)) = \partial_0^F f(t)g(t) + f(t+\Delta t)\partial_0^F g(t)$$

Switching between forward/backward differences

$$\partial_0^F f(t) = \partial_0^B f(t+\Delta t)$$

Example: planetary motion

Variation of the discrete action

$$
\begin{aligned}
\delta S[\vec{r}, \delta\vec{r}] &= \Delta t \sum_t \left(\partial_0^F \vec{r}(t) \cdot \partial_0^F \delta\vec{r}(t) - \nabla V(|\vec{r}(t)|) \cdot \delta\vec{r}(t) \right) \\
&= \Delta t \sum_t \Bigg[\left(-\partial_0^B \partial_0^F \vec{r}(t) - \nabla V(|\vec{r}(t)|) \right) \cdot \delta\vec{r}(t) \\
&+ \partial_0^F \left(\partial_0^F \vec{r}(t) \cdot \delta\vec{r}(t) \right) \Bigg] = 0
\end{aligned}
$$

Second term vanishes, because $\delta r(t)$ has "compact support". Introduce $\vec{p}(t) = \partial_0^F \vec{r}(t)$. The discrete EOM then read

$$
\begin{aligned}
\partial_0^B \vec{p}(t) &= -\nabla V(|\vec{r}(t)|) \\
\partial_0^F \vec{r}(t) &= \vec{p}(t)
\end{aligned}
$$

Note: use of backward difference in first EOM

Example: planetary motion

Infinitesimal rotation with angle ω

$$\vec{r}' = \vec{r} + \Omega\vec{r} + \mathcal{O}(\omega^2) = \vec{r} + \delta\vec{r} + \mathcal{O}(\omega^2), \qquad \Omega = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}$$

Variation of action due to rotation

$$\delta S[\vec{r}, \delta\vec{r}] = \Delta t \sum_t \Big[\Big(-\partial_0^B \partial_0^F \vec{r}(t) - \nabla V(|\vec{r}(t)|) \Big) \cdot \delta\vec{r}(t) + \partial_0^F \Big(\partial_0^F \vec{r}(t) \cdot \delta\vec{r}(t) \Big) \Big] = 0$$

- First term vanishes (EOM)
- Second term under the sum must vanish, but $\delta r(t)$ does not have compact support

Example: planetary motion

In order to get $\delta S[\vec{r}, \delta\vec{r}] = 0$, the discrete conservation law must hold:

$$\partial_0^F \left(\partial_0^F \vec{r}(t) \cdot \delta\vec{r}(t) \right) = 0$$

$\Rightarrow$ discrete angular momentum

$$L(t) = -\partial_0^F x(t) y(t) + \partial_0^F y(t) x(t) = -p_x(t) y(t) + p_y(t) x(t)$$

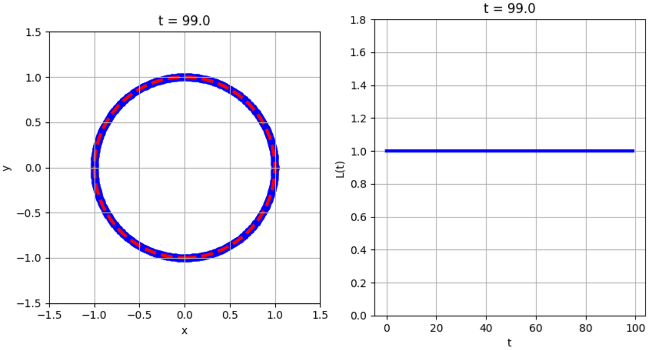
is conserved

$$\partial_0^F L(t) = 0$$

Everything completely analogous to the continuous model!

Example: planetary motion

Animation of simulation data: trajectory $\vec{r}(t)$ and angular momentum $L(t)$ as a function of time t from variational integrator



Trajectory stable, conserved angular momentum (up to numerical precision)

Example: planetary motion

Not all symmetries of the original (continuous) problem can be easily built into a discretized model.

Example: energy conservation

Energy conservation follows from the invariance under time translations $t' = t + \epsilon$.

$$\partial_0 E = \partial_0 \left(\frac{1}{2} (\partial_0 \vec{r}(t))^2 + V(|\vec{r}(t)|) \right) = 0$$

Discretizing the time coordinate breaks this symmetry and energy is not exactly conserved in the simulation.

Example: two-body problem

One more example: **the two body problem** $(m_1 = m_2 = 1)$

$$S[\vec{r}_1(t), \vec{r}_2(t)] = \int dt \left(\frac{1}{2} (\partial_0 \vec{r}_1)^2 + \frac{1}{2} (\partial_0 \vec{r}_2)^2 - V(|\vec{r}_1(t) - \vec{r}_2(t)|) \right)$$

Equations of motion from $\delta S = 0$:

$$\begin{aligned} \vec{p}_1 &\equiv \partial_0 \vec{r}_1 \\ \vec{p}_2 &\equiv \partial_0 \vec{r}_2 \\ \partial_0 \vec{p}_1 &= -\nabla_{(1)} V(|\vec{r}_1 - \vec{r}_2|) \\ \partial_0 \vec{p}_2 &= -\nabla_{(2)} V(|\vec{r}_1 - \vec{r}_2|) \end{aligned}$$

Example: two-body problem

$$S\left[\vec{r}_1(t), \vec{r}_2(t)\right] = \int dt \left(\frac{1}{2}\left(\partial_0 \vec{r}_1\right)^2 + \frac{1}{2}\left(\partial_0 \vec{r}_2\right)^2 - V(|\vec{r}_1(t) - \vec{r}_2(t)|)\right)$$

Symmetries and conservation laws:

- Invariance under rotations: $\vec{r}_i' = R\vec{r}_i$
  $\Rightarrow$ angular momentum conservation

$$\partial_0 L(t) = 0$$

- Invariance under spatial translations $\vec{r}_i' = \vec{r} + \vec{\epsilon}$
  $\Rightarrow$ linear momentum conservation

$$\partial_0(\vec{p}_1 + \vec{p}_2) = 0$$

- Invariance under time translations $t' = t + \epsilon$
  $\Rightarrow$ energy conservation

$$\partial_0 E = \partial_0 \left(\frac{1}{2}\vec{p}_1^2 + \frac{1}{2}\vec{p}_2^2 + V(|\vec{r}_1 - \vec{r}_2|)\right) = 0$$

Example: two-body problem

Discretized action for the two-body problem

$$S\left[\vec{r}_1(t), \vec{r}_2(t)\right] = \Delta t \sum_t \left(\frac{1}{2} \left(\partial_0^F \vec{r}_1\right)^2 + \frac{1}{2} \left(\partial_0^F \vec{r}_2\right)^2 - V(|\vec{r}_1(t) - \vec{r}_2(t)|) \right)$$

Symmetries and conservation laws:

- Invariance under rotations: $\vec{r}_i' = R\vec{r}_i$
  $\Rightarrow$ angular momentum conservation

$$\partial_0^F L(t) = 0$$

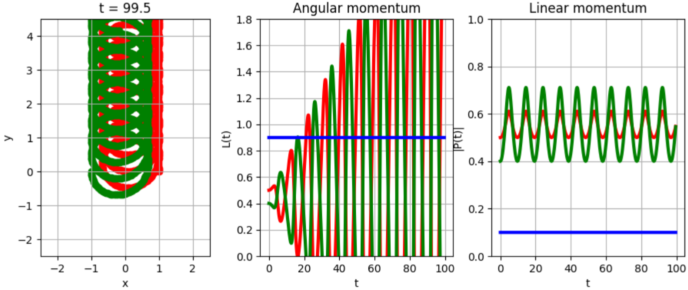
- Invariance under spatial translations $\vec{r}_i' = \vec{r} + \vec{\epsilon}$
  $\Rightarrow$ linear momentum conservation

$$\partial_0^F (\vec{p}_1(t) + \vec{p}_2(t)) = 0$$

- ~~Invariance under time translations $t' = t + \epsilon$~~
  ~~$\Rightarrow$ energy conservation~~

Example: two-body problem

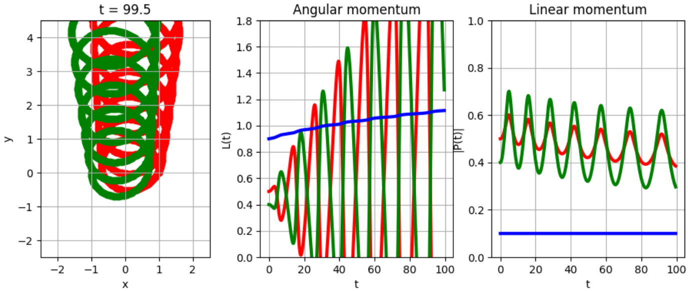
Motion of two bodies using variational integrator



Discrete angular momentum and linear momentum exactly conserved.

