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Geometry and Control of Mechanical Systems - An Eulerian, Lagrangian and Hamiltonian Approach

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Kurzfassung

Der erste Teil dieser Arbeit behandelt Punkt- und Kontinuumsmechanik mit den Mitteln, welche die Differentialgeometrie bereitstellt. Der Schwerpunkt dieser Dissertation liegt vor allem in der Interpretation bekannter Konzepte der Mechanik mithilfe geometrischer Methoden, und deren Verallgemeinerung basierend auf den Resultaten, welche aus dieser geometrischen Sichtweise herrühren. Ausgangspunkt der Untersuchungen bildet die Analyse der Newtonschen Gleichungen für einen Massenpunkt, der sich in einem Inertialsystem bewegt. Es stellt sich heraus, dass diesen wohlbekannten Relationen sehr tiefgreifende geometrische Strukturen zugrunde liegen, was sich vor allem bei der Definition der Beschleunigung eines Masseteilchens zeigt, wenn andere als Euklidische Koordinaten gewählt werden. Für die geometrische Darstellung der Newtonschen Gleichungen wird zur Beschreibung des Massepunktes eine Konfigurationsmannigfaltigkeit betrachtet und alle weiteren wesentlichen Größen wie zum Beispiel Energie, Geschwindigkeit, Impuls und Beschleunigung werden durch geeignete Bündelstrukturen sowie spezielle Ableitungsoperatoren, die sich durch die Wahl von bestimmen Zusammenhängen (Konnexionen) auf diesen Bündeln ergeben, dargestellt. Als wesentliches intrinsisches Objekt erlangt die Metrik auf der Konfigurationsmannigfaltigkeit jenen ausgezeichneten Stellenwert, als dass sich alle weiteren Konstrukte durch sie zwangsläufig ergeben.

Die Verallgemeinerung auf den Fall beschleunigter Bezugssysteme gelingt, indem man die Konfigurationsmannigfaltigkeit durch ein Bündel ersetzt, wobei nun der wesentliche Unterschied darin besteht, dass die Zeit nun kein Kurvenparameter sondern eine Koordinate ist. Die Beschleunigung des Koordinatensystems im Verhältnis zu einem Inertialsystem kann nun geometrisch wieder durch einen Zusammenhang dargestellt werden, welcher zusätzlich zur Metrik, die jetzt auch explizit zeitabhängig sein kann, eine wesentliche Größe zur intrinsischen Beschreibung ist. Weiters wird gezeigt, dass die Lagrange und die Hamiltonsche Betrachtungsweise, welche in der Regelungstheorie eine herausragende Rolle spielen, auch auf den Fall von Nichtinertialsystemen übertragen werden kann.

Ein weiterer essentieller Punkt dieser Arbeit ist die Analyse der Bewegung und Deformation eines Kontinuums aufbauend auf den Erkenntnissen der Punktmechanik. Hier spielen die Eulersche sowie die Lagrange Betrachtungsweise eine ausgezeichnete Rolle. Die Eulersche Betrachtung folgt unmittelbar aus der Punktmechanik, indem man anstatt von Vektoren und einer Punktmasse nun Massedichten und vektorwertige Formen betrachtet. Ausgehend von dieser Formulierung folgt die Lagrange Beschreibung indem man geometrische Objekte geeignet bezüglich einer Referenzkonfiguration beschreibt.

Der zweite Teil dieser Dissertation beschäftigt sich mit der geometrischen Analyse von zeitvarianten Hamiltonschen Systemen, wobei wieder die koordinatenfreie Darstellung eine wesentliche Rolle spielt. Diese Systeme treten in der Regelungstechnik beispielsweise auf, wenn man das Fehlersystem einer Trajektorienfolgeregelung in Hamiltonscher Schreibweise formulieren kann.

Abstract

The first part of this thesis discusses point and continuum mechanics using differential geometric methods. Special emphasis is placed on the interpretation of well known results using the geometric machinery and their generalization from a geometric point of view. The point of origin of the investigations are the well known equations from Newton describing how a mass point is moving in an inertial system. These well known equations possess a deep geometric structure, which is easily seen, when the definition of the acceleration of a mass point is given in non Euclidean coordinates. To describe the evolution of the mass point a configuration manifold is chosen and all other essential quantities such as the velocity, the momentum, the acceleration, and the energy are introduced with respect to adequate bundles as well as with respect to differential operators which stem from the choice of special connections. The main intrinsic object on the configuration manifold is the metric since all other objects essential for a coordinate free description depend on the metric, which is defined by the choice of a coordinate system.

The generalization to the case of accelerated coordinate systems can be performed by the replacement of the configuration manifold by a bundle, where the essential difference is given by the fact, that the time becomes a coordinate in contrast to the case where the time is only a curve parameter. The acceleration of the coordinate system with respect to an inertial system can be accomplished in this geometric formulation by a connection as well, which beside the metric that might be time dependent in this setting is now the key ingredient in this intrinsic description. Furthermore, it will be shown that the Lagrangian and the Hamiltonian point of view, which are also important concepts in control theory, can be formulated with respect to non inertial systems.

An essential demand of this thesis is the analysis of the motion and the deformation of a continuum based on the constructions gained when analyzing point mechanics. In this context the Eulerian and the Lagrangian formulation are important to mention. The Eulerian picture follows as a straightforward generalization from the case of particle mechanics, if instead of vectors and a point mass, now mass densities and vector valued forms are considered. Based on this formulation the Lagrangian picture is obtained, by considering geometric objects with respect to a so-called reference configuration.

The second part of this dissertation is focused on the geometric analysis of time variant Hamiltonian systems, where again the coordinate free description plays a key role. These systems arise in the context of control theory for example when the error system with respect to a certain trajectory can be expressed as a Hamiltonian system.

To Tereza

Preface

This thesis was developed at the Institute of Automatic Control and Control System Technology, which is part of the technical faculty of the university of Linz, within a DOC grant of the Austrian Academy of Sciences which mainly supported this work. Therefore, my special gratitude is devoted to the Academy for this scholarship, which gave me the opportunity to focus on science without having financial worries. Almost the same value as the financial background has of course the progress in research and in this context my deep thankfulness with respect to Professor Kurt Schlacher has to be emphasized, since he has introduced me to differential geometry and has raised my interest of combining geometry and physics. Additionally, I want to stress the fact that, although I was not a collaborator at the institute with a contract of employment, but self-employed, I was integrated by all the colleagues at the institute. The opportunity to teach at the university was given to me again by Professor Kurt Schlacher, which I enjoyed very much and which deserves again gratefulness. For teaching me control theory when I was an undergraduate student, I want to thank beside Professor Kurt Schlacher, also Professor Andreas Kugi for his exquisite style of teaching, which influenced me very much and which enlarged the desire in me to become a lecturer at the university as well - fortunately now I am.

Furthermore, I want to thank all the colleagues and former staff at the institute, namely Stefan Fuchshumer, Gernot Grabmair, Johann Holl, Karl Rieger, Bernhard Roider, Hannes Seyrkammer, Richard Stadlmayr, Martin Staudecker, and Kurt Zehetleitner for the good working atmosphere. My friends and former collaborators at the institute Helmut Ennsbrunner and Johannes Schröck deserve special thanks for everything they did for me from both a scientific and a personal view.

Finally my parents and my sister deserve my deep gratitude for all their support, love, and for the untroubled childhood. Especially, I want to express thankfulness to my parents for the cognition and the encouragement of my talents and for their enthusiasm they invested in sparking my interest in sports, literature, and music.

My last thought goes to Tereza, my beautiful better half.

Linz, April 2007 Markus Schöberl

Contents

Chapter 1

Introduction

Modeling of physical systems has a long tradition and the main tools are non relativistic theory, relativistic theory, and quantum theory. These concepts were developed driven by the human curiosity to analyze nature. Even more important is the fact that we do not only want to comprehend the physical systems but also want to manipulate them and this often leads to a control engineering problem. In both disciplines, modeling and control, geometric methods have become very popular during the years. The reason is obvious, since the desire of concepts that allow a general coordinate invariance can be satisfied using geometry in a comfortable way. In this context it is important to mention that modeling which is based on physical considerations often leads to mathematical representations, which possess several structural properties that are not dependent on the chosen coordinate system and therefore termed 'covariant'. Covariance is understood in the sense of Einstein, such that the laws of physics should be intrinsic with respect to the change of coordinates of space-time. However, since the non relativistic case is our objective we will consider only special morphisms of the space-time which preserve the fibering over the time.

The main part of this thesis is devoted to an important subclass of physical systems, namely non relativistic mechanics. Mechanics is of course a term of wide comprehension and lot of different theories and concepts could fit in this area. In this work we want to consider very elementary concepts using a modern mathematical language. More precisely, we want to focus on the problem to give a covariant description of particle mechanics and based on these investigations to generalize the concepts to a continuum, where the geometric structures that already arise analyzing a mass particle should also be exploited as much as possible for the case of continuum mechanics. With respect to classical mechanics let us quote for example the books [Abraham and Marsden, 1978, Arnold, 1989] and concerning continuum mechanics and field theories [Marsden and Hughes, 1994, Prastaro, 1996, Truesdell and Noll, $3rd$ ed. 2004] are noteworthy beside many others. Of course it has to be mentioned that the literature available seems exhaustless and here only some selected works have been cited. With respect to the demand that we want to use modern mathematical tools we refer to [Giachetta et al., 1997, Saunders, 1989] where most of the geometric concepts used in this thesis can be found. The goal of this work to bring together concepts of mechanics and the language of modern differential geometry is of course treated in the literature as well, for example in [Mangiarotti and Sardanashvily, 1998, Giachetta et al., 1997, Modugno et al., 2005, Jadczyk et al., 1998] but our approach differs in some constructions, especially in the choice of some connections and the rigorous use of the vertical machinery, but of course the equations for covariant particle mechanics coincide. The extension to continuum mechanics the way it is presented here is based on [Schlacher et al., 2004], which treats continuum mechanics in an inertial system with a curved space-time and this thesis generalizes the governing equations for a continuum in the Eulerian and the Lagrangian description such that they are independent of a splitting of space and time.

This geometric analysis is of course not restricted to mechanics and can be generalized to another important subclass of physical systems, namely Hamiltonian systems. These systems are well known in the literature, see for instance [Olver, 1986] and their extension for control purposes [Kugi, 2001, van der Schaft, 2000] and references therein. We will analyze time variant Hamiltonian control systems again in a covariant fashion and show how this formulation can be useful for a geometric interpretation of so-called error systems when tracking control is the demand.

In chapter 2 the relevant mathematical tools are summarized briefly, where most of those and detailed discussions and proofs can be found in [Giachetta et al., 1997, Saunders, 1989]. Then point mechanics is in the focus of chapter 3. We start with a rather general discussion of well known relations that are given in a geometric language needed for a generalization to a pure covariant description. Also the Lagrangian and the Hamiltonian approach are considered and compared to the relations obtained using differential operators, which are constructed with respect to several connections. Chapter 4 is devoted to the discussion of continuum mechanics where most of the topics that are considered are based on the results of chapter 3. The balance equations, namely, conservation of mass and energy, and balance of linear momentum are treated in the Eulerian and the Lagrangian description and furthermore a variational approach is proposed. Finally, chapter 5 is devoted to the discussion of an intrinsic description of time variant Hamiltonian control systems.

It is worth mentioning at this stage that during this thesis most of the proofs and coordinate calculations are omitted to increase the readability. Nevertheless, all these details can be found in a rather formal way in the Appendix and the reader is advised to consult this part of the thesis while reading the main chapters, since the usefulness of the geometric machinery becomes more clearly when examining the equations in detail. At first sight the equations also in the time variant setting look very familiar, but this is a consequence of the vertical machinery as a closer look on the derivation of the relations will show.

Chapter 2

Geometric Preliminaries

This chapter summarizes the relevant topics of standard differential geometric concepts, which will be needed in the sequel. Basic constructions of fibered manifolds, tangent and cotangent bundles will be assumed to be known. The notation used in the following is similar to [Giachetta et al., 1997] and [Saunders, 1989].

2.1 Exact Sequences

The splitting of bundles, which is one of the key geometric tools in the following, can be expressed using exact sequences, therefore we want to recall some basic facts here. We will only define the sequences for vector spaces since the generalization for vector bundles over the same base is straightforward. Given linear spaces U^k and the linear maps f^k the sequence of spaces

$$\dots \longrightarrow U^{k-1} \xrightarrow{f^{k-1}} U^k \xrightarrow{f^k} U^{k+1} \xrightarrow{f^{k+1}} \dots$$

is said to be a complex if the composition of any two neighboring arrows is the zero map, which means $f^k \circ f^{k-1} = 0$. By definition we have im$(f^{k-1}) \subset \ker(f^k)$. A complex is said to be exact (or acyclic) in degree k, if im$(f^{k-1}) = \ker(f^k)$. If a complex is exact in all degrees, it is called an exact sequence.

Example 2.1 *The sequence*

$$0 \longrightarrow U \xrightarrow{f} V \xrightarrow{g} W \longrightarrow 0$$

is a complex, if $g \circ f = 0$. Exactness at U means that f is injective and exactness at W means that g is surjective. In addition, if $\mathrm{im} f = \ker g$ then the sequence is called a short exact sequence. If there is another map $h : W \to V$ such that $g \circ h = \mathrm{id}_W$ is met then $V = U \oplus W$ and the sequence splits.

2.2 Vertical Bundles

We consider the bundle $\pi : \mathcal{E} \to \mathcal{X}$ with local coordinates x^i for $\mathcal{X}$ and coordinates (x^i, y^α) for $\mathcal{E}$. The tangent bundle $\tau_\mathcal{E} : \mathcal{T}(\mathcal{E}) \to \mathcal{E}$ is equipped with the corresponding induced coordinates $(x^i, y^\alpha, \dot{x}^i, \dot{y}^\alpha)$ and the cotangent bundle $\tau^*_\mathcal{E} : \mathcal{T}^*(\mathcal{E}) \to \mathcal{E}$ possesses the induced coordinates $(x^i, y^\alpha, \dot{x}_i, \dot{y}_\alpha)$ with respect to the holonomic bases $(\partial_i, \partial_\alpha)$ and $(\mathrm{d}x^i, \mathrm{d}y^\alpha)$. There is an important subbundle of the tangent bundle $\mathcal{T}(\mathcal{E}) \to \mathcal{E}$ which is called the vertical tangent bundle $\nu_\mathcal{E} : \mathcal{V}(\mathcal{E}) \to \mathcal{E}$ which meets $\mathcal{V}(\mathcal{E}) = \ker(\pi_*)$, where π_* denotes the tangent map. $\mathcal{V}(\mathcal{E})$ is provided with the induced coordinates $(x^i, y^\alpha, \dot{y}^\alpha)$ with respect to the holonomic fibre base ∂_α. Elements of $\mathcal{V}(\mathcal{E})$ are vectors which are tangent to the fibres of $\mathcal{E} \to \mathcal{X}$.

The vertical cotangent bundle $\nu^*_\mathcal{E} : \mathcal{V}^*(\mathcal{E}) \to \mathcal{E}$ is the bundle dual to $\nu_\mathcal{E} : \mathcal{V}(\mathcal{E}) \to \mathcal{E}$ but fails to be a subbundle of $\mathcal{T}^*(\mathcal{E}) \to \mathcal{E}$ as can be verified easily by the transition functions.

Remark 2.2 *Given a vector bundle $\pi : \mathcal{E} \to \mathcal{X}$ we have $\mathcal{V}(\mathcal{E}) \approx \mathcal{E} \times_\mathcal{X} \mathcal{E}$ by a natural isomorphism. To see this let us consider a change of coordinates $\bar{y}^{\bar{\alpha}} = \varphi^{\bar{\alpha}}_\beta(x)\, y^\beta$, $\bar{x}^{\bar{\imath}} = \phi^{\bar{\imath}}(x)$, since $\mathcal{E} \to \mathcal{X}$ is a vector bundle. The transition functions for $\mathcal{V}(\mathcal{E})$ read as*

$$
\begin{aligned}
\bar{x}^{\bar{\imath}} &= \phi^{\bar{\imath}}(x) \\
\bar{y}^{\bar{\alpha}} &= \varphi^{\bar{\alpha}}_\beta(x)\, y^\beta \\
\dot{\bar{y}}^{\bar{\alpha}} &= \varphi^{\bar{\alpha}}_\beta(x)\, \dot{y}^\beta
\end{aligned}
$$

because $\dot{x}^\alpha = 0$ and the isomorphism is readily observed by comparing the transition functions of $\bar{y}^{\bar{\alpha}}$ and $\dot{\bar{y}}^{\bar{\alpha}}$.

2.3 First-order Jet Bundles

Given the bundle $\pi : \mathcal{E} \to \mathcal{X}$ again with local coordinates x^i for $\mathcal{X}$ and coordinates (x^i, y^α) for $\mathcal{E}$ we are interested in the following in equivalence classes of sections s and $\acute{s}$ of π. We define two sections s and $\acute{s}$ to be *1-equivalent* at $\bar{x} \in \mathcal{X}$ if in some adapted coordinate system

$$
s(\bar{x}) = \acute{s}(\bar{x}) \, , \quad \partial_i s^\alpha|_{\bar{x}} = \partial_i \acute{s}^\alpha|_{\bar{x}} \ .
$$

This means that two sections are identified by their values and their values of the first partial derivatives at the point $\bar{x} \in \mathcal{X}$. The equivalence class containing s is called the *1-jet* and is denoted $j^1(s)$. The set of all the *1-jets* of local sections of $\mathcal{E} \to \mathcal{X}$ has a natural structure of a differentiable manifold which is denoted by $\mathcal{J}^1(\mathcal{E})$. Further bundles can be constructed, which are given by the following surjective submersions

$$
\begin{aligned}
\pi^1 &: \mathcal{J}^1(\mathcal{E}) \to \mathcal{X} \\
\pi^1_0 &: \mathcal{J}^1(\mathcal{E}) \to \mathcal{E} \ .
\end{aligned}
$$

For an extended discussion and especially for the case where $\mathcal{E} \to \mathcal{X}$ is only a fibred manifold we refer to [Giachetta et al., 1997, Saunders, 1989] and omit the technical details here.

Based on the bundle coordinates (x^i, y^α) one can construct the adapted coordinates for the manifold $\mathcal{J}^1(\mathcal{E})$ which follow to $(x^i, y^\alpha, y_i^\alpha)$ where the coordinates y_i^α are called the derivative coordinates

$$y_i^\alpha(j_{\bar{x}}^1(s)) = (\partial_i s^\alpha)|_{\bar{x}} \ .$$

They possess the transition functions

$$\begin{aligned} \bar{y}_{\bar{\imath}}^{\bar{\alpha}} &= \left(\partial_i \varphi^{\bar{\alpha}} + \partial_\beta \varphi^{\bar{\alpha}} y_i^\beta\right) \partial_{\bar{\imath}} \left(\phi^{-1}\right)^i \\ &= \left(\partial_i \varphi^{\bar{\alpha}} + \partial_\beta \varphi^{\bar{\alpha}} y_i^\beta\right) \partial_{\bar{\imath}} \hat{\phi}^i \end{aligned} \tag{2.1}$$

with respect to the bundle morphism $\bar{y}^{\bar{\alpha}} = \varphi^{\bar{\alpha}}(y, x)\, , \bar{x}^{\bar{\imath}} = \phi^{\bar{\imath}}(x)\, .$

From the transformation law (2.1) it is seen that the bundle $\mathcal{J}^1(\mathcal{E}) \to \mathcal{E}$ is an affine one, which is modeled over the vector bundle $\mathcal{T}^*(\mathcal{X}) \otimes_{\mathcal{E}} \mathcal{V}(\mathcal{E}) \to \mathcal{E}$. There exist two important canonical morphisms which allow us to identify jets as tangent-valued forms. In fact, there is a unique bundle monomorphism[1]

$$\begin{aligned} \lambda &: \mathcal{J}^1(\mathcal{E}) \hookrightarrow \mathcal{T}^*(\mathcal{X}) \otimes \mathcal{T}(\mathcal{E}) \\ \lambda &= \mathrm{d}x^i \otimes (\partial_i + y_i^\alpha \partial_\alpha) = \mathrm{d}x^i \otimes d_i \end{aligned}$$

and the complementary monomorphism

$$\begin{aligned} \theta &: \mathcal{J}^1(\mathcal{E}) \hookrightarrow \mathcal{T}^*(\mathcal{E}) \otimes \mathcal{V}(\mathcal{E}) \\ \theta &= \left(\mathrm{d}y^\alpha - y_i^\alpha \mathrm{d}x^i\right) \otimes \partial_\alpha = \theta^\alpha \otimes \partial_\alpha \end{aligned}$$

where d_i and θ^α are called the total derivative and contact form, respectively, see [Giachetta et al., 1997]. The morphisms λ and θ allow a canonical horizontal splitting of

$$\begin{aligned} \left(\pi_0^1\right)^* \mathcal{T}(\mathcal{E}) &= \mathcal{J}^1(\mathcal{E}) \times_{\mathcal{E}} \mathcal{T}(\mathcal{E}) = \lambda\left(\mathcal{T}(\mathcal{X})\right) \oplus_{\mathcal{J}^1(\mathcal{E})} \mathcal{V}(\mathcal{E}) \\ \left(\pi_0^1\right)^* \mathcal{T}^*(\mathcal{E}) &= \mathcal{J}^1(\mathcal{E}) \times_{\mathcal{E}} \mathcal{T}^*(\mathcal{E}) = \mathcal{T}^*(\mathcal{X}) \oplus_{\mathcal{J}^1(\mathcal{E})} \theta\left(\mathcal{V}^*(\mathcal{E})\right) \end{aligned}$$

which in coordinates reads as

$$\dot{x}^i \partial_i + \dot{y}^\alpha \partial_\alpha = \dot{x}^i \left(\partial_i + y_i^\alpha \partial_\alpha\right) + \left(\dot{y}^\alpha - \dot{x}^i y_i^\alpha\right) \partial_\alpha \tag{2.2}$$

$$\dot{x}_i \mathrm{d}x^i + \dot{y}_\alpha \mathrm{d}y^\alpha = \left(\dot{x}_i + \dot{y}_\alpha y_i^\alpha\right) \mathrm{d}x^i + \dot{y}_\alpha \left(\mathrm{d}y^\alpha - y_i^\alpha \mathrm{d}x^i\right) \ . \tag{2.3}$$

The importance of these constructions lies in the fact that $\mathcal{V}(\mathcal{E})$ has no distinguished complement in $\mathcal{T}(\mathcal{E})$ and that $\mathcal{H}^*(\mathcal{E})$ which is the annihilator of $\mathcal{V}(\mathcal{E})$ has no distinguished complement in $\mathcal{T}^*(\mathcal{E})$ without the specification of a connection. However, the pull backs $(\pi_0^1)^* \mathcal{T}(\mathcal{E})$ of $\mathcal{T}(\mathcal{E})$ and $(\pi_0^1)^* \mathcal{T}^*(\mathcal{E})$ of $\mathcal{T}^*(\mathcal{E})$ possess this splitting due to the tangent-valued forms λ and θ.

[1] In the sequel $\mathcal{J}^1(\mathcal{E}) \times_{\mathcal{E}} \mathcal{T}(\mathcal{E})$ which is equivalent to $\left(\pi_0^1\right)^* \mathcal{T}(\mathcal{E})$ is sometimes simply written as $\mathcal{T}(\mathcal{E})$ when the pull back is clear form the context. This convention is used for pull back bundles in general.

2.4 Connections

A connection on the bundle $\pi : \mathcal{E} \to \mathcal{X}$ is the choice of a splitting Γ of the exact sequence

$$0 \longrightarrow \mathcal{V}(\mathcal{E}) \hookrightarrow \mathcal{T}(\mathcal{E}) \overset{\Gamma}{\leftrightarrows} \mathcal{E} \times_{\mathcal{X}} \mathcal{T}(\mathcal{X}) \longrightarrow 0 \,. \tag{2.4}$$

Therefore a connection is a map

$$\begin{array}{rccc} \Gamma : & \mathcal{E} \times_{\mathcal{X}} \mathcal{T}(\mathcal{X}) & \to & \mathcal{T}(\mathcal{E}) \\ & (x^i, y^\alpha, \dot{x}^i) & \mapsto & (x^i, y^\alpha, \dot{x}^i, \Gamma_i^\alpha \dot{x}^i) \end{array}$$

where the local functions $\Gamma_i^\alpha \in \mathcal{C}^\infty(\mathcal{E})$ are called the components of the connection. This map Γ can also be represented as

$$\Gamma = \mathrm{d}x^i \otimes (\partial_i + \Gamma_i^\alpha \partial_\alpha)$$

and the image of $\mathcal{E} \times_{\mathcal{X}} \mathcal{T}(\mathcal{X})$ under Γ defines the horizontal subbundle $\mathcal{H}(\mathcal{E}) \to \mathcal{E}$ which splits $\mathcal{T}(\mathcal{E})$ as $\mathcal{T}(\mathcal{E}) = \mathcal{V}(\mathcal{E}) \oplus \mathcal{H}(\mathcal{E})$. The dual construction involves the splitting of the sequence

$$0 \longrightarrow \mathcal{E} \times_{\mathcal{X}} \mathcal{T}^*(\mathcal{X}) \hookrightarrow \mathcal{T}^*(\mathcal{E}) \overset{\Gamma}{\leftrightarrows} \mathcal{V}^*(\mathcal{E}) \longrightarrow 0$$

and in this case the map Γ is represented as

$$\Gamma = \left(\mathrm{d}y^\alpha - \Gamma_i^\alpha \mathrm{d}x^i\right) \otimes \partial_\alpha \,.$$

It follows that we have

$$\dot{x}^i \partial_i + \dot{y}^\alpha \partial_\alpha = \dot{x}^i (\partial_i + \Gamma_i^\alpha \partial_\alpha) + (\dot{y}^\alpha - \dot{x}_i \Gamma_i^\alpha)\, \partial_\alpha \tag{2.5}$$

$$\dot{x}_i \mathrm{d}x^i + \dot{y}_\alpha \mathrm{d}y^\alpha = (\dot{x}_i + \dot{y}_\alpha \Gamma_i^\alpha)\, \mathrm{d}x^i + \dot{y}_\alpha \left(\mathrm{d}y^\alpha - \Gamma_i^\alpha \mathrm{d}x^i\right) \tag{2.6}$$

and this suggests (compare (2.2) with (2.5) and (2.3) with (2.6)) that a connection can be defined as a section of the affine jet bundle $\mathcal{J}^1(\mathcal{E}) \to \mathcal{E}$ and consequently

$$\begin{array}{rccc} \Gamma : & \mathcal{E} & \to & \mathcal{J}^1(\mathcal{E}) \\ & (x^i, y^\alpha) & \mapsto & (x^i, y^\alpha, \Gamma_i^\alpha) \end{array}$$

is met.

Remark 2.3 *The transition functions for $\Gamma = \mathrm{d}x^i \otimes (\partial_i + \Gamma_i^\alpha \partial_\alpha)$ when the bundle morphism $\bar{y}^{\bar{\alpha}} = \varphi^{\bar{\alpha}}(y, x)\,,\ \bar{x}^{\bar{\imath}} = \phi^{\bar{\imath}}(x)$ is applied follow by elementary computations. To see this let us compute*

$$\begin{aligned} \dot{\bar{x}}^{\bar{\imath}} &= \partial_i \phi^{\bar{\imath}} \dot{x}^i \\ \dot{\bar{y}}^{\bar{\alpha}} &= \partial_i \varphi^{\bar{\alpha}} \dot{x}^i + \partial_\alpha \varphi^{\bar{\alpha}} \dot{y}^\alpha \end{aligned}$$

and therefore we obtain

$$\begin{aligned} \mathrm{d}x^i &\to \partial_{\bar{\imath}} \hat{\phi}^i \mathrm{d}\bar{x}^{\bar{\imath}} \\ \partial_i &\to \partial_i \phi^{\bar{\imath}} \partial_{\bar{\imath}} + \partial_i \varphi^{\bar{\alpha}} \partial_{\bar{\alpha}} \\ \Gamma_i^\alpha \partial_\alpha &\to \Gamma_i^\alpha \partial_\alpha \varphi^{\bar{\alpha}} \partial_{\bar{\alpha}} \end{aligned}$$

as well as

$$\partial_{\bar{\imath}}\hat{\phi}^i \mathrm{d}\bar{x}^{\bar{\imath}} \otimes \left(\partial_i \phi^{\bar{\imath}} \partial_{\bar{\imath}} + \partial_i \varphi^{\bar{\alpha}} \partial_{\bar{\alpha}} + \Gamma_i^\alpha \partial_\alpha \varphi^{\bar{\alpha}} \partial_{\bar{\alpha}}\right) \ .$$

This can be written as

$$\mathrm{d}\bar{x}^{\bar{\imath}} \otimes \left(\partial_{\bar{\imath}} + \partial_{\bar{\imath}}\hat{\phi}^i \left(\partial_i \varphi^{\bar{\alpha}} + \Gamma_i^\alpha \partial_\alpha \varphi^{\bar{\alpha}}\right) \partial_{\bar{\alpha}}\right) = \mathrm{d}\bar{x}^{\bar{\imath}} \otimes \left(\partial_{\bar{\imath}} + \bar{\Gamma}_{\bar{\imath}}^{\bar{\alpha}} \partial_{\bar{\alpha}}\right)$$

and we immediately obtain the transition functions for the connection coefficients

$$\bar{\Gamma}_{\bar{\imath}}^{\bar{\alpha}} = \partial_{\bar{\imath}}\hat{\phi}^i \left(\partial_i \varphi^{\bar{\alpha}} + \Gamma_i^\alpha \partial_\alpha \varphi^{\bar{\alpha}}\right) \tag{2.7}$$

where it is readily observed that this corresponds to the transition functions for the 1-jet variables. This is no coincidence but a consequence of the fact that the affine bundle $\mathcal{J}^1(\mathcal{E}) \to \mathcal{E}$ is modeled over the vector bundle $\mathcal{T}^*(\mathcal{X}) \otimes_{\mathcal{E}} \mathcal{V}(\mathcal{E}) \to \mathcal{E}$.

2.4.1 Covariant Differential and Covariant Derivative

Given a connection there exists a first order differential operator which is called covariant differential relative to the connection. Since a connection is a section of the affine bundle $\mathcal{J}^1(\mathcal{E}) \to \mathcal{E}$, it defines the morphism

$$\begin{aligned} \nabla^\Gamma &: \mathcal{J}^1(\mathcal{E}) \underset{\mathcal{E}}{\to} \mathcal{T}^*(\mathcal{X}) \otimes \mathcal{V}(\mathcal{E}) \\ \nabla^\Gamma &= \left(y_i^\alpha - \Gamma_i^\alpha\right) \mathrm{d}x^i \otimes \partial_\alpha \ . \end{aligned} \tag{2.8}$$

Let us consider a section $s : \mathcal{X} \to \mathcal{E}$ then we obtain the covariant derivative of s as

$$\begin{aligned} \nabla^\Gamma(s) &: \mathcal{X} \to \mathcal{T}^*(\mathcal{X}) \otimes \mathcal{V}(\mathcal{E}) \\ \nabla^\Gamma(s) &= \left(\partial_i s^\alpha - \Gamma_i^\alpha \circ s\right) \mathrm{d}x^i \otimes \partial_\alpha \ . \end{aligned} \tag{2.9}$$

Given a section $v : \mathcal{X} \to \mathcal{T}(\mathcal{X})$, $v = v^i(x)\partial_i$ we can define the covariant derivative of the section $s : \mathcal{X} \to \mathcal{E}$ along v which reads as

$$v \rfloor \nabla^\Gamma(s) = v^i \left(\partial_i s^\alpha - \Gamma_i^\alpha \circ s\right) \partial_\alpha \ .$$

2.4.2 Linear Connections

Let us consider a vector bundle $\mathcal{E} \to \mathcal{X}$ and its dual vector bundle $\mathcal{E}^* \to \mathcal{X}$. The connection Γ is a linear connection on $\mathcal{E} \to \mathcal{X}$ if it is a linear bundle morphism $\Gamma : \mathcal{E} \to \mathcal{J}^1(\mathcal{E})$ over $\mathcal{X}$, which means that the map $y_i^\alpha = \Gamma_i^\alpha(y^\beta, x^j)$ is linear in the fibre coordinates y^β. In coordinates the connection coefficients read

$$y_i^\alpha = \Gamma_i^\alpha = \Gamma_{i\beta}^\alpha y^\beta \ , \quad \Gamma_{i\beta}^\alpha \in \mathcal{C}^\infty(\mathcal{X}) \ .$$

From the canonical map $\langle , \rangle : \mathcal{E} \times_{\mathcal{X}} \mathcal{E}^* \to \mathcal{C}^\infty(\mathcal{X})$ given by $y^\alpha y_\alpha$ and its lift to the first jet $j^1(\langle , \rangle) : \mathcal{J}^1(\mathcal{E}) \times_{\mathcal{X}} \mathcal{J}^1(\mathcal{E}^*) \to \mathcal{T}^*(\mathcal{X}) \times \mathbb{R}$ with $\dot{x}_i = (y^\alpha)_i y_\alpha + y^\alpha (y_\alpha)_i$ we derive the dual connection Γ^* of Γ demanding $j^1(\langle \Gamma, \Gamma^* \rangle) = 0$. From

$$\left(\Gamma_i^\alpha y_\alpha + y^\alpha \Gamma_{\alpha i}^*\right) \mathrm{d}x^i = \left(\Gamma_{i\beta}^\alpha y_\alpha y^\beta + y^\alpha y_\beta \Gamma_{\alpha i}^{\beta *}\right) \mathrm{d}x^i = \left(\Gamma_{i\beta}^\alpha + \Gamma_{\beta i}^{\alpha *}\right) y_\alpha y^\beta \mathrm{d}x^i$$

it follows that

$$\Gamma_{i\beta}^{\alpha} = -\Gamma_{i\beta}^{\alpha *}$$

has to be met which is also illustrated in the following commutative diagram

$$\begin{tikzcd}
\mathcal{J}^1(\mathcal{E})\times_{\mathcal{X}}\mathcal{J}^1(\mathcal{E}^*) \arrow[r, "j^1(\langle,\rangle)"] & \mathcal{T}^*(\mathcal{X}) \times \mathbb{R} \\
\mathcal{E}\times_{\mathcal{X}}\mathcal{E}^* \arrow[u, "\Gamma\times\Gamma^*"] \arrow[r, "\langle,\rangle"] & \mathcal{X}\times\mathbb{R} \arrow[u, "0,\mathrm{Id}"']
\end{tikzcd}$$

where $\mathcal{X}\times\mathbb{R}$ corresponds to $\mathcal{C}^\infty(\mathcal{X})$.

2.4.3 Composite Connections

Let us consider the composite bundle structure $\mathcal{E}\to\mathcal{Z}\to\mathcal{X}$ with adapted coordinates (x^i, z^b, y^α) as well as

$$\Gamma = \mathrm{d}x^i \otimes \left(\partial_i + \Gamma_i^b \partial_b\right)$$

and

$$\Sigma = \mathrm{d}x^i \otimes (\partial_i + \Sigma_i^\alpha \partial_\alpha) + \mathrm{d}z^b \otimes (\partial_b + \Sigma_b^\alpha \partial_\alpha) \ .$$

The connection Γ splits $\mathcal{T}(\mathcal{Z})$ with respect to the bundle $\mathcal{Z}\to\mathcal{X}$ and Σ splits $\mathcal{T}(\mathcal{E})$ with respect to the bundle $\mathcal{E}\to\mathcal{Z}$.

Remark 2.4 *To simplify the notation with respect to composite bundles we denote the jet bundle with respect to $\mathcal{E}\to\mathcal{X}$ by $\mathcal{J}^1(\mathcal{E})$ and the jet bundle with respect to $\mathcal{E}\to\mathcal{Z}$ by $\mathcal{J}^1_{\mathcal{Z}}(\mathcal{E})$.*

A typical element of $\mathcal{T}(\mathcal{X})$ is written as $\dot{x}^i\partial_i$ and in the following we consider the horizontal lift of $\dot{x}^i\partial_i$ with respect to $\Sigma\circ\Gamma$. This gives

$$\begin{aligned}
\dot{x}^i\partial_i \rfloor \Gamma &= \dot{x}^i\left(\partial_i + \Gamma_i^b\partial_b\right) \\
\dot{x}^i\left(\partial_i + \Gamma_i^b\partial_b\right)\rfloor\Sigma &= \dot{x}^i\left(\partial_i + \Sigma_i^\alpha\partial_\alpha\right) + \dot{x}^i\Gamma_i^b\left(\partial_b + \Sigma_b^\alpha\partial_\alpha\right) \\
&= \dot{x}^i\left(\partial_i + \Gamma_i^b\partial_b + \left(\Sigma_i^\alpha + \Gamma_i^b\Sigma_b^\alpha\right)\partial_\alpha\right)
\end{aligned}$$

and therefore we observe that there is a connection that splits $\mathcal{E}\to\mathcal{X}$ which produces the same result and is called the composite connection

$$\Pi = \mathrm{d}x^i \otimes \left(\partial_i + \Gamma_i^b\partial_b + \left(\Sigma_i^\alpha + \Gamma_i^b\Sigma_b^\alpha\right)\partial_\alpha\right)$$

since

$$\dot{x}^i\partial_i\rfloor\Pi = \dot{x}^i\partial_i\rfloor\Gamma\rfloor\Sigma \ .$$

The covariant differential with respect to Π is a map

$$\begin{aligned}
\nabla^\Pi &: \mathcal{J}^1(\mathcal{E}) \to \mathcal{T}^*(\mathcal{X})\otimes\mathcal{V}(\mathcal{E}) \\
\nabla^\Pi &= \mathrm{d}x^i \otimes \left(\left(z_i^b - \Gamma_i^b\right)\partial_b + \left(y_i^\alpha - \Sigma_i^\alpha - \Gamma_i^b\Sigma_b^\alpha\right)\partial_\alpha\right)
\end{aligned} \tag{2.10}$$

and it is worth mentioning that due to the composite structure $\mathcal{E} \to \mathcal{Z} \to \mathcal{X}$ the vertical bundle $\mathcal{V}(\mathcal{E}) \to \mathcal{X}$ possesses the fibre base $(\partial_b, \partial_\alpha)$ and the adapted coordinates $(x^i, z^b, y^\alpha, \dot{z}^b, \dot{y}^\alpha)$. Let us consider the splitting of an element of $\mathcal{V}(\mathcal{E})$ with respect to the connection Σ

$$\dot{z}^b\partial_b + \dot{y}^\alpha\partial_\alpha = \dot{z}^b\left(\partial_b + \Sigma_b^\alpha\partial_\alpha\right) + \left(\dot{y}^\alpha - \dot{z}^b\Sigma_b^\alpha\right)\partial_\alpha \,.$$

If we apply this splitting to the vector part of (2.10), we obtain the vertical covariant differential relative to the connection Σ

$$\begin{aligned} \tilde{\nabla}^\Sigma &= \left(y_i^\alpha - \Sigma_i^\alpha - \Gamma_i^b\Sigma_b^\alpha - \left(z_i^b - \Gamma_i^b\right)\Sigma_b^\alpha\right)\mathrm{d}x^i \otimes \partial_\alpha \\ &= \left(y_i^\alpha - \Sigma_i^\alpha - \Sigma_b^\alpha z_i^b\right)\mathrm{d}x^i \otimes \partial_\alpha \,. \end{aligned} \tag{2.11}$$

2.5 Brackets and Differentials

Let us consider a manifold $\mathcal{M}$ together with the space of all vector valued forms on $\mathcal{M}$ which is denoted by $\wedge(\mathcal{M}) \otimes \mathcal{T}(\mathcal{M})$. The Frölicher Nijenhuis bracket (F-N bracket) is a map

$$[,]_{FN} : \wedge^r(\mathcal{M}) \otimes \mathcal{T}(\mathcal{M}) \times \wedge^s(\mathcal{M}) \otimes \mathcal{T}(\mathcal{M}) \to \wedge^{r+s}(\mathcal{M}) \otimes \mathcal{T}(\mathcal{M}) \,.$$

If $\alpha : \mathcal{M} \to \wedge^r(\mathcal{M})$, $\beta : \mathcal{M} \to \wedge^s(\mathcal{M})$, $u, v : \mathcal{M} \to \mathcal{T}(\mathcal{M})$ then the F-N bracket reads as

$$\begin{aligned} [\alpha \otimes u, \beta \otimes v]_{FN} &= \alpha \wedge \beta \otimes u(v) + (\alpha \wedge u(\beta)) \otimes v - (v(\alpha) \wedge \beta) \otimes u \\ &\quad + (-1)^r (\mathrm{d}\alpha \wedge u\rfloor\beta) \otimes v + (-1)^r (v\rfloor\alpha \wedge \mathrm{d}\beta) \otimes u \end{aligned}$$

where $u(v)$, $u(\beta)$ denote the Lie derivative of a field and a form, respectively. Given a vector valued form $\vartheta : \mathcal{M} \to \wedge(\mathcal{M}) \otimes \mathcal{T}(\mathcal{M})$ then the Nijenhuis differential is defined to be

$$\mathrm{d}_\vartheta : \nu \to \mathrm{d}_\vartheta\nu = [\vartheta, \nu]_{FN}$$

for $\nu : \mathcal{M} \to \wedge(\mathcal{M}) \otimes \mathcal{T}(\mathcal{M})$. Let us investigate again the bundle $\pi : \mathcal{E} \to \mathcal{X}$ and we assume that a connection $\gamma : \mathcal{E} \to \mathcal{J}^1(\mathcal{E})$ exists, which is a tensor

$$\gamma = \mathrm{d}x^i \otimes (\partial_i + \gamma_i^\alpha\partial_\alpha)$$

and therefore an object $\wedge^1(\mathcal{X}) \otimes \mathcal{P}(\mathcal{E})$, where $\mathcal{P}(\mathcal{E})$ denotes that the field $\partial_i + \gamma_i^\alpha\partial_\alpha$ is projectable. The F-N covariant differential associated with γ is the Nijenhuis differential

$$\mathrm{d}_\gamma : \wedge^r(\mathcal{X}) \otimes \mathcal{P}(\mathcal{E}) \to \wedge^{r+1}(\mathcal{X}) \otimes \mathcal{V}(\mathcal{E})$$

which plays a crucial role in continuum mechanics as will be seen. The coordinate expression for $\mathrm{d}_\gamma(\phi)$ with $\phi \in \wedge^r(\mathcal{X}) \otimes \mathcal{P}(\mathcal{E})$ reads as

$$\mathrm{d}_\gamma(\phi) = (-\gamma_k^\alpha\partial_i\phi_{i_1\ldots i_r}^k - \partial_k\gamma_i^\alpha\phi_{i_1\ldots i_r}^k + \partial_i\phi_{i_1\ldots i_r}^\alpha + \gamma_i^\beta\partial_\beta\phi_{i_1\ldots i_r}^\alpha - \partial_\beta\gamma_i^\alpha\phi_{i_1\ldots i_r}^\beta)\mathrm{d}x^i \wedge \mathrm{d}x^{i_1} \wedge \ldots \wedge \mathrm{d}x^{i_r} \otimes \partial_\alpha \,.$$

Example 2.5 *Let us consider the form $\phi \in \wedge^r(\mathcal{X}) \otimes \mathcal{V}(\mathcal{E})$ which in coordinates reads*

$$\phi_{i_1\ldots i_r}^\alpha \mathrm{d}x^{i_1} \wedge \ldots \wedge \mathrm{d}x^{i_r} \otimes \partial_\alpha$$

such that $\phi^{\alpha}_{i_1\dots i_r} \in \mathcal{C}^{\infty}(\mathcal{X})$ *and the linear connection*

$$\gamma = \mathrm{d}x^i \otimes \left(\partial_i + \gamma^{\alpha}_{i\beta} y^{\beta} \partial_{\alpha}\right) \; .$$

Then the F-N covariant differential associated with γ *of the vector valued form* ϕ *is given as*

$$\mathrm{d}_{\gamma}(\phi) = (\partial_i \phi^{\alpha}_{i_1\dots i_r} - \gamma^{\alpha}_{i\beta}\phi^{\beta}_{i_1\dots i_r})\mathrm{d}x^i \wedge \mathrm{d}x^{i_1} \wedge \dots \wedge \mathrm{d}x^{i_r} \otimes \partial_{\alpha}$$

since the coefficients $\phi^{k}_{i_1\dots i_r}$ *are all zero because in this example we consider a vertical valued form.*

2.6 Some Topics of Riemannian Geometry

Given an oriented manifold $\mathcal{M}$ with coordinates x^i we consider a non-degenerate metric $g : \mathcal{T}(\mathcal{M}) \to \mathcal{T}^*(\mathcal{M})$ that is represented by the tensor

$$g = g_{ij}\mathrm{d}x^i \otimes \mathrm{d}x^j$$

with $g_{ij} = g_{ji}$ and the inverse of the metric is a map $\hat{g} : \mathcal{T}^*(\mathcal{M}) \to \mathcal{T}(\mathcal{M})$ which is given as

$$\hat{g} = \hat{g}^{ij}\partial_i \otimes \partial_j \; .$$

Remark 2.6 *The tensors* g *and* $\hat{g}$ *can be interpreted as sections* $g : \mathcal{M} \to \mathcal{T}^*(\mathcal{M}) \vee \mathcal{T}^*(\mathcal{M})$ *and* $\hat{g} : \mathcal{M} \to \mathcal{T}(\mathcal{M}) \vee \mathcal{T}(\mathcal{M})$*, where* $\vee$ *denotes the symmetric tensor product.*

The associated volume form is given as

$$\mathrm{vol} = \sqrt{|\det{(g_{ij})}|}\mathrm{d}x^1 \wedge \dots \wedge \mathrm{d}x^n \; .$$

The components of the Levi-Civita connection, associated with g obey the relations

$$\Gamma^i_{jk} = -\frac{1}{2}\hat{g}^{il}\left(\partial_j g_{lk} + \partial_k g_{jl} - \partial_l g_{jk}\right) \tag{2.12}$$

$$\Gamma^k_{jk} = -\frac{1}{\sqrt{|\det{(g_{ij})}|}}\partial_j\sqrt{|\det{(g_{ij})}|} \tag{2.13}$$

and it is worth mentioning that the Levi-Civita connection is a linear connection that splits $\mathcal{T}\left(\mathcal{T}(\mathcal{M})\right)$ with respect to the bundle $\mathcal{T}(\mathcal{M}) \to \mathcal{M}$. The details will be discussed in the forthcoming sections with regard to mechanics.