Example: two-body problem

Comparison: simple forward Euler scheme



Discrete angular momentum not conserved. Linear momentum happens to be conserved.

Variational integrators: summary

- The method of variational integrators removes a lot of guesswork when deriving numerical schemes to solve initial value problems.
- Discretized actions can "keep" symmetries of their respective continuum analogues
- Symmetries of discretized actions lead to discretized conservation laws (Noether's theorem - discrete version)

Yang-Mills on the lattice and gauge symmetries

We will construct a discretized action for Yang-Mills theory, which "keeps" gauge symmetry.

$\Rightarrow$ Conserved Gauss constraint when solving Yang-Mills equations numerically

Variational integrators: summary

Literature:

- J. E. Marsden and M. West, "Discrete mechanics and variational integrators", Acta Numerica, 2001
- Adrián J. Lew, Pablo Mata A, "A Brief Introduction to Variational Integrators", chapter 5 of Peter Betsch (editor), "Structure-preserving Integrators in Nonlinear Structural Dynamics and Flexible Multibody Dynamics", CISM International Centre for Mechanical Sciences 2016, Springer, Cham

Wilson lines

Wilson lines: definition

Alternative degrees of freedom to A_μ: Wilson lines

Consider a continuous path $\mathcal{C}$ given by $x(s) : [0,1] \to \mathbf{M}$ with parameter $s \in [0,1]$ and a gauge field A_μ. The Wilson line $U_\mathcal{C} \in \mathrm{SU}(N_c)$ of the gauge field A_μ is given by

$$U_\mathcal{C}[A_\mu] \equiv \mathcal{P}\exp\left(-ig\int_0^1 ds\,\frac{dx^\mu(s)}{ds}A_\mu(x(s))\right),$$

where $\mathcal{P}$ is the path-ordering symbol. The Wilson line is also sometimes written as

$$U_\mathcal{C}[A_\mu] \equiv \mathcal{P}\exp\left(-ig\int_\mathcal{C} dx^\mu A_\mu\right).$$

The Wilson line maps a gauge field A_μ to an element in $\mathrm{SU}(N_c)$ given a path $\mathcal{C}$.

Wilson lines: definition

Path-ordered exponential as a series (with $A(s) = \frac{dx^\mu(s)}{ds} A_\mu(x(s))$)

$$\mathcal{P}\exp\left(-ig\int_0^1 ds A(s)\right) = \mathbf{1} + \sum_{n=1}^{\infty}\frac{1}{n!}\mathcal{P}\left[-ig\int_0^1 ds A(s)\right]^n$$

$$= \mathbf{1} + \sum_{n=1}^{\infty}\frac{1}{n!}(-ig)^n \int_0^1 ds_1 \int_0^1 ds_2 \cdots \int_0^1 ds_n \mathcal{P}\left[A(s_1)A(s_2)\ldots A(s_n)\right]$$

$$= \mathbf{1} + \sum_{n=1}^{\infty}(-ig)^n \int_0^1 ds_1 \int_0^{s_1} ds_2 \cdots \int_0^{s_{n-1}} ds_n A(s_1)A(s_2)\ldots A(s_n)$$

Wilson lines: definition

Path-ordered exponential as a product.

Discretize interval $s \in [0,1]$ as set: $s \in \{s_0, s_1, \cdots, s_n\}$ with $s_0 = 0$, $s_n = 1$ and $\Delta s = 1/n$.

$$\mathcal{P} \exp\left(-ig \int_0^1 ds A(s)\right) = \lim_{n\to\infty} \mathcal{P} \prod_{i=0}^{n} \left(\mathbf{1} - ig\Delta s A(s_i)\right)$$

$$= \lim_{n\to\infty} \left(\mathbf{1} - ig\Delta s A(s_n)\right)\left(\mathbf{1} - ig\Delta s A(s_{n-1})\right)\cdots\left(\mathbf{1} - ig\Delta s A(s_0)\right)$$

where

$$A(s) = \frac{dx^\mu(s)}{ds} A_\mu(x(s))$$

Wilson lines: products

Consider two continuous paths $\mathcal{C}_1$ and $\mathcal{C}_2$: $\mathcal{C}_1$ starts at z_1 and ends at z_2 (parameterized by $x_1(s)$), $\mathcal{C}_2$ starts at z_2 and ends at z_3 (parameterized by $x_2(s)$). Define the "glued together" path $\mathcal{C}$ $x(s) : [0,1] \to \mathbf{M}$:

$$x(s) = \begin{cases} x_1(2s) & 0 \leq s < \frac{1}{2}, \\ x_2(2(s-\frac{1}{2})) & \frac{1}{2} \leq s \leq 1. \end{cases}$$

The Wilson line $U_{\mathcal{C}}[A_\mu]$ is then given by the product of $U_{\mathcal{C}_1}[A_\mu]$ and $U_{\mathcal{C}_2}[A_\mu]$:

$$U_{\mathcal{C}}[A_\mu] = U_{\mathcal{C}_2}[A_\mu] U_{\mathcal{C}_1}[A_\mu].$$

(Use product definition of $U_{\mathcal{C}}$ for explicit proof)

Wilson lines: inverse

Consider the Wilson line $U_{\mathcal{C}}[A_\mu]$. The Wilson line is an element of $\mathrm{SU}(N_c)$. What's the inverse $(U_{\mathcal{C}}[A_\mu])^{-1} = U_{\mathcal{C}}^\dagger[A_\mu]$ of $U_{\mathcal{C}}[A_\mu]$?
Approximation using products:

$$U_{\mathcal{C}}^\dagger[A_\mu] = \left[\mathcal{P}\exp\left(-ig\int_0^1 dsA(s)\right)\right]^\dagger$$

$$\approx \left[\left(\mathbf{1} - ig\Delta sA(s_n)\right)\left(\mathbf{1} - ig\Delta sA(s_{n-1})\right)\cdots\left(\mathbf{1} - ig\Delta sA(s_0)\right)\right]^\dagger$$

$$= \left(\mathbf{1} + ig\Delta sA(s_0)\right)\cdots\left(\mathbf{1} + ig\Delta sA(s_{n-1})\right)\left(\mathbf{1} + ig\Delta sA(s_n)\right)$$

Wilson lines: inverse

Approximated inverse Wilson line:

$$U^\dagger_{\mathcal{C}}[A_\mu] \approx (\mathbf{1} + ig\Delta s A(s_0)) \cdots (\mathbf{1} + ig\Delta s A(s_{n-1}))\,(\mathbf{1} + ig\Delta s A(s_n))$$

This is simply the Wilson line along the reversed path $\mathcal{C}^{-1}$ parametrized by $x'(s) = x(1-s)$.

Reparametrize: $s' = 1 - s$, $\Delta s' = \frac{s'_n - s'_0}{n} = \frac{s_0 - s_n}{n} = -\Delta s$

$$U^\dagger_{\mathcal{C}}[A_\mu] \approx (\mathbf{1} - ig\Delta s' A(1 - s'_0))\,(\mathbf{1} - ig\Delta s' A(1 - s'_1)) \cdots$$
$$\cdots (\mathbf{1} - ig\Delta s' A(1 - s'_{n-1}))\,(\mathbf{1} - ig\Delta s' A(1 - s'_n))$$

Take limit $n \to \infty$:

$$U^\dagger_{\mathcal{C}}[A_\mu] = \mathcal{P} \exp\left(-ig \int_0^1 ds' \frac{dx'^\mu(s')}{ds'} A_\mu(x'(s')) \right) = U_{\mathcal{C}^{-1}}[A_\mu].$$

Wilson lines: gauge transformations

Consider a path $\mathcal{C}$, a gauge field A_μ and a gauge transformation Ω. The Wilson line $U_\mathcal{C}[A'_\mu]$ of the gauge transformed field

$$A'_\mu = \Omega \left(A_\mu + \frac{1}{ig} \partial_\mu \right) \Omega^\dagger$$

is given by

$$U_\mathcal{C}[A'_\mu] = \Omega(x(1)) U_\mathcal{C}[A_\mu] \Omega^\dagger(x(0))$$

where $x(1)$ and $x(0)$ are the end and start points of $\mathcal{C}$.

Wilson lines: gauge transformations

Proof of gauge transformation behavior:

$$A'_\mu = \Omega\left(A_\mu + \frac{1}{ig}\partial_\mu\right)\Omega^\dagger,$$
$$U_\mathcal{C}[A'_\mu] = \Omega(x(1))U_\mathcal{C}[A_\mu]\Omega^\dagger(x(0))$$

Define

$$U_\mathcal{C}[A_\mu](s,s_0) = \mathcal{P}\exp\left(-ig\int_{s_0}^{s} ds'\,\frac{dx^\mu(s')}{ds'}A_\mu(x(s'))\right).$$

such that $U_\mathcal{C}[A_\mu](1,0) = U_\mathcal{C}[A_\mu]$.

Wilson lines: gauge transformations

Take derivative with respect to parameter s at the end point:

$$\frac{dU_{\mathcal{C}}[A_\mu](s,s_0)}{ds} \equiv \lim_{\Delta s\to 0} \frac{U_{\mathcal{C}}[A_\mu](s+\Delta s, s_0) - U_{\mathcal{C}}[A_\mu](s,s_0)}{\Delta s}$$

$$= \lim_{\Delta s\to 0} \frac{U_{\mathcal{C}}[A_\mu](s+\Delta s, s) - \mathbf{1}}{\Delta s} U_{\mathcal{C}}[A_\mu](s,s_0)$$

$$= \lim_{\Delta s\to 0} \frac{\mathbf{1} - ig\int_s^{s+\Delta s} ds' \frac{dx^\mu}{ds'} A_\mu(x(s')) - \mathbf{1}}{\Delta s} U_{\mathcal{C}}[A_\mu](s,s_0)$$

$$= -ig\frac{dx^\mu}{ds} A_\mu(x(s)) U_{\mathcal{C}}[A_\mu](s,s_0)$$

Wilson lines: gauge transformations

Wilson line $U_{\mathcal{C}}$ along $\mathcal{C}$ fulfills differential equation

$$\left(\frac{d}{ds} + ig\frac{dx^\mu}{ds}A_\mu(x(s))\right) U_{\mathcal{C}}[A_\mu](s, s_0) = 0$$

Together with the boundary condition $U_{\mathcal{C}}[A_\mu](s_0, s_0) = \mathbf{1}$, this is an equivalent definition to the series and product definitions from before.

Wilson lines: gauge transformations

Now take (dropping "$[A_\mu]$" for a more compact notation)

$$U'_{\mathcal{C}}(s, s_0) = \Omega(x(s))U_{\mathcal{C}}(s, s_0)\Omega^\dagger(x(s_0)),$$

where $\Omega(s) = \Omega(x(s))$ is an arbitrary gauge transformation along $\mathcal{C}$ and compute derivative w.r.t. s:

$$\begin{aligned}
\frac{dU'_{\mathcal{C}}(s, s_0)}{ds} &= \frac{d\Omega(s)}{ds}U_{\mathcal{C}}(s, s_0)\Omega^\dagger(s_0) + \Omega(s)\frac{dU_{\mathcal{C}}(s, s_0)}{ds}\Omega^\dagger(s_0) \\
&= \partial_\mu\Omega(x)\frac{dx^\mu}{ds}U_{\mathcal{C}}(s, s_0)\Omega^\dagger(s_0) \\
&+ ig\Omega(s)\frac{dx^\mu}{ds}A_\mu(x(s))U_{\mathcal{C}}(s, s_0)\Omega^\dagger(s_0) \\
&= ig\frac{dx^\mu}{ds}\left(\Omega A_\mu\Omega^\dagger + \frac{1}{ig}\Omega\partial_\mu\Omega^\dagger\right)_{x=x(s)}\Omega(s)U_{\mathcal{C}}(s, s_0)\Omega^\dagger(s_0)
\end{aligned}$$

Wilson lines: gauge transformations

Continuation from last slide:

$$\frac{dU'_{\mathcal{C}}(s,s_0)}{ds} = ig\frac{dx^\mu}{ds}\left(\Omega A_\mu \Omega^\dagger + \frac{1}{ig}\Omega\partial_\mu\Omega^\dagger\right)_{x=x(s)}\Omega(s)U_{\mathcal{C}}(s,s_0)\Omega^\dagger(s_0)$$

$$= ig\frac{dx^\mu}{ds}A'_\mu(x(s))U'_{\mathcal{C}}(s,s_0)$$

Therefore, $U'_{\mathcal{C}}(s,s_0)$ fulfills the differential equation for Wilson lines with A'_μ in place of A_μ.

The boundary condition

$$U_{\mathcal{C}}(s_0,s_0) = \mathbf{1}$$

also holds for $U'_{\mathcal{C}}(s_0,s_0)$:

$$\begin{aligned} U'_{\mathcal{C}}(s_0,s_0) &= \Omega(s_0)U_{\mathcal{C}}(s_0,s_0)\Omega^\dagger(s_0) \\ &= \Omega(s_0)\Omega^\dagger(s_0) \\ &= \mathbf{1}. \end{aligned}$$

Wilson lines: gauge transformations

The Wilson line $U_\mathcal{C}$ along the path $\mathcal{C}$ is given by

$$U_\mathcal{C}[A_\mu] \equiv \mathcal{P} \exp \left(-ig \int_0^1 ds \frac{dx^\mu(s)}{ds} A_\mu(x(s)) \right),$$

and transforms according to

$$U'_\mathcal{C}[A_\mu] = \Omega(x(1)) U_\mathcal{C}[A_\mu] \Omega^\dagger(x(0)).$$

Note: the gauge transformation law for Wilson lines does not involve derivatives of $\Omega(x)$.

If all this was already familiar: in differential geometry Wilson lines are known as holonomies or parallel transport.

Wilson loops

Now consider closed paths (loops) $\mathcal{C}$ with $x_0 = x(1) = x(0)$, then we have

$$U'_{\mathcal{C}}[A_\mu] = \Omega(x_0) U_{\mathcal{C}}[A_\mu] \Omega^\dagger(x_0)$$

and in particular

$$\mathrm{tr}\left[U'_{\mathcal{C}}[A_\mu]\right] = \mathrm{tr}\left[U_{\mathcal{C}}[A_\mu]\right].$$

The trace of a Wilson loop is gauge invariant.

Traces of Wilson loops are physical observables (in principle).

Wilson action

Gauge links

Back to the lattice discretization of **M**:

$$\Lambda = \{x \in \mathbf{M} \,|\, x = \sum_{\mu=0}^{3} n_\mu \hat{a}^\mu, \quad n_\mu \in \mathbb{Z}\}, \qquad \hat{a}^\mu = a^\mu \hat{e}_\mu \quad \text{(no sum)},$$

The shortest possible arcs on this lattice connect nearest neighbors (e.g. x and $x + \hat{a}^\mu$). The Wilson lines along these shortest arcs are called gauge links.

Instead of A_μ we will use gauge links as degrees of freedom on the lattice.

From now on: no Einstein sum convention, only explicit sums

Gauge links

Consider a path from x to $x + \hat{a}^\mu$:

$$x^\nu(s) = x^\nu + s\, a^\mu \delta^\nu_\mu, \quad s \in [0,1] \quad \text{(no sum implied)}$$

Gauge link:

$$\begin{aligned}
U_{x \to x + \hat{a}^\mu} &= \mathcal{P} \exp\left(-ig \int\limits_0^1 ds \sum_{\nu=0}^{3} \frac{dx^\nu(s)}{ds} A_\nu(x(s)) \right) \\
&= \mathcal{P} \exp\left(-ig \int\limits_0^1 ds \sum_{\nu=0}^{3} a^\mu \delta^\nu_\mu A_\nu(x(s)) \right) \\
&= \mathcal{P} \exp\left(-ig \int\limits_0^1 ds a^\mu A_\mu(x(s)) \right)
\end{aligned}$$

Gauge links

Gauge link from x to $x + \hat{a}^\mu$:

$$U_{x \to x+\hat{a}^\mu} = \mathcal{P} \exp \left(-ig \int_0^1 ds a^\mu A_\mu(x(s)) \right)$$

Gauge transformations:

$$U'_{x \to x+\hat{a}^\mu} = \Omega(x + \hat{a}^\mu) U_{x \to x+\hat{a}^\mu} \Omega^\dagger(x)$$

If the lattice spacing a^μ goes to zero (continuum limit), we can use the mid-point rule to approximate the integrals:

$$\begin{aligned} U_{x \to x+\hat{a}^\mu} &\approx \exp \left(-iga^\mu A_\mu(x(\frac{1}{2})) + \mathcal{O}(a^3) \right) \\ &\approx \exp \left(-iga^\mu A_\mu(x + \frac{1}{2}\hat{a}^\mu) + \mathcal{O}(a^3)) \right) \end{aligned}$$

Gauge links

In lattice gauge theory, the most common convention is to define

$$U_{x,\mu} = [U_{x \to x+\hat{a}^\mu}]^\dagger \approx \exp\left(iga^\mu A_\mu(x + \frac{1}{2}\hat{a}^\mu) \right)$$

as the gauge link from x to $x + \hat{a}^\mu$.