Remark 2.7 *It is important to stress out that we use the sign convention which is compatible to [Giachetta et al., 1997], since in the physicist literature often a different one is used.*

The covariant derivative of the metric g with respect to the Levi-Civita connection reads as

$$\nabla^{\Gamma}(g) = \left(\partial_i g_{kl} + \Gamma^n_{ik} g_{nl} + \Gamma^m_{il} g_{mk}\right)\mathrm{d}x^i \otimes \dot{\partial}^k \otimes \dot{\partial}^l \; , \tag{2.14}$$

where $\mathcal{V}(\mathcal{T}^*(\mathcal{M}))$ is equipped with the induced base $(\dot{\partial}^k)$.

2.7 Integration on Manifolds

Furthermore integration on manifolds will be important in the sequel, but we only recapitulate the theorem of Stokes here and refer for all further constructions and details to [Abraham et al., 1988, Boothby, 1986].

Theorem 2.8 *Let $\mathcal{M}$ be an oriented manifold of dimension p with coherently oriented boundary $\partial\mathcal{M}$ and ω a form of degree $p-1$ of compact support. Then we have*

$$\int_{\mathcal{M}} \mathrm{d}\omega = \int_{\partial\mathcal{M}} \iota^* (\omega)$$

where ι denotes the inclusion $\iota : \partial\mathcal{M} \to \mathcal{M}$.

Chapter 3

Point Mechanics

Classical mechanics is one of the best analyzed scientific disciplines since centuries and most of the achievements made so far allow a wonderful geometric interpretation. Therefore, studying mechanics shows preeminently a fusion of geometry and physics. The intention of this chapter is to analyze Newton's famous second law. This seems surprising on first sight since this result is well known for a very long time, but the geometric ideas behind the equations are sophisticated already when other than Euclidean coordinates are used.

A standard construction to describe the evolution of a mass particle on a configuration manifold is the use of a covariant derivative with respect to the Levi-Civita connection. In the literature this is often interpreted as a map that assigns to two tangent vectors on the configuration manifold a third one. Of course, this is correct, but there is so much more geometry involved in this construction which will be essential, if the generalization to non inertial systems is the demand.

In this approach, close attention is paid to the splitting of several tangent bundles which can be accomplished by a connection and this is the key observation for a geometric analysis of mechanics.

In section 3.1 the equations describing the motion of a mass point on a configuration manifold regarded as an inertial system are recapitulated. This is accomplished in a rather geometric way, already introducing concepts which allow a generalization to the case where the chosen coordinate system does not qualify as an inertial system. This means for example that the motion of the mass point is observed from an accelerated coordinate system with respect to the inertial one. Section 3.2 presents point mechanics in a pure covariant fashion, such that the equations are formulated with respect to covariant derivatives and they will remain correct even if a time variant coordinate change is considered, which corresponds to the change of the observer. As an example we want to study the motion of a mass particle observed from a moving and a rotating coordinate system with respect to an Euclidean one. It is well known that in this case fictitious forces such as the Coriolis acceleration or the Centrifugal acceleration arise and it is our goal to give an interpretation of this classical example in terms of jet theory and connections. This intrinsic approach is definitely not limited to the case of rigid transformations, since any superposed motion with respect to an inertial coordinate system can be described by a connection and thus no restrictions arise.

Since the Lagrangian and the Hamiltonian description are well known concepts in mechanics, these topics are also presented compatible with the intrinsic formulations. Especially the Hamiltonian formulation allows some interesting new aspects. A connection will be presented that can be used to split the Hamiltonian vector field, which gives insights with respect to the energy balance of the system in the time variant case as well, see also [Schöberl and Schlacher, 2006a].

3.1 Classical Formulation

This part of the thesis is introductory and reviews some well known results of classical non-relativistic particle mechanics, where classical means that it is a kind of standard in the literature, as for example in [Abraham and Marsden, 1978]. However, we will reformulate these results using jet theory and connections to prepare for a discussion of the analogies and the differences to the case where the splitting into time and space is not fixed a priori.

3.1.1 Configuration Manifold, Motion, Velocity and Momentum

Let us consider the configuration manifold $\mathcal{M}$, where we use coordinates q^α, with $\alpha = 1\ldots\dim(\mathcal{M})$. It is important to stress out the fact that the time in this setting is used as a curve parameter on the manifold $\mathcal{M}$. Consequently, the time which is labeled t^0 for reasons that become obvious later, is not a coordinate and therefore changes of coordinates of the form $\bar{q}^{\bar{\alpha}} = \varphi^{\bar{\alpha}}(q^\beta)$ are considered only, which do not involve t^0. From standard constructions we are able to introduce the tangent bundle $\tau_\mathcal{M} : \mathcal{T}(\mathcal{M}) \to \mathcal{M}$ with coordinates $(q^\alpha, \dot{q}^\alpha)$ for $\mathcal{T}(\mathcal{M})$ and the cotangent bundle $\tau^*_\mathcal{M} : \mathcal{T}^*(\mathcal{M}) \to \mathcal{M}$ with coordinates $(q^\alpha, \dot{q}_\alpha)$ for $\mathcal{T}^*(\mathcal{M})$. A motion of a mass particle $m \in \mathbb{R}^+$ can be described by a curve $q^\alpha = s^\alpha(t^0)$ where t^0 is clearly used as the curve parameter. The velocity of a mass point is a tangent vector on the manifold $\mathcal{M}$ or equivalent a section of the bundle $\mathcal{T}(\mathcal{M}) \to \mathcal{M}$ and the momentum can be introduced as the dual object of the velocity with respect to a non-degenerate Riemannian metric on $\mathcal{M}$. Consequently the metric is a map $g : \mathcal{T}(\mathcal{M}) \to \mathcal{T}^*(\mathcal{M})$ represented by the tensor

$$g = g_{\alpha\beta}\mathrm{d}q^\alpha \otimes \mathrm{d}q^\beta$$

with $g_{\alpha\beta} = g_{\beta\alpha}$ and the inverse of the metric is a map $\hat{g} : \mathcal{T}^*(\mathcal{M}) \to \mathcal{T}(\mathcal{M})$ which is given as

$$\hat{g} = \hat{g}^{\alpha\beta}\partial_\alpha \otimes \partial_\beta$$

as introduced in section 2.6.

Remark 3.1 *In an inertial system with Euclidean coordinates we have by definition a trivial metric $g_{\alpha\beta} = \delta_{\alpha\beta}$, where δ denotes the Kronecker symbol.*

Remark 3.2 *The transition rules for the metric follow from*

$$\bar{q}^{\bar{\alpha}} = \phi^{\bar{\alpha}}(q^\beta) , \quad \mathrm{d}\bar{q}^{\bar{\alpha}} = \partial_\beta\phi^{\bar{\alpha}}\mathrm{d}q^\beta , \quad \mathrm{d}q^\alpha = \partial_{\bar{\alpha}}\hat{\phi}^\alpha\mathrm{d}\bar{q}^{\bar{\alpha}}$$

as

$$\bar{g}_{\bar{\alpha}\bar{\beta}} = g_{\alpha\beta}\partial_{\bar{\alpha}}\hat{\phi}^{\alpha}\partial_{\bar{\beta}}\hat{\phi}^{\beta} \ .$$

The momentum is a section of the bundle $\mathcal{T}^*(\mathcal{M}) \to \mathcal{M}$ and can be expressed as

$$p = m\,(v\rfloor g) \ , \quad p_\alpha = m g_{\alpha\beta} v^\beta \ .$$

Remark 3.3 *To be more precise we notice that there is an object dual to $v : \mathcal{M} \to \mathcal{T}(\mathcal{M})$, with respect to the metric which reads as $v^* = v\rfloor g$. The momentum is then introduced by an additional map $\mathcal{T}^*(\mathcal{M}) \to \mathcal{T}^*(\mathcal{M})$ such that $p = mv^*$.*

The velocity of a mass particle can be interpreted as a curve in the tangent bundle $\mathcal{T}(\mathcal{M})$ with $q^\alpha = s^\alpha(t^0)$, $\dot{q}^\alpha \circ s = (v^\alpha \circ s)(t^0)$. Consequently, the momentum is a curve in the cotangent bundle $\mathcal{T}^*(\mathcal{M})$ with $q^\alpha = s^\alpha(t^0)$, $\dot{q}_\alpha \circ s = (p_\alpha \circ s)\,(t^0) = m(g_{\alpha\beta} \circ s)(v^\beta \circ s)(t^0)$. It is worth mentioning that the change of the velocity is tangent to $\mathcal{T}(\mathcal{M})$. Therefore we introduce the bundle $\tau_{\mathcal{T}(\mathcal{M})} : \mathcal{T}(\mathcal{T}(\mathcal{M})) \to \mathcal{T}(\mathcal{M})$ with coordinates $(q^\alpha, \dot{q}^\alpha, \check{q}^\alpha, \check{\dot{q}}^\alpha)$ for $\mathcal{T}(\mathcal{T}(\mathcal{M}))$. To compute the change of the momentum we need the bundle $\tau_{\mathcal{T}^*(\mathcal{M})} : \mathcal{T}(\mathcal{T}^*(\mathcal{M})) \to \mathcal{T}^*(\mathcal{M})$ with coordinates $(q^\alpha, \dot{q}_\alpha, \check{q}^\alpha, \check{\dot{q}}_\alpha)$ for $\mathcal{T}(\mathcal{T}^*(\mathcal{M}))$. A typical element ξ of $\mathcal{T}(\mathcal{T}(\mathcal{M}))$ is then written in coordinates as

$$\xi = \check{q}^\alpha \partial_\alpha + \check{\dot{q}}^\alpha \dot{\partial}_\alpha$$

and in the same spirit we have for an element ξ^* of $\mathcal{T}(\mathcal{T}^*(\mathcal{M}))$ the coordinate expression

$$\xi^* = \check{q}^\alpha \partial_\alpha + \check{\dot{q}}_\alpha \dot{\partial}^\alpha \ ,$$

where we used the holonomic bases $\left(\partial_\alpha, \dot{\partial}_\alpha\right)$, $\left(\partial_\alpha, \dot{\partial}^\alpha\right)$, respectively. To compute the change of $v : \mathcal{M} \to \mathcal{T}(\mathcal{M})$ a connection that splits $\mathcal{T}(\mathcal{M}) \to \mathcal{M}$ is necessary. Such a connection is given by the tensor

$$\Lambda = \mathrm{d}q^\alpha \otimes \left(\partial_\alpha + \Lambda_\alpha^\beta \dot{\partial}_\beta\right) \ , \quad \Lambda_\alpha^\beta \in \mathcal{C}^\infty\left(\mathcal{T}(\mathcal{M})\right)$$

and the dual construction allows to determine the change of $p : \mathcal{M} \to \mathcal{T}^*(\mathcal{M})$ and thus a connection on $\mathcal{T}^*(\mathcal{M}) \to \mathcal{M}$ which is given as

$$\Lambda^* = \mathrm{d}q^\alpha \otimes \left(\partial_\alpha + \Lambda^*_{\alpha\beta} \dot{\partial}^\beta\right) \ , \quad \Lambda^*_{\alpha\beta} \in \mathcal{C}^\infty\left(\mathcal{T}^*(\mathcal{M})\right)$$

is needed. Roughly speaking these tensors are nothing else than a specification of a horizontal direction of $\mathcal{T}(\mathcal{T}(\mathcal{M}))$ at any point on $\mathcal{T}(\mathcal{M})$ or a horizontal direction of $\mathcal{T}(\mathcal{T}^*(\mathcal{M}))$ at any point on $\mathcal{T}^*(\mathcal{M})$.

Remark 3.4 *The transition functions for Λ follow from equation (2.7) and read as*

$$\bar{\Lambda}_{\bar{\alpha}}^{\bar{\beta}} = \partial_{\bar{\alpha}}\hat{\phi}^{\beta}\left(\partial_{\beta\alpha}\phi^{\bar{\beta}}\dot{q}^\alpha + \Lambda_\beta^\rho \partial_\rho \phi^{\bar{\beta}}\right) \tag{3.1}$$

where $\bar{q}^{\bar{\alpha}} = \phi^{\bar{\alpha}}(q^\beta)$, $\dot{\bar{q}}^{\bar{\alpha}} = \partial_\alpha \phi^{\bar{\alpha}} \dot{q}^\alpha = \varphi^{\bar{\alpha}}(q^\beta, \dot{q}^\beta)$ and the transition functions for Λ^* read as

$$\bar{\Lambda}^*_{\bar{\alpha}\bar{\beta}} = \partial_{\bar{\alpha}}\hat{\phi}^{\alpha}\left(\partial_{\bar{\rho}\bar{\beta}}\hat{\phi}^{\rho}\dot{q}_\rho \partial_\alpha \phi^{\bar{\rho}} + \Lambda^*_{\alpha\beta}\partial_{\bar{\beta}}\hat{\phi}^{\beta}\right) \tag{3.2}$$

where $\bar{q}^{\bar{\alpha}} = \phi^{\bar{\alpha}}(q^\beta)$, $\dot{\bar{q}}_{\bar{\alpha}} = \partial_{\bar{\alpha}}\hat{\phi}^{\beta}\dot{q}_\beta = \varphi_{\bar{\alpha}}(q^\beta, \dot{q}_\beta)$.

Remark 3.5 *For a linear connection we obtain from the relation (3.1)*

$$\begin{aligned}\bar{\Lambda}_{\bar{\alpha}}^{\bar{\beta}} &= \partial_{\bar{\alpha}}\hat{\phi}^{\beta}\partial_{\beta\alpha}\phi^{\bar{\beta}}\partial_{\bar{\rho}}\hat{\phi}^{\alpha}\dot{\bar{q}}^{\bar{\rho}} + \Lambda_{\beta\kappa}^{\rho}\partial_{\bar{\rho}}\hat{\phi}^{\kappa}\partial_{\bar{\alpha}}\hat{\phi}^{\beta}\partial_{\rho}\phi^{\bar{\beta}}\dot{\bar{q}}^{\bar{\rho}} \\ &= \bar{\Lambda}_{\bar{\alpha}\bar{\rho}}^{\bar{\beta}}\dot{\bar{q}}^{\bar{\rho}}\end{aligned}$$

with

$$\bar{\Lambda}_{\bar{\alpha}\bar{\rho}}^{\bar{\beta}} = \Lambda_{\beta\kappa}^{\rho}\partial_{\bar{\rho}}\hat{\phi}^{\kappa}\partial_{\bar{\alpha}}\hat{\phi}^{\beta}\partial_{\rho}\phi^{\bar{\beta}} - \partial_{\bar{\alpha}\bar{\rho}}\hat{\phi}^{\beta}\partial_{\beta}\phi^{\bar{\beta}}$$

where we used equation (A.2) from the Appendix and except for the sign this corresponds to [Marsden and Hughes, 1994]. From the equation (3.2) we have

$$\begin{aligned}\bar{\Lambda}_{\bar{\alpha}\bar{\beta}}^{*} &= \partial_{\bar{\alpha}\bar{\beta}}\hat{\phi}^{\rho}\partial_{\rho}\phi^{\bar{\tau}}\dot{\bar{q}}_{\bar{\tau}} + \partial_{\bar{\alpha}}\hat{\phi}^{\alpha}\partial_{\bar{\beta}}\hat{\phi}^{\beta}\Lambda_{\alpha\beta}^{\rho *}\partial_{\rho}\phi^{\bar{\tau}}\dot{\bar{q}}_{\bar{\tau}} \\ &= \bar{\Lambda}_{\bar{\alpha}\bar{\beta}}^{\bar{\tau}*}\dot{\bar{q}}_{\bar{\tau}}\end{aligned}$$

with

$$\bar{\Lambda}_{\bar{\alpha}\bar{\beta}}^{\bar{\tau}*} = \Lambda_{\alpha\beta}^{\rho *}\partial_{\bar{\alpha}}\hat{\phi}^{\alpha}\partial_{\bar{\beta}}\hat{\phi}^{\beta}\partial_{\rho}\phi^{\bar{\tau}} + \partial_{\bar{\alpha}\bar{\beta}}\hat{\phi}^{\rho}\partial_{\rho}\phi^{\bar{\tau}}\ .$$

From a mathematical point of view Λ is a splitting of the exact sequence (2.4) which in this case reads as

$$0 \longrightarrow \mathcal{V}(\mathcal{T}(\mathcal{M})) \hookrightarrow \mathcal{T}(\mathcal{T}(\mathcal{M}))\overset{\Lambda}{\leftrightarrows}\mathcal{T}(\mathcal{M})\times_{\mathcal{M}}\mathcal{T}(\mathcal{M}) \longrightarrow 0$$

and for Λ^* we have

$$0 \longrightarrow \mathcal{V}(\mathcal{T}^*(\mathcal{M})) \hookrightarrow \mathcal{T}(\mathcal{T}^*(\mathcal{M}))\overset{\Lambda^*}{\leftrightarrows}\mathcal{T}^*(\mathcal{M})\times_{\mathcal{M}}\mathcal{T}^*(\mathcal{M}) \longrightarrow 0\ .$$

Remark 3.6 *Let us apply remark 2.2 to this special situation and we immediately have*

$$\mathcal{V}(\mathcal{T}(\mathcal{M})) \approx \mathcal{T}(\mathcal{M})\times_{\mathcal{M}}\mathcal{T}(\mathcal{M})\ .$$

In coordinates this can be shown easily since for an element $\zeta \in \mathcal{V}(\mathcal{T}(\mathcal{M}))$ the transformation rules follow to

$$\bar{q}^{\bar{\alpha}} = \phi^{\bar{\alpha}}(q^{\beta})\ , \quad \dot{\bar{q}}^{\bar{\alpha}} = \partial_{\beta}\phi^{\bar{\alpha}}\dot{q}^{\beta}\ , \quad \dot{\check{\bar{q}}}^{\bar{\alpha}} = \partial_{\beta}\phi^{\bar{\alpha}}\dot{\check{q}}^{\beta}$$

because $\check{q}^{\alpha} = 0$ and the transformation rules of the two latter ones coincide. A similar argument holds for

$$\mathcal{V}(\mathcal{T}^*(\mathcal{M})) \approx \mathcal{T}^*(\mathcal{M})\times_{\mathcal{M}}\mathcal{T}^*(\mathcal{M})\ .$$

3.1.2 Connection Coefficients

It remains to choose the coefficients Λ_{α}^{β} and $\Lambda_{\alpha\beta}^{*}$. A connection on $\mathcal{T}(\mathcal{M}) \to \mathcal{M}$ that splits $\mathcal{T}(\mathcal{T}(\mathcal{M}))$ can be represented as the map

$$\Lambda : \mathcal{T}(\mathcal{M}) \to \mathcal{J}^1\left(\mathcal{T}(\mathcal{M})\right)$$

according to section 2.4. Following [Schlacher et al., 2004] we use the metric which is a map $g : \mathcal{T}(\mathcal{M}) \to \mathcal{T}^*(\mathcal{M})$ to obtain the coefficients Λ_{α}^{β} and $\Lambda_{\alpha\beta}^{*}$. The extension of the map g to its first jet

$$j^1(g) : \mathcal{J}^1\left(\mathcal{T}(\mathcal{M})\right) \to \mathcal{J}^1\left(\mathcal{T}^*(\mathcal{M})\right)$$

will turn out to be important for the following. The demand that $j^1(g)$ is a linear map leads to the desired connection Λ as will be shown. The map $j^1(g)$ is given in coordinates as

$$(\dot{q}_\alpha)_\rho = g_{\alpha\beta}(\dot{q}^\beta)_\rho + (\partial_\rho g_{\alpha\beta})\,\dot{q}^\beta$$

which is affine.

Remark 3.7 *The coordinates of the jet manifold $\mathcal{J}^1(\mathcal{T}(\mathcal{M}))$ are denoted $\left(q^\alpha, \dot{q}^\alpha, (\dot{q}^\alpha)_\rho\right)$ and the jet manifold $\mathcal{J}^1(\mathcal{T}^*(\mathcal{M}))$ possesses the coordinates $\left(q^\alpha, \dot{q}_\alpha, (\dot{q}_\alpha)_\rho\right)$.*

To make the map $j^1(g)$ linear we choose a linear connection Λ and its dual Λ^*, such that

$$(\dot{q}_\alpha)_\rho - \Lambda^*_{\alpha\rho} = g_{\alpha\beta}((\dot{q}^\beta)_\rho - \Lambda^\beta_\rho)$$

is met. Rewriting the expressions gives

$$\begin{aligned} g_{\alpha\beta}((\dot{q}^\beta)_\rho - \Lambda^\beta_\rho) + \Lambda^*_{\alpha\rho} &= g_{\alpha\beta}(\dot{q}^\beta)_\rho + (\partial_\rho g_{\alpha\beta})\,\dot{q}^\beta \\ -g_{\alpha\beta}\Lambda^\beta_{\rho\kappa}\dot{q}^\kappa + \Lambda^{\tau *}_{\alpha\rho}\dot{q}_\tau &= (\partial_\rho g_{\alpha\beta})\,\dot{q}^\beta \\ -g_{\alpha\beta}\Lambda^\beta_{\rho\kappa}\dot{q}^\kappa - \Lambda^\tau_{\alpha\rho}g_{\tau\kappa}\dot{q}^\kappa &= (\partial_\rho g_{\alpha\kappa})\,\dot{q}^\kappa \end{aligned}$$

and

$$(\partial_\rho g_{\alpha\varepsilon}) = -g_{\kappa\varepsilon}\Lambda^\kappa_{\alpha\rho} - g_{\alpha\beta}\Lambda^\beta_{\rho\varepsilon}\,. \tag{3.3}$$

Remark 3.8 *The relation (3.3) can be also interpreted differently, since it corresponds to $\nabla^\Lambda(g) = 0$.*

From the sum of the relations

$$\begin{aligned} (\partial_\rho g_{\alpha\varepsilon}) + g_{\kappa\varepsilon}\Lambda^\kappa_{\alpha\rho} + g_{\alpha\kappa}\Lambda^\kappa_{\rho\varepsilon} &= 0 \\ -(\partial_\varepsilon g_{\rho\alpha}) - g_{\kappa\alpha}\Lambda^\kappa_{\rho\varepsilon} - g_{\rho\kappa}\Lambda^\kappa_{\varepsilon\alpha} &= 0 \\ (\partial_\alpha g_{\varepsilon\rho}) + g_{\kappa\rho}\Lambda^\kappa_{\varepsilon\alpha} + g_{\varepsilon\kappa}\Lambda^\kappa_{\alpha\rho} &= 0 \end{aligned}$$

which is

$$(\partial_\rho g_{\alpha\varepsilon}) + g_{\kappa\varepsilon}\Lambda^\kappa_{\alpha\rho} - (\partial_\varepsilon g_{\rho\alpha}) + (\partial_\alpha g_{\varepsilon\rho}) + g_{\varepsilon\kappa}\Lambda^\kappa_{\alpha\rho} = 0$$

we finally obtain

$$2\Lambda^\kappa_{\alpha\rho} = -\hat{g}^{\kappa\varepsilon}\left(\partial_\alpha g_{\rho\varepsilon} + \partial_\rho g_{\varepsilon\alpha} - \partial_\varepsilon g_{\alpha\rho}\right)\,. \tag{3.4}$$

These are the Christoffel symbols of the second kind with respect to the metric g and this coincides with the relation (2.12).

3.1.3 Covariant Derivatives

The covariant differential, see equation (2.8), with respect to the linear connection

$$\Lambda = \mathrm{d}q^{\alpha} \otimes \left(\partial_{\alpha} + \Lambda^{\beta}_{\alpha\rho}\dot{q}^{\rho}\dot{\partial}_{\beta}\right) \ , \quad \Lambda^{\beta}_{\alpha\rho} \in \mathcal{C}^{\infty}\mathcal{M})$$

is a map

$$\begin{aligned}
\nabla^{\Lambda} \ &: \ \mathcal{J}^1\left(\mathcal{T}(\mathcal{M})\right) \to \mathcal{T}^*(\mathcal{M}) \otimes \mathcal{V}(\mathcal{T}(\mathcal{M})) \\
\nabla^{\Lambda} \ &: \ \left(\left(\dot{q}^{\beta}\right)_{\alpha} - \Lambda^{\beta}_{\alpha\rho}\dot{q}^{\rho}\right) \mathrm{d}q^{\alpha} \otimes \dot{\partial}_{\beta} \ .
\end{aligned}$$

Let us plug in a section $v : \mathcal{M} \to \mathcal{T}(\mathcal{M})$, $v = \dot{q}^{\alpha}\partial_{\alpha} = v^{\alpha}(q)\partial_{\alpha}$ and a contraction with the field $v^{\alpha}\partial_{\alpha}$ gives

$$v \rfloor \nabla^{\Lambda}(v) = v^{\alpha}\left(\partial_{\alpha}v^{\beta} - \Lambda^{\beta}_{\alpha\rho}v^{\rho}\right)\partial_{\beta} \ ,$$

where the isomorphism $\mathcal{V}(\mathcal{T}(\mathcal{M})) \approx \mathcal{T}(\mathcal{M}) \times_{\mathcal{M}} \mathcal{T}(\mathcal{M})$ is crucial.

Remark 3.9 *It is worth mentioning that alternatively also the notation $\nabla^{\Lambda}(j^1(v))$ is appropriate, however, for simplicity we will use the one presented above.*

Since in this case the time is the curve parameter we can set $q^{\alpha} = s^{\alpha}(t^0)$ and $v^{\alpha} \circ s = \partial_0 s^{\alpha}(t^0)$ and then we have

$$\begin{aligned}
\left(v \rfloor \nabla^{\Lambda}(v)\right) \circ s \ &= \ \left(\partial_0(v^{\beta} \circ s) - \left(\Lambda^{\beta}_{\alpha\rho} \circ s\right)\partial_0 s^{\alpha}\partial_0 s^{\rho}\right)\partial_{\beta} \\
&= \ \left(\partial_{00}s^{\beta} - \left(\Lambda^{\beta}_{\alpha\rho} \circ s\right)\partial_0 s^{\alpha}\partial_0 s^{\rho}\right)\partial_{\beta}
\end{aligned}$$

which is a standard result in point mechanics. The dual construction is based on the linear connection

$$\Lambda^* = \mathrm{d}q^{\alpha} \otimes \left(\partial_{\alpha} + \Lambda^{\rho *}_{\alpha\beta}\dot{q}_{\rho}\dot{\partial}^{\beta}\right)$$

and again the covariant differential with respect to Λ^* is

$$\begin{aligned}
\nabla^{\Lambda^*} \ &: \ \mathcal{J}^1\left(\mathcal{T}^*(\mathcal{M})\right) \to \mathcal{T}^*(\mathcal{M}) \otimes \mathcal{V}(\mathcal{T}^*(\mathcal{M})) \\
\nabla^{\Lambda^*} \ &: \ \left(\left(\dot{q}_{\beta}\right)_{\alpha} - \Lambda^{\rho *}_{\alpha\beta}\dot{q}_{\rho}\right) \mathrm{d}q^{\alpha} \otimes \dot{\partial}^{\beta} \ .
\end{aligned}$$

The change of $p : \mathcal{M} \to \mathcal{T}^*(\mathcal{M})$, $p = \dot{q}_{\alpha}\mathrm{d}q^{\alpha} = p_{\alpha}(q)\mathrm{d}q^{\alpha}$ along the field $v^{\alpha}\partial_{\alpha}$ consequently reads as

$$v \rfloor \nabla^{\Lambda^*}(p) = v^{\alpha}\left(\partial_{\alpha}p_{\beta} - \Lambda^{\rho *}_{\alpha\beta}p_{\rho}\right)\mathrm{d}q^{\beta}$$

where the isomorphism $\mathcal{V}(\mathcal{T}^*(\mathcal{M})) \approx \mathcal{T}^*(\mathcal{M}) \times_{\mathcal{M}} \mathcal{T}^*(\mathcal{M})$ is essential. This can be rewritten as

$$\left(v \rfloor \nabla^{\Lambda^*}(p)\right) \circ s = \left(\partial_0(p_{\beta} \circ s) - \frac{1}{m}\left(\hat{g}^{\alpha\kappa}\Lambda^{\rho *}_{\alpha\beta}p_{\rho}p_{\kappa}\right) \circ s\right)\mathrm{d}q^{\beta}$$

and will be discussed in more detail in the section 3.2.

3.1.4 Examples

Example 3.10 *Let us consider an inertial system with Euclidean coordinates* (q^α). *Then the connection* Λ *reads as* $\Lambda = \mathrm{d}q^\alpha \otimes \partial_\alpha$ *with* $\Lambda^\beta_\alpha = 0$. *A change of coordinates of the form* $\bar{q}^{\bar\alpha} = \phi^{\bar\alpha}(q^\beta)$ *leads with relation (3.1) to*

$$\bar\Lambda^{\bar\beta}_{\bar\alpha} = \partial_{\bar\alpha}\hat\phi^\beta \partial_{\beta\alpha}\phi^{\bar\beta}\dot{q}^\alpha = \partial_{\bar\alpha}\hat\phi^\beta \partial_{\beta\alpha}\phi^{\bar\beta}\partial_{\bar\rho}\hat\phi^\alpha \dot{\bar{q}}^{\bar\rho} = \bar\Lambda^{\bar\beta}_{\bar\alpha\bar\rho}\dot{\bar{q}}^{\bar\rho}$$

and it follows immediately from equation (A.2) that we have

$$\bar\Lambda^{\bar\beta}_{\bar\alpha\bar\rho} = \partial_{\bar\alpha}\hat\phi^\beta \partial_{\beta\alpha}\phi^{\bar\beta}\partial_{\bar\rho}\hat\phi^\alpha = -\partial_\beta\phi^{\bar\beta}\partial_{\bar\rho\bar\alpha}\hat\phi^\beta \ .$$

This is the standard result, see for example [Marsden and Hughes, 1994] except for the sign. From remark 3.5 we have

$$\bar\Lambda^{\bar\tau *}_{\bar\alpha\bar\beta} = \partial_{\bar\alpha\bar\beta}\hat\phi^\rho \partial_\rho \phi^{\bar\tau}$$

for the case $\Lambda^{\rho *}_{\alpha\beta} = 0$ *and it is immediately observed that*

$$\bar\Lambda^{\bar\beta}_{\bar\alpha\bar\rho} = -\bar\Lambda^{\bar\beta *}_{\bar\alpha\bar\rho}$$

is met, which is discussed in general in section 2.4.2. This of course applies here, since we are dealing with linear connections on vector bundles.

Example 3.11 *With the same situation as in the previous example the metric in the inertial system is given as* $g = \delta_{\alpha\beta}\mathrm{d}q^\alpha \otimes \mathrm{d}q^\beta$. *Then we evaluate the equation (3.4) and obtain in the new coordinates with the metric* $\bar{g}_{\bar\alpha\bar\beta} = \delta_{\alpha\beta}\partial_{\bar\alpha}\hat\phi^\alpha \partial_{\bar\beta}\hat\phi^\beta$

$$\begin{aligned} 2\bar\Lambda^{\bar\kappa}_{\bar\alpha\bar\rho} &= -\hat{\bar{g}}^{\bar\kappa\bar\varepsilon}\left(\partial_{\bar\alpha}\bar{g}_{\bar\rho\bar\varepsilon} + \partial_{\bar\rho}g_{\bar\varepsilon\bar\alpha} - \partial_{\bar\varepsilon}g_{\bar\alpha\bar\rho}\right) \\ &= -\delta^{\alpha\beta}\partial_\alpha\phi^{\bar\kappa}\partial_\beta\phi^{\bar\varepsilon}\left(\delta_{\kappa\tau}\partial_{\bar\alpha}\left(\partial_{\bar\rho}\hat\phi^\kappa\partial_{\bar\varepsilon}\hat\phi^\tau\right) + \delta_{\eta\lambda}\partial_{\bar\rho}\left(\partial_{\bar\alpha}\hat\phi^\eta\partial_{\bar\varepsilon}\hat\phi^\lambda\right) - \delta_{\mu\omega}\partial_{\bar\varepsilon}\left(\partial_{\bar\rho}\hat\phi^\mu\partial_{\bar\alpha}\hat\phi^\omega\right)\right) \\ &= -\delta^{\alpha\beta}\partial_\alpha\phi^{\bar\kappa}\partial_\beta\phi^{\bar\varepsilon}(\delta_{\kappa\tau}\partial_{\bar\alpha\bar\rho}\hat\phi^\kappa\partial_{\bar\varepsilon}\hat\phi^\tau + \delta_{\eta\lambda}\partial_{\bar\rho\bar\alpha}\hat\phi^\eta\partial_{\bar\varepsilon}\hat\phi^\lambda) \\ &= -2\partial_\kappa\phi^{\bar\kappa}\partial_{\bar\alpha\bar\rho}\hat\phi^\kappa \ . \end{aligned}$$

This obviously coincides with the result of the foregoing example.

3.2 Covariant Formulation

The purpose of this section is to generalize the formulas of section 3.1 to a pure covariant description. The key assumption of the previous section was that the time t^0 is a curve parameter on the configuration manifold $\mathcal{M}$. If we consider the time as a coordinate then an obvious procedure will be the choice of the product manifold $\mathcal{M} \times \mathcal{B}$, where $\mathcal{B}$ possesses the coordinate t^0. This self-evident selection has a disadvantage, namely the choice of a product manifold implies that a splitting into the spatial and temporal coordinates has been proposed implicitly. This may be reasonable from a physical point of view for a concrete problem, but from a geometric point of view such a splitting is not preserved by arbitrary bundle morphisms. Therefore the concept of a connection is indispensable for a pure covariant formulation, which of course should produce the correct equations also when coordinates are chosen which are not fixed to the inertial system. The main observation for this intrinsic setting will be that in place of $\mathcal{M} \times \mathcal{B}$, we use a bundle $\pi : \mathcal{Q} \to \mathcal{B}$. Then the splitting of time and spatial coordinates will be represented by a connection on $\mathcal{Q} \to \mathcal{B}$ which splits $\mathcal{T}(\mathcal{Q})$, see [Giachetta et al., 1997, Mangiarotti and Sardanashvily, 1998, Modugno et al., 2005]. At this point it is crucial to remark that due to the bundle structure the projection on the time manifold $\mathcal{Q} \to \mathcal{B}$ is well defined without a connection whereas a projection on the fibre $\mathcal{Q} \to \mathcal{Q}_{t^0}$ requires the definition of a horizontal direction, which is clearly given by a connection on $\mathcal{Q} \to \mathcal{B}$. In contrast to the foregoing section the space of the velocities will be $\mathcal{V}(\mathcal{Q})$ instead of $\mathcal{T}(\mathcal{M})$ and the space of the momentum will be $\mathcal{V}^*(\mathcal{Q})$ instead of $\mathcal{T}^*(\mathcal{M})$. To define the change of the velocity or the momentum connections on $\mathcal{V}(\mathcal{Q}) \to \mathcal{Q}$ and $\mathcal{V}^*(\mathcal{Q}) \to \mathcal{Q}$ will play the same role as the connections on $\mathcal{T}(\mathcal{M}) \to \mathcal{M}$ and $\mathcal{T}^*(\mathcal{M}) \to \mathcal{M}$ did before.

3.2.1 Space-Time and Reference Frame

Let us consider the bundle $\pi : \mathcal{Q} \to \mathcal{B}$ with coordinate t^0 for the time manifold $\mathcal{B}$ and for $\mathcal{Q}$ we introduce coordinates (t^0, q^α). The tangent bundle $\mathcal{T}(\mathcal{Q})$ is provided with the adapted coordinates $(t^0, q^\alpha, \dot{t}^0, \dot{q}^\alpha)$ and the dual vector bundle, namely the cotangent bundle $\mathcal{T}^*(\mathcal{Q})$, possesses the coordinates $(t^0, q^\alpha, \dot{t}_0, \dot{q}_\alpha)$. From the constructions in sections 2.2 and 2.4 we derive the vertical bundle $\mathcal{V}(\mathcal{Q}) \to \mathcal{Q}$ with the induced coordinates $(t^0, q^\alpha, \dot{q}^\alpha)$. To define the horizontal bundle $\mathcal{H}(\mathcal{Q}) \to \mathcal{Q}$ a connection γ that splits $\mathcal{T}(\mathcal{Q})$ is required. In the context with mechanics a connection on $\mathcal{Q}$ constitutes a reference frame, which can be interpreted as the velocity of an observer, see [Mangiarotti and Sardanashvily, 1998, Modugno et al., 2005], which can be stated as

$$\gamma = \mathrm{d}t^0 \otimes (\partial_0 + \gamma_0^\alpha \partial_\alpha) \ , \quad \gamma_0^\alpha \in \mathcal{C}^\infty(\mathcal{Q}) \ . \tag{3.5}$$

It follows that we have the splitting $\mathcal{T}(\mathcal{Q}) = \mathcal{V}(\mathcal{Q}) \oplus \mathcal{H}(\mathcal{Q})$ which can be visualized by the splitting of the exact sequence

$$0 \longrightarrow \mathcal{V}(\mathcal{Q}) \hookrightarrow \mathcal{T}(\mathcal{Q}) \overset{\gamma}{\leftrightarrows} \mathcal{Q} \times_{\mathcal{B}} \mathcal{T}(\mathcal{B}) \longrightarrow 0 \ .$$

These constructions can easily be assigned to $\mathcal{T}^*(\mathcal{Q})$. Here the choice of a connection gives us the possibility to construct $\mathcal{V}^*(\mathcal{Q})$, which is the annihilator of $\mathcal{H}(\mathcal{Q})$, where we use the induced coordinates $(t^0, q^\alpha, \dot{q}_\alpha)$ for $\mathcal{V}^*(\mathcal{Q})$ and in this case the sequence

$$0 \longrightarrow \mathcal{Q} \times_{\mathcal{B}} \mathcal{T}^*(\mathcal{B}) \hookrightarrow \mathcal{T}^*(\mathcal{Q}) \overset{\gamma}{\leftrightarrows} \mathcal{V}^*(\mathcal{Q}) \longrightarrow 0$$

is of importance. From remark 2.3 it follows that when we apply bundle morphisms of the form $\bar{q}^{\bar{\alpha}} = \varphi^{\bar{\alpha}}(q^\beta, t^0)$, $\bar{t}^{\bar{0}} = \phi^{\bar{0}}(t^0)$ the transition functions for the connection coefficients read as

$$\bar{\gamma}_{\bar{0}}^{\bar{\alpha}} = \partial_{\bar{0}} \hat{\phi}^0 \left(\partial_0 \varphi^{\bar{\alpha}} + \gamma_0^\alpha \partial_\alpha \varphi^{\bar{\alpha}} \right) \ . \tag{3.6}$$

Remark 3.12 *In an inertial system the connection γ reads as $\gamma = \mathrm{d}t^0 \otimes \partial_0$ and it follows from equation (3.6) that this trivial connection will be preserved only, if $\partial_0 \varphi^{\bar{\alpha}} = 0$ is met. This means that $\bar{q}^{\bar{\alpha}} = \varphi^{\bar{\alpha}}(q^\beta)$ and this corresponds to the fact that then the product manifold structure is preserved $\mathcal{M} \times \mathcal{B} \to \bar{\mathcal{M}} \times \bar{\mathcal{B}}$.*

Remark 3.13 *In the following we mostly do not consider time reparameterizations. This means that in the sequel* $t^{\bar{0}} = \delta_0^{\bar{0}} t^0$ *unless indicated differently.*

3.2.2 The Vertical Metric

A vertical metric is a map $g : \mathcal{V}(\mathcal{Q}) \to \mathcal{V}^*(\mathcal{Q})$, with $\dot{q}_\alpha = g_{\alpha\beta}\dot{q}^\beta$ which in coordinates is represented by a symmetric tensor

$$g = g_{\alpha\beta}(\mathrm{d}q^\alpha - \gamma_0^\alpha \mathrm{d}t^0) \otimes (\mathrm{d}q^\beta - \gamma_0^\beta \mathrm{d}t^0) \ , \tag{3.7}$$

with $g_{\alpha\beta}$, $\gamma_0^\alpha \in \mathcal{C}^\infty(\mathcal{Q})$ and $g_{\alpha\beta} = g_{\beta\alpha}$. The inverse of the metric will be denoted $\hat{g} : \mathcal{V}^*(\mathcal{Q}) \to \mathcal{V}(\mathcal{Q})$ which is represented as

$$\hat{g} = \hat{g}^{\alpha\beta}\partial_\alpha \otimes \partial_\beta \ .$$

Remark 3.14 *As tensors* g *and* $\hat{g}$ *can be interpreted as sections* $g : \mathcal{Q} \to \mathcal{V}^*(\mathcal{Q}) \vee \mathcal{V}^*(\mathcal{Q})$ *and* $\hat{g} : \mathcal{Q} \to \mathcal{V}(\mathcal{Q}) \vee \mathcal{V}(\mathcal{Q})$.

The metric allows us to measure the distance between two simultaneous events that lie in the same fibre $\mathcal{Q}_{t^0}$ for a given time t^0.

Example 3.15 *In an inertial system with Euclidean coordinates the metric reads as* $g = \delta_{\alpha\beta}\mathrm{d}q^\alpha \otimes \mathrm{d}q^\beta$. *A bundle morphism of the form* $\bar{q}^{\bar{\alpha}} = \varphi^{\bar{\alpha}}(q^\beta, t^0)$, $\bar{t}^{\bar{0}} = \delta_0^{\bar{0}} t^0$ *leads to the relation* $d\bar{q}^{\bar{\alpha}} = \partial_0\varphi^{\bar{\alpha}}\mathrm{d}t^0 + \partial_\beta\varphi^{\bar{\alpha}}\mathrm{d}q^\beta$ *and* $\mathrm{d}\bar{t}^{\bar{0}} = \delta_0^{\bar{0}}\mathrm{d}t^0$. *Therefore we obtain* $\mathrm{d}q^\beta = \partial_{\bar{\alpha}}\hat{\varphi}^\beta(d\bar{q}^{\bar{\alpha}} - \partial_0\varphi^{\bar{\alpha}}\mathrm{d}t^0)$ *and consequently the new metric reads as*

$$\bar{g} = \delta_{\alpha\beta}\partial_{\bar{\alpha}}\hat{\varphi}^\beta\partial_{\bar{\beta}}\hat{\varphi}^\alpha(\mathrm{d}\bar{q}^{\bar{\alpha}} - \delta_{\bar{0}}^0\partial_0\varphi^{\bar{\alpha}}\mathrm{d}\bar{t}^{\bar{0}}) \otimes (\mathrm{d}\bar{q}^{\bar{\beta}} - \delta_{\bar{0}}^0\partial_0\varphi^{\bar{\beta}}\mathrm{d}\bar{t}^{\bar{0}})$$

and this is exactly a metric of the form (3.7) with $\bar{g}_{\bar{\alpha}\bar{\beta}} = \delta_{\alpha\beta}\partial_{\bar{\alpha}}\hat{\varphi}^\beta\partial_{\bar{\beta}}\hat{\varphi}^\alpha$ *and* $\bar{\gamma}_{\bar{0}}^{\bar{\alpha}} = \delta_{\bar{0}}^0\partial_0\varphi^{\bar{\alpha}}$.

3.2.3 Motion, Velocity and Momentum

A motion of a mass point is defined by a section $s : \mathcal{B} \to \mathcal{Q}$ of the configuration bundle. In order to calculate the change of the motion we use a covariant differential with respect to the reference frame, see also [Mangiarotti and Sardanashvily, 1998],

$$\begin{array}{c} \nabla^\gamma : \mathcal{J}^1(\mathcal{Q}) \to \mathcal{T}^*(\mathcal{B}) \otimes \mathcal{V}(\mathcal{Q}) \\ \nabla^\gamma = \mathrm{d}t^0 \otimes (q_0^\alpha - \gamma_0^\alpha)\,\partial_\alpha \end{array} \tag{3.8}$$

and the covariant derivative of a section s with respect to the reference frame

$$\begin{array}{c} \nabla^\gamma(s) : \mathcal{B} \to \mathcal{T}^*(\mathcal{B}) \otimes \mathcal{V}(\mathcal{Q}) \\ \nabla^\gamma(s) = \mathrm{d}t^0 \otimes (\partial_0 s^\alpha - \gamma_0^\alpha \circ s)\,\partial_\alpha \end{array} . \tag{3.9}$$

The velocity v is a vertical vector field and can be interpreted as a section of the bundle $(\pi_0^1)^*(\mathcal{V}(\mathcal{Q})) \to \mathcal{J}^1(\mathcal{Q})$ where $\pi_0^1 : \mathcal{J}^1(\mathcal{Q}) \to \mathcal{Q}$ and follows as

$$v = \partial_0 \rfloor \nabla^\gamma = (q_0^\alpha - \gamma_0^\alpha)\,\partial_\alpha \ . \tag{3.10}$$

If we restrict the velocity to the motion we have

$$v \circ j^1(s) = \partial_0 \rfloor \nabla^\gamma(s) = (\partial_0 s^\alpha - \gamma_0^\alpha \circ s)\,\partial_\alpha \ .$$

These constructions are visualized in Figure (3.1).