Notation: $U_{x,\mu}$

- "x" denotes the starting point
- "μ" denotes that the gauge link is aligned with lattice axis μ

Shorthand: "$x + \mu$" denotes the point x shifted by one lattice spacing along axis μ, i.e. "$x + \mu$" is short for $x + \hat{a}^\mu$

Gauge transformations:

$$U'_{x,\mu} = \Omega_x U_{x,\mu} \Omega^\dagger_{x+\mu}$$

Plaquettes and field strength

The smallest possible Wilson loop that we can formulate is a 1×1 loop, known as a "plaquette".

The plaquette $U_{x,\mu\nu}$ is a Wilson loop starting at x given by

$$\begin{aligned} U_{x,\mu\nu} &\equiv U_{x,\mu} U_{x+\mu,\nu} U_{x+\mu+\nu,-\mu} U_{x+\nu,-\nu} \\ &= U_{x,\mu} U_{x+\mu,\nu} U^{\dagger}_{x+\nu,\mu} U^{\dagger}_{x,\nu} \end{aligned}$$

where we define $U_{x+\mu,-\mu} = U^{\dagger}_{x,\mu}$, etc.

Gauge transformation:

$$U'_{x,\mu\nu} = \Omega_x U_{x,\mu\nu} \Omega^{\dagger}_x$$

Trace of the plaquette is gauge invariant:

$$\mathrm{tr}[U'_{x,\mu\nu}] = \mathrm{tr}[U_{x,\mu\nu}]$$

Plaquettes and field strength

Plaquette in the continuum limit $a^\mu \to 0$:

Simple case first: assume that gauge field A_μ is Abelian, then all gauge links $U_{x,\mu}$ on the lattice commute.

$$U_{x,\mu} \approx \exp\left(iga^\mu A_\mu(x + \frac{1}{2}\hat{a}^\mu) + \mathcal{O}(a^3)\right)$$

Compute plaquette:

$$\begin{aligned}
U_{x,\mu\nu} &\equiv U_{x,\mu} U_{x+\mu,\nu} U_{x+\mu+\nu,-\mu} U_{x+\nu,-\nu} \\
&= U_{x,\mu} U_{x+\mu,\nu} U^\dagger_{x+\nu,\mu} U^\dagger_{x,\nu} \\
&\approx \exp\left(iga^\mu a^\nu \left(\partial^F_\mu A_\nu(x + \frac{1}{2}\hat{a}^\nu) - \partial^F_\nu A_\mu(x + \frac{1}{2}\hat{a}^\mu)\right)\right) \\
&\simeq \exp\left(iga^\mu a^\nu F_{\mu\nu}(x + \frac{1}{2}\hat{a}^\mu + \frac{1}{2}\hat{a}^\nu) + \mathcal{O}(a^4)\right)
\end{aligned}$$

Note: no Einstein summation over $\mu, \nu, \ldots$

Plaquettes and field strength

Use Baker-Campbell-Hausdorff formula derive that

$$U_{x,\mu\nu} \simeq \exp\left(iga^\mu a^\nu F_{\mu\nu}(x + \frac{1}{2}\hat{a}^\mu + \frac{1}{2}\hat{a}^\nu) + \mathcal{O}(a^4) \right)$$

also if A_μ is non-Abelian.

Baker-Campbell-Hausdorff: given two algebra elements $X, Y \in \mathfrak{su}(N_c)$, we have $Z \in \mathfrak{su}(N_c)$ such that

$$e^{iX} e^{iY} = e^{iZ}$$

and

$$Z = X + Y + \frac{i}{2}[X, Y] - \frac{1}{12}[X, [X, Y]] + \frac{1}{12}[Y, [X, Y]] + \dots$$

Plaquettes and field strength

Plaquette in the continuum limit $a^\mu \to 0$:

$$U_{x,\mu\nu} \simeq \exp\left(iga^\mu a^\nu F_{\mu\nu}(x + \frac{1}{2}\hat{a}^\mu + \frac{1}{2}\hat{a}^\nu) + \mathcal{O}(a^4) \right)$$
$$\simeq \mathbf{1} + iga^\mu a^\nu F_{\mu\nu} - \frac{1}{2}\left(ga^\mu a^\nu F_{\mu\nu}\right)^2 + \mathcal{O}(a^4)$$

Combine this to

$$\mathrm{tr}\left[\mathbf{1} - \frac{1}{2}U_{x,\mu\nu} - \frac{1}{2}U^\dagger_{x,\mu\nu}\right] \simeq \frac{1}{2}\left(ga^\mu a^\nu\right)^2 \mathrm{tr}\left[F^2_{\mu\nu}\right] + \mathcal{O}(a^6)$$

Note: order of the error term is not immediately obvious.
For a detailed derivation (and more), see [arXiv:hep-lat/0203008]

The Wilson action

Now we can construct an approximation of the Yang-Mills action using plaquettes.

1) Rewrite Yang-Mills action in "F^2" terms with lowered indices.

$$S[A_\mu] = \int d^4x \left(\sum_i \mathrm{tr}\left[F_{0i}^2\right] - \frac{1}{2} \sum_{i,j} \mathrm{tr}\left[F_{ij}^2\right] \right)$$

2) Approximate integral over **M** as sum over Λ

$$S[A_\mu] \approx V \sum_x \left(\sum_i \mathrm{tr}\left[F_{0i}^2\right] - \frac{1}{2} \sum_{i,j} \mathrm{tr}\left[F_{ij}^2\right] \right)$$

with space-time cell volume $V = \prod_\mu a^\mu$

The Wilson action

Yang-Mills action:

$$S[A_\mu] \approx V \sum_x \left(\sum_i \mathrm{tr}\left[F_{0i}^2\right] - \frac{1}{2} \sum_{i,j} \mathrm{tr}\left[F_{ij}^2\right] \right)$$

3) Replace "F^2" terms with plaquettes

$$S[A_\mu] \simeq V \sum_x \Bigg(\sum_i \frac{2}{(ga^0 a^i)^2} \mathrm{tr}\left[\mathbf{1} - \frac{1}{2} U_{x,0i} - \frac{1}{2} U^\dagger_{x,0i}\right] - \sum_{i,j} \frac{1}{(ga^i a^j)^2} \mathrm{tr}\left[\mathbf{1} - \frac{1}{2} U_{x,ij} - \frac{1}{2} U^\dagger_{x,ij}\right] \Bigg) + \mathcal{O}(a^2)$$

with $V = \prod_\mu a^\mu$.

This approximation of the Yang-Mills action with 1×1 loops is the Wilson action.