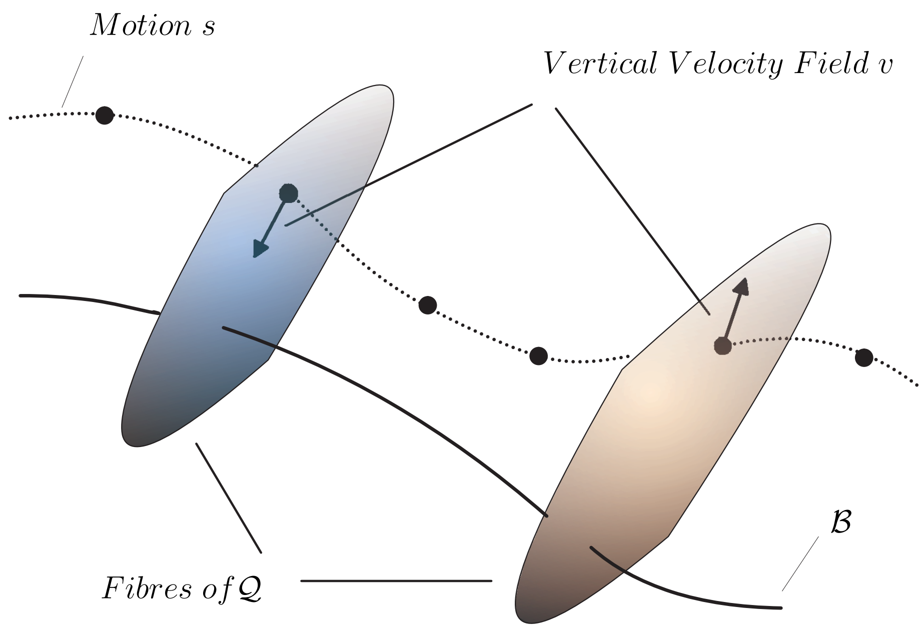

Figure 3.1: The Configuration Bundle

Remark 3.16 *It should be emphasized that this description of the velocity corresponds to a relative velocity with respect to an inertial reference frame which reads as* $\bar{\gamma}^{\bar{\alpha}}_{\bar{0}} = 0$ *in the bundle coordinates* $(\bar{t}^{\bar{0}} = t^0, \bar{q}^{\bar{\alpha}})$.

The dual construction is based on the momentum which is introduced as the dual object to the velocity with respect to the metric and is given by a section

$$p : \mathcal{J}^1(\mathcal{Q}) \to \left(\pi^1_0\right)^* (\mathcal{V}^*(\mathcal{Q}))$$

as

$$p = m(v \rfloor g) \ .$$

To be able to derive the change of the velocity along the motion we introduce a connection that splits $\mathcal{T}_{\mathcal{Q}}(\mathcal{V}(\mathcal{Q}))$. Such a connection is given by

$$\Lambda = \mathrm{d}t^0 \otimes \left(\partial_0 + \Lambda^\rho_0 \dot{\partial}_\rho\right) + \mathrm{d}q^\alpha \otimes \left(\partial_\alpha + \Lambda^\rho_\alpha \dot{\partial}_\rho\right) \ , \quad \Lambda^\rho_0, \Lambda^\rho_\alpha \in \mathcal{C}^\infty(\mathcal{V}(\mathcal{Q})) \tag{3.11}$$

where the standard holonomic base for $\mathcal{T}(\mathcal{V}(\mathcal{Q}))$ is given by $(\partial_0, \partial_\alpha, \dot{\partial}_\rho)$. The following commutative diagram illustrates the geometric constructions presented so far, involving the connections γ and Λ.

$$\begin{tikzcd}
\mathcal{V}(\mathcal{Q}) \arrow[d] & \mathcal{T}(\mathcal{V}(\mathcal{Q})) \arrow[l] \arrow[rr, equal] \arrow[d] & & \mathcal{V}(\mathcal{V}(\mathcal{Q})) \oplus \mathcal{H}(\mathcal{V}(\mathcal{Q})) \\
\mathcal{Q} \arrow[d] & \mathcal{T}(\mathcal{Q}) \arrow[l] \arrow[r, equal] \arrow[d] & \mathcal{H}(\mathcal{Q}) \oplus \mathcal{V}(\mathcal{Q}) \arrow[r, equal] & \mathcal{T}(\mathcal{Q}) \arrow[u, "\Lambda"'] \\
\mathcal{B} & \mathcal{T}(\mathcal{B}) \arrow[l] \arrow[r, equal] & \mathcal{T}(\mathcal{B}) \arrow[u, "\gamma"] &
\end{tikzcd}$$

The transition functions for the connection (3.11) read as

$$\begin{aligned} \bar{\Lambda}^{\bar{\rho}}_{\bar{\alpha}} &= \partial_{\bar{\alpha}}\hat{\varphi}^{\alpha}(\partial_{\alpha\tau}\varphi^{\bar{\rho}}\dot{q}^{\tau} + \partial_{\rho}\varphi^{\bar{\rho}}\Lambda^{\rho}_{\alpha}) \\ \bar{\Lambda}^{\bar{\rho}}_{\bar{0}} &= \partial_{\bar{0}}\hat{\phi}^{0}(\partial_{0\beta}\varphi^{\bar{\rho}}\dot{q}^{\beta} + \partial_{\beta}\varphi^{\bar{\rho}}\Lambda^{\beta}_{0} - \partial_{0}\varphi^{\bar{\alpha}}\bar{\Lambda}^{\bar{\rho}}_{\bar{\alpha}}) \end{aligned} \tag{3.12}$$

where the detailed computation can be found in the Appendix A.2.1. We will study only linear connections and so we have the relations

$$\Lambda^{\rho}_{0} = \Lambda^{\rho}_{0\varepsilon}\dot{q}^{\varepsilon} , \quad \Lambda^{\rho}_{\alpha} = \Lambda^{\rho}_{\alpha\varepsilon}\dot{q}^{\varepsilon} .$$

To study the change of the momentum a connection on $\mathcal{V}^*(\mathcal{Q}) \to \mathcal{Q}$ is of importance that splits $\mathcal{T}_{\mathcal{Q}}(\mathcal{V}^*(\mathcal{Q}))$ which is given by

$$\Lambda^* = \mathrm{d}t^0 \otimes \left(\partial_0 + \Lambda^*_{0\rho}\dot{\partial}^{\rho}\right) + \mathrm{d}q^{\alpha} \otimes \left(\partial_{\alpha} + \Lambda^*_{\alpha\rho}\dot{\partial}^{\rho}\right) \tag{3.13}$$

with the standard holonomic base $(\partial_0, \partial_\alpha, \dot{\partial}^\rho)$ for $\mathcal{T}(\mathcal{V}^*(\mathcal{Q}))$. The transition functions for the connection (3.13) read as

$$\begin{aligned} \bar{\Lambda}^*_{\bar{\alpha}\bar{\rho}} &= \partial_{\bar{\alpha}}\hat{\varphi}^{\alpha}(\partial_{\bar{\beta}\bar{\rho}}\hat{\varphi}^{\beta}\dot{q}_{\beta}\partial_{\alpha}\varphi^{\bar{\beta}} + \partial_{\bar{\rho}}\hat{\varphi}^{\beta}\Lambda^*_{\alpha\beta}) \\ \bar{\Lambda}^*_{\bar{0}\bar{\rho}} &= \partial_{\bar{0}\bar{\rho}}\hat{\varphi}^{\beta}\dot{q}_{\beta} + \partial_{\bar{0}}\hat{\phi}^{0}\left(\partial_{\bar{\tau}\bar{\rho}}\hat{\varphi}^{\beta}\partial_{0}\varphi^{\bar{\tau}}\dot{q}_{\beta} + \partial_{\bar{\rho}}\hat{\varphi}^{\beta}\Lambda^*_{0\beta} - \partial_{0}\varphi^{\bar{\alpha}}\bar{\Lambda}^*_{\bar{\alpha}\bar{\rho}}\right) \end{aligned} \tag{3.14}$$

where the detailed computation can be found in the Appendix A.2.2. Considering a linear connection, it is easily observed that the relations

$$\Lambda^*_{0\rho} = \Lambda^{\beta *}_{0\rho}\dot{q}_{\beta} , \quad \Lambda^*_{\rho\alpha} = \Lambda^{\beta *}_{\rho\alpha}\dot{q}_{\beta}$$

are met.

It is important to mention that Λ and Λ^* are linear connections on dual vector bundles and therefore they fulfill

$$\Lambda^{\rho}_{0\varepsilon} = -\Lambda^{\rho *}_{0\varepsilon} , \quad \Lambda^{\rho}_{\alpha\varepsilon} = -\Lambda^{\rho *}_{\alpha\varepsilon} ,$$

see section 2.4.2.

3.2.4 Covariant Derivatives

The covariant derivative of a section $w : \mathcal{Q} \to \mathcal{V}(\mathcal{Q})$ with respect to the linear connection Λ is given by

$$\begin{gathered} \nabla^{\Lambda}(w) : \mathcal{Q} \to \mathcal{T}^*(\mathcal{Q}) \otimes \mathcal{V}_{\mathcal{Q}}(\mathcal{V}(\mathcal{Q})) \\ \nabla^{\Lambda}(w) = (\partial_0 w^{\rho} - \Lambda^{\rho}_{0\alpha}w^{\alpha})\,\mathrm{d}t^0 \otimes \partial_{\rho} + (\partial_{\alpha}w^{\rho} - \Lambda^{\rho}_{\alpha\beta}w^{\beta})\mathrm{d}q^{\alpha} \otimes \partial_{\rho} \end{gathered} \tag{3.15}$$

where the natural isomorphism $\mathcal{V}_{\mathcal{Q}}(\mathcal{V}(\mathcal{Q})) \approx \mathcal{V}(\mathcal{Q}) \times_{\mathcal{Q}} \mathcal{V}(\mathcal{Q})$ was used. The covariant derivative of a section $\omega : \mathcal{Q} \to \mathcal{V}^*(\mathcal{Q})$ with respect to the linear connection Λ^* reads as

$$\begin{gathered} \nabla^{\Lambda^*}(\omega) : \mathcal{Q} \to \mathcal{T}^*(\mathcal{Q}) \otimes \mathcal{V}_{\mathcal{Q}}(\mathcal{V}^*(\mathcal{Q})) \\ \nabla^{\Lambda^*}(\omega) = \left((\partial_0\omega_{\beta} - \Lambda^{\rho *}_{0\beta}\omega_{\rho})\mathrm{d}t^0 + (\partial_{\alpha}\omega_{\beta} - \Lambda^{\rho *}_{\alpha\beta}\omega_{\rho})\mathrm{d}q^{\alpha}\right) \otimes \left(\mathrm{d}q^{\beta} - \gamma^{\beta}_{0}\mathrm{d}t^0\right) \end{gathered} \tag{3.16}$$

where the natural isomorphism $\mathcal{V}_{\mathcal{Q}}(\mathcal{V}^*(\mathcal{Q})) \approx \mathcal{V}^*(\mathcal{Q}) \times_{\mathcal{Q}} \mathcal{V}^*(\mathcal{Q})$ is of importance.

The Connection Coefficients

To determine the coefficients $\Lambda^{\rho}_{0\alpha}, \Lambda^{\rho}_{\alpha\beta}$ we consider the map $g : \mathcal{V}(\mathcal{Q}) \to \mathcal{V}^*(\mathcal{Q})$ and proceed as described in section 3.1.2 using the vertical metric. The extension of the map $\dot{q}_\alpha = g_{\alpha\beta}\dot{q}^\beta$ to its first jet

$$j^1(g) : \mathcal{J}^1_{\mathcal{Q}}(\mathcal{V}(\mathcal{Q})) \to \mathcal{J}^1_{\mathcal{Q}}(\mathcal{V}^*(\mathcal{Q}))$$

yields the following relations

$$(\dot{q}_\alpha)_0 = g_{\alpha\beta}(\dot{q}^\beta)_0 + (\partial_0 g_{\alpha\beta})\,\dot{q}^\beta \ , \quad (\dot{q}_\alpha)_\rho = g_{\alpha\beta}(\dot{q}^\beta)_\rho + (\partial_\rho g_{\alpha\beta})\,\dot{q}^\beta$$

which are affine.

Remark 3.17 *The first jet manifolds $\mathcal{J}^1_{\mathcal{Q}}(\mathcal{V}(\mathcal{Q}))$ and $\mathcal{J}^1_{\mathcal{Q}}(\mathcal{V}^*(\mathcal{Q}))$ possess the coordinates*

$$\begin{aligned}\mathcal{J}^1_{\mathcal{Q}}(\mathcal{V}(\mathcal{Q})) &\ : \quad (t^0, q^\alpha, \dot{q}^\alpha, (\dot{q}^\alpha)_0\,, (\dot{q}^\alpha)_\rho)\\ \mathcal{J}^1_{\mathcal{Q}}(\mathcal{V}^*(\mathcal{Q})) &\ : \quad (t^0, q^\alpha, \dot{q}_\alpha, (\dot{q}_\alpha)_0\,, (\dot{q}_\alpha)_\rho)\ .\end{aligned}$$

To make the map $j^1(g)$ linear, a linear connection Λ and its dual Λ^* is chosen such that

$$(\dot{q}_\alpha)_0 - \Lambda^*_{0\alpha} = g_{\alpha\beta}((\dot{q}^\beta)_0 - \Lambda^\beta_0) \ , \quad (\dot{q}_\alpha)_\rho - \Lambda^*_{\alpha\rho} = g_{\alpha\beta}((\dot{q}^\beta)_\rho - \Lambda^\beta_\rho)$$

is met. Rewriting the expressions we get

$$(\partial_0 g_{\alpha\varepsilon}) = -g_{\kappa\varepsilon}\Lambda^\kappa_{0\alpha} - g_{\alpha\beta}\Lambda^\beta_{0\varepsilon} \ , \quad (\partial_\rho g_{\alpha\varepsilon}) = -g_{\kappa\varepsilon}\Lambda^\kappa_{\alpha\rho} - g_{\alpha\beta}\Lambda^\beta_{\rho\varepsilon}\ . \tag{3.17}$$

Remark 3.18 *At this point again it is worth mentioning that the relations* (3.17) *can be interpreted as* $\nabla^\Lambda(g) = 0$ *for a time dependent metric.*

Remark 3.19 *We additionally have*

$$\frac{1}{\sqrt{|\det(g_{\alpha\beta})|}}\partial_0\sqrt{|\det(g_{\alpha\beta})|} = -\Lambda^\kappa_{0\kappa} \tag{3.18}$$

which is the analogy to the relation (2.13) for the time components, where the proof can be found in the Appendix A.3.

Remark 3.20 *Let us consider a motion $s^\alpha(t^0)$ and the corresponding generator $\partial_0 + \partial_0 s^\alpha\partial_\alpha$ together with the sections $u, v : \mathcal{Q} \to \mathcal{V}(\mathcal{Q})$, which are parallel along the motion $s^\alpha(t^0)$. A field $u : \mathcal{Q} \to \mathcal{V}(\mathcal{Q})$ is parallel along $\partial_0 + \partial_0 s^\alpha\partial_\alpha$ if*

$$(\partial_0 + \partial_0 s^\alpha\partial_\alpha)\rfloor\nabla^\Lambda(u\circ s) = \left(\partial_0(u^\rho\circ s) - \Lambda^\rho_{0\alpha}u^\alpha - \Lambda^\rho_{\alpha\beta}u^\beta\partial_0 s^\alpha\right)\partial_\rho = 0$$

is fulfilled. The formal expression

$$\partial_0\left(g_{\alpha\beta}u^\alpha v^\beta\circ s\right) = (\partial_0 g_{\alpha\beta} + \partial_0 s^\rho\partial_\rho g_{\alpha\beta})\,u^\alpha v^\beta + g_{\alpha\beta}v^\beta\partial_0(u^\alpha\circ s) + g_{\alpha\beta}u^\alpha\partial_0(v^\beta\circ s)$$

which is

$$\partial_0\left(g_{\alpha\beta}u^\alpha v^\beta\circ s\right) = \left(\partial_0 g_{\alpha\beta} + \partial_0 s^\rho\partial_\rho g_{\alpha\beta} + g_{\nu\beta}\left(\Lambda^\nu_{0\alpha} + \Lambda^\nu_{\rho\alpha}\partial_0 s^\rho\right) + g_{\alpha\mu}\left(\Lambda^\mu_{0\beta} + \Lambda^\mu_{\rho\beta}\partial_0 s^\rho\right)\right)u^\alpha v^\beta$$

shows that the vertical metric will preserve the inner product along a motion if the relations (3.17) *hold.*

The second equation of (3.17) is exactly the same as in section 3.1.2 and therefore we obtain the same result for the coefficients $\Lambda^{\kappa}_{\alpha\rho}$

$$2\Lambda^{\kappa}_{\alpha\rho} = -\hat{g}^{\kappa\beta}\left(\partial_\alpha g_{\rho\beta} + \partial_\rho g_{\beta\alpha} - \partial_\beta g_{\alpha\rho}\right) \ . \tag{3.19}$$

The equation (3.19) relates the connection coefficients $\Lambda^{\kappa}_{\alpha\rho}$ with the metric g and the demand to obtain a similar expression for $\Lambda^{\kappa}_{0\alpha}$ can not be achieved, since the first equation of (3.17) is much more delicate and we will show that to proceed one can obtain an expression that relates $\partial_0 g_{\alpha\varepsilon}$ with the connection γ and the metric g. Therefore let us consider an inertial system where the manifold $\mathcal{Q}$ possesses the coordinates (q^α, t^0) and a trivial connection $\gamma = \mathrm{d}t^0 \otimes \partial_0$ with $\gamma^\alpha_0 = 0$. Then the metric is given by $g = g_{\alpha\beta}\mathrm{d}q^\alpha \otimes \mathrm{d}q^\beta$. Again a bundle morphism of the type $\bar{q}^{\bar{\alpha}} = \varphi^{\bar{\alpha}}(q^\beta, t^0)$, $\bar{t}^{\bar{0}} = \phi^{\bar{0}}(t^0)$ is considered.

Remark 3.21 *In this case we use exceptionally $\bar{t}^{\bar{0}} = \phi^{\bar{0}}(t^0)$ instead of $\bar{t}^{\bar{0}} = \delta^{\bar{0}}_0 t^0$ in order to be more general.*

The computation proceeds as follows. Let us consider

$$\partial_{\bar{0}}(\bar{g}_{\bar{\alpha}\bar{\beta}}) = \partial_{\bar{0}}\left((g_{\alpha\beta}\circ\hat{\varphi})\left(\partial_{\bar{\alpha}}\hat{\varphi}^\alpha\right)\left(\partial_{\bar{\beta}}\hat{\varphi}^\beta\right)\right) \ . \tag{3.20}$$

The right hand side of equation (3.20) can be evaluated as

$$g_{\alpha\beta}\left(\left(\partial_{\bar{0}\bar{\alpha}}\hat{\varphi}^\alpha\right)\left(\partial_{\bar{\beta}}\hat{\varphi}^\beta\right) + \left(\partial_{\bar{\alpha}}\hat{\varphi}^\alpha\right)\left(\partial_{\bar{0}\bar{\beta}}\hat{\varphi}^\beta\right)\right) + \left(\partial_{\bar{\alpha}}\hat{\varphi}^\alpha\right)\left(\partial_{\bar{\beta}}\hat{\varphi}^\beta\right)\left(\partial_\tau g_{\alpha\beta}\right)\left(\partial_{\bar{0}}\hat{\varphi}^\tau\right)$$

and using equation (A.5) this consequently leads to

$$\begin{aligned} & g_{\alpha\beta}\Big((\partial_{\bar{\beta}}\hat{\varphi}^\beta)\partial_{\bar{\alpha}}(-\partial_0\varphi^{\bar{\rho}}\partial_{\bar{0}}\hat{\phi}^0\partial_{\bar{\rho}}\hat{\varphi}^\alpha) + \left(\partial_{\bar{\alpha}}\hat{\varphi}^\alpha\right)\left(\partial_{\bar{\beta}}(-\partial_0\varphi^{\bar{\rho}}\partial_{\bar{0}}\hat{\phi}^0\partial_{\bar{\rho}}\hat{\varphi}^\beta)\right)\Big) \\ & -\left(\partial_{\bar{\alpha}}\hat{\varphi}^\alpha\right)\left(\partial_{\bar{\beta}}\hat{\varphi}^\beta\right)\left(\partial_\tau g_{\alpha\beta}\right)\left(\partial_{\bar{\rho}}\hat{\varphi}^\tau\right)(\partial_0\varphi^{\bar{\rho}})\partial_{\bar{0}}\hat{\phi}^0 \ . \end{aligned} \tag{3.21}$$

The next step involves the transition function for the connection coefficients (3.6) and therefore we can use

$$\bar{\gamma}^{\bar{\alpha}}_{\bar{0}} = \partial_{\bar{0}}\hat{\phi}^0\partial_0\varphi^{\bar{\alpha}} \ .$$

This leads to

$$\partial_{\bar{0}}(\bar{g}_{\bar{\alpha}\bar{\beta}}) = -g_{\alpha\beta}\left((\partial_{\bar{\beta}}\hat{\varphi}^\beta)\partial_{\bar{\alpha}}(\bar{\gamma}^{\bar{\rho}}_{\bar{0}}\partial_{\bar{\rho}}\hat{\varphi}^\alpha) + \left(\partial_{\bar{\alpha}}\hat{\varphi}^\alpha\right)\partial_{\bar{\beta}}(\bar{\gamma}^{\bar{\rho}}_{\bar{0}}\partial_{\bar{\rho}}\hat{\varphi}^\beta)\right) - \left(\partial_{\bar{\alpha}}\hat{\varphi}^\alpha\right)\left(\partial_{\bar{\beta}}\hat{\varphi}^\beta\right)\left(\partial_\tau g_{\alpha\beta}\right)\left(\partial_{\bar{\rho}}\hat{\varphi}^\tau\right)\bar{\gamma}^{\bar{\rho}}_{\bar{0}} \ . \tag{3.22}$$

Let us consider the formal expression

$$\begin{aligned} \partial_{\bar{\rho}}\bar{g}_{\bar{\alpha}\bar{\beta}} &= \partial_{\bar{\rho}}\left((g_{\alpha\beta}\circ\hat{\varphi})\left(\partial_{\bar{\alpha}}\hat{\varphi}^\alpha\right)\left(\partial_{\bar{\beta}}\hat{\varphi}^\beta\right)\right) \\ &= g_{\alpha\beta}\left(\left(\partial_{\bar{\rho}\bar{\alpha}}\hat{\varphi}^\alpha\right)\left(\partial_{\bar{\beta}}\hat{\varphi}^\beta\right) + \left(\partial_{\bar{\alpha}}\hat{\varphi}^\alpha\right)\left(\partial_{\bar{\rho}\bar{\beta}}\hat{\varphi}^\beta\right)\right) + \left(\partial_\tau g_{\alpha\beta}\right)\left(\partial_{\bar{\rho}}\hat{\varphi}^\tau\right)\left(\partial_{\bar{\alpha}}\hat{\varphi}^\alpha\right)\left(\partial_{\bar{\beta}}\hat{\varphi}^\beta\right) \end{aligned} \tag{3.23}$$

and combining equation (3.23) with (3.22) we obtain

$$\partial_{\bar{0}}(\bar{g}_{\bar{\alpha}\bar{\beta}}) = -(\partial_{\bar{\rho}}\bar{g}_{\bar{\alpha}\bar{\beta}})\bar{\gamma}^{\bar{\rho}}_{\bar{0}} - g_{\bar{\beta}\bar{\rho}}(\partial_{\bar{\alpha}}\bar{\gamma}^{\bar{\rho}}_{\bar{0}}) - g_{\bar{\alpha}\bar{\rho}}(\partial_{\bar{\beta}}\bar{\gamma}^{\bar{\rho}}_{\bar{0}})$$

and omitting the bars we have the relation

$$\partial_0 g_{\alpha\beta} = -(\partial_\alpha\gamma^\rho_0)g_{\rho\beta} - (\partial_\beta\gamma^\rho_0)g_{\rho\alpha} - \left(\partial_\rho g_{\alpha\beta}\right)\gamma^\rho_0 \ . \tag{3.24}$$

Combining the first equation of (3.17) and (3.24) the desired result follows as

$$\Lambda^{\rho}_{0\alpha} = (\partial_\alpha \gamma^{\rho}_0) - \Lambda^{\rho}_{\beta\alpha}\gamma^{\beta}_0 \, . \tag{3.25}$$

We see that $\Lambda^{\kappa}_{\alpha\rho}$ are obviously the Christoffel symbols of the second kind and $\Lambda^{\rho}_{0\alpha}$ play a crucial role when the space-time connection γ is not trivial.

Remark 3.22 *The connection Λ in an inertial coordinate system with Euclidean coordinates (q^α, t^0) reads as*

$$\Lambda = \mathrm{d}t^0 \otimes \partial_0 + \mathrm{d}q^\alpha \otimes \partial_\alpha$$

and the transitions functions after a bundle morphism of the type $\bar{q}^{\bar{\alpha}} = \varphi^{\bar{\alpha}}(q^\beta, t^0)$, $\bar{t}^{\bar{0}} = \phi^{\bar{0}}(t^0)$ follow from the relations (3.12) to

$$\begin{aligned} \bar{\Lambda}^{\bar{\rho}}_{\bar{\alpha}} &= \partial_{\bar{\alpha}}\hat{\varphi}^{\alpha}\partial_{\alpha\tau}\varphi^{\bar{\rho}}\partial_{\bar{\kappa}}\varphi^{\tau}\dot{\bar{q}}^{\bar{\kappa}} = \bar{\Lambda}^{\bar{\rho}}_{\bar{\alpha}\bar{\kappa}}\dot{\bar{q}}^{\bar{\kappa}} \\ \bar{\Lambda}^{\bar{\rho}}_{\bar{0}} &= \partial_{\bar{0}}\hat{\phi}^{0}\partial_{0\beta}\varphi^{\bar{\rho}}\partial_{\bar{\kappa}}\hat{\varphi}^{\beta}\dot{\bar{q}}^{\bar{\kappa}} - \bar{\gamma}^{\bar{\alpha}}_{\bar{0}}\bar{\Lambda}^{\bar{\rho}}_{\bar{\alpha}} = \left(\partial_{\bar{\kappa}}\bar{\gamma}^{\bar{\rho}}_{\bar{0}} - \bar{\gamma}^{\bar{\alpha}}_{\bar{0}}\bar{\Lambda}^{\bar{\rho}}_{\bar{\alpha}\bar{\kappa}}\right)\dot{\bar{q}}^{\bar{\kappa}} = \bar{\Lambda}^{\bar{\rho}}_{\bar{0}\bar{\kappa}}\dot{\bar{q}}^{\bar{\kappa}} \end{aligned}$$

and this is exactly the result of relation (3.25). The dual result follows from the relations (3.14) as

$$\begin{aligned} \bar{\Lambda}^{*}_{\bar{\alpha}\bar{\rho}} &= \partial_{\bar{\alpha}\bar{\rho}}\hat{\varphi}^{\beta}\partial_{\beta}\varphi^{\bar{\kappa}}\dot{\bar{q}}_{\bar{\kappa}} = \bar{\Lambda}^{\bar{\kappa}*}_{\bar{\alpha}\bar{\rho}}\dot{\bar{q}}_{\bar{\kappa}} \\ \bar{\Lambda}^{*}_{\bar{0}\bar{\rho}} &= \partial_{\bar{0}\bar{\rho}}\hat{\varphi}^{\beta}\dot{q}_{\beta} + \partial_{\bar{0}}\hat{\phi}^{0}\left(\partial_{\bar{\tau}\bar{\rho}}\hat{\varphi}^{\beta}\partial_0\varphi^{\bar{\tau}}\dot{q}_{\beta} + \partial_{\bar{\rho}}\hat{\varphi}^{\beta}\Lambda^{*}_{0\beta} - \partial_0\varphi^{\bar{\alpha}}\bar{\Lambda}^{*}_{\bar{\alpha}\bar{\rho}}\right) \\ &= \left(\partial_{\bar{0}\bar{\rho}}\hat{\varphi}^{\beta}\partial_{\beta}\varphi^{\bar{\kappa}} + \partial_{\bar{0}}\hat{\phi}^{0}\partial_{\bar{\tau}\bar{\rho}}\hat{\varphi}^{\beta}\partial_0\varphi^{\bar{\tau}}\partial_{\beta}\varphi^{\bar{\kappa}} - \partial_{\bar{0}}\hat{\phi}^{0}\partial_0\varphi^{\bar{\alpha}}\bar{\Lambda}^{\bar{\kappa}*}_{\bar{\alpha}\bar{\rho}}\right)\dot{\bar{q}}_{\bar{\kappa}} \\ &= \left(\partial_{\bar{\rho}}(-\partial_0\varphi^{\bar{\alpha}}\partial_{\bar{0}}\hat{\phi}^{0}\partial_{\bar{\alpha}}\hat{\varphi}^{\beta})\partial_{\beta}\varphi^{\bar{\kappa}} + \partial_{\bar{0}}\hat{\phi}^{0}\partial_{\bar{\tau}\bar{\rho}}\hat{\varphi}^{\beta}\partial_0\varphi^{\bar{\tau}}\partial_{\beta}\varphi^{\bar{\kappa}} - \partial_{\bar{0}}\hat{\phi}^{0}\partial_0\varphi^{\bar{\alpha}}\bar{\Lambda}^{\bar{\kappa}*}_{\bar{\alpha}\bar{\rho}}\right)\dot{\bar{q}}_{\bar{\kappa}} \\ &= \left(-(\partial_{\bar{\rho}}\gamma^{\bar{\alpha}}_{\bar{0}})\partial_{\bar{\alpha}}\hat{\varphi}^{\beta}\partial_{\beta}\varphi^{\bar{\kappa}} - \gamma^{\bar{\alpha}}_{\bar{0}}\partial_{\bar{\rho}\bar{\alpha}}\hat{\varphi}^{\beta}\partial_{\beta}\varphi^{\bar{\kappa}} + \partial_{\bar{0}}\hat{\phi}^{0}\partial_0\varphi^{\bar{\tau}}\partial_{\bar{\tau}\bar{\rho}}\hat{\varphi}^{\beta}\partial_{\beta}\varphi^{\bar{\kappa}} - \partial_{\bar{0}}\hat{\phi}^{0}\partial_0\varphi^{\bar{\alpha}}\bar{\Lambda}^{\bar{\kappa}*}_{\bar{\alpha}\bar{\rho}}\right)\dot{\bar{q}}_{\bar{\kappa}} \\ &= \left(-\partial_{\bar{\rho}}\gamma^{\bar{\kappa}}_{\bar{0}} - \gamma^{\bar{\alpha}}_{\bar{0}}\bar{\Lambda}^{\bar{\kappa}*}_{\bar{\alpha}\bar{\rho}}\right)\dot{\bar{q}}_{\bar{\kappa}} = \bar{\Lambda}^{\bar{\kappa}*}_{\bar{0}\bar{\rho}}\dot{\bar{q}}_{\bar{\kappa}} \end{aligned}$$

where essential use of the equation (A.5) was made. This of course shows again the relationship

$$\bar{\Lambda}^{\bar{\rho}}_{\bar{\alpha}\bar{\kappa}} = -\bar{\Lambda}^{\bar{\rho}*}_{\bar{\alpha}\bar{\kappa}} \, , \quad \bar{\Lambda}^{\bar{\kappa}}_{\bar{0}\bar{\rho}} = -\bar{\Lambda}^{\bar{\kappa}*}_{\bar{0}\bar{\rho}}$$

where we used the equation (A.2)

$$\left(\partial_{\alpha\beta}\varphi^{\bar{\alpha}}\right)\left(\partial_{\bar{\beta}}\hat{\varphi}^{\beta}\right)\left(\partial_{\bar{\rho}}\hat{\varphi}^{\alpha}\right) = -\left(\partial_{\beta}\varphi^{\bar{\alpha}}\right)\left(\partial_{\bar{\rho}\bar{\beta}}\hat{\varphi}^{\beta}\right)$$

from the Appendix.

Some Important Formulas

This paragraph is devoted to the introduction of two useful relations involving covariant derivatives. The first one is given by

$$\nabla^{\Lambda}(w)\rfloor g = \nabla^{\Lambda^*}(w\rfloor g) \tag{3.26}$$

for any vertical field $w : \mathcal{Q} \to \mathcal{V}(\mathcal{Q})$ and the metric $g : \mathcal{V}(\mathcal{Q}) \to \mathcal{V}^*(\mathcal{Q})$ and this can be easily verified in coordinates since

$$\nabla^\Lambda(w)\rfloor g = g_{\rho\beta}\left((\partial_0 w^\rho - \Lambda^\rho_{0\kappa}w^\kappa)\,\mathrm{d}t^0 + (\partial_\alpha w^\rho - \Lambda^\rho_{\alpha\kappa}w^\kappa)\mathrm{d}q^\alpha\right) \otimes (\mathrm{d}q^\beta - \gamma^\beta_0\mathrm{d}t^0)$$

and

$$\begin{aligned}\nabla^{\Lambda^*}(w\rfloor g) &= \left((\partial_0(w^\tau g_{\tau\beta}) - \Lambda^{\rho*}_{0\beta}w^\tau g_{\tau\rho})\mathrm{d}t^0 + (\partial_\alpha(w^\tau g_{\tau\beta}) - \Lambda^{\rho*}_{\alpha\beta}w^\tau g_{\tau\rho})\mathrm{d}q^\alpha\right) \otimes \left(\mathrm{d}q^\beta - \gamma^\beta_0\mathrm{d}t^0\right)\\ &= (g_{\tau\beta}\partial_0 w^\tau + w^\tau\partial_0 g_{\tau\beta} - \Lambda^{\rho*}_{0\beta}w^\tau g_{\tau\rho})\mathrm{d}t^0 \otimes \left(\mathrm{d}q^\beta - \gamma^\beta_0\mathrm{d}t^0\right)\\ &\quad +(g_{\tau\beta}\partial_\alpha w^\tau + w^\tau\partial_\alpha g_{\tau\beta} - \Lambda^{\rho*}_{\alpha\beta}w^\tau g_{\tau\rho})\mathrm{d}q^\alpha \otimes \left(\mathrm{d}q^\beta - \gamma^\beta_0\mathrm{d}t^0\right)\ .\end{aligned}$$

With the relations (3.17) which were given as

$$(\partial_0 g_{\tau\beta}) = -g_{\kappa\beta}\Lambda^\kappa_{0\tau} - g_{\tau\kappa}\Lambda^\kappa_{0\beta}\ , \quad (\partial_\alpha g_{\tau\beta}) = -g_{\kappa\beta}\Lambda^\kappa_{\tau\alpha} - g_{\tau\kappa}\Lambda^\kappa_{\alpha\beta}$$

and the fact that

$$\Lambda^\rho_{0\varepsilon} = -\Lambda^{\rho*}_{0\varepsilon}\ , \quad \Lambda^\rho_{\alpha\varepsilon} = -\Lambda^{\rho*}_{\alpha\varepsilon}$$

is met, we obtain

$$\nabla^{\Lambda^*}(w\rfloor g) = g_{\tau\beta}\left((\partial_0 w^\tau - \Lambda^\tau_{0\kappa}w^\kappa)\mathrm{d}t^0 + (\partial_\alpha w^\tau - \Lambda^\tau_{\alpha\kappa}w^\kappa)\mathrm{d}q^\alpha\right) \otimes \left(\mathrm{d}q^\beta - \gamma^\beta_0\mathrm{d}t^0\right)$$

which proofs the equation (3.26).

The second construction is given by

$$\nabla^\Lambda(w)\rfloor\omega + w\rfloor\nabla^{\Lambda^*}(\omega) = \partial_0(w^\rho\omega_\rho)\mathrm{d}t^0 + \partial_\alpha(\omega_\rho w^\rho)\mathrm{d}q^\alpha$$

with $w : \mathcal{Q} \to\mathcal{V}(\mathcal{Q})$ and $\omega : \mathcal{Q} \to\mathcal{V}^*(\mathcal{Q})$. We derive the expression

$$\nabla^\Lambda(w)\rfloor\omega + w\rfloor\nabla^{\Lambda^*}(\omega) = \mathrm{d}(w\rfloor\omega) \tag{3.27}$$

since in coordinates it is obvious that we have

$$\begin{aligned}\nabla^\Lambda(w)\rfloor\omega + w\rfloor\nabla^{\Lambda^*}(\omega) &= \omega_\rho\left((\partial_0 w^\rho - \Lambda^\rho_{0\alpha}w^\alpha)\,\mathrm{d}t^0 + (\partial_\alpha w^\rho - \Lambda^\rho_{\alpha\beta}w^\beta)\mathrm{d}q^\alpha\right)\\ &\quad +w^\beta\left((\partial_0\omega_\beta - \Lambda^{\rho*}_{0\beta}\omega_\rho)\mathrm{d}t^0 + (\partial_\alpha\omega_\beta - \Lambda^{\rho*}_{\alpha\beta}\omega_\rho)\mathrm{d}q^\alpha\right)\end{aligned}$$

and since the dual connection coefficients cancel we obtain

$$\begin{aligned}\nabla^\Lambda(w)\rfloor\omega + w\rfloor\nabla^{\Lambda^*}(\omega) &= (\omega_\rho\partial_0 w^\rho + w^\beta\partial_0\omega_\beta)\mathrm{d}t^0 + (\omega_\rho\partial_\alpha w^\rho + w^\beta\partial_\alpha\omega_\beta)\mathrm{d}q^\alpha\\ &= \partial_0(w^\rho\omega_\rho)\mathrm{d}t^0 + \partial_\alpha(\omega_\rho w^\rho)\mathrm{d}q^\alpha\ .\end{aligned}$$

The Vertical Covariant Differential

To compute the change of the velocity of a mass point along a motion the vertical vector field v has to be restricted to the motion. This means that we have to focus on $\nabla^\Lambda(v\circ j^1\,(s))$

and we see that the bundle $\mathcal{V}(\mathcal{Q}) \to \mathcal{B}$ plays a key role in point mechanics. Let us consider the composite bundle $\mathcal{V}(\mathcal{Q}) \to \mathcal{Q} \to \mathcal{B}$ and the connection γ on $\mathcal{Q} \to \mathcal{B}$

$$\gamma = \mathrm{d}t^0 \otimes (\partial_0 + \gamma_0^\alpha \partial_\alpha)$$

as well as the connection Λ on $\mathcal{V}(\mathcal{Q}) \to \mathcal{Q}$ that splits $\mathcal{T}(\mathcal{V}(\mathcal{Q}))$

$$\Lambda = \mathrm{d}t^0 \otimes \left(\partial_0 + \Lambda^\rho_{0\beta}\dot{q}^\beta \dot{\partial}_\rho\right) + \mathrm{d}q^\alpha \otimes \left(\partial_\alpha + \Lambda^\rho_{\alpha\beta}\dot{q}^\beta \dot{\partial}_\rho\right) \ .$$

The composite connection as described in section 2.4.3 reads as

$$\Xi = \mathrm{d}t^0 \otimes \left(\partial_0 + \gamma_0^\alpha \partial_\alpha + (\partial_\alpha \gamma_0^\rho)\dot{q}^\alpha \dot{\partial}_\rho\right)$$

where we used

$$\Lambda^\rho_{0\alpha} = (\partial_\alpha \gamma_0^\rho) - \Lambda^\rho_{\beta\alpha}\gamma_0^\beta \ .$$

Thus the covariant derivative with respect to this composite connection is

$$\nabla^\Xi(s, \bar{s}) = \mathrm{d}t^0 \otimes \left((\partial_0 s^\alpha - \gamma_0^\alpha \circ s)\, \partial_\alpha + (\partial_0 \bar{s}^\rho - ((\partial_\alpha \gamma_0^\rho) \circ s)\bar{s}^\alpha)\, \dot{\partial}_\rho\right)$$

where $(s, \bar{s}) : \mathcal{B} \to \mathcal{V}(\mathcal{Q})$, in coordinates $q^\alpha = s^\alpha(t^0)$ and $\dot{q}^\alpha = \bar{s}^\alpha(t^0)$. Following the definitions of section 2.4.3, the vertical part with respect to the connection Λ is

$$\tilde{\nabla}^\Lambda(s, \bar{s}) = \mathrm{d}t^0 \otimes \left((\partial_0 \bar{s}^\rho - ((\partial_\alpha \gamma_0^\rho) \circ s)\bar{s}^\alpha) - (\Lambda^\rho_{\alpha\beta} \circ s)\bar{s}^\beta\, (\partial_0 s^\alpha - \gamma_0^\alpha \circ s)\right) \dot{\partial}_\rho$$

and setting $\bar{s} = v \circ j^1(s)$ delivers

$$\partial_0 \rfloor \tilde{\nabla}^\Lambda(s, v \circ j^1(s)) = \left(\partial_0 (v \circ j^1(s))^\rho - (v^\alpha \partial_\alpha \gamma_0^\rho) \circ j^1(s) - \left(\Lambda^\rho_{\alpha\beta} v^\alpha v^\beta\right) \circ j^1(s)\right) \dot{\partial}_\rho \ . \tag{3.28}$$

To compute the change of the momentum similar arguments as above hold. Now we have the composite bundle structure $\mathcal{V}^*(\mathcal{Q}) \to \mathcal{Q} \to \mathcal{B}$ and the connection γ on $\mathcal{Q} \to \mathcal{B}$ as before as well as the connection Λ^* on $\mathcal{V}^*(\mathcal{Q}) \to \mathcal{Q}$ that splits $\mathcal{T}(\mathcal{V}^*(\mathcal{Q}))$

$$\Lambda^* = \mathrm{d}t^0 \otimes \left(\partial_0 + \Lambda^{\tau *}_{0\rho}\dot{q}_\tau \dot{\partial}^\rho\right) + \mathrm{d}q^\alpha \otimes \left(\partial_\alpha + \Lambda^{\tau *}_{\alpha\rho}\dot{q}_\tau \dot{\partial}^\rho\right) \ .$$

The composite connection Ξ^* in this case splits $\mathcal{T}_\mathcal{B}(\mathcal{V}^*(\mathcal{Q}))$ and reads as

$$\Xi^* = \mathrm{d}t^0 \otimes \left(\partial_0 + \gamma_0^\alpha \partial_\alpha - (\partial_\alpha \gamma_0^\rho)\dot{q}_\rho \dot{\partial}^\alpha\right)$$

and the covariant derivative with respect to the connection Ξ^* is

$$\nabla^{\Xi^*}(s, \bar{s}) = \mathrm{d}t^0 \otimes \left((\partial_0 s^\alpha - \gamma_0^\alpha \circ s)\, \partial_\alpha + (\partial_0 \bar{s}_\alpha + ((\partial_\alpha \gamma_0^\rho) \circ s)\bar{s}_\rho)\, \dot{\partial}^\alpha\right)$$

where $(s, \bar{s}) : \mathcal{B} \to \mathcal{V}^*(\mathcal{Q})$, in coordinates $q^\alpha = s^\alpha(t^0)$ and $\dot{q}_\alpha = \bar{s}_\alpha(t^0)$. The vertical part with respect to the connection Λ^* follows as

$$\tilde{\nabla}^{\Lambda^*}(s, \bar{s}) = \mathrm{d}t^0 \otimes \left((\partial_0 \bar{s}_\alpha + ((\partial_\alpha \gamma_0^\rho) \circ s)\bar{s}_\rho) - (\Lambda^{\rho *}_{\alpha\beta} \circ s)\bar{s}_\rho \left(\partial_0 s^\beta - \gamma_0^\beta \circ s\right)\right) \dot{\partial}^\alpha$$

and setting $\bar{s} = p \circ j^1(s)$ gives

$$\partial_0 \rfloor \tilde{\nabla}^{\Lambda^*}(s, p \circ j^1(s)) = \left(\partial_0 \left(p_\alpha \circ j^1(s)\right) + (p_\rho \partial_\alpha \gamma_0^\rho) \circ j^1(s) - (\Lambda^{\rho *}_{\alpha\beta} p_\rho v^\beta) \circ j^1(s)\right) \dot{\partial}^\alpha \ . \tag{3.29}$$

It should be observed that in contrast to Λ and Λ^* the connections Ξ and Ξ^* are not dual since $\mathcal{V}(\mathcal{Q}) \to \mathcal{B}$ is not a vector bundle.