The Wilson action

Rearrange some terms, drop additive constant:

$$S[U] = -\frac{V}{g^2}\sum_x \left(\sum_i \frac{2}{(a^0 a^i)^2} \mathrm{Re}\,\mathrm{tr}\,[U_{x,0i}] - \sum_{i,j} \frac{1}{(a^i a^j)^2} \mathrm{Re}\,\mathrm{tr}\,[U_{x,ij}] \right)$$

Original papers:

- K. G. Wilson, "Confinement of Quarks", PRD 10 (1974), $\sim$ 4800 citations
- J. .B. Kogut and L. Susskind, "Hamiltonian Formulation of Wilson's Lattice Gauge Theories", PRD 11 (1975), $\sim$ 1700 citations

Lattice gauge invariance

The Wilson action is invariant under a discrete version of gauge transformations: lattice gauge transformations

Instead of $\Omega : \mathbf{M} \to \mathrm{SU}(N_c)$, we now have $\Omega : \Lambda \to \mathrm{SU}(N_c)$ with gauge links $U_{x,\mu}$ transforming as

$$U'_{x,\mu} = \Omega_x U_{x,\mu} \Omega^\dagger_{x+\mu},$$

and plaquettes transforming as

$$U'_{x,\mu\nu} = \Omega_x U_{x,\mu\nu} \Omega^\dagger_x.$$

The trace of the plaquette is invariant

$$\mathrm{tr}[U'_{x,\mu\nu}] = \mathrm{tr}[U_{x,\mu\nu}]$$

Lattice gauge invariance

The Wilson action

$$S[U] = -\frac{V}{g^2}\sum_x \left(\sum_i \frac{2}{(a^0 a^i)^2} \mathrm{Re\,tr}\,[U_{x,0i}] - \sum_{i,j} \frac{1}{(a^i a^j)^2} \mathrm{Re\,tr}\,[U_{x,ij}] \right)$$

is constructed from traces over plaquette and is invariant, i.e.

$$S[U'] = S[U].$$

We therefore have a discretized action

- with the correct continuum limit, up to errors $\mathcal{O}(a^2)$.
- with a discrete version of gauge invariance.

Even better approximations exist (increasing the order of the error term) and are gauge invariant as long as they are constructed from closed Wilson loops on the lattice.

Alternative form of the Wilson action

There is a different way of writing the Wilson action, where the continuum limit is easier to see.

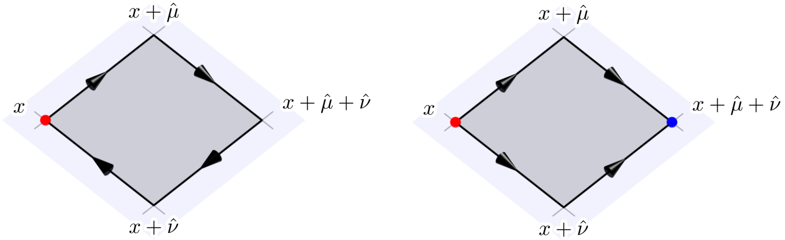
Introduce "L-shaped" variables

$$C_{x,\mu\nu} = U_{x,\mu} U_{x+\mu,\nu} - U_{x,\nu} U_{x+\nu,\mu}$$

which transform like

$$C'_{x,\mu\nu} = \Omega_x C_{x,\mu\nu} \Omega^\dagger_{x+\mu+\nu}$$

Alternative form of the Wilson action



a) path traced by $U_{x,\mu\nu}$

$$U_{x,\mu\nu} = U_{x,\mu} U_{x+\mu,\nu} U^{\dagger}_{x+\nu,\mu} U^{\dagger}_{x,\nu}$$

Gauge transformation

$$U'_{x,\mu\nu} = \Omega_x U_{x,\mu\nu} \Omega^{\dagger}_x$$

b) path traced by $C_{x,\mu\nu}$

$$C_{x,\mu\nu} = U_{x,\mu} U_{x+\mu,\nu} - U_{x,\nu} U_{x+\nu,\mu}$$

Gauge transformation

$$C'_{x,\mu\nu} = \Omega_x C_{x,\mu\nu} \Omega^{\dagger}_{x+\mu+\nu}$$

Alternative form of the Wilson action

A quick calculation shows:

$$\frac{1}{2} C_{x,\mu\nu} C^\dagger_{x,\mu\nu} = \mathbf{1} - \frac{1}{2} U_{x,\mu\nu} - \frac{1}{2} U^\dagger_{x,\mu\nu}$$

This is an exact relation.

The Wilson action can be written as

$$S[U] = \frac{V}{g^2} \sum_x \left(\sum_i \frac{1}{(a^0 a^i)^2} \mathrm{tr}\left[C_{x,0i} C^\dagger_{x,0i} \right] - \sum_{i,j} \frac{1}{2\,(a^i a^j)^2} \mathrm{tr}\left[C_{x,ij} C^\dagger_{x,ij} \right] \right)$$

Define $\tilde{C}_{x,\mu\nu} = \frac{1}{g a^\mu a^\nu} C_{x,\mu\nu}$:

$$S[U] = V \sum_x \left(\sum_i \mathrm{tr}\left[\tilde{C}_{x,0i} \tilde{C}^\dagger_{x,0i} \right] - \sum_{i,j} \frac{1}{2} \mathrm{tr}\left[\tilde{C}_{x,ij} \tilde{C}^\dagger_{x,ij} \right] \right)$$

Alternative form of the Wilson action

Wilson action:

$$S[U] = V \sum_x \left(\sum_i \mathrm{tr}\left[\tilde{C}_{x,0i} \tilde{C}^\dagger_{x,0i}\right] - \sum_{i,j} \frac{1}{2} \mathrm{tr}\left[\tilde{C}_{x,ij} \tilde{C}^\dagger_{x,ij}\right] \right)$$

Yang-Mills action:

$$S[A] = \int d^4x \left(\sum_i \mathrm{tr}\left[F_{0i} F^\dagger_{0i}\right] - \frac{1}{2} \sum_{i,j} \mathrm{tr}\left[F_{ij} F^\dagger_{ij}\right] \right)$$

The above form of the Wilson action can be a good starting point for making modifications.

- A. Ipp, DM, "Implicit schemes for real-time lattice gauge theory", [arXiv:1804.01995 [hep-lat]]

Variation of the Wilson action

Variation of the Wilson action

Obtain discretized equations of motion and discretized Gauss constraint from variation of the Wilson action:

$$S[U] = -\frac{V}{g^2} \sum_x \left(\sum_i \frac{2}{(a^0 a^i)^2} \mathrm{Re}\,\mathrm{tr}\,[U_{x,0i}] - \sum_{i,j} \frac{1}{(a^i a^j)^2} \mathrm{Re}\,\mathrm{tr}\,[U_{x,ij}] \right)$$

Degrees of freedom: gauge links $U_{x,\mu}$

Variation with respect to gauge links:

$$\delta S[U, \delta U] = 0$$

Note: since gauge links are elements of $\mathrm{SU}(N_c)$, we can't vary the matrix elements of $U_{x,\mu}$ independently.

$$U_{x,\mu} U^\dagger_{x,\mu} = \mathbf{1}, \qquad \det U_{x,\mu} = 1$$

Variation of the Wilson action

We need to make sure that we perform the variation of $S[U]$ "without leaving" $\mathrm{SU}(N_c)$, i.e. without violating the unitary constraint

$$U_{x,\mu} U^\dagger_{x,\mu} = \mathbf{1},$$

and the determinant constraint

$$\det U_{x,\mu} = 1.$$

Geometrical picture: $\mathrm{SU}(2)$ is isomorphic to S^3 (3-sphere)

Variation of the Wilson action

Two approaches to correctly varying $S[U]$:

1. Method of Lagrangian multipliers

Example: $U(1)$ lattice gauge theory

$$S'[U,\lambda] = -\frac{V}{g^2}\sum_x\left(\sum_i \frac{1}{(a^0 a^i)^2}\mathrm{Re}\,U_{x,0i} - \sum_{i,j}\frac{1}{2\,(a^i a^j)^2}\mathrm{Re}\,U_{x,ij}\right)$$

$$+ V\sum_{x,\mu}\lambda_{x,\mu}\left(|U_{x,\mu}|^2 - 1\right)$$

with

$$\delta S'[U,\lambda;\delta U,\delta\lambda] = 0$$

Potentially very tedious calculation, especially for $\mathrm{SU}(N_c)$

2. Construct constraint preserving perturbation $\delta U_{x,\mu}$

Variation of the Wilson action

Easier approach: choose $\delta U_{x,\mu}$ such that the perturbed gauge link $\tilde{U}_{x,\mu} = U_{x,\mu} + \delta U_{x,\mu}$ is still an element of $\mathrm{SU}(N_c)$ if $\delta U_{x,\mu}$ is infinitesimal, i.e.

$$\tilde{U}_{x,\mu}\tilde{U}^{\dagger}_{x,\mu} \simeq \mathbf{1} + \mathcal{O}(|\delta U|^2), \qquad \det \tilde{U}_{x,\mu} \simeq 1 + \mathcal{O}(|\delta U|^2).$$

Then, perturb action:

$$S[\tilde{U}] \simeq S[U] + \delta S[U, \delta U] + \mathcal{O}(|\delta U|^2)$$

This way $\delta S[U, \delta U]$ corresponds to the constrained variation of the action.

Variation of the Wilson action

Consider the perturbed matrix $\tilde{U} = U + \delta U$, where δU is a "small" perturbation. We have

- $\det \tilde{U} = \det U \det \left[\mathbf{1} + U^\dagger \delta U\right] \simeq 1 + \mathrm{tr}\left[U^\dagger \delta U\right] + \mathcal{O}(|\delta U|^2)$
- $\tilde{U}^\dagger \tilde{U} \simeq \mathbf{1} + \delta U^\dagger U + U^\dagger \delta U + \mathcal{O}(|\delta U|^2)$

The perturbation needs to satisfy

$$\delta U^\dagger + U^\dagger \delta U U^\dagger = 0,$$
$$\mathrm{tr}\left[U^\dagger \delta U\right] = 0.$$

These equations are satisfied by the following form:

$$\delta U \equiv i \delta A U,$$

where $\delta A \in \mathfrak{su}(N_c)$ is a "small", traceless, hermitian matrix.

Variation of the Wilson action

Procedure: perturb each link according to

$$U_{x,\mu} \to \tilde{U}_{x,\mu} = U_{x,\mu} + iga^{\mu}\delta A_{x,\mu} U_{x,\mu},$$

and compute the change of the action

$$S[\tilde{U}] \simeq S[U] + \delta S[U, \delta A] + \mathcal{O}(|\delta A|^2).$$

The variation δS is given by

$$\delta S[U, \delta A] = V \sum_{x,\mu,a} \text{``}\frac{\delta S}{\delta A^a_{x,\mu}}\text{''}\, \delta A^a_{x,\mu}.$$

The above form requires summation by parts.

We explicitly work through one example: the derivation of the discretized Gauss constraint

Wilson action:

$$S[U] = -\frac{V}{g^2}\sum_x \left(\sum_i \frac{2}{(a^0 a^i)^2} \mathrm{Re}\,\mathrm{tr}\,[U_{x,0i}] - \sum_{i,j} \frac{1}{(a^i a^j)^2} \mathrm{Re}\,\mathrm{tr}\,[U_{x,ij}] \right)$$

Variation w.r.t. $U_{x,0}$: Gauss constraint

$$U_{x,0i} = U_{x,0} U_{x+0,i} U^\dagger_{x+i,0} U^\dagger_{x,i}$$

Gauss constraint on the lattice

Variation of the relevant term:

$$\delta \sum_{x,i} \mathrm{tr}\left[U_{x,0i}\right] = \sum_{x,i} \mathrm{tr}\left[\delta U_{x,0} U_{x+0,i} U^\dagger_{x+i,0} U^\dagger_{x,i} + U_{x,0} U_{x+0,i} \delta U^\dagger_{x+i,0} U^\dagger_{x,i}\right]$$

$$= \sum_{x,i} \mathrm{tr}\left[\delta U_{x,0} U_{x+0,i} U^\dagger_{x+i,0} U^\dagger_{x,i} + U^\dagger_{x,i} U_{x,0} U_{x+0,i} \delta U^\dagger_{x+i,0}\right]$$

$$= \sum_{x,i} \mathrm{tr}\left[iga^0 \delta A_{x,0} U_{x,0i} - iga^0 U_{x+i,-i0} \delta A_{x+i,0}\right]$$

$$= iga^0 \sum_{x,i} \mathrm{tr}\left[\delta A_{x,0} \left(U_{x,0i} - U_{x,-i0}\right)\right]$$

Gauss constraint on the lattice

Variation of the relevant term:

$$\delta \sum_{x,i} \mathrm{tr}\left[U_{x,0i}\right] = iga^0 \sum_{x,i} \mathrm{tr}\left[\delta A_{x,0}\left(U_{x,0i} - U_{x,-i0}\right)\right]$$

Take real part:

$$\begin{aligned}
\delta \sum_{x,i} \mathrm{Re}\,\mathrm{tr}\left[U_{x,0i}\right] &= -ga^0 \sum_{x,i} \mathrm{Im}\,\mathrm{tr}\left[\delta A_{x,0}\left(U_{x,0i} - U_{x,-i0}\right)\right] \\
&= -ga^0 \sum_{x,i} \mathrm{Im}\,\mathrm{tr}\left[\delta A_{x,0}\left(U_{x,0i} + U_{x,0-i}\right)\right] \\
&= -ga^0 \sum_{x,i,a} \delta A^a_{x,0} \mathrm{Im}\,\mathrm{tr}\left[t^a\left(U_{x,0i} + U_{x,0-i}\right)\right] \\
&= -\frac{ga^0}{2} \sum_{x,a} \delta A^a_{x,0} \sum_i P^a\left(U_{x,0i} + U_{x,0-i}\right)
\end{aligned}$$

with $P^a(X) \equiv 2\,\mathrm{Im}\,\mathrm{tr}\left[t^a X\right]$.

Gauss constraint on the lattice

Variation of the Wilson action w.r.t. temporal links $U_{x,0}$:

$$\delta S[U,\delta A] = V \sum_{x,a} \delta A^a_{x,0} \sum_i \frac{1}{g a^0 (a^i)^2} P^a \left(U_{x,0i} + U_{x,0-i}\right)$$

Vary all $U_{x,0}$ independently and require $\delta S = 0$:

$$\sum_i \frac{1}{g a^0 \left(a^i\right)^2} P^a \left(U_{x,0i} + U_{x,0-i}\right) = 0.$$

This is the discrete Gauss constraint.

Compare to continuum limit:

$$\sum_i D_i F^{0i}(x) = \sum_i \left(\partial_i F^{0i}(x) + ig\left[A_i(x), F^{0i}(x)\right]\right) = 0$$

Gauss constraint on the lattice

Check the continuum limit for the discrete Gauss constraint:

$$\sum_i \frac{1}{ga^0\left(a^i\right)^2} P^a\left(U_{x,0i} + U_{x,0-i}\right)$$

$$= \sum_i \frac{1}{ga^0\left(ga^0 a^i\right)^2} P^a\left(U_{x,0i} + U_{x,-i} U_{x-i,i0} U_{x-i,i}\right)$$

$$= \sum_i \frac{1}{\left(a^i\right)^2} P^a\left(U_{x,0i} - U^\dagger_{x-i,i} U^\dagger_{x-i,i0} U^\dagger_{x,-i}\right)$$

$$= \sum_i \frac{1}{ga^0\left(a^i\right)^2} P^a\left(U_{x,0i} - U^\dagger_{x-i,i} U_{x-i,0i} U_{x-i,i}\right)$$

Then use

$$U_{x,0i} \simeq \exp\left(iga^0 a^i F_{0i}(\tilde{x}) + \mathcal{O}(a^4)\right),$$

where $\tilde{x} = x + \frac{1}{2}\hat{a}^0 + \frac{1}{2}\hat{a}^i$ is the center of the plaquette.

Gauss constraint on the lattice

Check the continuum limit for the discrete Gauss constraint:

$$
\begin{aligned}
P^a\left(U_{x,0i}\right) &\simeq P^a\left(\exp\left(iga^0a^iF_{0i}(\tilde{x})\right)\right)\\
&\simeq P^a\left(\mathbf{1}+iga^0a^iF_{0i}(\tilde{x})\right)\\
&\simeq ga^0a^iF^a_{0i}(\tilde{x})
\end{aligned}
$$

and

$$
\begin{aligned}
P^a\left(U^\dagger_{x-i,i}U_{x-i,0i}U_{x-i,i}\right) &\simeq ga^0a^iF^a_{0i}(\tilde{x}-\hat{a}^i)\\
&\quad+\sum_{b,c}\left(ga^i\right)^2a^0f^{abc}A^b_i(x-\frac{1}{2}\hat{a}^i)F^c_{0i}(\tilde{x}-\hat{a}^i)
\end{aligned}
$$

Gauss constraint on the lattice

Check the continuum limit for the discrete Gauss constraint:

Insert into original expression:

$$\sum_i \frac{1}{ga^0 \left(a^i\right)^2} P^a \left(U_{x,0i} - U^\dagger_{x-i,i} U_{x-i,0i} U_{x-i,i} \right)$$

$$\simeq \sum_i \frac{1}{a^i} \left(F^a_{0i}(\tilde{x}) - F^a_{0i}(\tilde{x} - \hat{a}^i) \right) + \sum_{i,b,c} g f^{abc} A^b_i (x - \frac{1}{2}\hat{a}^i) F^c_{0i}(\tilde{x} - \hat{a}^i)$$

$$\simeq \sum_i \partial_i F^a_{0i}(x) + \sum_{i,b,c} g f^{abc} A^b_i(x) F^c_{0i}(x)$$

$$= 0$$

The discrete Gauss constraint has the correct continuum limit. Determining the exact order of the error term takes more work: it's $\mathcal{O}(a^2)$ – same as the Wilson action.