3.2.5 Equations of Motion

Let us consider a motion $s : \mathcal{B} \to \mathcal{Q}$ together with $v_s = \partial_0 + \partial_0 s^\alpha \partial_\alpha$. The velocity of a motion s is a vertical field v and the momentum has been introduced as the object dual to v with respect to the metric. To derive the evolution of a mass particle we are interested in the change of the velocity or equivalently the momentum along the field v_s. Forces can be handled using Newton's postulate that the total change of the momentum of a mass point equals the force acting on it. The results here are given in a pure covariant fashion formulated with respect to a space time connection, which allows to consider non trivial observers.

The Change of the Velocity

We use the covariant derivative (3.15) and the demand that the vertical velocity field is parallel along the motion s, leads us to compute

$$v_s \rfloor \nabla^\Lambda (v \circ j^1(s)) = 0$$

which reads as

$$(\partial_0 + \partial_0 s^\alpha \partial_\alpha) \rfloor \left(\left(\partial_0 (v^\rho \circ j^1(s)) - \Lambda^\rho_{0\alpha}(v^\alpha \circ j^1(s))\right) \mathrm{d}t^0 - \Lambda^\rho_{\alpha\beta} \left(v^\beta \circ j^1(s)\right) \mathrm{d}q^\alpha \right) \otimes \partial_\rho = 0$$

and this results in the first order differential equations

$$\partial_0 (v^\rho \circ j^1(s)) - (\Lambda^\rho_{0\alpha} v^\alpha) \circ j^1(s) - \left(\left(\Lambda^\rho_{\alpha\beta} v^\beta \right) \circ j^1(s) \right) (\partial_0 s^\alpha) \ = 0 \ .$$

If we plug in the connection coefficients (3.25) we obtain

$$\partial_0 (v^\rho \circ j^1(s)) - (v^\alpha \partial_\alpha \gamma^\rho_0) \circ j^1(s) - \left(\left(\Lambda^\rho_{\alpha\beta} v^\beta \right) \circ j^1(s) \right) (\partial_0 s^\alpha - \gamma^\alpha_0 \circ s) = 0$$

and finally if we use $v^\alpha \circ j^1(s) = \partial_0 s^\alpha - \gamma^\alpha_0 \circ s$ we have

$$\partial_{00} s^\rho - \partial_0 (\gamma^\rho_0 \circ s) - (v^\alpha \partial_\alpha \gamma^\rho_0) \circ j^1(s) - \left(\Lambda^\rho_{\alpha\beta} v^\alpha v^\beta \right) \circ j^1(s) = 0 \tag{3.30}$$

and this result obviously coincides with the relation (3.28) which reads as

$$\partial_0 \rfloor \tilde{\nabla}^\Lambda (s, v \circ j^1(s)) = 0 \ .$$

Remark 3.23 *It is worth mentioning that there is a difference between $v_s \rfloor \nabla^\Lambda (v \circ j^1(s))$ and $\partial_0 \rfloor \tilde{\nabla}^\Lambda (s, v \circ j^1(s))$, although they produce the same set of equations. The connection Λ splits the bundle $\mathcal{T}(\mathcal{V}(\mathcal{Q}))$ with respect to $\mathcal{V}(\mathcal{Q}) \to \mathcal{Q}$ and therefore the covariant differential ∇^Λ is used to compute the change of sections $w : \mathcal{Q} \to \mathcal{V}(\mathcal{Q})$, which means $\dot{q}^\alpha = w^\alpha(t^0, q^\beta)$. In point mechanics this section w is identified with the velocity which has to be restricted to the motion of a single mass point, which explains the use of $\nabla^\Lambda (v \circ j^1(s))$. The vertical covariant differential already is appropriate for the bundle structure $\mathcal{V}(\mathcal{Q}) \to \mathcal{B}$, which is the desired form since in point mechanics the motion and the velocity are functions of time.*

The Change of the Momentum

The dual construction is based on the use of the covariant derivative (3.16) and we compute

$$v_s \rfloor \nabla^{\Lambda^*}(p \circ j^1(s)) = 0$$

which gives

$$\partial_0(p_\beta \circ j^1(s)) - (\Lambda^{\rho *}_{0\beta} p_\rho) \circ j^1(s) - \partial_0 s^\alpha (\Lambda^{\rho *}_{\alpha\beta} p_\rho) \circ j^1(s) = 0 \,.$$

With

$$\Lambda^\rho_{0\beta} = (\partial_\beta \gamma^\rho_0) - \Lambda^\rho_{\kappa\beta}\gamma^\kappa_0$$

this follows to

$$\partial_0(p_\beta \circ j^1(s)) + \left(\left(\partial_\beta \gamma^\rho_0 - \Lambda^\rho_{\kappa\beta}\gamma^\kappa_0\right) p_\rho\right) \circ j^1(s) + \partial_0 s^\alpha \left(\Lambda^\rho_{\alpha\beta} p_\rho\right) \circ j^1(s) = 0 \,.$$

Consequently

$$\partial_0(p_\beta \circ j^1(s)) + \left(p_\rho \left(\partial_\beta \gamma^\rho_0 + \Lambda^\rho_{\kappa\beta} v^\kappa\right)\right) \circ j^1(s) = 0$$

and this is exactly the equation (3.29) which was given as

$$\partial_0 \rfloor \tilde{\nabla}^{\Lambda^*}(s, p \circ j^1(s)) = 0$$

and furthermore we have

$$\partial_0(p_\beta \circ j^1(s)) + \left(p_\rho \partial_\beta \gamma^\rho_0 + \frac{1}{m}\Lambda^\rho_{\kappa\beta}\hat{g}^{\kappa\tau} p_\tau p_\rho\right) \circ j^1(s) = 0 \,.$$

From the relation

$$\left(\Lambda^\rho_{\kappa\beta}\hat{g}^{\kappa\tau} p_\rho p_\tau\right) \circ j^1(s) = s^* \left(\partial_\beta(\frac{1}{2}\left(p_\rho \circ j^1(s)\right)\hat{g}^{\rho\tau}\left(p_\tau \circ j^1(s)\right))\right)$$

which is given in full details in the Appendix A.4 we obtain

$$\partial_0(p_\beta \circ j^1(s)) + \left(\partial_\beta(\frac{1}{2m}\left(p_\rho \circ j^1(s)\right)\hat{g}^{\rho\tau}\left(p_\tau \circ j^1(s)\right)) + p_\rho \partial_\beta \gamma^\rho_0\right) \circ j^1(s) = 0 \,.$$

If we introduce the kinetic energy of the mass particle as

$$H \circ p = \left(\frac{1}{2m}\dot{q}_\rho \hat{g}^{\rho\tau}\dot{q}_\tau\right) \circ p$$

we can write

$$\partial_0(p_\beta \circ j^1(s)) + (p_\rho \partial_\beta \gamma^\rho_0) \circ j^1(s) = -\left((\partial_\beta H) \circ p\right) \circ j^1(s)$$

and furthermore from

$$\begin{aligned} \left(\left(\dot{\partial}^\beta H\right) \circ p\right) \circ j^1(s) &= \frac{1}{2m}\left(\hat{g}^{\beta\tau} p_\tau + p_\rho \hat{g}^{\rho\beta}\right) \circ j^1(s) \\ &= \frac{1}{m}\left(\hat{g}^{\beta\tau} p_\tau\right) \circ j^1(s) \end{aligned}$$

we additionally obtain

$$\partial_0 s^\beta - \gamma^\beta_0 \circ s = \left(\left(\dot{\partial}^\beta H\right) \circ p\right) \circ j^1(s) \,.$$

Forces

In the following, we treat the case where forces act on a point mass. Let $m \in \mathbb{R}^+$ denote the mass of the considered point mass and the force field that acts on the point mass is given by $F : \mathcal{Q} \rightarrow \mathcal{T}^*(\mathcal{Q})$ as $F = F_0 \mathrm{d}t^0 + F_\beta \mathrm{d}q^\beta$. The equations of motion follow to

$$m \left(v_s \rfloor \nabla^\Lambda (v \circ j^1(s)) \right) = (\hat{g} \rfloor F) \circ s \,. \tag{3.31}$$

Remark 3.24 *The case where the force field depends on the velocity for example can be accomplished by the choice of* $F : \mathcal{Q} \rightarrow (\nu_\mathcal{Q})^* (\mathcal{T}^*(\mathcal{Q}))$ *with* $\nu_\mathcal{Q} : \mathcal{V}(\mathcal{Q}) \rightarrow \mathcal{Q}$.

It is worth mentioning that $\hat{g} \rfloor F$ contains the vertical part of the force, only. From (3.31) and (3.26) we easily see that

$$v_s \rfloor \nabla^{\Lambda^*} (p \circ j^1(s)) = (\gamma_c \rfloor F) \circ s \tag{3.32}$$

is met with $\gamma_c = (\mathrm{d}q^\beta - \gamma_0^\beta \mathrm{d}t^0) \otimes \partial_\beta$. If the force field admits a potential, we can introduce a function $V \in \mathcal{C}^\infty(\mathcal{Q})$ such that $\mathrm{d}V = -F$ is met.

3.2.6 Energy

Let us consider the relation (3.27) which reads as

$$\mathrm{d}(v \rfloor p \circ j^1(s)) = \nabla^\Lambda (v \circ j^1(s)) \rfloor \left(p \circ j^1(s) \right) + \left(v \circ j^1(s) \right) \rfloor \nabla^{\Lambda^*} (p \circ j^1(s)) \,.$$

A contraction with v_s leads to

$$\frac{m}{2} \partial_0 \left(v \rfloor v \rfloor g \circ j^1(s) \right) = - \left(v \rfloor \mathrm{d}V \circ j^1(s) \right) \tag{3.33}$$

where the computation can be found in the Appendix A.5. The total energy of a mass point is given by

$$E = \frac{m}{2} v \rfloor v \rfloor g + V$$

and the change of the energy along the motion s can be expressed as

$$\partial_0 \left(E \circ j^1(s) \right) = \partial_0 \left(\frac{m}{2} v \rfloor v \rfloor g \circ j^1(s) + V \circ s \right) \,.$$

This can be rewritten with the help of (3.33) to obtain

$$\begin{aligned} \partial_0 \left(E \circ j^1(s) \right) &= - \left(v \rfloor \mathrm{d}V \circ j^1(s) \right) + \partial_0 (V \circ s) \\ &= (\partial_0 V + \gamma_0^\alpha \partial_\alpha V) \circ s \,. \end{aligned}$$

It is worth mentioning that in an inertial system with trivial connection $\gamma = \mathrm{d}t^0 \otimes \partial_0$ this reproduces a well known relation.

3.2.7 Application - Rigid Transformation

When the motion of a mass point is observed from a moving and rotating coordinate system it is well known that several fictitious forces appear, see for example [Arnold, 1989]. We want to study this classical example, using jet theory and connections and describe especially the acceleration of the coordinate system using the developed theory. Let us consider an inertial system with Euclidean coordinates (q^α, t^0) with trivial metric

$$g = \delta_{\alpha\beta}\mathrm{d}q^\alpha \otimes \mathrm{d}q^\beta$$

and a rotating and moving coordinate system with coordinates $(\bar{q}^{\bar{\alpha}}, t^0)$. The relation between these systems is given by

$$\bar{q}^{\bar{\alpha}} = R^{\bar{\alpha}}_{\beta}(t^0)\left(q^\beta - q^\beta_B(t^0)\right) = \varphi^{\bar{\alpha}}(q^\beta, t^0)$$

with

$$R^{\bar{\alpha}}_{\beta}\delta^{\alpha\beta}R^{\bar{\beta}}_{\alpha} = \delta^{\bar{\alpha}\bar{\beta}} \tag{3.34}$$

where $R^{\bar{\alpha}}_{\beta}$ is a classical rotation matrix and the functions $q^\beta_B(t^0)$ describe how the origin of the moving coordinate system is translating. It is worth mentioning that the metric stays constant, since we use a rigid transformation. The important fact is that we have to introduce a non trivial space time connection which in the moving system is given as

$$\bar{\gamma}^{\bar{\alpha}}_{\bar{0}} = \partial_0(\varphi^{\bar{\alpha}}(q^\beta, t^0))\delta^0_{\bar{0}} = \bar{\Omega}^{\bar{\alpha}}_{\bar{\beta}}(t^0)\bar{q}^{\bar{\beta}} - R^{\bar{\alpha}}_{\beta}(t^0)\partial_{\bar{0}}q^\beta_B$$

with $\bar{\Omega}^{\bar{\alpha}}_{\bar{\beta}} = \partial_0 R^{\bar{\alpha}}_{\beta} R^{\beta}_{\bar{\beta}}\delta^0_{\bar{0}}$. Now let us evaluate the equation (3.24) and since the metric in the rotating coordinate system stays trivial because of (3.34) as mentioned already, we obtain

$$\begin{aligned} 0_{\bar{\alpha}\bar{\beta}} &= -g_{\bar{\beta}\bar{\rho}}(\partial_{\bar{\alpha}}\bar{\gamma}^{\bar{\rho}}_{\bar{0}}) - g_{\bar{\alpha}\bar{\rho}}(\partial_{\bar{\beta}}\bar{\gamma}^{\bar{\rho}}_{\bar{0}}) \\ 0_{\bar{\alpha}\bar{\beta}} &= -\delta_{\bar{\beta}\bar{\rho}}\bar{\Omega}^{\bar{\rho}}_{\bar{\alpha}} - \delta_{\bar{\alpha}\bar{\rho}}\bar{\Omega}^{\bar{\rho}}_{\bar{\beta}} \end{aligned}$$

and this just says that $\bar{\Omega}$ is skew-symmetric as it should be since by a slight abuse of notation we obtain

$$\begin{aligned} \delta_{\bar{\beta}\bar{\rho}}\bar{\Omega}^{\bar{\rho}}_{\bar{\alpha}} &= -\delta_{\bar{\alpha}\bar{\rho}}\bar{\Omega}^{\bar{\rho}}_{\bar{\beta}} \\ \bar{\Omega}^{\bar{\beta}}_{\bar{\alpha}} &= -\bar{\Omega}^{\bar{\alpha}}_{\bar{\beta}}\ . \end{aligned}$$

This is easily verified also from

$$\begin{aligned} R^{\bar{\alpha}}_{\beta}\delta^{\alpha\beta}R^{\bar{\beta}}_{\alpha} &= \delta^{\bar{\alpha}\bar{\beta}} \\ \left(\partial_0 R^{\bar{\alpha}}_{\beta}\right)\delta^{\alpha\beta}R^{\bar{\beta}}_{\alpha} + R^{\bar{\alpha}}_{\beta}\delta^{\alpha\beta}\left(\partial_0 R^{\bar{\beta}}_{\alpha}\right) &= 0^{\bar{\alpha}\bar{\beta}} \\ \left(\partial_0 R^{\bar{\alpha}}_{\beta}\right)R^{\beta}_{\bar{\rho}}\delta^{\bar{\rho}\bar{\beta}} + \left(\partial_0 R^{\bar{\beta}}_{\beta}\right)R^{\beta}_{\bar{\rho}}\delta^{\bar{\rho}\bar{\alpha}} &= 0^{\bar{\alpha}\bar{\beta}} \\ \bar{\Omega}^{\bar{\alpha}}_{\bar{\rho}}\delta^{\bar{\rho}\bar{\beta}} + \bar{\Omega}^{\bar{\beta}}_{\bar{\rho}}\delta^{\bar{\rho}\bar{\alpha}} &= 0^{\bar{\alpha}\bar{\beta}} \end{aligned}$$

and this shows that the equation (3.24), which is crucial when time variant metrics appear, is also meaningful for this classical example.

The acceleration follows from the equation (3.30) with $t^0 = \delta^0_{\bar{0}} \bar{t}^{\bar{0}}$ and reads as

$$\partial_{\bar{0}\bar{0}} \bar{s}^{\bar{\rho}} - \partial_{\bar{0}} (\bar{\gamma}^{\bar{\rho}}_{\bar{0}} \circ \bar{s}) - \partial_{\bar{\alpha}} \bar{\gamma}^{\bar{\rho}}_{\bar{0}} (\partial_{\bar{0}} \bar{s}^{\bar{\alpha}} - \bar{\gamma}^{\bar{\alpha}}_{\bar{0}} \circ \bar{s})$$

since the Christoffel symbols $\bar{\Lambda}^{\bar{\rho}}_{\bar{\alpha}\bar{\beta}}$ vanish, because the metric is preserved trivial by our transformation, we obtain

$$\partial_{\bar{0}\bar{0}} \bar{s}^{\bar{\alpha}} - \partial_{\bar{0}} \left(\bar{\Omega}^{\bar{\alpha}}_{\bar{\beta}} \bar{s}^{\bar{\beta}} - R^{\bar{\alpha}}_{\beta} \partial_{\bar{0}} q^{\beta}_{B} \right) - \bar{\Omega}^{\bar{\alpha}}_{\bar{\beta}} \left(\partial_{\bar{0}} \bar{s}^{\bar{\beta}} - (\bar{\Omega}^{\bar{\beta}}_{\bar{\rho}} \bar{s}^{\bar{\rho}} - R^{\bar{\beta}}_{\beta} \partial_{\bar{0}} q^{\beta}_{B}) \right)$$

which follows to

$$\partial_{\bar{0}\bar{0}} \bar{s}^{\bar{\alpha}} - \partial_{\bar{0}} \left(\bar{\Omega}^{\bar{\alpha}}_{\bar{\beta}} \right) \bar{s}^{\bar{\beta}} - 2 \bar{\Omega}^{\bar{\alpha}}_{\bar{\beta}} \partial_{\bar{0}} \bar{s}^{\bar{\beta}} + \bar{\Omega}^{\bar{\alpha}}_{\bar{\beta}} \bar{\Omega}^{\bar{\beta}}_{\bar{\rho}} \bar{s}^{\bar{\rho}} + R^{\bar{\alpha}}_{\beta} \partial_{\bar{0}\bar{0}} q^{\beta}_{B} \,. \tag{3.35}$$

In expression (3.35) $\partial_{\bar{0}} \left(\bar{\Omega}^{\bar{\alpha}}_{\bar{\beta}} \right) \bar{s}^{\bar{\beta}}$ is the inertial force of rotation, $2\bar{\Omega}^{\bar{\alpha}}_{\bar{\beta}} \partial_0 \bar{s}^{\bar{\beta}}$ the Coriolis force and $\bar{\Omega}^{\bar{\alpha}}_{\bar{\beta}} \bar{\Omega}^{\bar{\beta}}_{\bar{\rho}} \bar{s}^{\bar{\rho}}$ the Centrifugal force with respect to a unit mass and this is a classical relation which in most books is written as

$$\bar{s}_{tt} + \bar{\omega}_t \times \bar{s} + 2\bar{\omega} \times \bar{s}_t + \bar{\omega} \times \bar{\omega} \times \bar{s} + R q_{B_{tt}}$$

where $\bar{\omega}$ is the associated vector to $\bar{\Omega}$. This example shows that the classical formulas of rigid body dynamics can easily be derived and interpreted by the use of a space time connection and the covariant formulation, which in this case involves the so-called angular velocity tensor $\bar{\Omega}$.

3.3 The Lagrangian Picture

3.3.1 The Euler Lagrange Operator

This section is devoted to the description of point mechanics in a Lagrangian formulation. This is a well known topic and the first part summarizes just a few familiar facts. Then it is our goal to find a formulation which fits to the covariant treatment of section 3.2. We again use the bundle $\mathcal{Q} \to \mathcal{B}$ and we consider a variational problem such that

$$0 = \frac{\mathrm{d}}{\mathrm{d}\varepsilon} \int_{\mathcal{D}} \left(j^1 \left(\varphi_{\eta_\varepsilon} \circ s \right) \right)^* \left(\mathcal{L} \mathrm{d} t^0 \right) \Big|_{\varepsilon = 0}$$

is met for variations of the motion $s : \mathcal{B} \to \mathcal{Q}$, with an admissible vertical vector field $\eta : \mathcal{Q} \to \mathcal{V}(\mathcal{Q})$, whose flow is denoted by $\varphi_{\eta_\varepsilon}$ with $\varepsilon \in (-a, a)$, $a \in \mathbb{R}$ and $\mathcal{D}$ denotes the interval $[t^0_{a,} t^0_b]$. This leads to (see [Saunders, 1989, Giachetta et al., 1997])

$$\begin{aligned} 0 &= \int_{\mathcal{D}} \left(j^1 (s) \right)^* \left(j^1 (\eta) (L) \right) \\ &= \int_{\mathcal{D}} \left(j^1 (s) \right)^* \left(j^1 (\eta) \rfloor \mathrm{d} L \right) + \int_{\partial \mathcal{D}} \left(j^1 (s) \right)^* \left(j^1 (\eta) \rfloor L \right) \end{aligned}$$

with the Lagrangian density $L = \mathcal{L} \mathrm{d} t^0$. We use a Lepagian equivalent for $L = \mathcal{L} \mathrm{d} t^0$ and choose the well known Poincaré-Cartan form

$$\rho_L = \mathcal{L} \mathrm{d} t^0 + \partial^0_\alpha \mathcal{L} (\mathrm{d} q^\alpha - q^\alpha_0 \mathrm{d} t^0) \,.$$

Remark 3.25 *For this particular example a Lepagian equivalent meets*

$$\int_{\mathcal{D}} j^1(s)^* (L) = \int_{\mathcal{D}} j^1(s)^* (\rho_L)$$

and the important fact with respect to Lepagian forms is that $j^1(\eta)\rfloor(\mathrm{d}\rho_L)$ *does only depend on the components of the vector field* η, *hence not on their derivatives.*

Given a bundle $\mathcal{E} \to \mathcal{X}$, exterior $n-$forms on $\mathcal{J}^r(\mathcal{E})$ are denoted by $\wedge^n(\mathcal{J}^r(\mathcal{E}))$ and the horizontal projection h_k then is a map $h_k : \wedge^n(\mathcal{J}^\infty(\mathcal{E})) \to \wedge^{k,n-k}(\mathcal{J}^\infty(\mathcal{E}))$, where elements of $\wedge^{k,n-k}(\mathcal{J}^\infty(\mathcal{E}))$ are denoted *k-contact* forms, see [Giachetta et al., 1997] for an extensive discussion. The horizontal projection h_0 for this particular case of a bundle with $1-$dimensional base and first order Lagrangian (see [Krupka, 1981]) is a map $h_0 : \wedge^1(\mathcal{J}^1(\mathcal{Q})) \to \wedge^{0,1}(\mathcal{J}^1(\mathcal{Q}))$ and reads

$$\mathrm{d}t^0 \longmapsto \mathrm{d}t^0 \ , \quad \mathrm{d}q^\alpha \longmapsto q_0^\alpha \mathrm{d}t^0 \ , \quad \mathrm{d}q_0^\alpha \longmapsto q_{00}^\alpha \mathrm{d}t^0 \ .$$

Thus we have

$$\begin{aligned} h_0\left(j^1(\eta)\rfloor(\mathrm{d}\rho_L)\right) &= \eta^\alpha(\delta_\alpha \mathcal{L})\mathrm{d}t^0 \\ &= \eta\rfloor\left((\delta_\alpha\mathcal{L})\theta^\alpha \wedge \mathrm{d}t^0\right) \end{aligned}$$

with the variational derivative

$$\delta_\alpha = \partial_\alpha - d_0\partial_\alpha^0 \ , \quad d_0 = \partial_0 + q_0^\alpha\partial_\alpha + q_{00}^\alpha\partial_\alpha^0$$

and the contact forms

$$\theta^\alpha = \mathrm{d}q^\alpha - q_0^\alpha \mathrm{d}t^0$$

where the detailed computations can be found in the Appendix A.6.1. The operator

$$\begin{aligned} \mathcal{E}_L &: \mathcal{J}^2(\mathcal{Q}) \to \mathcal{T}^*(\mathcal{Q}) \wedge \mathcal{T}^*(\mathcal{B}) \\ \mathcal{E}_L &= (\delta_\alpha\mathcal{L})\theta^\alpha \wedge \mathrm{d}t^0 \end{aligned}$$

is called the Euler-Lagrange operator associated with the Lagrangian density $\mathcal{L}\mathrm{d}t^0$, see [Giachetta et al., 1997].

3.3.2 Covariant Mechanics

In context with point mechanics we consider the Lagrangian density $L = \mathcal{L}\mathrm{d}t^0$ with Lagrangian $\mathcal{L} \in \mathcal{J}^1(\mathcal{Q})$). In particular we consider the special Lagrangian density

$$\begin{aligned} \mathcal{L}\mathrm{d}t^0 &= \left(\frac{1}{2}v\rfloor v\rfloor g\right)\mathrm{d}t^0 \\ &= \frac{1}{2}g_{\alpha\beta}(q_0^\alpha - \gamma_0^\alpha)(q_0^\beta - \gamma_0^\beta)\mathrm{d}t^0 \ , \quad g_{\alpha\beta},\ \gamma_0^\alpha \in \mathcal{C}^\infty(\mathcal{Q}) \ . \end{aligned} \tag{3.36}$$

From the calculations above we immediately observe that the variational problem has to satisfy the Lagrangian equations

$$j^2(s)^*\left(d_0\left(\partial_\alpha^0\mathcal{L}\right) - \partial_\alpha\mathcal{L}\right) = 0 \ . \tag{3.37}$$

If we plug in the Lagrangian from (3.36) into the equations (3.37), then a straightforward calculation, which is given in full details in the Appendix A.6.2, leads to

$$\begin{aligned} -j^2(s)^*(\delta_\alpha \mathcal{L}) &= j^2(s)^* \left(q^\phi_{00} - d_0\gamma^\phi_0 - (\partial_\alpha \gamma^\phi_0)\,(q^\alpha_0 - \gamma^\alpha_0) \right) - j^1(s)^* \left(\Lambda^\phi_{\alpha\sigma}\,(q^\alpha_0 - \gamma^\alpha_0)\,(q^\sigma_0 - \gamma^\sigma_0) \right) \\ &= \partial_{00} s^\rho - (d_0 \gamma^\rho_0) \circ j^1(s) - (v^\alpha \partial_\alpha \gamma^\rho_0) \circ j^1(s) - (\Lambda^\rho_{\alpha\sigma} v^\alpha v^\sigma) \circ j^1(s) \\ &= \partial_{00} s^\rho - \partial_0(\gamma^\rho_0 \circ s) - (v^\alpha \partial_\alpha \gamma^\rho_0) \circ j^1(s) - (\Lambda^\rho_{\alpha\sigma} v^\alpha v^\sigma) \circ j^1(s) \end{aligned}$$

with

$$j^1(s)^*(q^\alpha_0 - \gamma^\alpha_0) = \partial_0 s^\alpha - \gamma^\alpha_0 \circ s = v^\alpha \circ j^1(s)$$

and this is the same as the equation (3.30). Therefore it can be concluded that the covariant approach based on connections and the corresponding differential operators as described in section 3.2 is equivalent to a variational problem using the Lagrangian density (3.36).

Remark 3.26 *It is easy to verify that the Lagrangian*

$$\mathcal{L} = \frac{1}{2} m\,(v \rfloor v \rfloor g) - V$$

yields the same equations as the relations (3.31) with $dV = -F$.

3.4 The Hamiltonian Picture

Let us introduce the Liouville form $\Sigma : \mathcal{T}^*(\mathcal{Q}) \to \mathcal{T}^*(\mathcal{T}^*(\mathcal{Q}))$ which in coordinates follows to

$$\Sigma = \dot{t}_0 \mathrm{d}t^0 + \dot{q}_\alpha \mathrm{d}q^\alpha \ .$$

The Hamilton form ω_H is defined by $h^*(\Sigma)$ with a section $h : \mathcal{V}^*(\mathcal{Q}) \to \mathcal{T}^*(\mathcal{Q})$ and reads as

$$\omega_H = p_\alpha \mathrm{d}q^\alpha - (H + p_\alpha \gamma^\alpha_0)\,\mathrm{d}t^0$$

with the Hamiltonian function $H \in \mathcal{C}^\infty(\mathcal{V}^*(\mathcal{Q}))$.

Remark 3.27 *In this section we do not distinguish the coordinates* $\dot{q}_\alpha$ *and the functions* $p_\alpha \in C^\infty(\mathcal{Q})$ *as rigorously as in section 3.2.5. Furthermore, to obtain the results compatible with the standard literature the coordinates for* $\mathcal{V}^*(\mathcal{Q})$ *are denoted* $(t^0, q^\alpha, p_\alpha)$ *instead of* $(t^0, q^\alpha, \dot{q}_\alpha)$.

Example 3.28 *The classical Hamilton form as it can be found in [Abraham and Marsden, 1978, Frankel, 2nd ed. 2004] reads as*

$$\omega_H = p_\alpha \mathrm{d}q^\alpha - H \mathrm{d}t^0$$

and a change of coordinates $\bar{q}^{\bar{\alpha}} = \varphi^{\bar{\alpha}}(q^\beta, t^0)$, $\bar{t}^{\bar{0}} = \delta^{\bar{0}}_0 t^0$ *leads obviously to*

$$\omega_H = p_\alpha(\partial_{\bar{\alpha}} \hat{\varphi}^\alpha (\mathrm{d}\bar{q}^{\bar{\alpha}} - \delta^0_{\bar{0}} \partial_0 \varphi^{\bar{\alpha}} \mathrm{d}\bar{t}^{\bar{0}})) - H \delta^0_{\bar{0}} \mathrm{d}\bar{t}^{\bar{0}}$$

and therefore

$$\bar{\omega}_{\bar{H}} = \bar{p}_{\bar{\alpha}} \mathrm{d}\bar{q}^{\bar{\alpha}} - (\bar{H} + \bar{p}_{\bar{\alpha}} \bar{\gamma}^{\bar{\alpha}}_{\bar{0}}) \mathrm{d}\bar{t}^{\bar{0}}$$

which is of the desired form.

The Hamiltonian vector field $v_H : \mathcal{V}^*(\mathcal{Q}) \to \mathcal{T}(\mathcal{V}^*(\mathcal{Q})$ follows from the relations

$$v_H \rfloor \mathrm{d}\omega_H = 0\,, \quad v_H \rfloor \mathrm{d}t^0 = 1$$

as

$$v_H = \partial_0 + \dot{\partial}^\alpha (H + p_\beta \gamma_0^\beta)\partial_\alpha - \partial_\alpha (H + p_\beta \gamma_0^\beta)\dot{\partial}^\alpha\,. \tag{3.38}$$

To see this let us compute

$$\begin{aligned} \mathrm{d}\omega_H &= \mathrm{d}p_\alpha \wedge \mathrm{d}q^\alpha - \mathrm{d}H \wedge \mathrm{d}t^0 - \mathrm{d}(p_\alpha \gamma_0^\alpha) \wedge \mathrm{d}t^0 \\ &= \mathrm{d}p_\alpha \wedge \mathrm{d}q^\alpha - \big(\mathrm{d}H + \gamma_0^\alpha \mathrm{d}p_\alpha + p_\alpha (\partial_\beta \gamma_0^\alpha)\mathrm{d}q^\beta\big) \wedge \mathrm{d}t^0\,. \end{aligned}$$

The contraction with the field

$$v_H = \partial_0 + \dot{q}^\alpha \partial_\alpha - \dot{p}_\alpha \dot{\partial}^\alpha$$

leads to

$$\begin{aligned} v_H \rfloor \mathrm{d}\omega_H &= \partial_\alpha H \mathrm{d}q^\alpha + \dot{\partial}^\alpha H \mathrm{d}p_\alpha + \gamma_0^\alpha \mathrm{d}p_\alpha + p_\alpha (\partial_\beta \gamma_0^\alpha)\mathrm{d}q^\beta - \dot{p}_\alpha \mathrm{d}q^\alpha - \dot{q}^\alpha \mathrm{d}p_\alpha \\ &= \left(\partial_\alpha H + p_\tau (\partial_\alpha \gamma_0^\tau) - \dot{p}_\alpha\right) \mathrm{d}q^\alpha + \left(\dot{\partial}^\alpha H + \gamma_0^\alpha - \dot{q}^\alpha\right) \mathrm{d}p_\alpha \end{aligned}$$

and the result follows at once to

$$\begin{aligned} \dot{q}^\alpha &= \dot{\partial}^\alpha H + \gamma_0^\alpha = \dot{\partial}^\alpha \left(H + p_\beta \gamma_0^\beta\right) \\ \dot{p}_\alpha &= \partial_\alpha H + p_\tau (\partial_\alpha \gamma_0^\tau) = \partial_\alpha (H + p_\tau \gamma_0^\tau)\,. \end{aligned}$$

3.4.1 The Composite Bundle Structure

Now we discuss the composite bundle structure as it appears in this Hamiltonian setting. Therefore, let us inspect the composite bundle $\mathcal{V}^*(\mathcal{Q}) \to \mathcal{Q} \to \mathcal{B}$, as well as the bundles $\mathcal{Q} \to \mathcal{B}$, $\mathcal{V}^*(\mathcal{Q}) \to \mathcal{Q}$ and $\mathcal{V}^*(\mathcal{Q}) \to \mathcal{B}$ which are visualized in the following diagram.

$$\begin{tikzcd} & \mathcal{V}^*(\mathcal{Q})\ (t^0, q^\alpha, p_\alpha) \arrow[d, "\nu_{\mathcal{Q}}^*"'] & \mathcal{V}^*(\mathcal{Q})\ (t^0, q^\alpha, p_\alpha) \arrow[dd, "\nu_{\mathcal{Q}\mathcal{B}}^*"'] \\ \mathcal{Q}\ (t^0, q^\alpha) \arrow[d, "\pi"'] & \mathcal{Q}\ (t^0, q^\alpha) \arrow[d, "\pi"'] & \\ \mathcal{B}\ (t^0) & \mathcal{B}\ (t^0) & \mathcal{B}\ (t^0) \end{tikzcd}$$

We have a connection γ on $\mathcal{Q} \to \mathcal{B}$ that splits $\mathcal{T}(\mathcal{Q})$

$$\gamma = \mathrm{d}t^0 \otimes (\partial_0 + \gamma_0^\alpha \partial_\alpha)$$

and we have another connection Λ^* on $\mathcal{V}^*(\mathcal{Q}) \to \mathcal{Q}$ that splits $\mathcal{T}(\mathcal{V}^*(\mathcal{Q}))$

$$\Lambda^* = \mathrm{d}t^0 \otimes \left(\partial_0 + \Lambda_{0\rho}^{\tau *} \dot{q}_\tau \dot{\partial}^\rho\right) + \mathrm{d}q^\alpha \otimes \left(\partial_\alpha + \Lambda_{\alpha\rho}^{\tau *} \dot{q}_\tau \dot{\partial}^\rho\right)\,.$$

From section 2.4.3 we immediately obtain the composite connection on $\mathcal{V}^*(\mathcal{Q}) \to \mathcal{B}$ as

$$\Gamma_H = \mathrm{d}t^0 \otimes \left(\partial_0 + \gamma_0^\alpha \partial_\alpha + (\Lambda_{0\rho}^{\tau *}\dot{q}_\tau + \gamma_0^\alpha \Lambda_{\alpha\rho}^{\tau *}\dot{q}_\tau)\dot{\partial}^\rho\right)$$

and with the equation (3.25) which can be written as

$$\Lambda_{0\rho}^{\tau *} = -(\partial_\rho \gamma_0^\tau) - \Lambda_{\beta\rho}^{\tau *}\gamma_0^\beta$$

it follows that

$$\Gamma_H = \mathrm{d}t^0 \otimes \left(\partial_0 + \gamma_0^\alpha \partial_\alpha - (\partial_\rho \gamma_0^\beta) p_\beta \dot{\partial}^\rho\right) \tag{3.39}$$

is met, which will be called the Hamiltonian connection.

Remark 3.29 *Of course the Hamiltonian connection (3.39) already appeared in the section 3.2.4 and was termed there Ξ^* and used to construct a covariant derivative. Since we will use the connection here to split the Hamiltonian field, we have decided to call it Hamiltonian connection.*

3.4.2 The Splitting of the Hamiltonian field

This connection (3.39) enables us to split the Hamiltonian vector field (3.38) into a vertical and horizontal part, respectively. We easily see that

$$\begin{aligned} v_{H,\mathcal{V}} &= \dot{\partial}^\alpha H \partial_\alpha - \partial_\alpha H \dot{\partial}^\alpha & (3.40)\\ v_{H,\mathcal{H}} &= \partial_0 + \gamma_0^\alpha \partial_\alpha - (\partial_\rho \gamma_0^\beta) p_\beta \dot{\partial}^\rho & (3.41) \end{aligned}$$

is met according to the Hamiltonian connection (3.39). The computation of the change of the form $H\mathrm{d}t^0$ along the Hamiltonian vector field is based on the very useful formula

$$\begin{aligned} v_H(H\mathrm{d}t^0) &= \mathrm{d}(v_H \rfloor H\mathrm{d}t^0) + v_H \rfloor \mathrm{d}(H\mathrm{d}t^0) \\ &= v_{H,\mathcal{H}}(H)\mathrm{d}t^0 & (3.42) \end{aligned}$$

with $v_{H,\mathcal{H}}$ from (3.41), where the detailed calculation can be found in the Appendix A.7.1. Now we consider an extended Hamiltonian of the form

$$H = H_0 - H_\rho u^\rho\ , \quad H_0,\ H_\rho \in C^\infty(\mathcal{V}^*(\mathcal{Q})) \tag{3.43}$$

with the input functions $u^\rho \in \mathcal{C}^\infty(\mathcal{B})$. From the relation (3.42) together with (3.43) we see that

$$\begin{aligned} v_H(H_0\mathrm{d}t^0) &= \mathrm{d}(v_H \rfloor H_0\mathrm{d}t^0) + v_H \rfloor \mathrm{d}(H_0\mathrm{d}t^0) \\ &= \left(v_{H,\mathcal{H}}(H_0) + v_{H,\mathcal{V}}(H_\varepsilon)u^\varepsilon\right)\mathrm{d}t^0 \end{aligned}$$

is met, where the details are omitted and can be found in the Appendix A.7.2. Obviously the choice of the output $y_\varepsilon = v_{H,\mathcal{V}}(H_\varepsilon)$ allows a physical interpretation of the power flows of the system, since $v_{H,\mathcal{H}}(H_0)$ corresponds to the power caused by the free Hamiltonian H_0 and the product $y_\varepsilon u^\varepsilon$ describes the power flow into the system caused by the input.

Remark 3.30 *Furthermore, from the Liouville form $\Sigma : \mathcal{T}^*(\mathcal{Q}) \to \mathcal{T}^*(\mathcal{T}^*(\mathcal{Q}))$ one derives the symplectic form*

$$\Omega = \mathrm{d}\dot{t}_0 \wedge \mathrm{d}t^0 + \mathrm{d}\dot{q}_\alpha \wedge \mathrm{d}q^\alpha$$

and the corresponding Poisson bracket

$$\{f, g\} = \dot{\partial}^0 f \partial_0 g - \dot{\partial}^0 g \partial_0 f + \dot{\partial}^\alpha f \partial_\alpha g - \dot{\partial}^\alpha g \partial_\alpha f \tag{3.44}$$

for $f, g \in C^\infty(\mathcal{T}^*(\mathcal{Q}))$. In [Giachetta et al., 1997] it is shown that there exists a vertical restriction of the relation (3.44) which reads as

$$\{f, g\}_\mathcal{V} = \dot{\partial}^\alpha f \partial_\alpha g - \dot{\partial}^\alpha g \partial_\alpha f$$

which will be used in the following. Let us compute the Lie derivative of a function $f \in C^\infty(\mathcal{V}^*(\mathcal{Q}))$ along the field v_H which reads as

$$v_H(f) = \left(\partial_0 + \gamma_0^\alpha \partial_\alpha - (\partial_\rho \gamma_0^\beta)\dot{q}_\beta \dot{\partial}^\rho\right) f + \{H, f\}_\mathcal{V}$$

therefore the change of the Hamiltonian along v_H follows to

$$v_H(H) = \left(\partial_0 + \gamma_0^\alpha \partial_\alpha - (\partial_\rho \gamma_0^\beta)\dot{q}_\beta \dot{\partial}^\rho\right) H \tag{3.45}$$

since $\{H, H\}_\mathcal{V} = 0$. This obviously corresponds to the formula (3.42). It is worth mentioning that Γ_H is a Hamiltonian connection in the sense of [Giachetta et al., 1997], since the form

$$\Gamma_H \rfloor \left(\mathrm{d}\dot{q}_\alpha \wedge \mathrm{d}q^\alpha \wedge \mathrm{d}t^0 \otimes \partial_0\right) \tag{3.46}$$

is closed. In this concrete example we evaluate (3.46) and obtain

$$\mathrm{d}t^0 \otimes \left(\partial_0 + \gamma_0^\alpha \partial_\alpha - (\partial_\rho \gamma_0^\beta)\dot{q}_\beta \dot{\partial}^\rho\right) \rfloor \left(\mathrm{d}\dot{q}_\alpha \wedge \mathrm{d}q^\alpha \wedge \mathrm{d}t^0 \otimes \partial_0\right)$$

which is

$$\mathrm{d}t^0 \otimes \left(\mathrm{d}\dot{q}_\alpha \wedge \mathrm{d}q^\alpha - \gamma_0^\alpha \mathrm{d}\dot{q}_\alpha \wedge \mathrm{d}t^0 - (\partial_\alpha \gamma_0^\beta)\dot{q}_\beta \mathrm{d}q^\alpha \wedge \mathrm{d}t^0\right) \otimes \partial_0 \,. \tag{3.47}$$

The requirement that the form part of (3.47) is closed leads us to compute

$$-\mathrm{d}\gamma_0^\alpha \wedge \mathrm{d}\dot{q}_\alpha \wedge \mathrm{d}t^0 - \mathrm{d}\left((\partial_\alpha \gamma_0^\beta)\dot{q}_\beta\right) \wedge \mathrm{d}q^\alpha \wedge \mathrm{d}t^0$$

which is

$$-\partial_\beta \gamma_0^\alpha \mathrm{d}q^\beta \wedge \mathrm{d}\dot{q}_\alpha \wedge \mathrm{d}t^0 - (\partial_\alpha \gamma_0^\beta)\mathrm{d}\dot{q}_\beta \wedge \mathrm{d}q^\alpha \wedge \mathrm{d}t^0 = 0$$

and shows the assertion.

Example 3.31 *Let us consider an inertial system with Euclidean coordinates (q^α, t^0) and a rotating coordinate system with respect to the inertial one such that the relation*

$$\begin{aligned} \bar{q}^{\bar{\alpha}} &= R_\beta^{\bar{\alpha}}(t^0) q^\beta = \varphi^{\bar{\alpha}}(q^\beta, t^0) \\ \bar{t}^{\bar{0}} &= \delta_0^{\bar{0}} t^0 \end{aligned}$$

with

$$R^{\bar\alpha}_{\beta}\delta^{\alpha\beta}R^{\bar\beta}_{\alpha}=\delta^{\bar\alpha\bar\beta}$$

is given. The Hamiltonian of a mass particle in the non moving system is given as

$$H=\frac{1}{2m}p_\alpha\delta^{\alpha\beta}p_\beta$$

and the space time connection follows to

$$\bar\gamma^{\bar\alpha}_{\bar 0}=\partial_0(R^{\bar\alpha}_{\beta}(t^0)q^\beta)\delta^0_{\bar 0}=\partial_0(R^{\bar\alpha}_{\beta})\delta^0_{\bar 0}q^\beta=\partial_0(R^{\bar\alpha}_{\beta})R^{\beta}_{\bar\rho}\delta^0_{\bar 0}\bar q^{\bar\rho}=\Omega^{\bar\alpha}_{\bar\rho}\bar q^{\bar\rho}$$

which should be compared with section 3.2.7. The Hamiltonian in the rotating coordinate system is

$$\bar H=\frac{1}{2m}\partial_\alpha\varphi^{\bar\beta}\bar p_{\bar\beta}\delta^{\alpha\beta}\partial_\beta\varphi^{\bar\alpha}\bar p_{\bar\alpha}=\frac{1}{2m}\bar p_{\bar\beta}R^{\bar\beta}_{\alpha}\delta^{\alpha\beta}R^{\bar\alpha}_{\beta}\bar p_{\bar\alpha}=\frac{1}{2m}\bar p_{\bar\beta}\delta^{\bar\alpha\bar\beta}\bar p_{\bar\alpha}$$

and it is worth mentioning that

$$\partial_{\bar 0}\bar H=0$$

is met because of the rigidity of the transformation. The change of the Hamiltonian computed in the moving system gives

$$v_{\bar H}(\bar H\mathrm{d}\bar t^{\bar 0})=v_{\bar H,\mathcal{H}}(\bar H)\mathrm{d}\bar t^{\bar 0}=\left(\partial_{\bar 0}\bar H+\bar\gamma^{\bar\alpha}_{\bar 0}\left(\partial_{\bar\alpha}\bar H\right)-\bar p_{\bar\alpha}\partial_{\bar\rho}\bar\gamma^{\bar\alpha}_{\bar 0}\left(\dot\partial^{\bar\rho}\bar H\right)\right)\mathrm{d}\bar t^{\bar 0}$$

and in this concrete example

$$v_{\bar H,\mathcal{H}}(\bar H)=-\bar p_{\bar\alpha}\partial_{\bar\rho}\bar\gamma^{\bar\alpha}_{\bar 0}\frac{1}{m}\bar p_{\bar\alpha}\delta^{\bar\alpha\bar\rho}=-\bar p_{\bar\alpha}\Omega^{\bar\alpha}_{\bar\rho}\frac{1}{m}\bar p_{\bar\beta}\delta^{\bar\beta\bar\rho}=-\frac{1}{m}\bar p_{\bar\alpha}\Omega^{\bar\alpha}_{\bar\rho}\bar p_{\bar\beta}\delta^{\bar\beta\bar\rho}=0$$

because Ω is skew symmetric.