Equations of motion on the lattice

We find the equations of motion (EOM) by varying $S[U]$ with respect to spatial links $U_{x,i}$.

Discrete equations of motion

$$\frac{1}{(a^0 a^i)^2} P^a \left(U_{x,i,0} + U_{x,i,-0}\right) = \sum_j \frac{1}{(a^i a^j)^2} P^a \left(U_{x,i,j} + U_{x,i,-j}\right)$$

Discrete Gauss constraint

$$\sum_i \frac{1}{(a^0 a^i)^2} P^a \left(U_{x,0,i} + U_{x,0,-i}\right) = 0$$

$\Rightarrow$ Visualization of the EOM and the constraint

Gauss constraint conservation on the lattice

The discrete Gauss constraint is conserved by the discrete EOM.

This can be checked directly using the explicit forms of the constraint and the EOM (not very interesting) or more generally by making an argument based on lattice gauge invariance.

The Wilson action $S[U]$ is invariant under lattice gauge transformations.

$$S[U'] = S[U]$$

with

$$U'_{x,\mu} = \Omega_x U_{x,\mu} \Omega^\dagger_{x+\mu,\mu}$$

Independent of the exact form of $S[U]$, the Gauss constraint is conserved by the discrete EOM.

Gauss constraint conservation on the lattice

Lattice gauge invariance also implies invariance under infinitesimal transformations. We write

$$\Omega_x = \exp\left(ig\alpha_x\right) \simeq \mathbf{1} + ig\alpha_x + \mathcal{O}(|\alpha|^2)$$

A gauge link transforms according to

$$\begin{aligned} U'_{x,\mu} &= \Omega_x U_{x,\mu} \Omega^\dagger_{x+\mu} \\ &\simeq \left(\mathbf{1} + ig\alpha_x\right) U_{x,\mu} \left(\mathbf{1} - ig\alpha_{x+\mu}\right) + \mathcal{O}(|\alpha|^2) \\ &\simeq U_{x,\mu} - ig\left(U_{x,\mu}\alpha_{x+\mu}U^\dagger_{x,\mu} - \alpha_x\right) U_{x,\mu} + \mathcal{O}(|\alpha|^2) \\ &\simeq U_{x,\mu} - iga^\mu D^F_\mu \alpha_x U_{x,\mu} + \mathcal{O}(|\alpha|^2) \end{aligned}$$

The infinitesimal gauge transformation is of the form

$$U'_{x,\mu} = U_{x,\mu} + \delta U_{x,\mu} = U_{x,\mu} + iga^\mu \delta A_{x,\mu} U_{x,\mu}$$

with $\delta A_{x,\mu} = -D^F_\mu \alpha_x$.

Gauss constraint conservation on the lattice

Apply infinitesimal transformation to action $S[U]$

$$S[U'] = S[U] + \delta S[U, \delta A] + \mathcal{O}(|\delta A|^2)$$

where

$$\delta S[U, \delta A] = V \sum_{x,\mu,a} \frac{\delta S}{\delta A^a_{x,\mu}} \delta A^a_{x,\mu}$$

with $\delta A^a_{x,\mu} = -\left(D^F_\mu \alpha_x\right)^a$.

Due to gauge invariance $S[U'] = S[U]$ it must hold that

$$\sum_{x,\mu,a} \frac{\delta S}{\delta A^a_{x,\mu}} \left(D^F_\mu \alpha_x\right)^a = 0.$$

Gauss constraint conservation on the lattice

Gauge invariance implies the relation

$$\sum_{x,\mu,a} \frac{\delta S}{\delta A^a_{x,\mu}} \left(D^F_\mu \alpha_x\right)^a = 0.$$

where $\left(D^F_\mu \alpha_x\right)^a = 2\,\mathrm{tr}\left[t^a D^F_\mu \alpha_x\right]$. We find

$$\begin{aligned}
\left(D^F_\mu \alpha_x\right)^a &= 2\,\mathrm{tr}\left[t^a \left(\frac{U_{x,\mu}\alpha_{x+\mu}U^\dagger_{x,\mu} - \alpha_x}{a^\mu}\right)\right] \\
&= \sum_b \frac{1}{a^\mu}\left(U^{ab}_{x,\mu}\alpha^b_{x+\mu} - \alpha^a_x\right) \\
&= \sum_b D^{F,ab}_\mu \alpha^b_x
\end{aligned}$$

where the adjoint representation matrix $U^{ab}_{x,\mu}$ of $U_{x,\mu}$ is given by

$$U^{ab}_{x,\mu} = 2\,\mathrm{tr}\left[t^a U_{x,\mu} t^b U^\dagger_{x,\mu}\right].$$

Gauss constraint conservation on the lattice

Expression from previous slide

$$0 = \sum_{x,\mu,a} \frac{\delta S}{\delta A^a_{x,\mu}} \left(D^F_\mu \alpha_x\right)^a = \sum_{x,\mu,a,b} \frac{\delta S}{\delta A^a_{x,\mu}} D^{F,ab}_\mu \alpha^b_x$$

Using summation by parts we find

$$\sum_{x,\mu,a,b} \frac{\delta S}{\delta A^a_{x,\mu}} D^{F,ab}_\mu \alpha^b_x = - \sum_{x,\mu,a,b} D^{B,ab}_\mu \frac{\delta S}{\delta A^b_{x,\mu}} \alpha^a_x = 0.$$

This must vanish for arbitrary α^a_x, therefore

$$\sum_{\mu,b} D^{B,ab}_\mu \frac{\delta S}{\delta A^b_{x,\mu}} = 0$$

Gauss constraint conservation on the lattice

Lattice gauge invariance implies the conservation law

$$\sum_{\mu,b} D_\mu^{B,ab} \frac{\delta S}{\delta A_{x,\mu}^b} = 0,$$

which holds even if the Euler-Lagrange eqs. are not fulfilled.

Recall from Yang-Mills theory (continuum limit):

$$\sum_\mu D_\mu \frac{\delta S[A]}{\delta A_\mu(x)} = 0.$$

We can use this to show that if the EOM $\frac{\delta S}{\delta A_{x,i}^a}$ are satisfied, the Gauss constraint $\frac{\delta S}{\delta A_{x,0}^a}$ is conserved.

$\Rightarrow$ The constraint also holds on the lattice!

Temporal gauge

Discrete equation of motion

$$\frac{1}{(a^0 a^i)^2} P^a \left(U_{x,i,0} + U_{x,i,-0}\right) = \sum_j \frac{1}{(a^i a^j)^2} P^a \left(U_{x,i,j} + U_{x,i,-j}\right)$$

Discrete Gauss constraint

$$\sum_i \frac{1}{(a^0 a^i)^2} P^a \left(U_{x,0,i} + U_{x,0,-i}\right) = 0$$

Same as in the continuum, the discrete EOM require gauge fixing to become solvable initial value problems.

Temporal gauge condition:

$$A_0(x) = 0, \quad \forall x \in \mathbf{M}, \quad \Rightarrow \quad U_{x,0} = \mathbf{1}, \qquad \forall x \in \Lambda$$

Temporal gauge

Realizability of a gauge condition $G[A_\mu] = 0$:

Suppose A_μ does not satisfy the gauge condition $G[A_\mu] \neq 0$. G is realizable if there exists a gauge transformation Ω such that $A'_\mu = \Omega(A_\mu + \frac{1}{ig}\partial_\mu)\Omega^\dagger$ satisfies $G[A'_\mu] = 0$.

Temporal gauge on the lattice is realizable as well.