Example 3.32 *Let us consider a similar situation as in the foregoing example but we do not use a rigid transformation but*

$$\bar q^{\bar\alpha}=\varphi^{\bar\alpha}(q,t^0)=R^{\bar\alpha}_{\beta}(t^0)q^\beta$$

with

$$R^{\bar\alpha}_{\beta}\delta^{\alpha\beta}R^{\bar\beta}_{\alpha}\neq\delta^{\bar\alpha\bar\beta}\ .$$

The space time connection follows to

$$\bar\gamma^{\bar\alpha}_{\bar 0}=\partial_{\bar 0}(R^{\bar\alpha}_{\beta}(t^0)q^\beta)\delta^0_{\bar 0}=\partial_0(R^{\bar\alpha}_{\beta})\delta^0_{\bar 0}q^\beta=\partial_0(R^{\bar\alpha}_{\beta})R^{\beta}_{\bar\rho}\delta^0_{\bar 0}\bar q^{\bar\rho}$$

and the Hamiltonian in new coordinates is

$$\bar H=\frac{1}{2m}\partial_\alpha\varphi^{\bar\beta}\bar p_{\bar\beta}\delta^{\alpha\beta}\partial_\beta\varphi^{\bar\alpha}\bar p_{\bar\alpha}=\frac{1}{2m}\bar p_{\bar\beta}R^{\bar\beta}_{\alpha}\delta^{\alpha\beta}R^{\bar\alpha}_{\beta}\bar p_{\bar\alpha}$$

where it is worth mentioning that now

$$\partial_{\bar 0}\bar H\neq 0$$

is met. The change of the Hamiltonian in this example reads as

$$v_{\bar{H},\mathcal{H}}(\bar{H})) = \frac{1}{2m}\left(\partial_{\bar{0}}(R^{\bar{\beta}}_{\alpha})\bar{p}_{\bar{\beta}}\delta^{\alpha\beta}R^{\bar{\alpha}}_{\beta}\bar{p}_{\bar{\alpha}} + R^{\bar{\beta}}_{\alpha}\bar{p}_{\bar{\beta}}\delta^{\alpha\beta}\partial_{\bar{0}}(R^{\bar{\alpha}}_{\beta})\bar{p}_{\bar{\alpha}} - 2\bar{p}_{\bar{\alpha}}\partial_{\bar{0}}(R^{\bar{\alpha}}_{\beta})\delta^{\alpha\beta}R^{\bar{\kappa}}_{\alpha}\bar{p}_{\bar{\kappa}}\right) = 0$$

where the usability of the operator (3.42) is shown and the essential conclusion is that although $\bar{H}$ seems time dependent, the conservation of energy is obtained, because of the special intrinsic construction of the operator (3.42).

Remark 3.33 *The case of a general transformation*

$$\bar{q}^{\bar{\alpha}} = \varphi^{\bar{\alpha}}(q, t^0)$$

can be treated in a similar way.

Chapter 4

Continuum Mechanics

The analysis of the deformation and the motion of a continuum is rich of geometric ideas. In literature multilinear algebra is necessarily introduced for the consideration of continuum mechanics, but in many cases only the components of the involved tensors are considered in the manipulations and a detailed description of the spaces, where these tensors are defined, is missing. An exception with regard to these circumstances are beside others of course [Marsden and Hughes, 1994, Prastaro, 1996] who use advanced differential geometric methods and self evident treat tensor algebra in a much more detailed way.

The goal of this chapter is to extend the constructions gained when examining the motion of a mass point moving in space to the case of a continuum, where we want to pay attention to a rigorous geometric specification of all the involved maps and tensorial objects. We will use the configuration bundle $\pi : \mathcal{Q} \to \mathcal{B}$ presented in section 3.2 and it is worth mentioning that all the results concerning the metric, especially the covariant derivatives, can be used to describe continuum mechanics. The main difference to the case of point mechanics is the consideration of mass densities and the appearance of vector valued forms instead of vectors. The covariant derivatives we have constructed so far will be used in the forthcoming, however, in the Eulerian formulation $\nabla^{\Lambda}(v)$ is used instead of $\nabla^{\Lambda}(v \circ j^1(s))$ as for point mechanics. This leads to a construction which is well known as the material time derivative. The main purpose of this chapter, which is based on [Schöberl and Schlacher, 2006b], is the derivation of the conservation of mass and the balance of linear momentum. The balance of moment of momentum will not be considered by making the constitutive assumption that the Cauchy stress tensor (or equivalent the 2nd Piola tensor) is symmetric. This guarantees that the balance of moment of momentum is fulfilled.

In section 4.1 the Eulerian picture of a continuum is analyzed. This part shows how the geometric ideas presented so far can be generalized when mass densities and vector valued forms appear. An important tool will be the Nijenhuis differential which is a covariant derivative applicable to vector valued forms and was presented already in section 2.5. Beside the balance of mass and linear momentum also an energy relation will be derived in section 4.1.3. Section 4.2 is devoted to the Lagrangian picture of a continuum. This part is dominated by the description of the so-called Piola transformation which allows to express the spatial quantities used in the Eulerian setting in material quantities. In addition, a variational approach and the Hamiltonian description of a continuum are presented in section 4.2.5 and 4.2.6.

4.1 The Eulerian Picture

The Eulerian description of a continuum is based on the ideas presented for point mechanics in section 3.2. We use the same configuration bundle as in Figure (3.1) but we replace the point mass by a mass density as shown in Figure (4.1). The Eulerian picture describes the motion of a continuum in a spatial description and this is very popular in fluid dynamics.

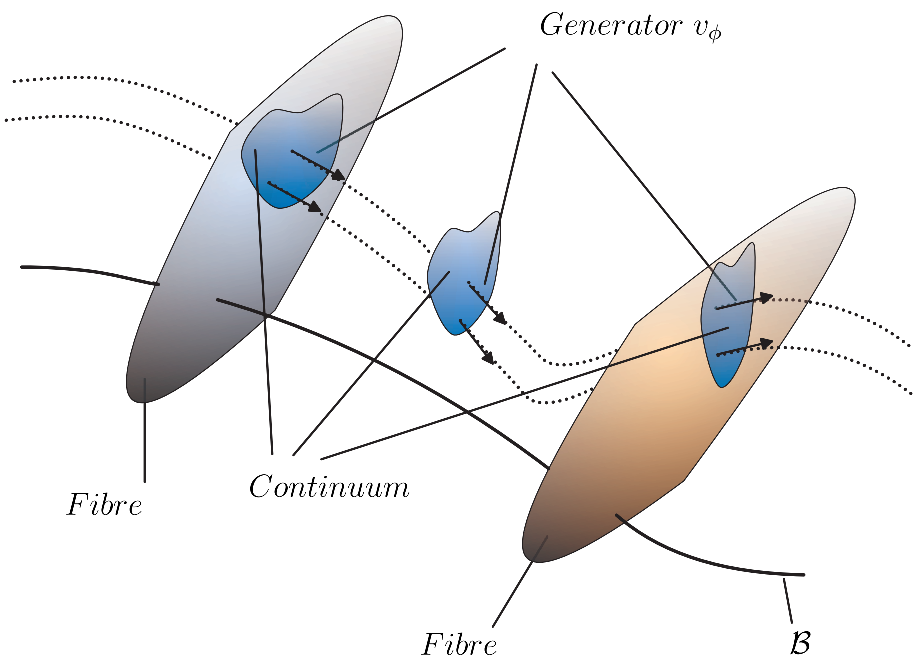

Figure 4.1: The Configuration Bundle in the Eulerian Picture

An alternative approach concerning fluid dynamics can be found in [Luo and Bewely, 2004], where the authors consider the Navier stokes equations in general time variant curvilinear coordinate systems. In contrast to the approach proposed in this thesis they do not use the splitting of vector bundles to derive the differential operators.

The same vertical metric is considered as in section 3.2

$$g = g_{\alpha\beta}\left(\mathrm{d}q^\alpha - \gamma_0^\alpha \mathrm{d}t^0\right) \otimes \left(\mathrm{d}q^\beta - \gamma_0^\beta \mathrm{d}t^0\right), \quad \gamma_0^\alpha,\ g_{\alpha\beta} \in \mathcal{C}^\infty(\mathcal{Q})$$

and a volume form $\mathrm{vol} : \mathcal{Q} \rightarrow \Lambda^n \left(\mathcal{V}^*(\mathcal{Q})\right)$ is introduced as

$$\mathrm{vol} = \sqrt{|\det(g_{\alpha\beta})|}\left(\mathrm{d}q^1 - \gamma_0^1 \mathrm{d}t^0\right) \wedge \ldots \wedge \left(\mathrm{d}q^n - \gamma_0^n \mathrm{d}t^0\right), \quad \gamma_0^\alpha,\ g_{\alpha\beta} \in \mathcal{C}^\infty(\mathcal{Q}) \tag{4.1}$$

in order to be able to carry out integrations over the fibres. A motion in the Eulerian picture is an isomorphism $\phi_\tau : \mathcal{Q} \rightarrow \mathcal{Q}$ that maps a configuration at t^0 to a configuration at $\phi_\tau^0 = t^0 + \tau$. The infinitesimal generator of ϕ_τ is the field $v_\phi = \partial_0 + v_\phi^\alpha \partial_\alpha$ with $v_\phi^\alpha \in \mathcal{C}^\infty(\mathcal{Q})$.

Remark 4.1 *The construction of the velocity is based on the observations introduced in section 3.2.5. The field* $v_\phi = \partial_0 + v_\phi^\alpha \partial_\alpha$*, which generates the flow, leads to a vertical velocity field* $v_\phi^\alpha - \gamma_0^\alpha$*. This is the desired analogy since from* $v_s = \partial_0 + \partial_0 s^\alpha \partial_\alpha$ *we constructed* $\partial_0 s^\alpha - \gamma_0^\alpha \circ s$*. The main difference is that in point mechanics the velocity field is restricted to the motion of a single mass point. This is the reason why we used* $\nabla^\Lambda (v \circ j^1(s))$ *or the corresponding vertical covariant derivative* $\tilde{\nabla}^\Lambda (s, v \circ j^1(s))$*.*

4.1.1 The Mass Balance

The principle of conservation of mass states that the mass of a material region is constant with respect to time. We assume that there exists a function $\rho \in \mathcal{C}^\infty(\mathcal{Q})$ called the mass density. The mass of the continuum is defined as

$$m = \int_{\mathcal{U}} \rho \mathrm{vol} = c$$

with $c \in \mathbb{R}^+$, where this integral has to be evaluated at a fixed time t^0 and $\mathcal{U} \subset \mathcal{C} \subset \mathcal{Q}_{t^0}$ where $\mathcal{U}$ is a nice domain of integration and $\mathcal{C}$ corresponds to a configuration of the continuum. The conservation of mass states that

$$\frac{\mathrm{d}}{\mathrm{d}\tau} \int_{\phi_\tau(\mathcal{U}, t^0)} \rho \mathrm{vol} \wedge \mathrm{d}t^0 \bigg|_{\tau=0} = 0 \tag{4.2}$$

has to be met.

Remark 4.2 *Roughly speaking the wedge product with* $\mathrm{d}t^0$ *is equivalent to the restriction of* $\rho\mathrm{vol}$ *to* $\mathrm{d}t^0 = 0$*, which is a fixed fibre of* $\mathcal{Q}$*.*

The relation (4.2) is equivalent to

$$\int_{\phi_\tau(\mathcal{U}, t^0)} v_\phi(\rho \mathrm{vol} \wedge \mathrm{d}t^0) = 0 \ ,$$

see for example [Frankel, 2nd ed. 2004]. Finally the conservation of mass reads as

$$v_\phi(\rho) + \rho \mathrm{div}(v_\phi) = 0 \tag{4.3}$$

since $\mathcal{U}$ has to be arbitrary with

$$\mathrm{div}(v_\phi) = \frac{1}{\sqrt{|\det(g_{\alpha\beta})|}} \left(\partial_0 \sqrt{|\det(g_{\alpha\beta})|} + \partial_\alpha (v_\phi^\alpha \sqrt{|\det(g_{\alpha\beta})|}) \right) \tag{4.4}$$

where a detailed computation can be found in the Appendix A.8.1.

Remark 4.3 *We want to point out here that the equation* (4.3) *is the standard equation as can be found for example in [Marsden and Hughes, 1994] for the case of a trivial space time connection* $\gamma = \mathrm{d}t^0 \otimes \partial_0$ *and the novelty considering the relation (4.4) is the fact that the mass balance can be formulated with respect to a non rigid coordinate system, i.e. the metric becomes time dependent. This is easily verified by the equation (4.4) since the time variance of the metric can be taken into account.*

Remark 4.4 *With*

$$\Lambda^{\rho}_{\alpha\rho} = -\frac{1}{\sqrt{|\det(g_{\alpha\beta})|}}\partial_\alpha\sqrt{|\det(g_{\alpha\beta})|}\ , \quad \Lambda^{\rho}_{0\rho} = -\frac{1}{\sqrt{|\det(g_{\alpha\beta})|}}\partial_0\sqrt{|\det(g_{\alpha\beta})|}$$

the equation (4.3) follows as

$$v_\phi(\rho) + \rho\frac{1}{\sqrt{|\det(g_{\alpha\beta})|}}\left(-\Lambda^{\rho}_{0\rho}\sqrt{|\det(g_{\alpha\beta})|} + \partial_\alpha(v^\alpha_\phi)\sqrt{|\det(g_{\alpha\beta})|} + v^\alpha_\phi\partial_\alpha(\sqrt{|\det(g_{\alpha\beta})|})\right) = 0$$

or

$$v_\phi(\rho) + \rho\left(\partial_\alpha v^\alpha_\phi - \Lambda^{\rho}_{0\rho} - v^\alpha_\phi\Lambda^{\rho}_{\alpha\rho}\right) = 0\ .$$

Then by applying

$$\Lambda^{\rho}_{0\rho} = (\partial_\rho\gamma^\rho_0) - \Lambda^{\rho}_{\beta\rho}\gamma^\beta_0$$

we have

$$\begin{aligned} v_\phi(\rho) + \rho\left(\partial_\alpha v^\alpha_\phi - (\partial_\rho\gamma^\rho_0) + \Lambda^{\rho}_{\beta\rho}\gamma^\beta_0 - v^\alpha_\phi\Lambda^{\rho}_{\alpha\rho}\right) &= 0 \\ v_\phi(\rho) + \rho\left(\partial_\alpha(v^\alpha_\phi - \gamma^\alpha_0) - (v^\alpha_\phi - \gamma^\alpha_0)\Lambda^{\rho}_{\alpha\rho}\right) &= 0\ . \end{aligned}$$

The final result reads as

$$v_\phi(\rho) + \rho\left(\partial_\alpha v^\alpha - v^\alpha\Lambda^{\rho}_{\alpha\rho}\right) = 0\ , \tag{4.5}$$

where it is worth mentioning that both v_ϕ and $v^\alpha = v^\alpha_\phi - \gamma^\alpha_0$ appear.

4.1.2 The Balance of Linear Momentum

In contrast to point mechanics the equations for the balance of momentum are much more intricate since we are confronted with vector valued forms as can be seen from the relation

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\int_{\phi_\tau(\mathcal{U},t^0)}\rho\mathrm{vol}\otimes v^\alpha\partial_\alpha\right) = \int_{\phi_\tau(\mathcal{U},t^0)}\rho\mathrm{vol}\otimes b^\alpha\partial_\alpha + \int_{\phi_\tau(\partial\mathcal{U},t^0)}\sigma^{\alpha\beta}\partial_\alpha\rfloor\mathrm{vol}\otimes\partial_\beta\ , \tag{4.6}$$

which is evaluated at a fixed point of time t^0. The equation (4.6) states that the total time change of the momentum equals the force acting on $\mathcal{U}$, where $v : \mathcal{Q} \to \mathcal{V}(\mathcal{Q})$ is again the vertical part of the the field v_ϕ and meets $v^\alpha = v^\alpha_\phi - \gamma^\alpha_0$. The right hand side of equation (4.6) corresponds to the forces, where

$$\rho\mathrm{vol}\otimes b = \rho\mathrm{vol}\otimes b^\alpha\partial_\alpha$$

represents the volume density of the body forces and

$$\sigma = \sigma^{\alpha\beta}\partial_\alpha\rfloor\mathrm{vol}\otimes\partial_\beta$$

is the Cauchy stress form, which describes the density of the surface forces. The symmetry $\sigma^{\alpha\beta} = \sigma^{\beta\alpha}$ is assumed.

Remark 4.5 *In chapter 3 the momentum and the forces have been considered as elements of $\mathcal{V}^*(\mathcal{Q})$. It is important to stress out the fact that in equation (4.6) the metric $\hat{g} : \mathcal{V}^*(\mathcal{Q}) \to \mathcal{V}(\mathcal{Q})$ was used to convert the covector valued forms, which appear naturally being conform with our previous definitions, into vector valued forms, using $\dot{q}^\alpha = \hat{g}^{\alpha\beta}\dot{q}_\beta$.*

As already mentioned the difficulty here is the treatment of vector valued forms. To overcome this problem we follow the approach proposed in [Schlacher et al., 2004]. We choose a section $\bar{q} : \mathcal{Q} \to \mathcal{V}^*(\mathcal{Q})$ that meets $\nabla^{\Lambda^*}(\bar{q}) = 0$ which enables us to convert the vector valued forms into true forms

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\int_{\phi_\tau(\mathcal{U},t^0)} \rho \mathrm{vol}(v \rfloor \bar{q}) = \int_{\phi_\tau(\mathcal{U},t^0)} \rho \mathrm{vol}(b \rfloor \bar{q}) + \int_{\phi_\tau(\partial\mathcal{U},t^0)} \bar{q}_\beta \sigma^{\alpha\beta}\partial_\alpha \rfloor \mathrm{vol}\ . \tag{4.7}$$

Remark 4.6 *It is worth mentioning that the condition $\nabla^{\Lambda^*}(\bar{q}) = 0$ is important to guarantee that the total time change of the momentum is computed correctly, i.e. it is not affected by $\bar{q}$. This fact will be discussed later on in this section.*

The surface integral of equation (4.7) can be reformulated by applying the theorem of Stokes, see (2.8), to get the relation

$$\int_{\phi_\tau(\partial\mathcal{U},t^0)} \bar{q}_\beta \sigma^{\alpha\beta}\partial_\alpha \rfloor \mathrm{vol} = \int_{\phi_\tau(\mathcal{U},t^0)} \mathrm{d}\left(\bar{q}_\beta \sigma^{\alpha\beta}\partial_\alpha \rfloor \mathrm{vol}\right)\ . \tag{4.8}$$

The Nijenhuis Differential of the stress form

Let us consider the vertical bundle $\mathcal{V}(\mathcal{Q}) \to \mathcal{Q}$ and the connection Λ that splits $\mathcal{T}_\mathcal{Q}(\mathcal{V}(\mathcal{Q}))$. In the forthcoming we want to apply the Nijenhuis differential, which was introduced in section 2.5, to continuum mechanics. For this bundle construction it is a map

$$\mathrm{d}_\Lambda : (\wedge^r \mathcal{T}^*(\mathcal{Q})) \otimes \mathcal{P}_\mathcal{Q}(\mathcal{V}(\mathcal{Q})) \to (\wedge^{r+1}\mathcal{T}^*(\mathcal{Q})) \otimes \mathcal{V}_\mathcal{Q}(\mathcal{V}(\mathcal{Q}))$$

where $\mathcal{P}_\mathcal{Q}(\mathcal{V}(\mathcal{Q}))$ is a projectable vector field on $\mathcal{V}(\mathcal{Q}) \to \mathcal{Q}$. We consider the vector valued form

$$\sigma = \sigma^{\alpha\beta}\partial_\alpha \rfloor \mathrm{vol} \otimes \partial_\beta \in (\wedge^{n-1}\mathcal{V}^*(\mathcal{Q})) \otimes \mathcal{V}(\mathcal{Q})$$

together with the natural isomorphism $\mathcal{V}_\mathcal{Q}(\mathcal{V}(\mathcal{Q})) \approx \mathcal{V}(\mathcal{Q}) \times_\mathcal{Q} \mathcal{V}(\mathcal{Q})$. It is worth mentioning that this case was treated in example (2.5). The Nijenhuis differential applied to σ is a map

$$\mathrm{d}_\Lambda : (\wedge^{n-1}\mathcal{V}^*(\mathcal{Q})) \otimes \mathcal{V}(\mathcal{Q}) \to (\wedge^n \mathcal{T}^*(\mathcal{Q})) \otimes \mathcal{V}(\mathcal{Q})$$

and in coordinates we have

$$\mathrm{d}_\Lambda(\sigma) = \left[\mathrm{d}(\sigma^{\alpha\beta}\partial_\alpha \rfloor \mathrm{vol}) - \left(\Lambda^\beta_{0\tau}\sigma^{\alpha\tau}\mathrm{d}t^0 + \Lambda^\beta_{\rho\tau}\sigma^{\alpha\tau}\mathrm{d}q^\rho\right) \wedge \partial_\alpha \rfloor \mathrm{vol}\right] \otimes \partial_\beta\ . \tag{4.9}$$

The goal is to rewrite the right hand side of the equation (4.8) and with the presented map (4.9) we obtain the result as

$$\mathrm{d}\left(\sigma \rfloor \bar{q}\right) = \hat{\otimes}\left(\sigma \rfloor \nabla^{\Lambda^*}(\bar{q})\right) + \mathrm{d}_\Lambda(\sigma) \rfloor \bar{q} \tag{4.10}$$

where $\hat{\otimes}(r)$ denotes the replacement of $\otimes$ by $\wedge$ in the expression r. Equation (4.10) follows easily from

$$\mathrm{d}(\sigma\rfloor\bar{q}) = \mathrm{d}(\bar{q}_\beta\sigma^{\alpha\beta}\partial_\alpha\rfloor\mathrm{vol}) = \mathrm{d}\bar{q}_\beta \wedge \sigma^{\alpha\beta}\partial_\alpha\rfloor\mathrm{vol} + \bar{q}_\beta\left(\mathrm{d}\left(\sigma^{\alpha\beta}\partial_\alpha\rfloor\mathrm{vol}\right)\right) \tag{4.11}$$

and

$$\hat{\otimes}\left(\sigma\rfloor\nabla^{\Lambda^*}(\bar{q})\right) = \hat{\otimes}\left(\sigma^{\alpha\beta}\partial_\alpha\rfloor\mathrm{vol}\otimes\left((\partial_0\bar{q}_\beta - \Lambda^{\rho*}_{0\beta}\bar{q}_\rho)\mathrm{d}t^0 + (\partial_\nu\bar{q}_\beta - \Lambda^{\rho*}_{\nu\beta}\bar{q}_\rho)\mathrm{d}q^\nu\right)\right) \ .$$

The equation (4.10) will turn out to be one of the key relations for an intrinsic formulation of continuum mechanics.

Dynamic Equations

With the latter results the balance of momentum relation (4.6) can be analyzed closer. We again integrate for fixed time t^0, thus we consider

$$\int_{\phi_\tau(\mathcal{U},t^0)} v_\phi\left(\rho\mathrm{vol}\wedge\mathrm{d}t^0(v\rfloor\bar{q})\right) = \int_{\phi_\tau(\mathcal{U},t^0)} \rho\mathrm{vol}\wedge\mathrm{d}t^0(b\rfloor\bar{q}) + \mathrm{d}\left(\sigma\rfloor\bar{q}\right)\wedge\mathrm{d}t^0 \ . \tag{4.12}$$

Remark 4.7 *The wedge product with* $\mathrm{d}t^0$ *has the same interpretation as already discussed in remark 4.2.*

The left hand side of equation (4.12) can be expressed as

$$\int_{\phi_\tau(\mathcal{U},t^0)} v_\phi\left(\rho\mathrm{vol}\wedge\mathrm{d}t^0(v\rfloor\bar{q})\right) = \int_{\phi_\tau(\mathcal{U},t^0)} v_\phi(v\rfloor\bar{q})\rho\mathrm{vol}\wedge\mathrm{d}t^0 \tag{4.13}$$

since the balance of mass implies

$$\int_{\phi_\tau(\mathcal{U},t^0)} (v\rfloor\bar{q})v_\phi\left(\rho\mathrm{vol}\wedge\mathrm{d}t^0\right) = 0 \ .$$

To analyze the right hand side of (4.13) we make use of the equation (3.27) and derive

$$v_\phi(v\rfloor\bar{q}) = v_\phi\rfloor\mathrm{d}(v\rfloor\bar{q}) = v_\phi\rfloor\left(\nabla^\Lambda(v)\rfloor\bar{q} + v\rfloor\nabla^{\Lambda*}(\bar{q})\right) \ .$$

Therefore, the balance of momentum relation can be rewritten as

$$\int_{\phi_\tau(\mathcal{U},t^0)} \left(v_\phi\rfloor\nabla^\Lambda(v)\rfloor\bar{q}\right)\rho\mathrm{vol}\wedge\mathrm{d}t^0 = \int_{\phi_\tau(\mathcal{U},t^0)} \left(\rho\mathrm{vol}\wedge\mathrm{d}t^0\otimes b + \mathrm{d}_\Lambda(\sigma)\wedge\mathrm{d}t^0\right)\rfloor\bar{q} \tag{4.14}$$

where $\nabla^{\Lambda*}(\bar{q}) = 0$ was used. Let us rewrite the equation (4.14) as

$$\int_{\phi_\tau(\mathcal{U},t^0)} \underbrace{\left[\left(v_\phi\rfloor\nabla^\Lambda(v) - b\right)\otimes\left(\rho\mathrm{vol}\wedge\mathrm{d}t^0\right) - \mathrm{d}_\Lambda(\sigma)\wedge\mathrm{d}t^0\right]}_{\bar{v}}\rfloor\bar{q} = 0 \tag{4.15}$$

where

$$\bar{v}\in\mathcal{V}(\mathcal{Q})\otimes\left(\wedge^n\left(\mathcal{V}^*(\mathcal{Q})\right)\wedge\mathcal{T}^*(\mathcal{B})\right)$$

is met. If we plug in a solution, the expression $\bar{v}$ has to be annihilated by all admissible $\bar{q} \in \mathcal{V}^*(\mathcal{Q})$ which meet $\nabla^{\Lambda *}(\bar{q}) = 0$ or, equivalently, $\bar{v}$ has to lie in the kernel with respect to the associated inner product $\mathcal{V}(\mathcal{Q}) \times \mathcal{V}(\mathcal{Q}) \to \mathcal{C}^\infty(\mathcal{Q})$ where we used the inverse of the metric $\hat{g} : \mathcal{V}^*(\mathcal{Q}) \to \mathcal{V}(\mathcal{Q})$. In this context it is worth mentioning that the admissible $\bar{q}$ can be interpreted as geodesics with respect to the covariant derivative $\nabla^{\Lambda *}$. Since the relation (4.15) has to hold for any $\mathcal{U}$ we conclude that $\bar{v}$ has to be annihilated by the geodesics $\bar{q}$ implying that

$$\left(\rho \mathrm{vol} \wedge \mathrm{d}t^0\right) \otimes v_\phi \rfloor \nabla^\Lambda(v) - \left(\rho \mathrm{vol} \wedge \mathrm{d}t^0\right) \otimes b - \mathrm{d}_\Lambda(\sigma) \wedge \mathrm{d}t^0 = 0 \tag{4.16}$$

has to be met. This equation (4.16) is the balance of momentum relation which is valid also when the reference frame is changed, since we use an intrinsic definition of the acceleration term and the volume form and the Nijenhuis differential of the stress $\mathrm{d}_\Lambda(\sigma)$ are formulated with respect to the connections γ and Λ.

The coordinate expression of equation (4.16) can be computed as follows

$$v_\phi \rfloor \nabla^\Lambda(v) = \left(\partial_0 v^\rho + v_\phi^\alpha \partial_\alpha v^\rho - v^\alpha \partial_\alpha \gamma_0^\rho - \Lambda_{\alpha\beta}^\rho v^\alpha v^\beta\right) \partial_\rho$$

and the more intricate expression $\mathrm{d}_\Lambda(\sigma) \wedge \mathrm{d}t^0$ reads as

$$\begin{aligned}
\mathrm{d}_\Lambda(\sigma) \wedge \mathrm{d}t^0 &= \left(\mathrm{d}(\sigma^{\alpha\beta}) \wedge \partial_\alpha \rfloor \mathrm{vol} + \sigma^{\alpha\beta} \mathrm{d}(\partial_\alpha \rfloor \mathrm{vol}) + \Lambda_{\rho\tau}^{\beta *} \sigma^{\alpha\tau} \mathrm{d}q^\rho \wedge \partial_\alpha \rfloor \mathrm{vol}\right) \wedge \mathrm{d}t^0 \otimes \partial_\beta \\
&= \left(\partial_\alpha(\sigma^{\alpha\beta}) + \sigma^{\alpha\beta} \partial_\alpha \sqrt{|\det(g_{\alpha\beta})|}\left(|\det(g_{\alpha\beta})|\right)^{-\frac{1}{2}} + \Lambda_{\alpha\tau}^{\beta *} \sigma^{\alpha\tau}\right) \mathrm{vol} \wedge \mathrm{d}t^0 \otimes \partial_\beta \ .
\end{aligned}$$

From the formula (2.13)

$$\Lambda_{\alpha\rho}^\rho = -\frac{1}{\sqrt{|\det(g_{\alpha\beta})|}} \partial_\alpha \sqrt{|\det(g_{\alpha\beta})|} \tag{4.17}$$

we finally have

$$\mathrm{d}_\Lambda(\sigma) \wedge \mathrm{d}t^0 = \left[\left(\partial_\alpha \sigma^{\alpha\beta} - \sigma^{\alpha\beta} \Lambda_{\alpha\kappa}^\kappa - \Lambda_{\alpha\tau}^\beta \sigma^{\alpha\tau}\right) \mathrm{vol} \wedge \mathrm{d}t^0\right] \otimes \partial_\beta \ .$$

In the end the balance relation in coordinates is given by

$$\rho\left(\partial_0 v^\rho + v_\phi^\alpha \partial_\alpha v^\rho - v^\alpha \partial_\alpha \gamma_0^\rho - \Lambda_{\alpha\beta}^\rho v^\alpha v^\beta\right) = \rho b^\rho + \partial_\alpha \sigma^{\alpha\rho} - \sigma^{\alpha\rho} \Lambda_{\alpha\kappa}^\kappa - \Lambda_{\alpha\tau}^\rho \sigma^{\alpha\tau} \ .$$

Example 4.8 *In order to show that this equation reproduces the standard equations for a non flat metric let us consider the case of an inertial system with $\gamma = \mathrm{d}t^0 \otimes \partial_0$ and $v^\alpha = v_\phi^\alpha$. We then obtain*

$$\left(\partial_0 v^\rho + v^\alpha \partial_\alpha v^\rho - \Lambda_{\alpha\beta}^\rho v^\beta v^\alpha\right) \rho = \rho b^\rho + \partial_\alpha \sigma^{\alpha\rho} - \sigma^{\alpha\rho} \Lambda_{\alpha\kappa}^\kappa - \Lambda_{\alpha\tau}^\rho \sigma^{\alpha\tau}$$

which coincides with the results of [Marsden and Hughes, 1994].

4.1.3 The Balance of Energy

In order to derive an energy relation let us consider the vertical metric $g : \mathcal{V}(\mathcal{Q}) \to \mathcal{V}^*(\mathcal{Q})$ which can be interpreted as a map $g : \mathcal{V}(\mathcal{Q}) \times \mathcal{V}(\mathcal{Q}) \to \mathcal{C}^\infty(\mathcal{Q})$. We use this map to rewrite the equation of balance of momentum (4.16) such that

$$\left(\rho \mathrm{vol} \wedge \mathrm{d}t^0\right) \otimes \left(v_\phi \rfloor \nabla^\Lambda(v) - b\right) \rfloor v \rfloor g = (\mathrm{d}_\Lambda(\sigma) \rfloor v \rfloor g) \wedge \mathrm{d}t^0$$

is met. By means of (4.10) the right hand side can be rewritten as

$$(\mathrm{d}_\Lambda(\sigma) \rfloor v \rfloor g) \wedge \mathrm{d}t^0 = \left(\mathrm{d}\left(\sigma \rfloor v \rfloor g\right) - \hat{\otimes}\left(\sigma \rfloor \nabla^{\Lambda^*}(v \rfloor g)\right)\right) \wedge \mathrm{d}t^0 \ .$$

A calculation in coordinates, see the Appendix A.8.2, gives

$$\hat{\otimes}\left(\sigma \rfloor \nabla^{\Lambda^*}(v \rfloor g)\right) \wedge \mathrm{d}t^0 = \hat{\otimes}\left(\sigma \rfloor \left(\frac{1}{2} v_\phi(g)\right)\right) \wedge \mathrm{d}t^0 \ , \tag{4.18}$$

where $v_\phi(g)$ denotes the Lie derivative of the metric along the field v_ϕ. Therefore, we obtain the relation

$$\left(\rho \mathrm{vol} \wedge \mathrm{d}t^0\right) \otimes \left(v_\phi \rfloor \nabla^\Lambda(v) - b\right) \rfloor v \rfloor g = \left(\mathrm{d}\left(\sigma \rfloor v \rfloor g\right) - \hat{\otimes}\left(\sigma \rfloor \frac{1}{2} v_\phi(g)\right)\right) \wedge \mathrm{d}t^0 \ . \tag{4.19}$$

In coordinates, see again the Appendix A.8.2, one can verify that

$$\left(v_\phi \rfloor \nabla^\Lambda(v)\right) \rfloor v \rfloor g = v_\phi \left(\frac{1}{2} v \rfloor v \rfloor g\right)$$

is met. Finally the equation describing the energy flows can be expressed as

$$\begin{aligned} \int_{\phi_\tau(\mathcal{U},t^0)} \left(v_\phi \left(\frac{1}{2} v^\alpha g_{\alpha\beta} v^\beta\right) \rho + \sigma^{\rho\tau} d_{\rho\tau}\right) \mathrm{vol} \wedge \mathrm{d}t^0 \quad &= \int_{\phi_\tau(\mathcal{U},t^0)} \left(b^\alpha g_{\alpha\beta} v^\beta\right) \rho \mathrm{vol} \wedge \mathrm{d}t^0 \\ &+ \int_{\phi_\tau(\partial\mathcal{U},t^0)} \left(v^\rho g_{\rho\beta} \sigma^{\alpha\beta} \partial_\alpha \rfloor \mathrm{vol}\right) \wedge \mathrm{d}t^0 \end{aligned}$$

with the components of the rate of the deformation

$$d_{\alpha\beta} = \frac{1}{2}\left(g_{\varepsilon\beta}\left(\partial_\alpha v^\varepsilon - \Lambda^\varepsilon_{\alpha\rho} v^\rho\right) + g_{\alpha\varepsilon}\left(\partial_\beta v^\varepsilon - \Lambda^\varepsilon_{\rho\beta} v^\rho\right)\right) \ .$$

Some remarks on the rate of the deformation tensor and the proof of

$$\sigma^{\rho\tau} d_{\rho\tau} \mathrm{vol} \wedge \mathrm{d}t^0 = \hat{\otimes}\left(\sigma \rfloor \frac{1}{2} v_\phi(g)\right) \wedge \mathrm{d}t^0$$

can be found in the Appendix A.8.2.

4.1.4 Application - Stagnation Point Flow

Motivated by the work of [Luo and Bewely, 2004] let us show how the derived relations look like for the problem of a stagnation point flow. We consider a configuration bundle $\mathcal{E} \to \mathcal{B}$ with Euclidean coordinates (t^0, q^1, q^2) for $\mathcal{E}$ and t^0 for $\mathcal{B}$, with trivial metric and trivial space time connection

$$g = \delta_{\alpha\beta} \mathrm{d}q^\alpha \otimes \mathrm{d}q^\beta \ , \quad \gamma = \mathrm{d}t^0 \otimes \partial_0 \ .$$

The Euler equations, see [Marsden and Hughes, 1994], read as

$$\left(\partial_0 v^\rho + v^\alpha \partial_\alpha v^\rho\right) \rho = -\partial_\alpha \left(p \hat{g}^{\alpha\rho}\right)$$

with the pressure function p, where $\sigma^{\alpha\rho} = -p\hat{g}^{\alpha\rho}$ was used. For

$$p = -\rho \frac{\alpha^2}{2} \left(\left(q^1\right)^2 + \left(q^2\right)^2 \right) \ , \quad \alpha \in \mathbb{R}$$

it is verified that

$$v^1 = \alpha q^1 \ , \quad v^2 = -\alpha q^2$$

holds and the balance of mass is fulfilled with $\rho \in \mathbb{R}$. The flow ϕ is given as

$$q^1 = e^{\alpha\tau} q^1_s \ , \quad q^2 = e^{-\alpha\tau} q^2_s \ , \quad t^0 = \tau + t^0_s \ .$$

Let us consider a diffeomorphic change of coordinates of the form $\bar{q}^{\bar{\alpha}} = \varphi^{\bar{\alpha}}(q^\beta, t^0)$ with

$$\begin{aligned} \bar{q}^{\bar{1}} &= \varphi^{\bar{1}}(q^1, q^2, t^0) = (q^1 + \gamma q^2) e^{-\beta t^0} \\ \bar{q}^{\bar{2}} &= \varphi^{\bar{2}}(q^1, q^2, t^0) = (q^2 - \gamma q^1) e^{\beta t^0} \\ \bar{t}^{\bar{0}} &= t^0 \end{aligned}$$

and $\gamma, \beta \in \mathbb{R}$, which coincides with the one presented in[Luo and Bewely, 2004] for $\alpha = \beta$, $\gamma = 0$. Then it is verified from the transformation law for the connection coefficients (3.6) that we have

$$\bar{\gamma}^{\bar{1}}_{\bar{0}} = \left(\partial_0 \varphi^{\bar{1}}\right) \circ \hat{\varphi} = -\beta \bar{q}^{\bar{1}} \ , \quad \bar{\gamma}^{\bar{2}}_{\bar{0}} = \left(\partial_0 \varphi^{\bar{2}}\right) \circ \hat{\varphi} = \beta \bar{q}^{\bar{2}}$$

since $\gamma^\alpha_0 = 0$, which means, that in contrast to the inertial space we have a non trivial space time connection $\bar{\gamma}$. From the trivial metric in the inertial system $g = \delta_{\alpha\beta} \mathrm{d}q^\alpha \otimes \mathrm{d}q^\beta$ and the transformation law

$$\bar{g}_{\bar{\alpha}\bar{\beta}} = g_{\alpha\beta} \partial_{\bar{\alpha}} \hat{\varphi}^\alpha \partial_{\bar{\beta}} \hat{\varphi}^\beta$$

we obtain the nonzero metric elements in the new coordinates

$$\bar{g}_{\bar{1}\bar{1}} = \frac{e^{2\beta t^0}}{(1 + \gamma^2)} \ , \quad \bar{g}_{\bar{2}\bar{2}} = \frac{e^{-2\beta t^0}}{(1 + \gamma^2)}$$

which are time dependent. The flow $\bar{\phi}$ in the new coordinates meets $\bar{\phi} = (\varphi \circ \phi) \circ \hat{\varphi}_s$ and reads with $\bar{q}^{\bar{\alpha}}_s = \varphi^{\bar{\alpha}}_s(q^1_s, q^2_s, t^0_s)$ as

$$\begin{aligned} \bar{q}^{\bar{1}} = \bar{\phi}^{\bar{1}}_\tau(\bar{q}^{\bar{1}}_s, \bar{q}^{\bar{2}}_s, t^0_s) &= \tfrac{e^{(-\beta(\tau + t^0_s))}}{1+\gamma^2} \left(\bar{q}^{\bar{1}}_s (e^{\alpha\tau + \beta t^0_s} + \gamma^2 e^{-\alpha\tau + \beta t^0_s}) + \gamma \bar{q}^{\bar{2}}_s (e^{-\alpha\tau - \beta t^0_s} - e^{\alpha\tau - \beta t^0_s}) \right) \\ \bar{q}^{\bar{2}} = \bar{\phi}^{\bar{1}}_\tau(\bar{q}^{\bar{1}}_s, \bar{q}^{\bar{2}}_s, t^0_s) &= \tfrac{e^{(\beta(\tau + t^0_s))}}{1+\gamma^2} \left(\gamma \bar{q}^{\bar{1}}_s (e^{-\alpha\tau + \beta t^0_s} - e^{\alpha\tau + \beta t^0_s}) + \bar{q}^{\bar{2}}_s (e^{-\alpha\tau - \beta t^0_s} + \gamma^2 e^{\alpha\tau - \beta t^0_s}) \right) \\ \bar{t}^{\bar{0}} &= \bar{\phi}^{\bar{0}}_\tau(t^0_s) = \tau + t^0_s \ . \end{aligned} \tag{4.20}$$

We have to verify that

$$\rho\left(\partial_{\bar{0}}\bar{v}^{\bar{1}} + \bar{v}_{\bar{\phi}}^{\bar{\alpha}}\partial_{\bar{\alpha}}\bar{v}^{\bar{1}} - \bar{v}^{\bar{\alpha}}\partial_{\bar{\alpha}}\gamma_{\bar{0}}^{\bar{1}}\right) = -\partial_{\bar{\beta}}(\bar{p}\bar{g}^{\bar{\beta}\bar{1}}) \tag{4.21}$$

$$\rho\left(\partial_{\bar{0}}\bar{v}^{\bar{2}} + \bar{v}_{\bar{\phi}}^{\bar{\alpha}}\partial_{\bar{\alpha}}\bar{v}^{\bar{2}} - \bar{v}^{\bar{\alpha}}\partial_{\bar{\alpha}}\gamma_{\bar{0}}^{\bar{2}}\right) = -\partial_{\bar{\beta}}(\bar{p}\bar{g}^{\bar{\beta}\bar{2}}) \tag{4.22}$$

is met with

$$\bar{v}_{\bar{\phi}}^{\bar{1}} = \partial_\tau \bar{\phi}_\tau^{\bar{1}}\Big|_{\tau=0}\ , \quad \bar{v}_{\bar{\phi}}^{\bar{2}} = \partial_\tau \bar{\phi}_\tau^{\bar{2}}\Big|_{\tau=0}\ , \quad \bar{v}^{\bar{1}} = \bar{v}_{\bar{\phi}}^{\bar{1}} - \bar{\gamma}_{\bar{0}}^{\bar{1}}\ , \quad \bar{v}^{\bar{2}} = \bar{v}_{\bar{\phi}}^{\bar{2}} - \bar{\gamma}_{\bar{0}}^{\bar{2}}\ .$$

The pressure function in the new coordinates is given as

$$\bar{p} = -\rho\frac{\alpha^2 e^{2\beta t^0}((\bar{q}^{\bar{1}})^2 + (\bar{q}^{\bar{2}})^2 e^{-4\beta t^0})}{2(1+\gamma^2)}$$

and therefore

$$-\partial_{\bar{\beta}}(\bar{p}\bar{g}^{\bar{\beta}\bar{1}}) = \rho\alpha^2\bar{q}^{\bar{1}}\ , \quad -\partial_{\bar{\beta}}(\bar{p}\bar{g}^{\bar{\beta}\bar{2}}) = \rho\alpha^2\bar{q}^{\bar{2}}$$

follows. The expressions for the velocities read as

$$\begin{aligned}\bar{v}^{\bar{1}} &= -\alpha\frac{(\bar{q}^{\bar{1}}\gamma^2 + 2\bar{q}^{\bar{2}}\gamma e^{-2\beta\bar{t}^{\bar{0}}} - \bar{q}^{\bar{1}})}{1+\gamma^2}\\ \bar{v}^{\bar{2}} &= -\alpha\frac{(-\bar{q}^{\bar{2}}\gamma^2 + 2\bar{q}^{\bar{1}}\gamma e^{2\beta\bar{t}^{\bar{0}}} + \bar{q}^{\bar{2}})}{1+\gamma^2}\end{aligned}$$

and it can be verified that we have

$$\begin{aligned}\partial_{\bar{0}}\bar{v}^{\bar{1}} + \bar{v}_{\bar{\phi}}^{\bar{\alpha}}\partial_{\bar{\alpha}}\bar{v}^{\bar{1}} - \bar{v}^{\bar{\alpha}}\partial_{\bar{\alpha}}\gamma_{\bar{0}}^{\bar{1}} &= \alpha^2\bar{q}^{\bar{1}}\\ \partial_{\bar{0}}\bar{v}^{\bar{2}} + \bar{v}_{\bar{\phi}}^{\bar{\alpha}}\partial_{\bar{\alpha}}\bar{v}^{\bar{2}} - \bar{v}^{\bar{\alpha}}\partial_{\bar{\alpha}}\gamma_{\bar{0}}^{\bar{2}} &= \alpha^2\bar{q}^{\bar{2}}\end{aligned}$$

and therefore the equations (4.21) and (4.22) hold. The mass balance is fulfilled also, since (4.5) is met with $\bar{\Lambda}_{\bar{\alpha}\bar{\rho}}^{\bar{\rho}} = 0$ as well as

$$\partial_{\bar{1}}\bar{v}^{\bar{1}} + \partial_{\bar{2}}\bar{v}^{\bar{2}} = 0$$

and $\rho \in \mathbb{R}$.

Remark 4.9 *From the transformation (4.20) one easily sees that the transition functions for vertical vectors hold because*

$$\bar{v}^{\bar{\alpha}} = \partial_\alpha\varphi^{\bar{\alpha}}v^\alpha$$

is met. Therefore, let us stress out again the fact that in Euclidean coordinates q^α the components of the velocities v^α are tangent to the flow lines and measure the velocity of the fluid particles. In the coordinates $\bar{q}^{\bar{\alpha}}$ the vectors tangent to the flow lines $\bar{v}_{\bar{\phi}}^{\bar{\alpha}}$ do not measure the velocity of the fluid particles in general and one has to take into account the velocity of the coordinate system itself to obtain the correct relations. It is worth mentioning again that only for a trivial space time connection one has $v^\alpha = v_\phi^\alpha$ and the importance of this example lies in the fact to demonstrate the validity of the equations (4.3) and (4.16) when the metric becomes time dependent and the space time connection is not trivial. The case where the Christoffel symbols of the second kind are nonzero can be treated in the same manner. Of course the computations are much more extensive then but nothing essential changes.

4.2 The Lagrangian Picture

In contrast to the Eulerian picture the Lagrangian approach makes explicit use of the reference manifold which is used to identify material points. The Eulerian point of view allows the interpretation that the configuration and the reference bundle are identical. This perception will be abandoned in the Lagrangian formulation, which means that the equations of motions are expressed in material quantities. This can be achieved by a special transformation, the Piola transformation which will be discussed in the sequel. The reference bundle $\mathcal{R} \to \mathcal{B}$ is introduced with coordinates (t^0, X^i) for $\mathcal{R}$. The configuration bundle $\mathcal{Q} \to \mathcal{B}$ will be extended to the bundle $\mathcal{C}_e \to \mathcal{R}$, with

$$\mathcal{C}_e = \mathcal{Q} \times_{\mathcal{B}} \mathcal{R}$$

and coordinates (t^0, X^i, q^α) for $\mathcal{C}_e$. A motion in the Lagrangian setting is a map $\Phi : \mathcal{R} \to \mathcal{C}_e$ with

$$q^\alpha = \Phi^\alpha \left(t^0, X^i\right) \ .$$

Furthermore, we introduce a vertical metric

$$G = G_{ij}(\mathrm{d}X^i - \Gamma^i_0 \mathrm{d}t^0) \otimes (\mathrm{d}X^j - \Gamma^j_0 \mathrm{d}t^0) \ , \quad \Gamma^i_0, \ G_{ij} \in \mathcal{C}^\infty(\mathcal{R})$$

and a volume form

$$\mathrm{VOL} = \sqrt{|\det(G_{ij})|}(\mathrm{d}X^1 - \Gamma^1_0 \mathrm{d}t^0) \wedge \ldots \wedge (\mathrm{d}X^n - \Gamma^n_0 \mathrm{d}t^0) \ , \quad \Gamma^i_0, \ G_{ij} \in \mathcal{C}^\infty(\mathcal{R}) \ .$$

Remark 4.10 *The choice of a coordinate system apparently leads to a metric and a volume form on the fibres of $\mathcal{R}$. The connection coefficients Γ^i_0 have computational reasons and their appearance will be discussed thereinafter.*

For simplicity we only discuss the case $\dim(\mathcal{Q}) = \dim(\mathcal{R})$. The tangent map of $\Phi : \mathcal{R} \to \mathcal{C}_e$

$$\begin{aligned} \mathcal{T}(\Phi) \quad &: \quad \mathcal{T}(\mathcal{R}) \to \mathcal{T}(\mathcal{C}_e) \\ \mathcal{T}(\Phi) \quad &= \quad \mathrm{d}t^0 \otimes (\partial_0 + V^\alpha_0 \partial_\alpha) + \mathrm{d}X^i \otimes (\partial_i + F^\alpha_i \partial_\alpha) \end{aligned}$$

involves the well known quantity of the deformation gradient $F^\alpha_i = \partial_i \Phi^\alpha$. Care must be taken since $V^\alpha_0 = \partial_0 \Phi^\alpha$ does not correspond to the material velocity due to similar arguments as presented in section 3.2. Consequently, the velocity of a material point is defined by

$$V^\alpha = V^\alpha_0 - \gamma^\alpha_0 \circ \Phi = \partial_0 \Phi^\alpha - \gamma^\alpha_0 \circ \Phi \tag{4.23}$$

and it can be seen that V^α is a material quantity.

Remark 4.11 *From the equation (4.23) it is seen that the definition of the velocity corresponds to the one used for point mechanics in section 3.2. This is obvious since in the Lagrangian description one is interested in the evolution of a fixed distinguished mass point selected from the continuum.*

If the map Φ is invertible, the vertical field $v : \mathcal{Q} \to \mathcal{V}(\mathcal{Q})$

$$v^\alpha \partial_\alpha = \left(V^\alpha \circ \Phi^{-1}\right) \partial_\alpha = \left(V_0^\alpha \circ \Phi^{-1} - \gamma_0^\alpha\right) \partial_\alpha$$

can be constructed. With the help of the map $\Phi : \mathcal{R} \to \mathcal{C}_e$ we can pull back the form $\rho\mathrm{vol}$ to

$$\rho_{\mathcal{R}} \mathrm{VOL} = \Phi^*(\rho \mathrm{vol})$$

where $\rho_{\mathcal{R}}$ is the mass density in the reference configuration meeting

$$\rho_{\mathcal{R}} = \left(\rho \frac{\sqrt{|\det(g_{\alpha\beta})|}}{\sqrt{|\det(G_{ij})|}} \circ \Phi\right) \det[F_i^\alpha] \,. \tag{4.24}$$

The Figure (4.2) illustrates the constructions concerning the Lagrangian approach.

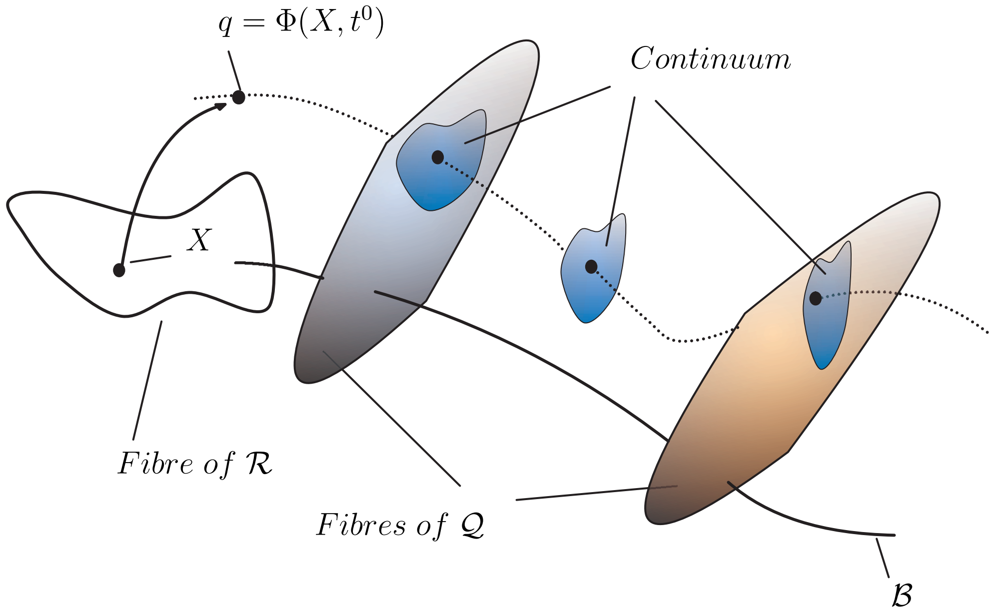

Figure 4.2: The Configuration Bundle in the Lagrangian Picture

Remark 4.12 *Let us consider the two dimensional case with*

$$\begin{aligned}
\mathrm{vol} &= \sqrt{|\det(g_{\alpha\beta})|}(\mathrm{d}q^1 - \gamma_0^1 \mathrm{d}t^0) \wedge (\mathrm{d}q^2 - \gamma_0^2 \mathrm{d}t^0) \\
&= \sqrt{|\det(g_{\alpha\beta})|}(\mathrm{d}q^1 \wedge \mathrm{d}q^2 - \gamma_0^1 \mathrm{d}t^0 \wedge \mathrm{d}q^2 - \mathrm{d}q^1 \wedge \gamma_0^2 \mathrm{d}t^0)
\end{aligned}$$

and

$$q^1 = \Phi^1(X^1, X^2, t^0)\,, \quad q^2 = \Phi^2(X^1, X^2, t^0)\,.$$

Consequently we obtain

$$\begin{aligned}
\mathrm{d}q^1 &= \partial_{X_1} \Phi^1 \mathrm{d}X^1 + \partial_{X_2} \Phi^1 \mathrm{d}X^2 + \partial_0 \Phi^1 \mathrm{d}t^0 \\
\mathrm{d}q^2 &= \partial_{X_1} \Phi^2 \mathrm{d}X^1 + \partial_{X_2} \Phi^2 \mathrm{d}X^2 + \partial_0 \Phi^2 \mathrm{d}t^0
\end{aligned}$$

and

$$\begin{aligned}\Phi^*(\mathrm{vol}) &= \Phi^*\sqrt{|\det(g_{\alpha\beta})|}\,\big((\partial_{X_1}\Phi^1\partial_{X_2}\Phi^2-\partial_{X_2}\Phi^1\partial_{X_1}\Phi^2)\mathrm{d}X^1\wedge\mathrm{d}X^2\\ &\quad+(\partial_{X_1}\Phi^1(\partial_0\Phi^2-\gamma_0^2)-\partial_{X_1}\Phi^2(\partial_0\Phi^1-\gamma_0^1))\mathrm{d}X^1\wedge\mathrm{d}t^0\\ &\quad+(\partial_{X_2}\Phi^1(\partial_0\Phi^2-\gamma_0^2)-\partial_{X_2}\Phi^2(\partial_0\Phi^1-\gamma_0^1))\mathrm{d}X^2\wedge\mathrm{d}t^0\big)\ .\end{aligned}$$

Furthermore, we can write

$$\Phi^*(\mathrm{vol})=\left(\frac{\sqrt{|\det(g_{\alpha\beta})|}}{\sqrt{|\det(G_{\alpha\beta})|}}\circ\Phi\right)\det[F_i^\alpha]\mathrm{VOL}$$

with

$$\mathrm{VOL}=\sqrt{|\det(G_{\alpha\beta})|}(\mathrm{d}X^1-\Gamma_0^1\mathrm{d}t^0)\wedge(\mathrm{d}X^2-\Gamma_0^2\mathrm{d}t^0)$$

and

$$\begin{aligned}\Gamma_0^1 &= \frac{V^2\partial_{X_2}\Phi^1-V^1\partial_{X_2}\Phi^2}{\det[F_i^\alpha]}\\ \Gamma_0^2 &= \frac{-V^2\partial_{X_1}\Phi^1+V^1\partial_{X_1}\Phi^2}{\det[F_i^\alpha]}\ .\end{aligned}$$

It is worth mentioning that the reference bundle is used to bookmark material points at a fixed time and therefore the connection coefficients Γ_0^i are not used in calculations. They only appear formally due to the presented construction. This construction can be generalized such that we have

$$\Gamma_0^i=-\hat{F}_\alpha^i V^\alpha\ ,\quad \hat{F}_\alpha^i F_k^\alpha=\delta_k^i$$

but will not be discussed here lengthy since the coefficients Γ_0^i are dispensable for all further investigations.

4.2.1 The Piola Transformation

The Piola transformation is an important concept that allows to express spatial quantities in a material form.

Piola Tensors and Cauchy Green Tensor

Let us consider the Cauchy stress form

$$\sigma=\sigma^{\alpha\beta}\partial_\alpha\rfloor\mathrm{vol}\otimes\partial_\beta \tag{4.25}$$

together with the map $\Phi:\mathcal{R}\to\mathcal{C}_e$ that allows to pull back the form part of (4.25). This leads to the 1st Piola stress tensor

$$P=\Phi^*\left(\sigma^{\alpha\beta}\partial_\alpha\rfloor\mathrm{vol}\right)\otimes\partial_\beta=P^{i\beta}\partial_i\rfloor\mathrm{VOL}\otimes\partial_\beta \tag{4.26}$$

with

$$P^{i\beta}=\left(\sigma^{\alpha\beta}\frac{\sqrt{|\det(g_{\alpha\beta})|}}{\sqrt{|\det(G_{ij})|}}\hat{F}_\alpha^i\det[F_i^\alpha]\right)\circ j^1\Phi,\quad \hat{F}_\alpha^i F_j^\alpha=\delta_j^i\ . \tag{4.27}$$

Remark 4.13 *In order to explain how ∂_α is pulled back along Φ let us consider the map*

$$X^i = \hat{\Phi}^i\left(q^\alpha, t^0\right)$$

as well as

$$\dot{X}^i = \partial_\beta \hat{\Phi}^i \dot{q}^\beta + \partial_0 \hat{\Phi}^i \dot{t}^0 \ .$$

The push forward $\hat{\Phi}_*$ of ∂_α yields $\partial_\alpha \to \partial_\alpha \hat{\Phi}^i \partial_i$ which is equivalent to $\partial_\alpha \to \hat{F}^i_\alpha \partial_i$.

The 2nd Piola stress tensor is given as

$$S = \Phi^*\left(\sigma^{\alpha\beta}\partial_\alpha \rfloor \mathrm{vol} \otimes \partial_\beta\right) = S^{ij}\partial_i \rfloor \mathrm{VOL} \otimes \partial_j$$

with

$$S^{ij} = \left(\sigma^{\alpha\beta}\frac{\sqrt{|\det(g_{\alpha\beta})|}}{\sqrt{|\det(G_{kl})|}}\hat{F}^i_\alpha \hat{F}^j_\beta \det[F^\alpha_i]\right) \circ j^1\Phi$$

and in coordinates one has the relation

$$S^{ij} = P^{i\beta}\hat{F}^j_\beta \ .$$

The Cauchy Green tensor is obtained by pulling back the metric g by the map $\Phi : \mathcal{R} \to \mathcal{C}_e$. By means of

$$\mathrm{d}q^\alpha = \partial_i \Phi^\alpha \mathrm{d}X^i + \partial_0 \Phi^\alpha \mathrm{d}t^0$$

we obtain

$$C = \Phi^*(g) = (g_{\alpha\beta} \circ \Phi)\left(\partial_i \Phi^\alpha \mathrm{d}X^i + (\partial_0\Phi^\alpha - \gamma^\alpha_0)\mathrm{d}t^0\right) \otimes \left(\partial_j \Phi^\beta \mathrm{d}X^j + (\partial_0\Phi^\beta - \gamma^\beta_0)\mathrm{d}t^0\right)$$

and

$$C = (g_{\alpha\beta} \circ \Phi)\, \partial_i \Phi^\alpha \partial_j \Phi^\beta (\mathrm{d}X^i + \hat{F}^i_\alpha V^\alpha \mathrm{d}t^0) \otimes (\mathrm{d}X^j + \hat{F}^j_\beta V^\beta \mathrm{d}t^0) \ .$$

From remark 4.12 we finally have

$$C = C_{ij}(\mathrm{d}X^i - \Gamma^i_0 \mathrm{d}t^0) \otimes (\mathrm{d}X^j - \Gamma^j_0 \mathrm{d}t^0)$$

with

$$C_{ij} = (g_{\alpha\beta} \circ \Phi) F^\alpha_i F^\beta_j \ .$$

Remark 4.14 *We have to introduce a bit more of notation especially for the variational approach which follows. Since the motion $\Phi : \mathcal{R} \to \mathcal{C}_e$ is the wanted term, the pull back operation carried out to obtain the first Piola tensor P, the second Piola tensor S and the Cauchy Green tensor C can be accomplished only when the solution Φ of the problem is known. Therefore the following quantities are adopted which do not require the knowledge of Φ. We have*

$$\begin{aligned} \breve{P}^{i\beta} \circ j^1\Phi &= P^{i\beta} \\ \breve{S}^{ij} \circ j^1\Phi &= S^{ij} \\ \breve{C}_{ij} \circ j^1\Phi &= C_{ij} \end{aligned}$$

which means that $\breve{P}^{i\beta}, \breve{S}^{ij}, \breve{C}_{ij} \in C^\infty(\mathcal{J}^1(\mathcal{C}_e))$. Then the relation

$$P^{i\beta} = S^{ij} F^\beta_j$$

reads

$$\breve{P}^{i\beta} = \breve{S}^{ij} q^\beta_j \ .$$

Remark 4.15 *In the following the composition with the function Φ resulting from the pull back is only indicated when necessary, for example when derivations appear. The quantities P, S, and C are treated as described above and, of course, the Christoffel symbols $\Lambda^{\rho}_{\alpha\beta}$ and $\Lambda^{\rho *}_{\alpha\beta}$ and all other quantities will be evaluated by plugging in the motion Φ. This is omitted sometimes to enhance the readability.*

Pull back of the Nijenhuis Differential of the Stress Form

To deal with the stress in the material picture we have to pull back the Nijenhuis differential of the stress form which was given in section 4.1.2 and for convenience we give here again the coordinate relation which is

$$\mathrm{d}_\Lambda(\sigma) = \Big[\mathrm{d}(\sigma^{\alpha\beta}\partial_\alpha \rfloor \mathrm{vol}) - \Big(\Lambda^{\beta}_{0\tau}\sigma^{\alpha\tau}\mathrm{d}t^0 + \Lambda^{\beta}_{\rho\tau}\sigma^{\alpha\tau}\mathrm{d}q^\rho\Big) \wedge \partial_\alpha \rfloor \mathrm{vol}\Big] \otimes \partial_\beta \ .$$

The pull back the form part of $\mathrm{d}_\Lambda(\sigma)$ along the map Φ is

$$\Phi^*\left(\mathrm{d}_\Lambda(\sigma)\right) = \mathrm{d}^\Phi_\Lambda(P) \ .$$

From

$$\Phi^*\left(\mathrm{d}(\sigma^{\alpha\beta}\partial_\alpha \rfloor \mathrm{vol})\right) = \mathrm{d}\left(\Phi^*(\sigma^{\alpha\beta}\partial_\alpha \rfloor \mathrm{vol})\right) = \mathrm{d}(P^{i\beta}\partial_i \rfloor \mathrm{VOL})$$

and

$$\Phi^*\left(\left(\Lambda^{\beta}_{0\tau}\sigma^{\alpha\tau}\mathrm{d}t^0 + \Lambda^{\beta}_{\rho\tau}\sigma^{\alpha\tau}\mathrm{d}q^\rho\right) \wedge \partial_\alpha \rfloor \mathrm{vol}\right) = P^{i\tau}\left(\Lambda^{\beta}_{\rho\tau}F^\rho_i \mathrm{d}X^i + \left(\Lambda^{\beta}_{\rho\tau}V^\rho_0 + \Lambda^{\beta}_{0\tau}\right)\mathrm{d}t^0\right) \wedge \partial_i \rfloor \mathrm{VOL}$$

the coordinate expression

$$\mathrm{d}^\Phi_\Lambda(P) = \left(P^{i\tau}\left(\Lambda^{\beta *}_{\rho\tau}F^\rho_i \mathrm{d}X^i + \left(\Lambda^{\beta *}_{\rho\tau}V^\rho_0 + \Lambda^{\beta *}_{0\tau}\right)\mathrm{d}t^0\right) \wedge \partial_i \rfloor \mathrm{VOL} + \mathrm{d}(P^{i\beta}\partial_i \rfloor \mathrm{VOL})\right) \otimes \partial_\beta \quad (4.28)$$

is obtained. Furthermore, from the equation (4.28) one has

$$\mathrm{d}^\Phi_\Lambda(P) \wedge \mathrm{d}t^0 = \left(P^{i\tau}\Lambda^{\beta *}_{\rho\tau}F^\rho_i + \frac{1}{\sqrt{|\det(G_{kl})|}}\partial_i\left(\sqrt{|\det(G_{kl})|}P^{i\beta}\right)\right)\mathrm{VOL} \wedge \mathrm{d}t^0 \otimes \partial_\beta$$

as well as

$$\mathrm{d}^\Phi_\Lambda(P) \wedge \mathrm{d}t^0 = \left(\partial_i P^{i\beta} - P^{i\beta}\Lambda^r_{ir} - P^{i\tau}\Lambda^{\beta}_{\rho\tau}F^\rho_i\right)\mathrm{VOL} \wedge \mathrm{d}t^0 \otimes \partial_\beta \quad (4.29)$$

where we used the equation (2.13) and Λ^r_{ir} denote the Christoffel symbols with respect to the metric G.

Material Covariant Derivatives

Let us inspect how the covariant derivatives (3.15) and (3.16) can be pulled back along the map $\Phi : \mathcal{R} \to \mathcal{C}_e$. Therefore the notation

$$\nabla^\Lambda_\Phi(w) = \Phi^*\left(\nabla^\Lambda(w)\right) \quad (4.30)$$

and

$$\nabla^{\Lambda^*}_{\Phi}(\omega) = \Phi^*\left(\nabla^{\Lambda^*}(\omega)\right) \tag{4.31}$$

will be used. In coordinates we obtain for the relation (4.30)

$$\nabla^{\Lambda}_{\Phi}(w) = \left((\partial_0 w^\rho - \Lambda^\rho_{0\alpha}w^\alpha)\,\mathrm{d}t^0 + (\partial_\alpha w^\rho - \Lambda^\rho_{\alpha\beta}w^\beta)(F^\alpha_i \mathrm{d}X^i + V^\alpha_0 \mathrm{d}t^0)\right)\otimes \hat{F}^i_\rho\partial_i$$

where the right hand side can be rewritten as

$$\left(\left(\partial_0\left(w^\rho\circ\Phi\right) - \left(\Lambda^\rho_{0\beta} + \Lambda^\rho_{\alpha\beta}V^\alpha_0\right)w^\beta\right)\mathrm{d}t^0 + \left(\partial_i\left(w^\rho\circ\Phi\right) - \Lambda^\rho_{\alpha\beta}F^\alpha_i w^\beta\right)\mathrm{d}X^i\right)\otimes\hat{F}^i_\rho\partial_i \tag{4.32}$$

and for the relation (4.31)

$$\nabla^{\Lambda^*}_{\Phi}(\omega) = \left((\partial_0\omega_\beta - \Lambda^{\rho *}_{0\beta}\omega_\rho)\mathrm{d}t^0 + (\partial_\alpha\omega_\beta - \Lambda^{\rho *}_{\alpha\beta}\omega_\rho)(F^\alpha_i\mathrm{d}X^i + V^\alpha_0\mathrm{d}t^0)\right)\otimes\left(F^\beta_i\mathrm{d}X^i + V^\beta\mathrm{d}t^0\right)$$

where the right hand side follows to

$$\begin{aligned}&\left(\partial_0(\omega_\beta\circ\Phi) - \left(\Lambda^{\rho *}_{0\beta} + V^\alpha_0\Lambda^{\rho *}_{\alpha\beta}\right)\omega_\rho\right)\mathrm{d}t^0\otimes\left(F^\beta_i\mathrm{d}X^i + V^\beta\mathrm{d}t^0\right)\\&+\left(\left(\partial_i\left(\omega_\beta\circ\Phi\right) - \Lambda^{\rho *}_{\alpha\beta}F^\alpha_i\omega_\rho\right)\mathrm{d}X^i\right)\otimes\left(F^\beta_i\mathrm{d}X^i + V^\beta\mathrm{d}t^0\right)\ .\end{aligned} \tag{4.33}$$

The Nijenhuis Relation

The next step is to construct a counterpart to the expression (4.10) which can be done in the following way. We first analyze the relation

$$\mathrm{d}(P\rfloor\bar{q}) = \mathrm{d}(\bar{q}_\beta P^{i\beta}\partial_i\rfloor\mathrm{VOL}) = \mathrm{d}\bar{q}_\beta\wedge P^{i\beta}\partial_i\rfloor\mathrm{VOL} + \bar{q}_\beta\left(\mathrm{d}\left(P^{i\beta}\partial_i\rfloor\mathrm{VOL}\right)\right)$$

where the section $\bar{q}:\mathcal{Q}\to\mathcal{V}^*(\mathcal{Q})$ is evaluated along the map Φ, which means that we have $\bar{q} = (\bar{q}_\alpha\circ\Phi)\left(\mathrm{d}q^\alpha - \gamma^\alpha_0\mathrm{d}t^0\right)$. It follows from the equation (4.28) and the expression

$$\begin{aligned}\hat{\otimes}\left(S\rfloor\nabla^{\Lambda^*}_{\Phi}(\bar{q})\right) = \;&\hat{\otimes}\left(P^{\beta j}\partial_j\rfloor\mathrm{VOL}\otimes\left((\partial_0(\bar{q}_\beta\circ\Phi) - \left(\Lambda^{\rho *}_{0\beta} + V^\alpha_0\Lambda^{\rho *}_{\alpha\beta}\right)\bar{q}_\rho)\mathrm{d}t^0\right)\right)\\&+\hat{\otimes}\left(P^{\beta j}\partial_j\rfloor\mathrm{VOL}\otimes\left((\partial_i\left(\bar{q}_\beta\circ\Phi\right) - \Lambda^{\rho *}_{\alpha\beta}F^\alpha_i\bar{q}_\rho)\mathrm{d}X^i\right)\right)\end{aligned}$$

that

$$\mathrm{d}(P\rfloor\bar{q}) = \mathrm{d}^{\Phi}_{\Lambda}(P)\rfloor\bar{q} + \hat{\otimes}\left(S\rfloor\nabla^{\Lambda^*}_{\Phi}(\bar{q})\right) \tag{4.34}$$

is met. Furthermore, the counterpart to the relation (4.18) in the Lagrangian setting is given as

$$\hat{\otimes}\left(S\rfloor\nabla^{\Lambda^*}_{\Phi}(V\rfloor g)\right)\wedge\mathrm{d}t^0 = \hat{\otimes}\left(S\rfloor\frac{1}{2}\left(\partial_0 C\right)\right)\wedge\mathrm{d}t^0\ . \tag{4.35}$$

The details of the derivation are omitted and can be found in the Appendix A.8.3, where also the rate of the deformation tensor is presented in the material form in remark A.2.

4.2.2 The Mass Balance

The balance of mass, as stated in the section 4.1.1, can also be formulated with respect to material quantities. Therefore, the mass of a continuous region in the Lagrangian description is given as

$$m = \int_{\mathcal{S}} \rho_{\mathcal{R}} \mathrm{VOL} = c \tag{4.36}$$

with $c \in \mathbb{R}^+$ where $\mathcal{S} \subset \mathcal{R}$ is a nice domain of integration. The conservation of mass states that

$$\int_{\mathcal{S}} \partial_0 \left(\rho_{\mathcal{R}} \mathrm{VOL} \wedge \mathrm{d}t^0\right) = 0 \tag{4.37}$$

has to be met and since $\mathcal{S}$ is arbitrary we conclude that $\partial_0(\rho_{\mathcal{R}}\sqrt{|\det(G_{ij})|}) = 0$ holds, where the detailed computation is given in the Appendix A.8.4.

4.2.3 The Balance of Linear Momentum

The formulation of the equations (4.14) and (4.16) in the material picture requires to express the acceleration term and the force term with respect to material quantities. This is easy for the forces since we have $B^\rho \partial_\rho = (b^\rho \circ \Phi)\, \partial_\rho$ and for the expression involving the covariant derivative of the vertical velocity field $\Phi^* \left(v_\phi \rfloor \nabla^\Lambda(v)\right) = \partial_0 \rfloor \nabla^\Lambda_\Phi(V)$ holds. In coordinates

$$\partial_0 \rfloor \nabla^\Lambda_\Phi(V) = (\partial_0 V^\rho - (V^\alpha \Lambda^\rho_{0\alpha} \circ \Phi) - \left(V^\alpha_0 \Lambda^\rho_{\alpha\beta} V^\beta\right) \circ \Phi)\partial_\rho$$

is met, where the equation (4.32) was used.

Remark 4.16 *We used the expression*

$$\nabla^\Lambda_\Phi(w) = \left((\partial_0 w^\rho - \Lambda^\rho_{0\alpha} w^\alpha)\,\mathrm{d}t^0 + (\partial_\alpha w^\rho - \Lambda^\rho_{\alpha\beta} w^\beta)(F^\alpha_i \mathrm{d}X^i + V^\alpha_0 \mathrm{d}t^0)\right) \otimes \partial_\rho$$

which differs from (4.32) by the fact that we did not pull back the vertical part ∂_ρ.

Again, using the equation (3.25) it follows that

$$\begin{aligned}
\partial_0 \rfloor \nabla^\Lambda_\Phi(V) &= (\partial_0 V^\rho - \left(V^\alpha \left(\partial_\alpha \gamma^\rho_0 - \Lambda^\rho_{\beta\alpha}\gamma^\beta_0\right) \circ \Phi\right) - \left(V^\alpha_0 \Lambda^\rho_{\alpha\beta} V^\beta\right) \circ \Phi)\partial_\rho \\
&= (\partial_0 V^\rho - (V^\alpha \partial_\alpha \gamma^\rho_0) \circ \Phi - (V^\alpha_0 - \gamma^\alpha_0 \circ \Phi)\left(V^\beta \Lambda^\rho_{\alpha\beta} \circ \Phi\right))\partial_\rho \\
&= (\partial_0 V^\rho - (V^\alpha \partial_\alpha \gamma^\rho_0) \circ \Phi - \left(V^\alpha V^\beta \Lambda^\rho_{\alpha\beta} \circ \Phi\right))\partial_\rho
\end{aligned}$$

is met, where it is worth mentioning that we have

$$V^\rho = V^\rho_0 - \gamma^\rho_0 \circ \Phi\ .$$

Only in the case of a trivial space time connection the standard expression follows, because then we have $V^\rho = V^\rho_0 = \partial_0 \Phi^\rho$. Finally we end up with

$$\int_{\mathcal{S}} \partial_0 \rfloor \nabla^\Lambda_\Phi(V) \otimes \rho_{\mathcal{R}} \mathrm{VOL} \wedge \mathrm{d}t^0 = \int_{\mathcal{S}} \left(B^\rho \partial_\rho \otimes \rho_{\mathcal{R}} \mathrm{VOL} + \mathrm{d}^\Phi_\Lambda(P)\right) \wedge \mathrm{d}t^0 \tag{4.38}$$

and since $\mathcal{S}$ is arbitrary it follows

$$\partial_0 \rfloor \nabla^{\Lambda}_{\Phi}(V) \otimes \rho_{\mathcal{R}} \mathrm{VOL} \wedge \mathrm{d}t^0 = \left(B^{\rho}\partial_{\rho} \otimes \rho_{\mathcal{R}}\mathrm{VOL} + \mathrm{d}^{\Phi}_{\Lambda}(P)\right) \wedge \mathrm{d}t^0 \ . \tag{4.39}$$

The right hand side of the equation (4.39) can be expressed as

$$\left(B^{\beta}\rho_{\mathcal{R}} + \partial_i P^{i\beta} - P^{i\beta}\Lambda^{r}_{ir} - P^{i\tau}\Lambda^{\beta}_{\rho\tau}F^{\rho}_{i}\right)\mathrm{VOL} \wedge \mathrm{d}t^0 \otimes \partial_{\beta}$$

where the equation (4.29) was used.

Example 4.17 *The case of a trivial space time connection follows easily again by setting $\gamma^{\rho}_{0} = 0$. Then, the equations of motion read as*

$$\rho_{\mathcal{R}}\left(\partial_0 V^{\rho} - V^{\alpha}V^{\beta}\Lambda^{\rho}_{\alpha\beta}\right) = B^{\rho}\rho_{\mathcal{R}} + \partial_i P^{i\rho} - P^{i\rho}\Lambda^{r}_{ir} - P^{i\tau}\Lambda^{\rho}_{\kappa\tau}F^{\kappa}_{i}$$

with $V^{\rho} = V^{\rho}_{0} = \partial_0\Phi^{\rho}$ which can be found in [Marsden and Hughes, 1994].

Remark 4.18 *From remark 4.14 it is clear that we have*

$$\partial_i P^{i\beta} = d_i \breve{P}^{i\beta}$$

and furthermore

$$\mathrm{d}(P^{i\beta}\partial_i \rfloor \mathrm{VOL}) = j^2(\Phi)^* \left(\mathrm{d}_H(\breve{P}^{i\beta}\partial_i \rfloor \mathrm{VOL})\right)$$

with the horizontal differential d_H, see for example [Giachetta et al., 1997].

4.2.4 The Balance of Energy

To derive an energy relation in the same spirit as in section 4.1.3 we start again with the equation of motion (4.39) and use the metric as a map $g : \mathcal{V}(\mathcal{Q}) \times \mathcal{V}(\mathcal{Q}) \to \mathcal{C}^{\infty}(\mathcal{Q})$. Consequently, we obtain

$$\partial_0 \rfloor \nabla^{\Lambda}_{\Phi}(V) \rfloor V \rfloor g \otimes \rho_{\mathcal{R}}\mathrm{VOL} \wedge \mathrm{d}t^0 = \left((B \rfloor V \rfloor g)\,\rho_{\mathcal{R}}\mathrm{VOL} + \mathrm{d}^{\Phi}_{\Lambda}(P) \rfloor V \rfloor g\right) \wedge \mathrm{d}t^0$$

and from the relation (4.34) given as

$$\mathrm{d}(P \rfloor \bar{q}) = \mathrm{d}^{\Phi}_{\Lambda}(P) \rfloor \bar{q} + \hat{\otimes}\left(S \rfloor \nabla^{\Lambda^*}_{\Phi}(\bar{q})\right)$$

the expression

$$\left(\partial_0 \rfloor \nabla^{\Lambda}_{\Phi}(V) \rfloor V \rfloor g \otimes \rho_{\mathcal{R}}\mathrm{VOL} - (B \rfloor V \rfloor g)\,\rho_{\mathcal{R}}\mathrm{VOL}\right) \wedge \mathrm{d}t^0 = \left(\mathrm{d}(P \rfloor V \rfloor g) - \hat{\otimes}\left(S \rfloor \nabla^{\Lambda^*}_{\Phi}(V \rfloor g)\right)\right) \wedge \mathrm{d}t^0$$

follows. Using the equation (4.35) we get

$$\left(\partial_0 \rfloor \nabla^{\Lambda}_{\Phi}(V) \rfloor V \rfloor g \otimes \rho_{\mathcal{R}}\mathrm{VOL} + \hat{\otimes}\left(S \rfloor \frac{1}{2}(\partial_0 C)\right)\right) \wedge \mathrm{d}t^0 = \left((B \rfloor V \rfloor g)\,\rho_{\mathcal{R}}\mathrm{VOL} + \mathrm{d}(P \rfloor V \rfloor g)\right) \wedge \mathrm{d}t^0$$

which is the analogy to the relation (4.19) in the Eulerian description. The next step is to introduce the stored energy function, see [Marsden and Hughes, 1994]

$$2\rho_{\mathcal{R}}\frac{\partial}{\partial C_{ij}}E_{el} = S^{ij} \ .$$

Then we have

$$\int_{\mathcal{S}} \partial_0 \left(\frac{1}{2} V \rfloor V \rfloor (g \circ \Phi) + E_{el} \right) \rho_{\mathcal{R}} \mathrm{VOL} \wedge \mathrm{d}t^0 = \int_{\mathcal{S}} (B \rfloor V \rfloor g)\, \rho_{\mathcal{R}} \mathrm{VOL} \wedge \mathrm{d}t^0 + \int_{\partial \mathcal{S}} (P \rfloor V \rfloor g) \wedge \mathrm{d}t^0 \,.$$

This comes from the fact that

$$\partial_0 \left(E_{el}\right) \rho_{\mathcal{R}} = \rho_{\mathcal{R}} \frac{\partial}{\partial C_{ij}} E_{el} \left(\partial_0 C_{ij}\right) = S^{ij} \frac{1}{2} \left(\partial_0 C_{ij}\right)$$

and

$$\partial_0 \rfloor \nabla^{\Lambda}_{\Phi}(V) \rfloor V \rfloor g = \partial_0 \left(\frac{1}{2} V \rfloor V \rfloor (g \circ \Phi) \right)$$

which is a similar calculation as in the Appendix A.8.2. With the help of the rate of the deformation tensor whose components are given, as described in remark A.2 of the Appendix A.8.3, as

$$D_{ij} = \frac{1}{2} g_{\alpha\beta} \left(\left(\partial_i V^{\alpha} - V^{\rho} \Lambda^{\alpha}_{\nu\rho} F^{\nu}_i \right) F^{\beta}_j + \left(\partial_j V^{\beta} - V^{\rho} \Lambda^{\beta}_{\rho\mu} F^{\mu}_j \right) F^{\alpha}_i \right)$$

we have the coordinate expression

$$\int_{\mathcal{S}} \partial_0 \left(\frac{1}{2} V \rfloor V \rfloor (g \circ \Phi) + E_{el} \right) \rho_{\mathcal{R}} \mathrm{VOL} \wedge \mathrm{d}t^0 = \int_{\mathcal{S}} \left(\rho_{\mathcal{R}} \frac{1}{2} \partial_0 \left(V^{\alpha} V^{\beta} g_{\alpha\beta} \circ \Phi \right) + S^{ij} D_{ij} \right) \mathrm{VOL} \wedge \mathrm{d}t^0 \,.$$

4.2.5 A Variational Approach

Let us recapitulate the bundle structure which was used in the Lagrangian setting. We obviously considered the bundle $\mathcal{C}_e \to \mathcal{R}$ with coordinates (t^0, X^i, q^α) for $\mathcal{C}_e$ and (t^0, X^i) for $\mathcal{R}$, which is visualized in the following diagram.

$$\begin{tikzcd}
\mathcal{C}_e \arrow[r, equal] \arrow[d] & \mathcal{Q} \times_{\mathcal{B}} \mathcal{R} \arrow[d] & (t^0, X^i, q^\alpha) \arrow[d] \\
\mathcal{R} \arrow[r, equal] & \mathcal{R} & (t^0, X^i)
\end{tikzcd}$$

We investigate only first order Lagrangians and the variational derivative for this setting looks in coordinates as

$$\delta_\alpha = \partial_\alpha - d_i \partial^i_\alpha - d_0 \partial^0_\alpha$$

with

$$d_i = \partial_i + q^\alpha_i \partial_\alpha \,, \quad d_0 = \partial_0 + q^\alpha_0 \partial_\alpha \,.$$

Remark 4.19 *In the following essential use is made of the quantities $\breve{C}, \breve{S}$ and $\breve{P}$ instead of C, S and P. In particular the components of the Cauchy Green tensor are given as $\breve{C}_{ij}\left(q, t^0\right) = q^\alpha_i g_{\alpha\beta} q^\beta_j \left(q, t^0\right)$.*

Let us consider the density of the kinetic energy

$$E_{k_d} = \rho_{\mathcal{R}} \frac{1}{2} (q_0^\alpha - \gamma_0^\alpha) g_{\alpha\beta} (q_0^\beta - \gamma_0^\beta) \mathrm{VOL} \wedge \mathrm{d}t^0$$

with the corresponding kinetic energy function

$$E_k = \frac{1}{2} (q_0^\alpha - \gamma_0^\alpha) g_{\alpha\beta} (q_0^\beta - \gamma_0^\beta)$$

and the stored energy function which meets

$$2\rho_{\mathcal{R}} \frac{\partial}{\partial \breve{C}_{ij}} E_{el} = \breve{S}^{ij} \; .$$

We use the variational principle and compute

$$\delta_\alpha \left(E_k \sqrt{|\det(G_{ij})|} \rho_{\mathcal{R}} - E_{el} \sqrt{|\det(G_{ij})|} \rho_{\mathcal{R}} \right) + \rho_{\mathcal{R}} \sqrt{|\det(G_{ij})|} g_{\alpha\rho} B^\rho = 0 \; . \tag{4.40}$$

The expressions

$$\delta_\alpha \left(\rho_{\mathcal{R}} E_{el} \sqrt{|\det(G_{ij})|} \right) = -\sqrt{|\det(G_{ij})|} \breve{P}^{k\tau} g_{\eta\alpha} \Lambda^{\eta *}_{\tau\beta} q_k^\beta - d_k (\sqrt{|\det(G_{ij})|} \breve{P}^{k\tau}) g_{\alpha\tau}$$

and

$$\begin{aligned} \delta_\alpha \left(\rho_{\mathcal{R}} E_k \sqrt{|\det(G_{ij})|} \right) &= \\ -\rho_{\mathcal{R}} \sqrt{|\det(G_{ij})|} g_{\phi\alpha} &\left[q_{00}^\phi - \partial_0 \left(\gamma_0^\phi \circ \Phi \right) - (q_0^\kappa - \gamma_0^\kappa) \, \partial_\kappa \gamma_0^\phi - \Lambda^\phi_{\beta\sigma} \left(q_0^\beta - \gamma_0^\beta \right) (q_0^\sigma - \gamma_0^\sigma) \right] \end{aligned}$$

as shown in the Appendix A.8.5, consequently lead to

$$\begin{aligned} \rho_{\mathcal{R}} \left[q_{00}^\eta - \partial_0 \left(\gamma_0^\eta \circ \Phi \right) - (q_0^\kappa - \gamma_0^\kappa) \, \partial_\kappa \gamma_0^\eta - \Lambda^\eta_{\beta\sigma} \left(q_0^\beta - \gamma_0^\beta \right) (q_0^\sigma - \gamma_0^\sigma) \right] &= \\ \breve{P}^{k\tau} \Lambda^{\eta *}_{\tau\beta} q_k^\beta + \frac{1}{\sqrt{|\det(G_{ij})|}} d_k (\sqrt{|\det(G_{ij})|} \breve{P}^{k\eta}) + \rho_{\mathcal{R}} B^\eta & \end{aligned}$$

which can be rewritten using the pullback by Φ as

$$\rho_{\mathcal{R}} \partial_0 \rfloor \nabla^\Lambda_\Phi (V) = \rho_{\mathcal{R}} B^\eta + \partial_k P^{k\eta} - P^{k\tau} \Lambda^\eta_{\tau\beta} F_k^\beta - P^{k\eta} \Lambda^j_{kj} \tag{4.41}$$

since

$$j^1 (\Phi)^* \left(q_0^\beta - \gamma_0^\beta \right) = V_0^\beta - \gamma_0^\beta \circ \Phi = \partial_0 \Phi^\beta - \gamma_0^\beta \circ \Phi = V^\beta$$

and this corresponds to the relation (4.39). Therefore, it can be concluded that the equations of motion

$$\delta_\alpha (\mathcal{L}) + \rho_{\mathcal{R}} \sqrt{|\det(G_{ij})|} g_{\alpha\rho} B^\rho = 0$$

with

$$\mathcal{L} = \rho_{\mathcal{R}} \sqrt{|\det(G_{ij})|} \, (E_k - E_{el})$$

are represented as a variational problem in the Lagrangian picture.

Remark 4.20 *This should be compared with section 3.3 where in contrast to* $L = \mathcal{L} \mathrm{d}t^0$ *we are dealing with* $L = \mathcal{L} (\mathrm{d}X^1 \wedge \ldots \wedge \mathrm{d}X^n) \wedge \mathrm{d}t^0$*, which is obvious, since the independent quantities are* (t^0, X^i)*.*

4.2.6 The Hamiltonian Structure

Combining the results of section 3.2.5 and 4.2.5 we are able to give a Hamiltonian interpretation of the governing equations of a continuum in the Lagrangian setting. Let us start to introduce the momentum in the spatial description which can be given as

$$\begin{aligned} p_\alpha &= g_{\alpha\beta}(q_0^\beta - \gamma_0^\beta)\rho_{\mathcal{R}}\sqrt{|\det(G_{ij})|} \\ &= g_{\alpha\beta}v^\beta\rho_{\mathcal{R}}\sqrt{|\det(G_{ij})|}\,. \end{aligned} \tag{4.42}$$

In the material description we obtain

$$\begin{aligned} P_\alpha &= p_\alpha \circ j^1(\Phi) \\ &= (g_{\alpha\beta}\circ\Phi)\left(\partial_0\Phi^\beta - \gamma_0^\beta\circ\Phi\right)\rho_{\mathcal{R}}\sqrt{|\det(G_{ij})|} \\ &= (g_{\alpha\beta}\circ\Phi)\,V^\beta\rho_{\mathcal{R}}\sqrt{|\det(G_{ij})|}\,, \end{aligned}$$

where the symbol of the momentum P should not be confused with the one of the Piola tensor. The total energy is the sum of the kinetic and the stored energy function. It is termed the Hamiltonian and reads as

$$H = \frac{1}{2}\frac{\hat{g}^{\rho\kappa}p_\kappa p_\rho}{\rho_{\mathcal{R}}\sqrt{|\det(G_{ij})|}} + E_{el}\rho_{\mathcal{R}}\sqrt{|\det(G_{ij})|}\,.$$

The equations of motion, as given in equation (4.39), can be written as

$$\begin{aligned} \partial_0\Phi^\beta - \gamma_0^\beta\circ\Phi &= \left(\delta^\beta H\right)\circ j^1(\Phi) \\ \partial_0(P_\beta) + P_\rho(\partial_\beta\gamma_0^\rho)\circ\Phi &= -\left(\delta_\beta H\right)\circ j^1(\Phi) + \sqrt{|\det(G_{ij})|}\rho_{\mathcal{R}}g_{\beta\eta}B^\eta \end{aligned} \tag{4.43}$$

where the balance of mass was used.