Temporal gauge

Consider a configuration of links $U_{x,\mu}$ on Λ such that $U_{x,0} \neq \mathbf{1}$.

Perform lattice gauge transformation Ω_x:

$$U'_{x,\mu} = \Omega_x U_{x,\mu} \Omega^\dagger_{x+\mu}$$

Enforce temporal gauge:

$$U'_{x,0} = \Omega_x U_{x,0} \Omega^\dagger_{x+0} = \mathbf{1}$$

Solve for Ω_{x+0}:

$$\begin{aligned}
\Omega_{x+0} &= \Omega_x U_{x,0} \\
&= \Omega_{x-0} U_{x-0,0} U_{x,0} \\
&= \ldots U_{x-3\cdot\hat{0},0} U_{x-2\cdot\hat{0},0} U_{x-\hat{0},0} U_{x,0},
\end{aligned}$$

which is a discretization of the temporal Wilson line used in the continuum version of temporal gauge.

Temporal gauge

Temporal gauge simplifies Wilson loops involving a time direction:

$$\begin{aligned} U_{x,0i} &= U_{x,0} U_{x+0,i} U^\dagger_{x+i,0} U^\dagger_{x,i} \\ &= U_{x+0,i} U^\dagger_{x,i} \end{aligned}$$

For example: in the discrete Gauss constraint we now have

$$\sum_i \frac{1}{(a^0 a^i)^2} P^a \left(U_{x+0,i} U^\dagger_{x,i} + U_{x+0,-i} U^\dagger_{x,-i} \right) = 0,$$

which relates the spatial gauge links of one spatial layer of the lattice (a "time slice") to the next time slice.

Numerical time evolution

Procedure to perform a numerical time evolution:

Specify initial data in two consecutive "time slices":
$U_{x,i}$ $\forall x \in \Lambda$ with $x^0 = t_0$ and $x^0 = t_0 + a^0$

1. Compute $P^a(U_{x,0,i})$ from EOM

$$P^a\left(U_{x,i,0}\right) = \sum_j \left(\frac{a^0}{a^j}\right)^2 P^a\left(U_{x,i,j} + U_{x,i,-j}\right) - P^a\left(U_{x,i,-0}\right)$$

2. Compute plaquette $U_{x,i,0}$ from $P^a\left(U_{x,i,0}\right)$
3. Compute $U_{x+0,i}$ from $U_{x,i,0}$ using

$$U_{x+0,i} = U_{x,0,i} U_{x,i}$$

4. Repeat with step 1 until final time t_1

Numerical time evolution

Step 2: Compute plaquette $U_{x,i,0}$ from $P^a\left(U_{x,i,0}\right)$

The EOM provide $N_c^2 - 1$ real numbers:
$P^a\left(U_{x,i,0}\right) \in \mathbb{R}$ for $a \in \{1, 2, \ldots, N_c^2 - 1\}$

$\Rightarrow$ Enough information to reconstruct the plaquette $U_{x,i,0}$ because every element in $\mathrm{SU}(N_c)$ is determined by $N_c^2 - 1$ real parameters

Numerical time evolution

Step 2: Compute plaquette $U_{x,i,0}$ from $P^a(U_{x,i,0})$

Example: $\mathrm{SU}(2)$ lattice gauge theory

- Use S^3-parametrization: Every element $U \in \mathrm{SU}(2)$ can be written as a complex $\mathbb{C}^{2\times 2}$ matrix

$$U = u_0 \mathbf{1} + i \sum_a \sigma_a u_a,$$

with four real-valued parameters u_0, u_1, u_2, u_3 which satisfy

$$1 = u_0^2 + u_1^2 + u_2^2 + u_3^2$$

and Pauli matrices σ^a.

Numerical time evolution

Step 2: Compute plaquette $U_{x,i,0}$ from $P^a\left(U_{x,i,0}\right)$

Recall: $P^a(U) \equiv 2\,\mathrm{Im}\,\mathrm{tr}\,[t^a U]$

Using $U = u_0\mathbf{1} + i\sum_a \sigma^a u_a$ and $t^a = \sigma^a/2$ we find

$$P^a(U) = 2\,\mathrm{Im}\,\mathrm{tr}\left[t^a\left(u_0\mathbf{1} + i\sum_b \sigma^b u_b\right)\right] = 2u_a$$

Need to compute u_0 from constraint $1 = u_0^2 + u_1^2 + u_2^2 + u_3^2$.

For sufficiently small time step a^0, the plaquette $U_{x,0,i}$ is "close" to the unit matrix $\mathbf{1}$ and the solution for u_0 is given by

$$u_0 = \sqrt{1 - \left(u_1^2 + u_2^2 + u_3^2\right)}.$$

Numerical time evolution

Step 2: Compute plaquette $U_{x,i,0}$ from $P^a\left(U_{x,i,0}\right)$

For $\mathrm{SU}(2)$, if a^0 is sufficiently small then we can reconstruct $U_{x,0,i}$ via

$$U_{x,0,i} = \sqrt{1 - \frac{1}{4}\sum_a \left(P^a(U_{x,0,i})\right)^2}\,\mathbf{1} + \frac{i}{2}\sum_a P^a(U_{x,0,i})\sigma_a,$$

where $P^a(U_{x,0,i})$ is given by the discrete EOM.

Numerical time evolution

Step 2: Compute plaquette $U_{x,i,0}$ from $P^a\left(U_{x,i,0}\right)$

For $\mathrm{SU}(N_c)$: I'm not aware of any general, analytical solution.

Numerical approach: fixed point iteration

Given $P^a(U)$, start with initial guess

$$U^{(0)} = \exp\left(i \sum_a t^a P^a(U)\right)$$

Update guess according to

$$U^{(k+1)} = \exp\left(i \sum_a t^a \delta^a_{(k+1)}\right) U^{(k)}, \quad \delta^a_{(k+1)} = P^a(U) - P^a(U^{(k)})$$

until some convergence criterion is met.

Numerical time evolution

Step 3: Compute $U_{x+0,i}$ from $U_{x,i,0}$

This is simple due to temporal gauge $U_{x,0} = \mathbf{1}$.

Plaquette in temporal gauge:

$$U_{x,0,i} = U_{x,0} U_{x+0,i} U^{\dagger}_{x+i,0} U^{\dagger}_{x,i} = U_{x+0,i} U^{\dagger}_{x,i}$$

We can solve for the unknown link in the next "time slice"

$$U_{x+0,i} = U_{x,0,i} U_{x,i}$$

Numerical time evolution

Specify initial data in two consecutive "time slices":
$U_{x,i}\ \forall x \in \Lambda$ with $x^0 = t_0$ and $x^0 = t_0 + a^0$ at initial time t_0.

1. Compute $P^a(U_{x,0,i})$ from EOM

$$P^a\left(U_{x,i,0}\right) = \sum_j \left(\frac{a^0}{a^j}\right)^2 P^a\left(U_{x,i,j} + U_{x,i,-j}\right) - P^a\left(U_{x,i,-0}\right)$$

2. Compute plaquette $U_{x,i,0}$ from $P^a\left(U_{x,i,0}\right)$
3. Compute $U_{x+0,i}$ from $U_{x,i,0}$ using

$$U_{x+0,i} = U_{x,0,i} U_{x,i}$$

4. Repeat with step 1 until final time t_1

A note on stability

Stability for finite difference schemes: Von Neumann stability

- "Are plane wave solutions stable?"
- Works for finite difference discretizations of linear PDEs
- Yang-Mills eqs. become linear for Abelian limit, small amplitudes, small coupling g, ...
- Von Neumann stability analysis of linearized discrete EOM yields

$$\sum_i \left(a^0/a^i\right)^2 \leq 1$$

However, numerical time evolution also becomes unstable for large amplitudes.

- A. Ipp, DM, "Implicit schemes for real-time lattice gauge theory", [arXiv:1804.01995 [hep-lat]]

Lattice gauge theory: summary

- Using gauge links $U_{x,\mu}$ instead of gauge fields $A_{x,\mu}$ we can formulate the Wilson action $S[U]$: a lattice gauge invariant discretization of the Yang-Mills action
- Using constrained variation we can derive the discretized equations of motion and the Gauss constraint
- Lattice gauge invariance guarantees the conservation of the Gauss constraint
- Temporal gauge (which is realizable on the lattice) allows us to perform a numerical time evolution from initial data