Remark 4.21 *In the set of equations (4.43) the variational derivatives read as*

$$\begin{aligned} \delta^\beta &= \dot{\partial}^\beta \\ \delta_\beta &= \partial_\beta - d_i\partial_\beta^i\,, \end{aligned}$$

in addition, a direct computation in coordinates gives

$$\begin{aligned} \left(\delta^\beta H\right)\circ j^1(\Phi) &= \frac{\hat{g}^{\rho\beta}p_\rho}{\rho_{\mathcal{R}}\sqrt{|\det(G_{ij})|}}\circ j^1(\Phi) \\ &= v^\beta\circ j^1(\Phi) \end{aligned}$$

where the relation (4.42) was used. Thus the desired result is obtained by

$$\partial_0\Phi^\beta - \gamma_0^\beta\circ\Phi = \frac{\hat{g}^{\rho\beta}P_\rho}{\rho_{\mathcal{R}}\sqrt{|\det(G_{ij})|}} = V^\beta\,.$$

The expression

$$(\delta_\beta H) \circ j^1(\Phi) = \left(\left(\partial_\beta - d_i \partial^i_\beta\right) H\right) \circ j^1(\Phi)$$

can be decomposed into

$$\begin{aligned}(\delta_\beta H) \circ j^1(\Phi) &= \left(\partial_\beta \frac{1}{2} \frac{\hat{g}^{\rho\kappa} p_\kappa p_\rho}{\rho_{\mathcal{R}} \sqrt{|\det(G_{ij})|}}\right) \circ j^1(\Phi) \\ &\quad + \left(\left(\partial_\beta - d_i \partial^i_\beta\right) E_{el} \rho_{\mathcal{R}} \sqrt{|\det(G_{ij})|}\right) \circ j^1(\Phi) \ .\end{aligned}$$

From section 3.2.5 we have

$$\begin{aligned}\left(\partial_\beta \frac{1}{2} \frac{\hat{g}^{\rho\kappa} p_\kappa p_\rho}{\rho_{\mathcal{R}} \sqrt{|\det(G_{ij})|}}\right) \circ j^1(\Phi) &= \left(\frac{1}{\rho_{\mathcal{R}} \sqrt{|\det(G_{ij})|}} \hat{g}^{\tau\kappa} p_\tau p_\rho \Lambda^\rho_{\kappa\beta}\right) \circ j^1(\Phi) \\ &= \left(v^\kappa p_\rho \Lambda^\rho_{\kappa\beta}\right) \circ j^1(\Phi)\end{aligned}$$

which in material coordinates gives

$$\left(\partial_\beta \frac{1}{2} \frac{\hat{g}^{\rho\kappa} p_\kappa p_\rho}{\rho_{\mathcal{R}} \sqrt{|\det(G_{ij})|}}\right) \circ j^1(\Phi) = V^\kappa P_\rho \Lambda^\rho_{\kappa\beta}$$

and from section 4.2.5 we have

$$\left(\partial_\beta - d_i \partial^i_\beta\right)\left(E_{el} \rho_{\mathcal{R}} \sqrt{|\det(G_{ij})|}\right) = -\sqrt{|\det(G_{ij})|} \breve{P}^{k\tau} g_{\eta\alpha} \Lambda^{\eta *}_{\tau\beta} q^\beta_k - d_k(\sqrt{|\det(G_{ij})|} \breve{P}^{k\tau}) g_{\alpha\tau} \ .$$

Therefore the second equation of (4.43) is

$$\begin{aligned}\partial_0(P_\beta) + P_\rho(\partial_\beta \gamma^\rho_0) + V^\kappa P_\rho \Lambda^\rho_{\kappa\beta} &= \sqrt{|\det(G_{ij})|} \breve{P}^{k\tau} g_{\eta\alpha} \Lambda^{\eta *}_{\tau\beta} q^\beta_k + d_k(\sqrt{|\det(G_{ij})|} \breve{P}^{k\tau}) g_{\alpha\tau} \\ &\quad + \sqrt{|\det(G_{ij})|} \rho_{\mathcal{R}} g_{\beta\eta} B^\eta\end{aligned}$$

where the left hand side is clearly the covariant derivative of the momentum in the Lagrangian description

$$\begin{aligned}\partial_0 \rfloor \nabla^\Lambda_\Phi(P) &= \left(\partial_0 P_\beta + \left(\partial_\beta \gamma^\rho_0 - \Lambda^\rho_{\kappa\beta} \gamma^\kappa_0 + V^\alpha_0 \Lambda^\rho_{\alpha\beta}\right) P_\rho\right)(\mathrm{d}q^\beta - \gamma^\beta_0 \mathrm{d}t^0) \\ &= \left(\partial_0 P_\beta + P_\rho \partial_\beta \gamma^\rho_0 + P_\rho V^\alpha \Lambda^\rho_{\alpha\beta}\right)(\mathrm{d}q^\beta - \gamma^\beta_0 \mathrm{d}t^0) \ ,\end{aligned}$$

which should be compared with the results of section (3.2.5). Then, the equations of motion follow to

$$\left(\partial_0 P_\beta + P_\rho \partial_\beta \gamma^\rho_0 + P_\rho V^\alpha \Lambda^\rho_{\alpha\beta}\right) = \sqrt{|\det(G_{ij})|} g_{\beta\tau}\left(\partial_k P^{k\tau} - P^{k\rho} \Lambda^\tau_{\rho\beta} F^\beta_k - \Lambda^l_{kl} P^{k\tau} + \rho_{\mathcal{R}} B^\tau\right)$$

which are the counterpart to the relations (4.41).

Chapter 5

Time Variant Hamiltonian Systems

Hamiltonian systems are well known in the literature in the finite dimensional as well as in the infinite dimensional case in the context with modeling and control, see for instance [van der Schaft, 2000, Kugi, 2001, Olver, 1986, Schlacher, 2006]. This chapter is devoted to a generalization of these systems for the lumped parameter case, such that they are covariant with respect to the change of the frame of reference. We will use the same mathematical machinery as in the previous chapters, which means that the covariance is achieved by an adequate formulation of the mathematical objects with respect to connections and the appropriate covariant derivatives.

5.1 Geometric Analysis

Let us consider the bundle $\pi : \mathcal{E} \to \mathcal{B}$ where we introduce the coordinates (t^0, x^α) for $\mathcal{E}$ and the coordinate t^0 for the time manifold $\mathcal{B}$. In order to cope with the situation of a control system, we additionally need the vector bundle $\rho : \mathcal{Z} \to \mathcal{E}$ with coordinates $(t^0, x^\alpha, u^\varsigma)$ for $\mathcal{Z}$.

Remark 5.1 *The fact that $\rho : \mathcal{Z} \to \mathcal{E}$ is a vector bundle has the consequence that the following bundle morphism is appropriate*

$$\bar{t}^{\bar{0}} = \phi^{\bar{0}}(t^0) \ , \quad \bar{x}^{\bar{\alpha}} = \varphi^{\bar{\alpha}}(t^0, x^\beta) \ , \quad \bar{u}^{\bar{\varsigma}} = \psi^{\bar{\varsigma}}_{\varsigma}(t^0, x^\beta) u^\varsigma$$

where the fibres of the vector bundle are linear spaces and are denoted by $\mathcal{U}$. The case where $\mathcal{Z} \to \mathcal{E}$ is an affine bundle will be treated with respect to control theory later on in this chapter.

A connection on the bundle $\mathcal{E} \to \mathcal{B}$ follows in coordinates as

$$\mathrm{d}t^0 \otimes (\partial_0 + \Gamma^\alpha_0 \partial_\alpha) \ , \quad \Gamma^\alpha_0 \in \mathcal{C}^\infty(\mathcal{E}) \ .$$

We consider a lumped parameter Hamiltonian control system which is given as

$$(x^\alpha_0 - \Gamma^\alpha_0)\, \partial_\alpha = v^\alpha_H \partial_\alpha \tag{5.1}$$

with

$$v_H^\alpha = \left(J^{\alpha\beta} - R^{\alpha\beta}\right)\partial_\beta H + G_\varsigma^\alpha u^\varsigma$$

and $H,\ G_\varsigma^\alpha,\ J^{\alpha\beta},\ R^{\alpha\beta} \in \mathcal{C}^\infty(\mathcal{E})$. Furthermore

$$\left[J^{\alpha\beta}\right] = -\left[J^{\beta\alpha}\right]\ , \quad \left[R^{\alpha\beta}\right] = \left[R^{\beta\alpha}\right] \geq 0$$

is met, which means that the matrix representation of the operator J is skew symmetric and of the operator R symmetric and positive semi definite.

Remark 5.2 *This special form of the control system (5.1) is considered, since its structure is invariant with respect to the bundle morphism given in remark 5.1 except for a time reparameterization, but this will be discussed in more detail in section 5.1.1.*

Some additional comments on the representation (5.1) are necessary. Let us consider the splitting of the total differential of the Hamiltonian $\mathrm{d}H$. From the relation (2.6) we immediately have

$$\partial_0 H \mathrm{d}t^0 + \partial_\beta H \mathrm{d}x^\beta = \partial_\beta H\left(\mathrm{d}x^\beta - \Gamma_0^\beta \mathrm{d}t^0\right) + \left(\partial_0 H + \Gamma_0^\beta \partial_\beta H\right)\mathrm{d}t^0\ .$$

Since the field $\left(x_0^\alpha - \Gamma_0^\alpha\right)\partial_\alpha$ is a section of $\left(\pi_0^1\right)^* \mathcal{V}(\mathcal{E})$ we have the interpretation that the maps J and R can be expressed as

$$J,\ R : \mathcal{V}^*(\mathcal{E}) \rightarrow \mathcal{V}(\mathcal{E})$$

with

$$J = J^{\alpha\beta}\partial_\alpha \otimes \partial_\beta\ , \quad R = R^{\alpha\beta}\partial_\alpha \otimes \partial_\beta\ .$$

Remark 5.3 *Furthermore*

$$G = G_\varsigma^\alpha e^\varsigma \otimes \partial_\alpha\ , \quad \mathcal{Y} = \mathrm{span}\left\{e^\varsigma\right\}$$

is fulfilled, where $\mathcal{Y}$ is the dual space to the input space $\mathcal{U}$ and meets $\mathcal{Y}^* = \mathcal{U}$.

Let us consider now the right hand side of the relation (5.1). For an input which is a section $\gamma : \mathcal{E} \rightarrow \mathcal{Z}$ we have $v_H : \mathcal{E} \rightarrow \mathcal{V}(\mathcal{E})$ but control theory has to deal with underdetermined systems and this allows the interpretation $v_H : \mathcal{Z} \rightarrow \rho^*(\mathcal{V}(\mathcal{E}))$. Let us consider a map $\gamma : \mathcal{E} \rightarrow \mathcal{Z}$, such that $u^\varsigma = \gamma^\varsigma\left(t^0, x^\beta\right)$, then a solution of the system (5.1) is a section $c : \mathcal{B} \rightarrow \mathcal{E}$ such that we have

$$j^1(c)^*\left(x_0^\alpha - \Gamma_0^\alpha\right) = c^*\left(\left(J^{\alpha\beta} - R^{\alpha\beta}\right)\partial_\beta H + G_\varsigma^\alpha \gamma^\varsigma\right)$$

which is

$$\partial_0 c^\alpha - \Gamma_0^\alpha \circ c = \left(\left(J^{\alpha\beta} - R^{\alpha\beta}\right)\partial_\beta H + G_\varsigma^\alpha \gamma^\varsigma\right) \circ c\ .$$

Furthermore, this suggests

$$\partial_0 \rfloor \nabla^\Gamma(c) = \left(\left(J - R\right)\rfloor \mathrm{d}H + \left(G \rfloor u\right) \circ \gamma\right) \circ c\ , \tag{5.2}$$

which is the intrinsic definition of a Hamiltonian control system.

5.1.1 Transformations

If we apply a bundle morphism of the form

$$\bar{t}^{\bar{0}} = \phi^{\bar{0}}(t^0) = \delta_0^{\bar{0}} t^0 + a \;, \quad \bar{x}^{\bar{\alpha}} = \varphi^{\bar{\alpha}}(t^0, x^\beta) \;, \quad \bar{u}^{\bar{\varsigma}} = \psi_\varsigma^{\bar{\varsigma}}(t^0, x^\beta) u^\varsigma, \quad a \in \mathbb{R}$$

to the system (5.1) which means that we do not consider time reparameterization then the transformed system follows as

$$\bar{x}_{\bar{0}}^{\bar{\alpha}} = \left(\partial_0 \varphi^{\bar{\alpha}} + \partial_\alpha \varphi^{\bar{\alpha}} x_0^\alpha\right) \delta_{\bar{0}}^0$$

as well as

$$\bar{x}_{\bar{0}}^{\bar{\alpha}} = \left(\partial_0 \varphi^{\bar{\alpha}} + \partial_\alpha \varphi^{\bar{\alpha}} \Gamma_0^\alpha\right) \delta_{\bar{0}}^0 + \partial_\alpha \varphi^{\bar{\alpha}} \left(\left(J^{\alpha\beta} - R^{\alpha\beta}\right) \partial_\beta H + G_\varsigma^\alpha \hat{\psi}_{\bar{\varsigma}}^\varsigma \bar{u}^{\bar{\varsigma}} \right) \delta_{\bar{0}}^0 \;.$$

From the relation (2.7) we have

$$\bar{x}_{\bar{0}}^{\bar{\alpha}} - \bar{\Gamma}_{\bar{0}}^{\bar{\alpha}} = \left(\bar{J}^{\bar{\alpha}\bar{\beta}} - \bar{R}^{\bar{\alpha}\bar{\beta}}\right) \partial_{\bar{\beta}} \bar{H} + \bar{G}_{\bar{\varsigma}}^{\bar{\alpha}} \bar{u}^{\bar{\varsigma}} \tag{5.3}$$

where we used

$$\partial_{\bar{\beta}} \left(H \circ \hat{\varphi}\right) \partial_\beta \varphi^{\bar{\beta}} = \left(\partial_\beta H\right) \circ \hat{\varphi} \;.$$

This points out once again the tensorial transformation properties of J, R, G and Γ, since we have

$$\begin{aligned} \bar{J}^{\bar{\alpha}\bar{\beta}} &= \left(\partial_\alpha \varphi^{\bar{\alpha}} J^{\alpha\beta} \partial_\beta \varphi^{\bar{\beta}}\right) \circ \hat{\varphi} \\ \bar{R}^{\bar{\alpha}\bar{\beta}} &= \left(\partial_\alpha \varphi^{\bar{\alpha}} R^{\alpha\beta} \partial_\beta \varphi^{\bar{\beta}}\right) \circ \hat{\varphi} \\ \bar{\Gamma}_{\bar{0}}^{\bar{\alpha}} &= \left(\left(\partial_0 \varphi^{\bar{\alpha}} + \partial_\alpha \varphi^{\bar{\alpha}} \Gamma_0^\alpha\right) \delta_{\bar{0}}^0\right) \circ \hat{\varphi} \end{aligned}$$

as well as

$$\bar{G}_{\bar{\varsigma}}^{\bar{\alpha}} = \left(\partial_\alpha \varphi^{\bar{\alpha}} G_\varsigma^\alpha \hat{\psi}_{\bar{\varsigma}}^\varsigma\right) \circ \hat{\varphi} \;.$$

It is worth to mention that the system (5.3) is a Hamiltonian system that describes the evolution of the system (5.1) with respect to a coordinate system that possesses the connection

$$\mathrm{d}\bar{t}^{\bar{0}} \otimes \left(\partial_{\bar{0}} + \bar{\Gamma}_{\bar{0}}^{\bar{\alpha}} \partial_{\bar{\alpha}}\right) \;.$$

To be more precise, this means for instance, if the system (5.1) is modeled in an inertial system with $\Gamma = \mathrm{d}t^0 \otimes \partial_0$ such that $\Gamma_0^\alpha = 0$, then (5.3) will describe the observed evolution in a Hamiltonian formulation. It is essential to take into account the effect of a non trivial connection, which in this case reads $\bar{\Gamma}_{\bar{0}}^{\bar{\alpha}} = \left(\delta_{\bar{0}}^0 \partial_0 \varphi^{\bar{\alpha}}\right) \circ \hat{\varphi}$. The most general case of a time variant coordinate transformation of course also includes time reparameterization, such that additionally $\bar{t}^{\bar{0}} = \phi^{\bar{0}}(t^0)$ is met. In this case the system (5.1) has to be identified with the tensor

$$\mathrm{d}t^0 \otimes \left(x_0^\alpha - \Gamma_0^\alpha\right) \partial_\alpha = \mathrm{d}t^0 \otimes v_H^\alpha \partial_\alpha \tag{5.4}$$

and it is easily seen that a transformation $\bar{x}^{\bar{\alpha}} = \varphi^{\bar{\alpha}}(t^0, x^\beta)$, $\bar{t}^{\bar{0}} = \phi^{\bar{0}}(t^0)$ preserves this tensor. The left hand side of (5.4) corresponds to the morphism (2.8) which corroborates the use of the covariant derivative in the relation (5.2). In the sequel we only consider the case $\bar{t}^{\bar{0}} = \delta_0^{\bar{0}} t^0$ for simplicity.

5.1.2 Change of the Hamiltonian

The total time change of the Hamiltonian H along the motion c is computed as $\partial_0 (H \circ c)$ and this can be written as

$$(d_0 H) \circ c = (\partial_0 H + x_0^\alpha \partial_\alpha H) \circ c$$

which follows to

$$d_0 H = (\partial_0 + \Gamma_0^\alpha \partial_\alpha) H - (\partial_\alpha H) R^{\alpha\beta} (\partial_\beta H) + \partial_\alpha H G_\varsigma^\alpha u^\varsigma \ . \tag{5.5}$$

The relation (5.5) is well known for the case of $\Gamma_0^\alpha = 0$, which corresponds to an inertial system and the choice of $y_\varsigma = \partial_\alpha H G_\varsigma^\alpha$ shows that the product of input and collocated output affects the power flows of the system. Furthermore, we have the decomposition of the relation (5.5) as

$$d_0 H = w_0^{\mathcal{H}}(H) + v_H (H) \tag{5.6}$$

where $w_0^{\mathcal{H}} = \partial_0 + \Gamma_0^\alpha \partial_\alpha$ corresponds to the horizontal derivative induced by the connection Γ and v_H represents a vertical derivative in this sense of course.

Example 5.4 *Let us consider the system modeled on the trivial bundle $\mathcal{E} = \mathcal{X} \times \mathbb{R} \to \mathbb{R}$, with $H,\ G_\varsigma^\alpha,\ J^{\alpha\beta},\ R^{\alpha\beta} \in \mathcal{C}^\infty(\mathcal{X})$ and*

$$x_0^\alpha = \left(J^{\alpha\beta} - R^{\alpha\beta}\right) \partial_\beta H + G_\varsigma^\alpha u^\varsigma \ .$$

When we apply a bundle morphism of the form

$$\bar{x}^{\bar{\alpha}} = \varphi^{\bar{\alpha}}(t^0, x^\beta) \ , \quad \bar{t}^{\bar{0}} = \delta_0^{\bar{0}} t^0$$

the transformed system reads as

$$\bar{x}_{\bar{0}}^{\bar{\alpha}} - \bar{\Gamma}_{\bar{0}}^{\bar{\alpha}} = \left(\bar{J}^{\bar{\alpha}\bar{\beta}} - \bar{R}^{\bar{\alpha}\bar{\beta}}\right) \partial_{\bar{\beta}} \bar{H} + \bar{G}_\varsigma^{\bar{\alpha}} u^\varsigma \ .$$

The horizontal derivative leads to

$$\begin{aligned} w_{\bar{0}}^{\mathcal{H}} \bar{H} &= \left(\partial_{\bar{0}} + \bar{\Gamma}_{\bar{0}}^{\bar{\alpha}} \partial_{\bar{\alpha}}\right) (H \circ \hat{\varphi}) \\ &= \delta_{\bar{0}}^0 \left(\left(\partial_0 H + \partial_\beta H \partial_0 \hat{\varphi}^\beta\right) \circ \hat{\varphi} + \bar{\Gamma}_{\bar{0}}^{\bar{\alpha}} \partial_{\bar{\alpha}} (H \circ \hat{\varphi})\right) \ . \end{aligned}$$

With the formula (A.5) from the Appendix we obtain

$$\begin{aligned} w_{\bar{0}}^{\mathcal{H}} \bar{H} &= \delta_{\bar{0}}^0 \left(\left(\partial_0 H - \partial_\beta H \partial_0 \varphi^{\bar{\alpha}} \partial_{\bar{\alpha}} \hat{\varphi}^\beta\right) \circ \hat{\varphi} + \bar{\Gamma}_{\bar{0}}^{\bar{\alpha}} \partial_{\bar{\alpha}} (H \circ \hat{\varphi})\right) \\ &= \delta_{\bar{0}}^0 (\partial_0 H) \circ \hat{\varphi} \end{aligned}$$

which shows that the horizontal derivative $w_0^{\mathcal{H}}$ can be interpreted as an intrinsic time derivative which takes into account the case where the connection Γ is not trivial.

Remark 5.5 *It is seen from the previous example that the total change of a function along the motion of the system is, of course, independent of the chosen coordinates, but care must be taken if this result is used for stability arguments as the next section will show.*

5.1.3 Stability Analysis

Stability of nonautonomous systems of the form

$$x_0^\alpha = f^\alpha(t^0, x^\beta) \tag{5.7}$$

is well analyzed in the literature, see for example [H.K. Khalil, 1996] and references therein. In context with systems of the form

$$\left(x_0^\alpha - \Gamma_0^\alpha\right) = \left(J^{\alpha\beta} - R^{\alpha\beta}\right)\partial_\beta H + G_\varsigma^\alpha \gamma^\varsigma$$

as in (5.1) care must be taken, because if the connection Γ is not trivial, these equations are not of the form (5.7) unless the interpretation as a Hamiltonian system is discarded. However, if we choose coordinates such that in these coordinates the connection Γ is trivial, then the coordinates will be called adapted to the frame. In these adapted coordinates the covariant differential corresponds to the classical time derivative and therefore the equilibrium equation is $v_H = 0$, and the stability analysis can be accomplished as for example described in [H.K. Khalil, 1996].

Remark 5.6 *If in the coordinates which are adapted to the frame, which means that $\Gamma_0^\alpha = 0$, the equilibrium does not meet $x^\alpha = 0$ a transformation $\bar{x}^{\bar{\alpha}} = \varphi^{\bar{\alpha}}(x^\beta) = \delta_\alpha^{\bar{\alpha}}\left(x^\alpha - \mathring{c}^\alpha\right)$, $\bar{u}^{\bar{\varsigma}} = \delta_\varsigma^{\bar{\varsigma}}\left(u^\varsigma - \mathring{\gamma}^\varsigma\right)$ can be used which does not necessarily destroy the Hamiltonian structure as will be shown in the forthcoming section 5.2.1.*

If the coordinates are not adapted to the frame, the condition $v_H = 0$ expresses the equilibrium obtained in an adapted set of coordinates formulated in the non inertial frame. Therefore, if one is interested in a stability analysis with respect to the reference frame care must be taken which equilibrium is considered, since the system

$$x_0^\alpha = \Gamma_0^\alpha + \left(J^{\alpha\beta} - R^{\alpha\beta}\right)\partial_\beta H + G_\varsigma^\alpha \gamma^\varsigma \tag{5.8}$$

together with $x_0^\alpha = 0$, describes the equilibrium condition with respect to the frame of reference.

Remark 5.7 *A special case of this construction arises when the stability with respect to a trajectory is in the focus. Roughly speaking the stability analysis of a solution of a system of ordinary differential equations is treated by examining the stability of the origin of a system in transformed coordinates.*

But from (5.8) it is seen that this interpretation does not allow the exploitation of the pleasing properties of a Hamiltonian system. A strategy to overcome this problem will be discussed in section 5.2.2, where the effect of the nontrivial frame of reference will be hidden in a modified Hamiltonian such that the classical stability analysis can be used.

Example 5.8 *It is easily seen that the equilibrium condition $x_0^\alpha = 0$ in an inertial system with trivial connection Γ_0^α is transformed to*

$$\begin{aligned}
\bar{x}_{\bar{0}}^{\bar{\alpha}} &= \left(\partial_0 \varphi^{\bar{\alpha}} + \partial_\alpha \varphi^{\bar{\alpha}} x_0^\alpha\right)\delta_{\bar{0}}^0 \\
&= \partial_0 \varphi^{\bar{\alpha}} \delta_{\bar{0}}^0 \\
&= \bar{\Gamma}_{\bar{0}}^{\bar{\alpha}} .
\end{aligned}$$

Remark 5.9 *It is worth mentioning that the horizontal derivative $w_0^{\mathcal{H}}$ presented in the previous section is important for instance when the analysis of the change of the energy is of importance. However, one has to be aware that the energy is always formulated with respect to a frame as described in section 3.2.6 where the kinetic energy of a mass particle was introduced as*

$$E = \frac{1}{2} g_{\alpha\beta} \left(q_0^\alpha - \gamma_0^\alpha\right) \left(q_0^\beta - \gamma_0^\beta\right) \ .$$

It is obvious that in these coordinates the energy has a minimum for $q_0^\alpha = \gamma_0^\alpha$, whereas if the coordinates are adapted to the frame we have $\gamma_0^\alpha = 0$, which obviously leads to $q_0^\alpha = 0$.

5.2 Control Theoretic Aspects

From the control theoretic point of view it is sometimes beneficial to describe a time invariant Hamiltonian system with respect to an equilibrium point or with respect to so-called error or displacement coordinates.

5.2.1 Equilibrium Points

Let us consider a control system modeled on the trivial bundle $\mathcal{E} = \mathcal{X} \times \mathbb{R} \to \mathbb{R}$ with the trivial connection $\Gamma = \mathrm{d}t^0 \otimes \partial_0$ that reads as

$$x_0^\alpha = \left(J^{\alpha\beta} - R^{\alpha\beta}\right) \partial_\beta H + G_\varsigma^\alpha u^\varsigma \tag{5.9}$$

such that $H,\ G_\varsigma^\alpha,\ J^{\alpha\beta},\ R^{\alpha\beta} \in \mathcal{C}^\infty(\mathcal{X})$ is met. The equilibrium condition for the system (5.9) follows as

$$0 = \left(\left(J^{\alpha\beta} - R^{\alpha\beta}\right) \partial_\beta H + G_\varsigma^\alpha \mathring{\gamma}^\varsigma\right) \circ \mathring{c} \ . \tag{5.10}$$

A change of coordinates of the form

$$\bar{x}^{\bar{\alpha}} = \varphi^{\bar{\alpha}}(x^\beta) = \delta_\alpha^{\bar{\alpha}} \left(x^\alpha - \mathring{c}^\alpha\right) \tag{5.11}$$

where we again do not consider time reparameterization, as well as an input transformation

$$\bar{u}^{\bar{\varsigma}} = \delta_\varsigma^{\bar{\varsigma}} \left(u^\varsigma - \mathring{\gamma}^\varsigma\right) \tag{5.12}$$

lead to a system of the form

$$\bar{x}_{\bar{0}}^{\bar{\alpha}} = \left(\delta_\alpha^{\bar{\alpha}} \left(J^{\alpha\beta} - R^{\alpha\beta}\right) \delta_\beta^{\bar{\beta}} \partial_{\bar{\beta}} \bar{H} + \delta_\alpha^{\bar{\alpha}} G_\varsigma^\alpha \left(\bar{u}^{\bar{\varsigma}} \delta_{\bar{\varsigma}}^\varsigma + \mathring{\gamma}^\varsigma\right)\right) \circ \hat{\varphi} \ .$$

Remark 5.10 *It is worth mentioning that a transformation of the type as in equation (5.12) has the consequence that the bundle $\mathcal{Z} \to \mathcal{E}$ is an affine one.*

For systems that meet $J^{\alpha\beta}, R^{\alpha\beta}, G_\varsigma^\alpha \in \mathbb{R}$ one can take into account the equilibrium relation (5.10) rather easily in order to obtain

$$\begin{aligned}
\bar{x}_{\bar{0}}^{\bar{\alpha}} &= \left(\delta_\alpha^{\bar{\alpha}} \left(J^{\alpha\beta} - R^{\alpha\beta}\right) \delta_\beta^{\bar{\beta}} \partial_{\bar{\beta}} \bar{H} + \delta_\alpha^{\bar{\alpha}} G_\varsigma^\alpha \left(\bar{u}^{\bar{\varsigma}} \delta_{\bar{\varsigma}}^\varsigma\right) - \delta_\alpha^{\bar{\alpha}} \left(\left(J^{\alpha\beta} - R^{\alpha\beta}\right) \partial_\beta H \circ \mathring{c}\right)\right) \circ \hat{\varphi} \\
&= \left(\bar{J}^{\bar{\alpha}\bar{\beta}} - \bar{R}^{\bar{\alpha}\bar{\beta}}\right) \partial_{\bar{\beta}} \hat{H} + \bar{G}_{\bar{\varsigma}}^{\bar{\alpha}} \bar{u}^{\bar{\varsigma}}
\end{aligned}$$

with

$$\begin{aligned}\hat{H} &= \bar{H} - \bar{x}^{\bar{\alpha}} \delta^{\beta}_{\bar{\alpha}} (\partial_\beta H) \circ \mathring{c} \\ \bar{J}^{\bar{\alpha}\bar{\beta}} &= \delta^{\bar{\alpha}}_{\alpha} J^{\alpha\beta} \delta^{\bar{\beta}}_{\beta} \,, \quad \bar{R}^{\bar{\alpha}\bar{\beta}} = \delta^{\bar{\alpha}}_{\alpha} R^{\alpha\beta} \delta^{\bar{\beta}}_{\beta} \,, \quad \bar{G}^{\bar{\alpha}}_{\bar{\varsigma}} = \delta^{\bar{\alpha}}_{\alpha} G^{\alpha}_{\varsigma} \delta^{\varsigma}_{\bar{\varsigma}} \,.\end{aligned}$$

An application can be found in [Schlacher and Kugi, 2002], where also a more detailed discussion of this topic is presented.

Remark 5.11 *For systems that do not have the pleasing property that $J^{\alpha\beta}, R^{\alpha\beta}, G^{\alpha}_{\varsigma} \in \mathbb{R}$ is met, further investigations are necessary.*

Remark 5.12 *It is worth mentioning at this stage that the transformation (5.11) preserves the trivial connection, such that we have $\bar{\Gamma} = \mathrm{d}\bar{t}^{\bar{0}} \otimes \partial_{\bar{0}}$. This is obvious since $\partial_0 \varphi^{\bar{\alpha}} = 0$. A transformation with respect to displacement coordinates changes the connection as the next section will show.*

5.2.2 Reference Trajectories

The analysis of the dynamics of a control system with respect to error or so-called displacement coordinates leads to a time variant Hamiltonian system and fits exactly in this geometric setting as the following shows. Let us consider the system (5.9) and a bundle morphism of the form

$$\bar{x}^{\bar{\alpha}} = \varphi^{\bar{\alpha}}(t^0, x^\beta) = \delta^{\bar{\alpha}}_{\alpha} \left(x^\alpha - c^\alpha_d\right) \tag{5.13}$$

with $c^\alpha_d \in \mathcal{C}^\infty(\mathcal{B})$. By a straightforward calculation we obtain the system

$$\bar{x}^{\bar{\alpha}}_{\bar{0}} + \left(\partial_0 c^\alpha_d\right) \delta^{\bar{\alpha}}_{\alpha} \delta^0_{\bar{0}} = \delta^{\bar{\alpha}}_{\alpha} \left(J^{\alpha\beta} - R^{\alpha\beta}\right) \delta^{\bar{\beta}}_{\beta} \partial_{\bar{\beta}} \bar{H} + \delta^{\bar{\alpha}}_{\alpha} G^{\alpha}_{\varsigma} u^{\varsigma} \tag{5.14}$$

which of course corresponds to the system (5.3) with

$$\bar{\Gamma}^{\bar{\alpha}}_{\bar{0}} = -\left(\partial_0 c^\alpha_d\right) \delta^{\bar{\alpha}}_{\alpha} \delta^0_{\bar{0}} \,, \quad \bar{J}^{\bar{\alpha}\bar{\beta}} = \delta^{\bar{\alpha}}_{\alpha} J^{\alpha\beta} \delta^{\bar{\beta}}_{\beta} \,, \quad \bar{R}^{\bar{\alpha}\bar{\beta}} = \delta^{\bar{\alpha}}_{\alpha} R^{\alpha\beta} \delta^{\bar{\beta}}_{\beta} \,, \quad \bar{G}^{\bar{\alpha}}_{\varsigma} = \delta^{\bar{\alpha}}_{\alpha} G^{\alpha}_{\varsigma} \,.$$

A solution of the partial differential equations

$$\left(\bar{J}^{\bar{\alpha}\bar{\beta}} - \bar{R}^{\bar{\alpha}\bar{\beta}}\right) \partial_{\bar{\beta}} \hat{H} = -\left(\partial_0 c^\alpha_d\right) \delta^{\bar{\alpha}}_{\alpha}$$

allows a different representation of the system (5.14), since then we obtain a Hamiltonian system which describes the evolution of the error coordinates $\bar{x}^{\bar{\alpha}}$ and reads as

$$\bar{x}^{\bar{\alpha}}_{\bar{0}} = \left(\bar{J}^{\bar{\alpha}\bar{\beta}} - \bar{R}^{\bar{\alpha}\bar{\beta}}\right) \partial_{\bar{\beta}} \left(\bar{H} + \hat{H}\right) + \bar{G}^{\bar{\alpha}}_{\varsigma} u^{\varsigma}$$

which corresponds to a system modeled on a trivial bundle with the connection $\mathrm{d}\bar{t}^{\bar{0}} \otimes \partial_{\bar{0}}$. This is obvious, because the effect of the nontrivial connection is now hidden in a modified Hamiltonian. Additionally, if the functions $c^\alpha_d \in \mathcal{C}^\infty(\mathcal{B})$ correspond to a solution of the system such that

$$\partial_0 c^\alpha_d = \left(\left(J^{\alpha\beta} - R^{\alpha\beta}\right) \partial_\beta H\right) \circ c^\alpha_d + \left(G^\alpha_\varsigma \gamma^\varsigma_d \circ c^\alpha_d\right) \tag{5.15}$$

is met, where $\gamma_d^\varsigma : \mathcal{E} \to \mathcal{Z}$ correspond to the input functions then it may be of interest to consider an input transformation as well. Therefore, let us examine the transformation

$$\bar{u}^{\bar{\varsigma}} = \varphi_\varsigma^{\bar{\varsigma}} \left(u^\varsigma - \gamma_d^\varsigma \right) \tag{5.16}$$

with $\varphi_\varsigma^{\bar{\varsigma}} \in C^\infty(\mathcal{X})$, where again $\mathcal{Z} \to \mathcal{E}$ is now understood as an affine bundle. Then the system (5.14) reads as

$$\bar{x}_{\bar{0}}^{\bar{\alpha}} + \left(\partial_0 c_d^\alpha\right) \delta_\alpha^{\bar{\alpha}} \delta_{\bar{0}}^0 = \delta_\alpha^{\bar{\alpha}} \left(J^{\alpha\beta} - R^{\alpha\beta}\right) \delta_\beta^{\bar{\beta}} \partial_{\bar{\beta}} \bar{H} + \delta_\alpha^{\bar{\alpha}} G_\varsigma^\alpha \left(\bar{u}^{\bar{\varsigma}} \hat{\varphi}_{\bar{\varsigma}}^\varsigma + \gamma_d^\varsigma \right)$$

and again a solution of the partial differential equation

$$\left(\bar{J}^{\bar{\alpha}\bar{\beta}} - \bar{R}^{\bar{\alpha}\bar{\beta}}\right) \partial_{\bar{\beta}} \breve{H} = -\left(\partial_0 c_d^\alpha\right) \delta_\alpha^{\bar{\alpha}} + \delta_\alpha^{\bar{\alpha}} G_\varsigma^\alpha \gamma_d^\varsigma \tag{5.17}$$

allows a representation as

$$\bar{x}_{\bar{0}}^{\bar{\alpha}} = \left(\bar{J}^{\bar{\alpha}\bar{\beta}} - \bar{R}^{\bar{\alpha}\bar{\beta}}\right) \partial_{\bar{\beta}} \left(\bar{H} + \breve{H}\right) + \bar{G}_{\bar{\varsigma}}^{\bar{\alpha}} \bar{u}^{\bar{\varsigma}} \tag{5.18}$$

with $\bar{G}_{\bar{\varsigma}}^{\bar{\alpha}} = \delta_\alpha^{\bar{\alpha}} G_\varsigma^\alpha \hat{\varphi}_{\bar{\varsigma}}^\varsigma$. The Hamiltonian system (5.18) describes the evolution of the error coordinates with respect to a desired solution of the system c_d^α. The special case, where the functions G_ς^α are constant and $\gamma_d^\varsigma \in C^\infty(\mathcal{B})$ can be treated easily since then it can be verified that plugging in the equation (5.15) in (5.17) leads to

$$\left(\bar{J}^{\bar{\alpha}\bar{\beta}} - \bar{R}^{\bar{\alpha}\bar{\beta}}\right) \partial_{\bar{\beta}} \breve{H} = -\left(\delta_\alpha^{\bar{\alpha}} \left(J^{\alpha\beta} - R^{\alpha\beta}\right) \partial_\beta H\right) \circ c_d^\alpha$$

which arises in many applications.

5.3 Application

As a simple academic example we consider the magnetically levitated ball as shown in Figure 5.1, where the control problem is the demand to track the position of the ball s. The main focus in this example lies in the fact that we want to show how the time variant coordinate transformation changes the connection and, additionally, how this nontrivial connection can be included in an extended Hamiltonian to obtain the error system in a Hamiltonian formulation modeled on a bundle whose horizontal derivative is again trivial. The system can be described in a straightforward fashion on a bundle $\mathcal{X} \times \mathbb{R} \to \mathbb{R}$ with a trivial connection on $\mathcal{E} = \mathcal{X} \times \mathbb{R}$, which reads as $\Gamma = \mathrm{d}t^0 \otimes \partial_0$. The equations of motion with the momentum p and the flux ψ can be written as a Hamiltonian system with $x = (\psi, s, p)$, the control input u, which is the input voltage and

$$\begin{aligned} \left[J^{\alpha\beta}\right] &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \quad \left[R^{\alpha\beta}\right] = \begin{bmatrix} R_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & d \end{bmatrix} \\ \left[g_\varsigma^\alpha\right] &= \delta_\varsigma^\alpha, \quad H = \frac{p^2}{2m_B} + m_B g s + \frac{\psi^2}{2L(s)}, \end{aligned}$$

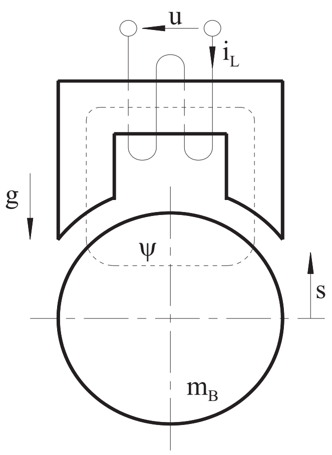

Figure 5.1: Magnetic Levitated Ball

with the inductance $L(s) = k/(1-s)$, the mass of the ball m_B, the resistance in the electrical circuit R_1, and the damping, which takes into account eddy currents and possible mechanical damping in a first order approximation for simplicity, with damping constant d. The theory of flatness based control, see for example [Fliess et al., 1995] now suggests to parameterize the system by means of the flat output $s = s_d$. By an easy calculation one finds that

$$\begin{aligned} \psi_d &= \sqrt{2k\left(m_B\left(g + \partial_{00}s_d\right) + d\partial_0 s_d\right)} \\ p_d &= m_B\partial_0 s_d \end{aligned}$$

and $u_d = u_d(s_d)$, which can be calculated rather easily. A bundle morphism $\mathcal{E} \to \bar{\mathcal{E}}$, $\bar{x}^{\bar{\alpha}} = \varphi^{\bar{\alpha}}(x^\alpha, t^0)$, $\bar{x} = \left(\bar{\psi}, \bar{s}, \bar{p}\right)$ of the type as in the relations (5.13) and (5.16) leads to non trivial connection coefficients

$$\bar{\Gamma}_{\bar{0}}^{\bar{\alpha}}\partial_{\bar{\alpha}} = -k\frac{m_B\partial_{000}s_d + d\partial_{00}s_d}{\sqrt{2k\left(m_B\left(g + \partial_{00}s_d\right) + d\partial_0 s_d\right)}}\partial_{\bar{\psi}} - \partial_0 s_d \partial_{\bar{s}} - m_B\partial_{00}s_d\partial_{\bar{p}}$$

and it is straightforward to verify that a solution of the pde (5.17) is given as

$$\breve{H} = -\left(dH \rfloor \delta_{\bar{\alpha}}^{\beta}\bar{x}^{\bar{\alpha}}\partial_\beta\right) \circ \hat{\varphi}\ ,$$

therefore the error system describing the evolution of $\bar{x}^{\bar{\alpha}}$ in a Hamiltonian description on a bundle with trivial connection $\mathrm{d}\bar{t}^{\bar{0}} \otimes \partial_{\bar{0}}$ is given with

$$\left[\bar{J}^{\bar{\alpha}\bar{\beta}}\right] = \left[J^{\alpha\beta}\right]\ ,\quad \left[\bar{R}^{\bar{\alpha}\bar{\beta}}\right] = \left[R^{\alpha\beta}\right]\ ,\quad \left[\bar{g}_{\bar{\varsigma}}^{\bar{\alpha}}\right] = \left[g_{\varsigma}^{\alpha}\right]$$

and a Hamiltonian $\bar{H} = H \circ \hat{\varphi} + \breve{H}$. The error system can be stabilized for example by means of the IDA-PBC approach, see [Ortega et al., 2002].

Remark 5.13 *This approach allows some flexibility in the design, since it may turn out that not all the states have to be measured because the control is designed in such a way that it does not depend on them. This can avoid the necessity of the velocity measurement for instance in some applications.*

Let us assume there exist constants $c_1, c_2, c_3, c_4 \in \mathbb{R}^+$ such that

$$|s_d| < c_1 \ , \quad |\partial_0 s_d| < c_2 \ , \quad |\partial_{00} s_d| < c_3| \ , \quad \partial_{000} s_d| < c_4$$

and the desired trajectory s_d fulfills $m_B \left(g + \partial_{00} s_d\right) + d\partial_0 s_d > 0$. Then the classical IDA-PBC approach with the desired structure and dissipation matrices

$$\left[\bar{J}_d^{\bar{\alpha}\bar{\beta}}\right] = \begin{bmatrix} 0 & 0 & -\beta \\ 0 & 0 & \alpha \\ \beta & -\alpha & 0 \end{bmatrix} , \quad \left[\bar{R}_d^{\bar{\alpha}\bar{\beta}}\right] = \begin{bmatrix} r_{11} & 0 & -\rho \\ 0 & r_{22} & 0 \\ -\rho & 0 & \alpha d \end{bmatrix}$$

leads to an asymptotically stable closed loop system. It can be verified after some lengthy calculation, which will not be presented here in full details, that locally $W_1(\bar{x}) \geq \bar{H}_d \geq W_2(\bar{x}) > 0$, and $d_{\bar{0}} \bar{H}_d \leq -W_3(\bar{x})$, with $W_3(\bar{x}) > 0$ can be achieved by proper choice of c_1, c_2, c_3, c_4 and a proper adjustment of $\alpha, \beta, r_{11,} r_{22}, \rho > 0$, where $\bar{H}_d$ is a solution of the matching pde's of the IDA-PBC approach which was chosen as

$$\begin{aligned} \bar{H}_d \ = \ & \left(\frac{(\beta+\rho)\,\bar{p}\,r_{22} + \ (\beta+\rho)\,\bar{s}\,\alpha + \alpha^2 \bar{\psi} + \alpha\,\bar{\psi}\,d\,r_{22}}{r_{22}\,(\beta+\rho)} \right)^2 + \frac{\bar{\psi}^3}{6(\beta+\rho)k} \\ & + \frac{1}{2(\beta+\rho)k}\,\bar{\psi}^2 \sqrt{2}\sqrt{k\,(mg + d\,\partial_0 s_d + m\partial_{00} s_d)} \\ & - \frac{1}{(\beta+\rho)^2 m r_{22}^2} \,\left(\bar{s}\,(\beta+\rho) + \ \bar{\psi}\ (\alpha + d\,r_{22})\right) \times \\ & \left(\left(\frac{1}{2}\bar{s}\,\alpha + \ \bar{p}\,r_{22} \right) (\beta+\rho) + \frac{1}{2}\,\bar{\psi}\,\alpha\ (\alpha + d\,r_{22}) \right) \ . \end{aligned}$$

Remark 5.14 *In order to show $W_1(\bar{x}) \geq \bar{H}_d \geq W_2(\bar{x}) > 0$ locally a Taylor series expansion of $\bar{H}_d$ was accomplished and the analysis of the Hessian at the point $\bar{x} = 0$, yields the conditions*

$$\begin{aligned} \frac{\alpha\sqrt{2}\sqrt{k\,(mg + d\,\partial_0 s_d + m\partial_{00} s_d)}\,(2\alpha m - 1)}{(\beta+\rho)kmr_{22}^2} \ &> \ 0 \\ \frac{\sqrt{2}\sqrt{k\,(mg + d\,\partial_0 s_d + m\partial_{00} s_d)}\,(2\alpha m - 1)}{(\beta+\rho)kmr_{22}^2} \ &> \ 0 \\ \frac{\alpha\,(2\alpha m - 1)\,(\alpha + dr_{22})^2}{(\beta+\rho)^2 m r_{22}^2} + \frac{\sqrt{2}\sqrt{k\,(mg + d\,\partial_0 s_d + m\partial_{00} s_d)}}{(\beta+\rho)k} \ &> \ 0 \end{aligned}$$

which are easy to fulfill. The functions W_i ,$i = 1, 2$ follow from the fact that the time variance only arises from the trajectory s_d and its time derivatives and therefore for a given trajectory the bounds c_i , $i = 1 \ldots 4$ are known and can be used to derive W_i locally. A similar approach can be used to show $d_{\bar{0}} \bar{H}_d \leq -W_3(\bar{x})$ but the expressions are very complicated and therefore only a numeric analysis was enforced with the parameter values given in the sequel.

5.3.1 Simulation Results

In Figure 5.2 a simulation result is shown for the parameter values $m_B = 0.5$, $k = 1$, $R_1 = 2$, $d = 0.1$ and $\alpha = 2$, $\beta = 1$, $r_{11} = 10$, $r_{22} = 4$, $\rho = 1$. After 15 seconds an additive disturbance $0.3(\sigma(t-15)-\sigma(t-15.5))$ acts on the input voltage of the system and the reference trajectory was chosen as $s_d(t^0) = 0.5\sin(t^0)$. In the Figure 5.2 u corresponds to the flatnessbased control signal, $\bar{u} = u_c$ is the tracking controller which stabilizes the error system and $e = (e_1, e_{2,}, e_3) = (\bar{\psi}, \bar{s}, \bar{p})$.

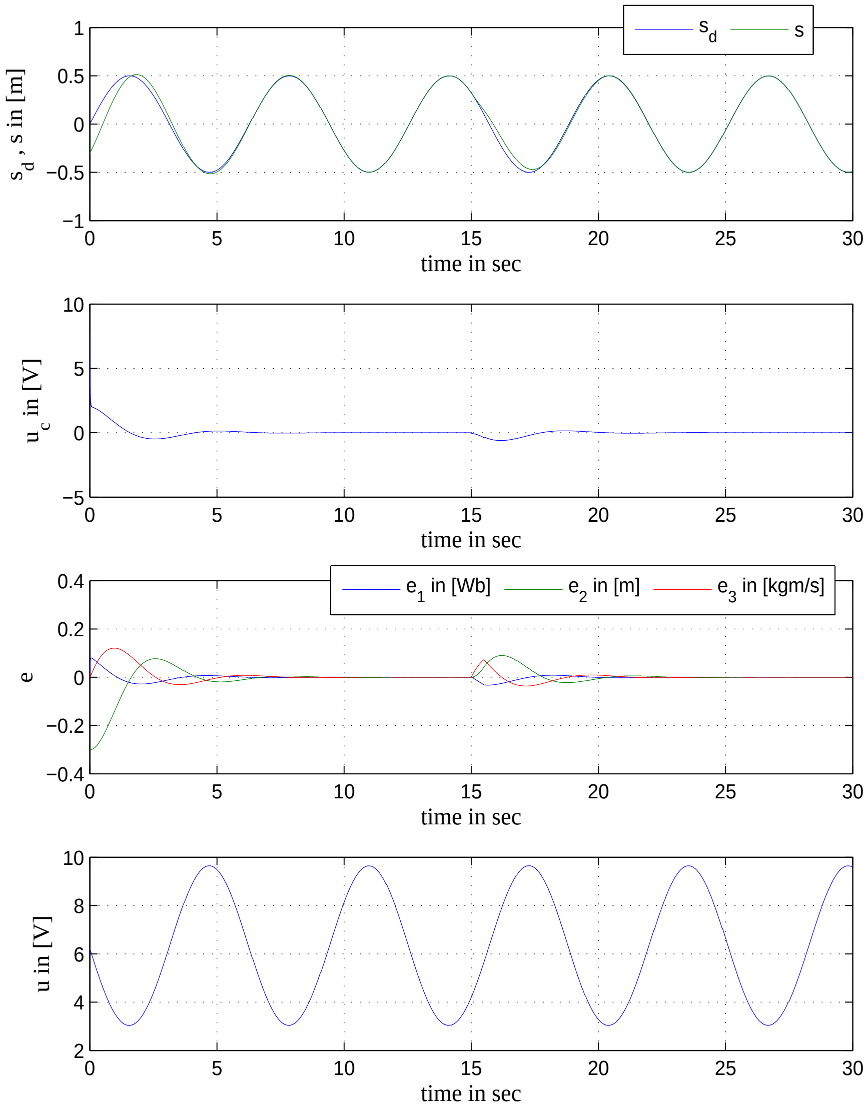

Figure 5.2: Simulation Results

5.4 Hamiltonian Mechanics

Let us recapitulate the observations of section 3.4, where we used the same bundle structures. We have the correspondence that the analogy to the bundle $\mathcal{E} \to \mathcal{B}$ is now $\mathcal{V}^*(\mathcal{Q}) \to \mathcal{B}$. A simple mechanical control system without dissipation in canonical representation reads as

$$\begin{aligned} \partial_0 s^\alpha - \gamma_0^\alpha &= \dot{\partial}^\alpha H && (5.19)\\ \partial_0(p_\beta \circ s) + p_\rho \partial_\beta \gamma_0^\rho &= -\partial_\beta H + G_{\beta\xi} u^\xi && (5.20) \end{aligned}$$

with $s : \mathcal{B} \to \mathcal{Q}$ and $p : \mathcal{Q} \to \mathcal{V}^*(\mathcal{Q})$. The Hamiltonian corresponds to the total energy which is

$$H = \frac{1}{2}\dot{q}_\alpha \hat{m}^{\alpha\beta}\dot{q}_\beta + V$$

where the tensor $m : \mathcal{V}(\mathcal{Q}) \to \mathcal{V}^*(\mathcal{Q})$ is the mass metric. The connection that splits the state bundle is in this case of course not the space-time connection γ, but the Hamilton connection that splits $\mathcal{T}_\mathcal{B}(\mathcal{V}^*(\mathcal{Q}))$ and reads as

$$\Gamma_H = \mathrm{d}t^0 \otimes \left(\partial_0 + \gamma_0^\alpha \partial_\alpha - (\partial_\rho \gamma_0^\beta)\dot{q}_\beta \dot{\partial}^\rho\right) \ , \tag{5.21}$$

where the holonomic base for $\mathcal{T}(\mathcal{V}^*(\mathcal{Q}))$ is given as $(\partial_0, \partial_\alpha, \dot{\partial}^\rho)$. This is readily observed from the relations (5.19) and (5.20) by a comparison with (5.1). The connection coefficients of (5.21) can be also derived easily considering a mechanical system modeled on the trivial bundle $\mathcal{Q} = \mathcal{M} \times \mathbb{R}$, with a connection on $\mathcal{V}^*(\mathcal{Q}) \to \mathcal{B}$ that reads as $\mathrm{d}t^0 \otimes \partial_0$. A change of coordinates of the form

$$\begin{aligned} \bar{q}^{\bar{\alpha}} &= \varphi^{\bar{\alpha}}(t^0, q^\beta)\\ \dot{\bar{q}}_{\bar{\alpha}} &= \dot{q}_\beta \left(\partial_{\bar{\alpha}} \hat{\varphi}^\beta\right) \circ \varphi = \psi_{\bar{\alpha}}\left(t^0, q^\beta, \dot{q}_\beta\right) \end{aligned}$$

leads to the nontrivial connection as stated in (5.21), where we again do not consider time reparameterization. To see this let us evaluate the equation (2.7) which leads to

$$\left(\delta_0^0 \partial_0 \varphi^{\bar{\alpha}}\right) \circ \hat{\varphi} = \bar{\gamma}_{\bar{0}}^{\bar{\alpha}}$$

and from

$$\begin{aligned} \delta_{\bar{0}}^0 \partial_0 \psi_{\bar{\alpha}} &= \delta_{\bar{0}}^0 \dot{q}_\beta \left(\partial_{0\bar{\alpha}} \hat{\varphi}^\beta + \partial_{\bar{\beta}\bar{\alpha}} \hat{\varphi}^\beta \partial_0 \varphi^{\bar{\beta}}\right)\\ &= \delta_{\bar{0}}^0 \dot{q}_\beta \left(\partial_{\bar{\alpha}}(-\partial_0 \varphi^{\bar{\rho}} \partial_{\bar{\rho}} \hat{\varphi}^\beta) + \partial_{\bar{\beta}\bar{\alpha}} \hat{\varphi}^\beta \partial_0 \varphi^{\bar{\beta}}\right)\\ &= -\dot{\bar{q}}_{\bar{\rho}} \left(\partial_{\bar{\alpha}} \bar{\gamma}_{\bar{0}}^{\bar{\rho}}\right) \end{aligned}$$

we obtain exactly the coefficients of the tensor (5.21).

Remark 5.15 *The formula (5.6) should be compared with the expression (3.42) in section 3.4. Due to the previous discussion it is clear that the horizontal part of the derivative $w_0^\mathcal{H}(H)$ in (5.6) corresponds to $v_{H,\mathcal{H}}(H)$ in (3.42) since the connection*

$$\mathrm{d}t^0 \otimes (\partial_0 + \Gamma_0^\alpha \partial_\alpha)$$

that splits $\mathcal{T}(\mathcal{E})$ with respect to $\mathcal{E} \to \mathcal{B}$ is in mechanics represented as

$$\mathrm{d}t^0 \otimes \left(\partial_0 + \gamma_0^\alpha \partial_\alpha - (\partial_\rho \gamma_0^\beta)\dot{q}_\beta \dot{\partial}^\rho\right)$$

and splits $\mathcal{T}(\mathcal{V}^*(\mathcal{Q}))$ with respect to $\mathcal{V}^*(\mathcal{Q}) \to \mathcal{B}$.

Appendix A

Proofs and Detailed Computations

The aim of this part is to present in some detail computations which are used in this thesis.

A.1 Frequently used Relations

Many calculations are based on some tricky indices manipulations and indices shiftings and some of them which were often used are presented in this section. Let us start with a well known formula. Suppose we have two manifolds $\mathcal{M}$ and $\bar{\mathcal{M}}$ with coordinates q^α and $\bar{q}^{\bar{\alpha}}$, respectively and a diffeomorphism

$$\bar{q}^{\bar{\alpha}} = \varphi^{\bar{\alpha}}(q^\beta) \ .$$

From

$$\begin{aligned} \varphi^{\bar{\alpha}}(q^\beta) &= \varphi^{\bar{\alpha}}(\hat{\varphi}^\beta(\bar{q})) = \bar{q}^{\bar{\alpha}} \\ \partial_{\bar{\beta}}\bar{q}^{\bar{\alpha}} &= \delta^{\bar{\alpha}}_{\bar{\beta}} = \left(\partial_\beta \varphi^{\bar{\alpha}}\right)\left(\partial_{\bar{\beta}}\hat{\varphi}^\beta\right) \end{aligned}$$

the result follows immediately to

$$\delta^{\bar{\alpha}}_{\bar{\beta}} = \left(\partial_\beta \varphi^{\bar{\alpha}}\right)\left(\partial_{\bar{\beta}}\hat{\varphi}^\beta\right) \ . \tag{A.1}$$

From equation (A.1) it follows by another differentiation that we have

$$\begin{aligned} \partial_\alpha(\delta^{\bar{\alpha}}_{\bar{\beta}}) &= \partial_\alpha((\partial_\beta \varphi^{\bar{\alpha}})\left(\partial_{\bar{\beta}}\hat{\varphi}^\beta\right)) \\ 0 &= \left(\partial_{\alpha\beta}\varphi^{\bar{\alpha}}\right)\left(\partial_{\bar{\beta}}\hat{\varphi}^\beta\right) + \left(\partial_\beta\varphi^{\bar{\alpha}}\right)\partial_\alpha\left(\partial_{\bar{\beta}}\hat{\varphi}^\beta\right) \\ -\left(\partial_{\alpha\beta}\varphi^{\bar{\alpha}}\right)\left(\partial_{\bar{\beta}}\hat{\varphi}^\beta\right) &= \left(\partial_\beta\varphi^{\bar{\alpha}}\right)\partial_\alpha\left(\partial_{\bar{\beta}}\hat{\varphi}^\beta\right) \\ -\left(\partial_{\alpha\beta}\varphi^{\bar{\alpha}}\right)\left(\partial_{\bar{\beta}}\hat{\varphi}^\beta\right)\left(\partial_{\bar{\rho}}\hat{\varphi}^\alpha\right) &= \left(\partial_{\bar{\rho}}\hat{\varphi}^\alpha\right)\left(\partial_\beta\varphi^{\bar{\alpha}}\right)\left(\partial_{\bar{\nu}\bar{\beta}}\hat{\varphi}^\beta\right)\left(\partial_\alpha\varphi^{\bar{\nu}}\right) \\ \left(\partial_{\alpha\beta}\varphi^{\bar{\alpha}}\right)\left(\partial_{\bar{\beta}}\hat{\varphi}^\beta\right)\left(\partial_{\bar{\rho}}\hat{\varphi}^\alpha\right) &= -\delta^{\bar{\nu}}_{\bar{\rho}}\left(\partial_\beta\varphi^{\bar{\alpha}}\right)\left(\partial_{\bar{\nu}\bar{\beta}}\hat{\varphi}^\beta\right) \end{aligned}$$

and consequently

$$\left(\partial_{\alpha\beta}\varphi^{\bar{\alpha}}\right)\left(\partial_{\bar{\beta}}\hat{\varphi}^\beta\right)\left(\partial_{\bar{\rho}}\hat{\varphi}^\alpha\right) = -\left(\partial_\beta\varphi^{\bar{\alpha}}\right)\left(\partial_{\bar{\rho}\bar{\beta}}\hat{\varphi}^\beta\right) \ . \tag{A.2}$$

The next calculation is essentially used in the case of time variant transformations. Therefore we consider the bundles $\mathcal{Q} \to \mathcal{B}$ and $\bar{\mathcal{Q}} \to \bar{\mathcal{B}}$ with the coordinates (t^0, q^β) and $(\bar{t}^{\bar{0}}, \bar{q}^{\bar{\alpha}})$, respectively. We consider the bundle morphism

$$\begin{aligned} \bar{q}^{\bar{\alpha}} &= \varphi^{\bar{\alpha}}\left(q^\beta, t^0\right) , \quad q^\beta = \hat{\varphi}^\beta(\bar{q}^{\bar{\alpha}}, \bar{t}^{\bar{0}}) \\ \bar{t}^{\bar{0}} &= \phi^{\bar{0}}(t^0) , \quad t^0 = \hat{\phi}^0(\bar{t}^{\bar{0}}) \end{aligned}$$

and successively we obtain

$$\bar{q}_{\bar{0}}^{\bar{\alpha}} = \left(\partial_0 \varphi^{\bar{\alpha}} + \partial_\beta \varphi^{\bar{\alpha}} q_0^\beta\right) \partial_{\bar{0}} \hat{\phi}^0 \tag{A.3}$$

and

$$q_0^\beta = \left(\partial_{\bar{0}} \hat{\varphi}^\beta + \partial_{\bar{\sigma}} \hat{\varphi}^\beta \bar{q}_{\bar{0}}^{\bar{\sigma}}\right) \partial_0 \phi^{\bar{0}} . \tag{A.4}$$

Substituting equation (A.4) in equation (A.3) we get

$$\begin{aligned} \bar{q}_{\bar{0}}^{\bar{\alpha}} &= \left(\partial_0 \varphi^{\bar{\alpha}} + \partial_\beta \varphi^{\bar{\alpha}} \left(\partial_{\bar{0}} \hat{\varphi}^\beta + \partial_{\bar{\sigma}} \hat{\varphi}^\beta \bar{q}_{\bar{0}}^{\bar{\sigma}}\right) \partial_0 \phi^{\bar{0}}\right) \partial_{\bar{0}} \hat{\phi}^0 \\ \bar{q}_{\bar{0}}^{\bar{\alpha}} &= \partial_0 \varphi^{\bar{\alpha}} \partial_{\bar{0}} \hat{\phi}^0 + \partial_\beta \varphi^{\bar{\alpha}} \partial_{\bar{0}} \hat{\varphi}^\beta + \delta_{\bar{\sigma}}^{\bar{\alpha}} \bar{q}_{\bar{0}}^{\bar{\sigma}} . \end{aligned}$$

Therefore the result follows to

$$\begin{aligned} \partial_\beta \varphi^{\bar{\alpha}} \partial_{\bar{0}} \hat{\varphi}^\beta &= -\partial_0 \varphi^{\bar{\alpha}} \partial_{\bar{0}} \hat{\phi}^0 \\ \partial_{\bar{0}} \hat{\varphi}^\beta &= -\partial_0 \varphi^{\bar{\alpha}} \partial_{\bar{0}} \hat{\phi}^0 \partial_{\bar{\alpha}} \hat{\varphi}^\beta . \end{aligned} \tag{A.5}$$

A.2 Transition Functions

A.2.1 The Connection Λ

Let us consider the connection Λ on $\mathcal{V}(\mathcal{Q}) \to \mathcal{Q}$ where we have coordinates $(t^0, q^\alpha, \dot{q}^\alpha)$ for $\mathcal{V}(\mathcal{Q})$ and the holonomic base $(\partial_0, \partial_\alpha, \dot{\partial}_\alpha)$ for $\mathcal{T}(\mathcal{V}(\mathcal{Q}))$ which reads as

$$\Lambda = \mathrm{d}t^0 \otimes \left(\partial_0 + \Lambda_0^\rho \dot{\partial}_\rho\right) + \mathrm{d}q^\alpha \otimes \left(\partial_\alpha + \Lambda_\alpha^\rho \dot{\partial}_\rho\right) \tag{A.6}$$

as it appears in section 3.2 and consider a change of coordinates of the form

$$\bar{t}^{\bar{0}} = \phi^{\bar{0}}(t^0) , \quad \bar{q}^{\bar{\alpha}} = \varphi^{\bar{\alpha}}(q^\beta, t^0) , \quad \dot{\bar{q}}^{\bar{\alpha}} = \partial_\beta \varphi^{\bar{\alpha}} \dot{q}^\beta .$$

Successively we obtain

$$\begin{aligned} \check{\bar{t}}^{\bar{0}} &= \partial_0 \phi^{\bar{0}} \check{t}^0 \\ \check{\bar{q}}^{\bar{\alpha}} &= \partial_0 \varphi^{\bar{\alpha}} \check{t}^0 + \partial_\beta \varphi^{\bar{\alpha}} \check{q}^\beta \\ \check{\dot{\bar{q}}}^{\bar{\alpha}} &= \partial_{0\beta} \varphi^{\bar{\alpha}} \dot{q}^\beta \check{t}^0 + \partial_{\alpha\beta} \varphi^{\bar{\alpha}} \dot{q}^\beta \check{q}^\alpha + \partial_\beta \varphi^{\bar{\alpha}} \check{\dot{q}}^\beta \end{aligned}$$

and consequently

$$\begin{aligned} \mathrm{d}\bar{t}^{\bar{0}} &= \partial_0 \phi^{\bar{0}} \mathrm{d}t^0 , \quad \mathrm{d}t^0 = \partial_{\bar{0}} \hat{\phi}^0 \mathrm{d}\bar{t}^{\bar{0}} \\ \mathrm{d}\bar{q}^{\bar{\alpha}} &= \partial_0 \varphi^{\bar{\alpha}} \mathrm{d}t^0 + \partial_\beta \varphi^{\bar{\alpha}} \mathrm{d}q^\beta , \quad \mathrm{d}q^\alpha = \partial_{\bar{\alpha}} \hat{\varphi}^\alpha \left(\mathrm{d}\bar{q}^{\bar{\alpha}} - \partial_0 \varphi^{\bar{\alpha}} \partial_{\bar{0}} \hat{\phi}^0 \mathrm{d}\bar{t}^{\bar{0}}\right) \end{aligned} \tag{A.7}$$

as well as

$$\begin{array}{rcl}\partial_0 & \rightarrow & \partial_0\phi^{\bar{0}}\partial_{\bar{0}} + \partial_0\varphi^{\bar{\alpha}}\partial_{\bar{\alpha}} + \partial_{0\beta}\varphi^{\bar{\rho}}\dot{q}^{\beta}\dot{\partial}_{\bar{\rho}} \\ \partial_\alpha & \rightarrow & \partial_\alpha\varphi^{\bar{\alpha}}\partial_{\bar{\alpha}} + \partial_{\alpha\tau}\varphi^{\bar{\rho}}\dot{q}^{\tau}\dot{\partial}_{\bar{\rho}} \\ \Lambda_0^{\beta}\dot{\partial}_\beta & \rightarrow & \partial_\beta\varphi^{\bar{\rho}}\Lambda_0^{\beta}\dot{\partial}_{\bar{\rho}} \\ \Lambda_\alpha^{\rho}\dot{\partial}_\rho & \rightarrow & \partial_\rho\varphi^{\bar{\rho}}\Lambda_\alpha^{\rho}\dot{\partial}_{\bar{\rho}}\end{array} \tag{A.8}$$

is met. Plugging in the expressions (A.7) and (A.8) into (A.6) we obtain

$$\begin{aligned}&\partial_{\bar{0}}\hat{\phi}^{0}\mathrm{d}\bar{t}^{\bar{0}} \otimes \left(\partial_0\phi^{\bar{0}}\partial_{\bar{0}} + \partial_0\varphi^{\bar{\alpha}}\partial_{\bar{\alpha}} + \partial_{0\beta}\varphi^{\bar{\rho}}\dot{q}^{\beta}\dot{\partial}_{\bar{\rho}} + \partial_\beta\varphi^{\bar{\rho}}\Lambda_0^{\beta}\dot{\partial}_{\bar{\rho}}\right) \\ &+\partial_{\bar{\alpha}}\hat{\varphi}^{\alpha}\left(\mathrm{d}\bar{q}^{\bar{\alpha}} - \partial_0\varphi^{\bar{\alpha}}\partial_{\bar{0}}\hat{\phi}^{0}\mathrm{d}\bar{t}^{\bar{0}}\right) \otimes \left(\partial_\alpha\varphi^{\bar{\kappa}}\partial_{\bar{\kappa}} + \partial_{\alpha\tau}\varphi^{\bar{\rho}}\dot{q}^{\tau}\dot{\partial}_{\bar{\rho}} + \partial_\rho\varphi^{\bar{\rho}}\Lambda_\alpha^{\rho}\dot{\partial}_{\bar{\rho}}\right) \;.\end{aligned}$$

Rearranging the expressions we have

$$\begin{aligned}&\mathrm{d}\bar{t}^{\bar{0}} \otimes \left(\partial_{\bar{0}} + \partial_{\bar{0}}\hat{\phi}^{0}\partial_0\varphi^{\bar{\alpha}}\partial_{\bar{\alpha}} + \partial_{\bar{0}}\hat{\phi}^{0}\left(\partial_{0\beta}\varphi^{\bar{\rho}}\dot{q}^{\beta} + \partial_\beta\varphi^{\bar{\rho}}\Lambda_0^{\beta}\right)\dot{\partial}_{\bar{\rho}}\right) \\ &-\mathrm{d}\bar{t}^{\bar{0}} \otimes \left(\partial_0\varphi^{\bar{\alpha}}\partial_{\bar{0}}\hat{\phi}^{0}\partial_{\bar{\alpha}} + \partial_{\bar{\alpha}}\hat{\varphi}^{\alpha}\partial_0\varphi^{\bar{\alpha}}\partial_{\bar{0}}\hat{\phi}^{0}\left(\partial_{\alpha\tau}\varphi^{\bar{\rho}}\dot{q}^{\tau} + \partial_\rho\varphi^{\bar{\rho}}\Lambda_\alpha^{\rho}\right)\dot{\partial}_{\bar{\rho}}\right) \\ &+\mathrm{d}\bar{q}^{\bar{\alpha}} \otimes \left(\partial_{\bar{\alpha}} + \partial_{\bar{\alpha}}\hat{\varphi}^{\alpha}(\partial_{\alpha\tau}\varphi^{\bar{\rho}}\dot{q}^{\tau} + \partial_\rho\varphi^{\bar{\rho}}\Lambda_\alpha^{\rho})\dot{\partial}_{\bar{\rho}}\right)\end{aligned}$$

and finally

$$\begin{aligned}&\mathrm{d}\bar{t}^{\bar{0}} \otimes \left(\partial_{\bar{0}} + \left(\partial_{\bar{0}}\hat{\phi}^{0}\partial_{0\beta}\varphi^{\bar{\rho}}\dot{q}^{\beta} + \partial_{\bar{0}}\hat{\phi}^{0}\partial_\beta\varphi^{\bar{\rho}}\Lambda_0^{\beta} - \partial_{\bar{\alpha}}\hat{\varphi}^{\alpha}\partial_0\varphi^{\bar{\alpha}}\partial_{\bar{0}}\hat{\phi}^{0}\left(\partial_{\alpha\tau}\varphi^{\bar{\rho}}\dot{q}^{\tau} + \partial_\rho\varphi^{\bar{\rho}}\Lambda_\alpha^{\rho}\right)\right)\dot{\partial}_{\bar{\rho}}\right) \\ &+\mathrm{d}\bar{q}^{\bar{\alpha}} \otimes \left(\partial_{\bar{\alpha}} + \partial_{\bar{\alpha}}\hat{\varphi}^{\alpha}(\partial_{\alpha\tau}\varphi^{\bar{\rho}}\dot{q}^{\tau} + \partial_\rho\varphi^{\bar{\rho}}\Lambda_\alpha^{\rho})\dot{\partial}_{\bar{\rho}}\right) \;.\end{aligned}$$

This result can be written

$$\bar{\Lambda} = \mathrm{d}\bar{t}^{\bar{0}} \otimes \left(\partial_{\bar{0}} + \bar{\Lambda}_{\bar{0}}^{\bar{\rho}}\dot{\partial}_{\bar{\rho}}\right) + \mathrm{d}\bar{q}^{\bar{\alpha}} \otimes \left(\partial_{\bar{\alpha}} + \bar{\Lambda}_{\bar{\alpha}}^{\bar{\rho}}\dot{\partial}_{\bar{\rho}}\right)$$

with

$$\begin{array}{rcl}\bar{\Lambda}_{\bar{\alpha}}^{\bar{\rho}} & = & \partial_{\bar{\alpha}}\hat{\varphi}^{\alpha}(\partial_{\alpha\tau}\varphi^{\bar{\rho}}\dot{q}^{\tau} + \partial_\rho\varphi^{\bar{\rho}}\Lambda_\alpha^{\rho}) \\ \bar{\Lambda}_{\bar{0}}^{\bar{\rho}} & = & \partial_{\bar{0}}\hat{\phi}^{0}(\partial_{0\beta}\varphi^{\bar{\rho}}\dot{q}^{\beta} + \partial_\beta\varphi^{\bar{\rho}}\Lambda_0^{\beta} - \partial_0\varphi^{\bar{\alpha}}\bar{\Lambda}_{\bar{\alpha}}^{\bar{\rho}}) \;.\end{array}$$

A.2.2 The Connection Λ^*

Let us consider the connection Λ^* on $\mathcal{V}^*(\mathcal{Q}) \rightarrow \mathcal{Q}$ where we have coordinates $(t^0, q^\alpha, \dot{q}_\alpha)$ for $\mathcal{V}^*(\mathcal{Q})$ and the holonomic base $(\partial_0, \partial_\alpha, \dot{\partial}^\alpha)$ for $\mathcal{T}(\mathcal{V}^*(\mathcal{Q}))$ which reads as

$$\Lambda^* = \mathrm{d}t^0 \otimes \left(\partial_0 + \Lambda^*_{0\rho}\dot{\partial}^{\rho}\right) + \mathrm{d}q^\alpha \otimes \left(\partial_\alpha + \Lambda^*_{\alpha\rho}\dot{\partial}^{\rho}\right) \tag{A.9}$$

as it appears in section 3.2 and consider a change of coordinates of the form

$$\bar{t}^{\bar{0}} = \phi^{\bar{0}}(t^0) \;, \quad \bar{q}^{\bar{\alpha}} = \varphi^{\bar{\alpha}}(q^\beta, t^0) \;, \quad \dot{\bar{q}}_{\bar{\alpha}} = \partial_{\bar{\alpha}}\hat{\varphi}^{\beta}\dot{q}_\beta \;.$$

We obtain successively

$$\begin{aligned}\check{\bar{t}}^{\bar{0}} &= \partial_0\phi^{\bar{0}}\check{t}^0\\ \check{\bar{q}}^{\bar{\alpha}} &= \partial_0\varphi^{\bar{\alpha}}\check{t}^0+\partial_\beta\varphi^{\bar{\alpha}}\check{q}^\beta\\ \check{\dot{\bar{q}}}_{\bar{\alpha}} &= \partial_0\left((\partial_{\bar{\alpha}}\hat{\varphi}^\beta)\circ\varphi\circ\phi\right)\dot{q}_\beta\check{t}^0+\partial_{\bar{\beta}\bar{\alpha}}\hat{\varphi}^\beta\dot{q}_\beta\partial_\alpha\varphi^{\bar{\beta}}\check{q}^\alpha+\partial_{\bar{\alpha}}\hat{\varphi}^\beta\check{\dot{q}}_\beta\end{aligned}$$

and consequently

$$\begin{aligned}\mathrm{d}\bar{t}^{\bar{0}} &= \partial_0\phi^{\bar{0}}\mathrm{d}t^0\;,\quad \mathrm{d}t^0=\partial_{\bar{0}}\hat{\phi}^0\mathrm{d}\bar{t}^{\bar{0}}\\ \mathrm{d}\bar{q}^{\bar{\alpha}} &= \partial_0\varphi^{\bar{\alpha}}\mathrm{d}t^0+\partial_\beta\varphi^{\bar{\alpha}}\mathrm{d}q^\beta\;,\quad \mathrm{d}q^\alpha=\partial_{\bar{\alpha}}\hat{\varphi}^\alpha\left(\mathrm{d}\bar{q}^{\bar{\alpha}}-\partial_0\varphi^{\bar{\alpha}}\partial_{\bar{0}}\hat{\phi}^0\mathrm{d}\bar{t}^{\bar{0}}\right)\end{aligned}\tag{A.10}$$

as well as

$$\begin{aligned}\partial_0 &\to \partial_0\phi^{\bar{0}}\partial_{\bar{0}}+\partial_0\varphi^{\bar{\alpha}}\partial_{\bar{\alpha}}+\partial_0\left((\partial_{\bar{\rho}}\hat{\varphi}^\beta)\circ\varphi\circ\phi\right)\dot{q}_\beta\dot{\partial}^{\bar{\rho}}\\ &\to \partial_0\phi^{\bar{0}}\partial_{\bar{0}}+\partial_0\varphi^{\bar{\alpha}}\partial_{\bar{\alpha}}+\partial_{\bar{0}\bar{\rho}}\hat{\varphi}^\beta\partial_0\phi^{\bar{0}}\dot{q}_\beta\dot{\partial}^{\bar{\rho}}+\partial_{\bar{\tau}\bar{\rho}}\hat{\varphi}^\beta\partial_0\varphi^{\bar{\tau}}\dot{q}_\beta\dot{\partial}^{\bar{\rho}}\\ \partial_\alpha &\to \partial_\alpha\varphi^{\bar{\alpha}}\partial_{\bar{\alpha}}+\partial_{\bar{\beta}\bar{\rho}}\hat{\varphi}^\beta\dot{q}_\beta\partial_\alpha\varphi^{\bar{\beta}}\dot{\partial}^{\bar{\rho}}\\ \Lambda^*_{0\beta}\dot{\partial}^\beta &\to \partial_{\bar{\rho}}\hat{\varphi}^\beta\Lambda^*_{0\beta}\dot{\partial}^{\bar{\rho}}\\ \Lambda^*_{\alpha\rho}\dot{\partial}^\rho &\to \partial_{\bar{\rho}}\hat{\varphi}^\beta\Lambda^*_{\alpha\beta}\dot{\partial}^{\bar{\rho}}\end{aligned}\tag{A.11}$$

is met. Plugging in the expressions (A.10) and (A.11) into (A.9) we obtain

$$\begin{aligned}&\partial_{\bar{0}}\hat{\phi}^0\mathrm{d}\bar{t}^{\bar{0}}\otimes\left(\partial_0\phi^{\bar{0}}\partial_{\bar{0}}+\partial_0\varphi^{\bar{\alpha}}\partial_{\bar{\alpha}}+\partial_{\bar{0}\bar{\rho}}\hat{\varphi}^\beta\partial_0\phi^{\bar{0}}\dot{q}_\beta\dot{\partial}^{\bar{\rho}}+\partial_{\bar{\tau}\bar{\rho}}\hat{\varphi}^\beta\partial_0\varphi^{\bar{\tau}}\dot{q}_\beta\dot{\partial}^{\bar{\rho}}+\partial_{\bar{\rho}}\hat{\varphi}^\beta\Lambda^*_{0\beta}\dot{\partial}^{\bar{\rho}}\right)\\ &+\partial_{\bar{\alpha}}\hat{\varphi}^\alpha\left(\mathrm{d}\bar{q}^{\bar{\alpha}}-\partial_0\varphi^{\bar{\alpha}}\partial_{\bar{0}}\hat{\phi}^0\mathrm{d}\bar{t}^{\bar{0}}\right)\otimes\left(\partial_\alpha\varphi^{\bar{\kappa}}\partial_{\bar{\kappa}}+\partial_{\bar{\beta}\bar{\rho}}\hat{\varphi}^\beta\dot{q}_\beta\partial_\alpha\varphi^{\bar{\beta}}\dot{\partial}^{\bar{\rho}}+\partial_{\bar{\rho}}\hat{\varphi}^\beta\Lambda^*_{\alpha\beta}\dot{\partial}^{\bar{\rho}}\right)\;.\end{aligned}$$

Rearranging the expressions we have

$$\begin{aligned}&\mathrm{d}\bar{t}^{\bar{0}}\otimes\left(\partial_{\bar{0}}+\partial_{\bar{0}}\hat{\phi}^0\partial_0\varphi^{\bar{\alpha}}\partial_{\bar{\alpha}}+\partial_{\bar{0}}\hat{\phi}^0\left(\partial_{\bar{0}\bar{\rho}}\hat{\varphi}^\beta\partial_0\phi^{\bar{0}}\dot{q}_\beta+\partial_{\bar{\tau}\bar{\rho}}\hat{\varphi}^\beta\partial_0\varphi^{\bar{\tau}}\dot{q}_\beta+\partial_{\bar{\rho}}\hat{\varphi}^\beta\Lambda^*_{0\beta}\right)\dot{\partial}^{\bar{\rho}}\right)\\ &-\mathrm{d}\bar{t}^{\bar{0}}\otimes\left(\partial_0\varphi^{\bar{\alpha}}\partial_{\bar{0}}\hat{\phi}^0\partial_{\bar{\alpha}}+\partial_{\bar{\alpha}}\hat{\varphi}^\alpha\partial_0\varphi^{\bar{\alpha}}\partial_{\bar{0}}\hat{\phi}^0(\partial_{\bar{\beta}\bar{\rho}}\hat{\varphi}^\beta\dot{q}_\beta\partial_\alpha\varphi^{\bar{\beta}}+\partial_{\bar{\rho}}\hat{\varphi}^\beta\Lambda^*_{\alpha\beta})\dot{\partial}^{\bar{\rho}}\right)\\ &+\mathrm{d}\bar{q}^{\bar{\alpha}}\otimes\left(\partial_{\bar{\alpha}}+\partial_{\bar{\alpha}}\hat{\varphi}^\alpha(\partial_{\bar{\beta}\bar{\rho}}\hat{\varphi}^\beta\dot{q}_\beta\partial_\alpha\varphi^{\bar{\beta}}+\partial_{\bar{\rho}}\hat{\varphi}^\beta\Lambda^*_{\alpha\beta})\dot{\partial}^{\bar{\rho}}\right)\end{aligned}$$

and finally

$$\begin{aligned}&\mathrm{d}\bar{t}^{\bar{0}}\otimes\left(\partial_{\bar{0}}+\left(\partial_{\bar{0}}\hat{\phi}^0\partial_{\bar{0}\bar{\rho}}\hat{\varphi}^\beta\partial_0\phi^{\bar{0}}\dot{q}_\beta\dot{\partial}^{\bar{\rho}}+\partial_{\bar{0}}\hat{\phi}^0\partial_{\bar{\tau}\bar{\rho}}\hat{\varphi}^\beta\partial_0\varphi^{\bar{\tau}}\dot{q}_\beta\dot{\partial}^{\bar{\rho}}+\partial_{\bar{0}}\hat{\phi}^0\partial_{\bar{\rho}}\hat{\varphi}^\beta\Lambda^*_{0\beta}\right)\dot{\partial}^{\bar{\rho}}\right)\\ &-\mathrm{d}\bar{t}^{\bar{0}}\otimes\left(\partial_{\bar{\alpha}}\hat{\varphi}^\alpha\partial_0\varphi^{\bar{\alpha}}\partial_{\bar{0}}\hat{\phi}^0(\partial_{\bar{\beta}\bar{\rho}}\hat{\varphi}^\beta\dot{q}_\beta\partial_\alpha\varphi^{\bar{\beta}}+\partial_{\bar{\rho}}\hat{\varphi}^\beta\Lambda^*_{\alpha\beta})\dot{\partial}^{\bar{\rho}}\right)\\ &+\mathrm{d}\bar{q}^{\bar{\alpha}}\otimes\left(\partial_{\bar{\alpha}}+\partial_{\bar{\alpha}}\hat{\varphi}^\alpha(\partial_{\bar{\beta}\bar{\rho}}\hat{\varphi}^\beta\dot{q}_\beta\partial_\alpha\varphi^{\bar{\beta}}+\partial_{\bar{\rho}}\hat{\varphi}^\beta\Lambda^*_{\alpha\beta})\dot{\partial}^{\bar{\rho}}\right)\;.\end{aligned}$$

This result can be written

$$\bar{\Lambda}^*=\mathrm{d}\bar{t}^{\bar{0}}\otimes\left(\partial_{\bar{0}}+\bar{\Lambda}^*_{\bar{0}\bar{\rho}}\dot{\partial}^{\bar{\rho}}\right)+\mathrm{d}\bar{q}^{\bar{\alpha}}\otimes\left(\partial_{\bar{\alpha}}+\bar{\Lambda}^*_{\bar{\alpha}\bar{\rho}}\dot{\partial}^{\bar{\rho}}\right)$$

with

$$\begin{aligned}\bar{\Lambda}^*_{\bar{\alpha}\bar{\rho}} &= \partial_{\bar{\alpha}}\hat{\varphi}^\alpha(\partial_{\bar{\beta}\bar{\rho}}\hat{\varphi}^\beta\dot{q}_\beta\partial_\alpha\varphi^{\bar{\beta}}+\partial_{\bar{\rho}}\hat{\varphi}^\beta\Lambda^*_{\alpha\beta})\\ \bar{\Lambda}^*_{\bar{0}\bar{\rho}} &= \partial_{\bar{0}\bar{\rho}}\hat{\varphi}^\beta\dot{q}_\beta+\partial_{\bar{0}}\hat{\phi}^0\left(\partial_{\bar{\tau}\bar{\rho}}\hat{\varphi}^\beta\partial_0\varphi^{\bar{\tau}}\dot{q}_\beta+\partial_{\bar{\rho}}\hat{\varphi}^\beta\Lambda^*_{0\beta}-\partial_0\varphi^{\bar{\alpha}}\bar{\Lambda}^*_{\bar{\alpha}\bar{\rho}}\right)\;.\end{aligned}$$

A.3 Christoffel Symbols Relation

We want to proof the equation (3.18) and start with

$$\begin{aligned}\hat{g}^{\alpha\varepsilon}\left(\partial_0 g_{\alpha\varepsilon}\right) &= -\hat{g}^{\alpha\varepsilon} g_{\kappa\varepsilon}\Lambda^{\kappa}_{0\alpha} - \hat{g}^{\alpha\varepsilon} g_{\alpha\beta}\Lambda^{\beta}_{0\varepsilon} \\ \hat{g}^{\alpha\varepsilon}\left(\partial_0 g_{\alpha\varepsilon}\right) &= -2\Lambda^{\kappa}_{0\kappa}\end{aligned}$$

as well as

$$\begin{aligned}\hat{g}^{\alpha\varepsilon}\left(\partial_0 g_{\alpha\varepsilon}\right) &= -2\Lambda^{\kappa}_{0\kappa} \\ &= \frac{\partial_0\left(\det(g_{\alpha\beta})\right)}{\det(g_{\alpha\beta})}\ ,\end{aligned}$$

where we used the well known formula

$$\partial_x(\det Y) = \det(Y)\cdot tr(Y^{-1}\partial_x Y)$$

for an invertible matrix Y. Furthermore, from

$$\partial_0\sqrt{|\det(g_{\alpha\beta})|} = \frac{\partial_0\left(|\det(g_{\alpha\beta})|\right)}{2\sqrt{|\det(g_{\alpha\beta})|}}$$

we have

$$\frac{\partial_0\left(\det(g_{\alpha\beta})\right)}{\det(g_{\alpha\beta})} = -2\Lambda^{\kappa}_{0\kappa}$$

and

$$\begin{aligned}\frac{\partial_0\sqrt{|\det(g_{\alpha\beta})|}\sqrt{|\det(g_{\alpha\beta})|}}{\det(g_{\alpha\beta})} &= -\Lambda^{\kappa}_{0\kappa} \\ \frac{1}{\sqrt{|\det(g_{\alpha\beta})|}}\partial_0\sqrt{|\det(g_{\alpha\beta})|} &= -\Lambda^{\kappa}_{0\kappa}\end{aligned}$$

where it is worth mentioning that we only consider the case of a Riemannian metric, which is positive definite by definition.

A.4 Computation with Respect to the Momentum

We have to show that

$$\left(\Lambda^{\rho}_{\kappa\beta}\hat{g}^{\kappa\tau}p_\rho p_\tau\right)\circ j^1(s) = s^*\left(\partial_\beta(\frac{1}{2}\left(p_\rho\circ j^1(s)\right)\hat{g}^{\rho\tau}\left(p_\tau\circ j^1(s)\right))\right)$$

is met. We have

$$\begin{aligned}j^1(s)^*\left(\Lambda^{\rho}_{\kappa\beta}\hat{g}^{\kappa\tau}p_\tau p_\rho\right) &= j^1(s)^*\left(-\frac{1}{2}p_\rho\hat{g}^{\rho\nu}\left(\partial_\kappa g_{\beta\nu} + \partial_\beta g_{\nu\kappa} - \partial_\nu g_{\kappa\beta}\right)\hat{g}^{\kappa\tau}p_\tau\right) \\ &= j^1(s)^*\left(-\frac{1}{2}p_\rho\hat{g}^{\rho\nu}\left(\partial_\beta g_{\nu\kappa}\right)\hat{g}^{\kappa\tau}p_\tau\right)\ .\end{aligned}$$

A simple calculation gives

$$\begin{aligned}\hat{g}^{\kappa\tau} g_{\nu\kappa} &= \delta^\tau_\nu \\ (\hat{g}^{\kappa\tau})(\partial_\beta g_{\nu\kappa}) + (\partial_\beta \hat{g}^{\kappa\tau})(g_{\nu\kappa}) &= 0\end{aligned}$$

and then

$$\begin{aligned} j^1(s)^* \left(\Lambda^\rho_{\kappa\beta} \hat{g}^{\kappa\tau} p_\tau p_\rho\right) &= j^1(s)^* \left(\frac{1}{2} p_\rho \hat{g}^{\rho\nu} (\partial_\beta \hat{g}^{\kappa\tau}) g_{\nu\kappa} p_\tau\right) \\ &= j^1(s)^* \left(\frac{1}{2} p_\rho (\partial_\beta \hat{g}^{\rho\tau}) p_\tau\right) \\ &= s^* \left(\partial_\beta \left(\frac{1}{2} (p_\rho \circ j^1(s)) \hat{g}^{\rho\tau} (p_\tau \circ j^1(s))\right)\right)\end{aligned}$$

which is the desired result.

A.5 Energy

The relation to be shown is that a contraction of

$$\mathrm{d}(v \rfloor p \circ j^1(s)) = \nabla^\Lambda (v \circ j^1(s)) \rfloor \left(p \circ j^1(s)\right) + \left(v \circ j^1(s)\right) \rfloor \nabla^{\Lambda^*} (p \circ j^1(s))$$

with $v_s = \partial_0 + \partial_0 s^\alpha \partial_\alpha$ leads to

$$\frac{m}{2} \partial_0 \left(v \rfloor v \rfloor g \circ j^1(s)\right) = -\left(v \rfloor \mathrm{d}V \circ j^1(s)\right) \ .$$

The left hand side is rather easy, since

$$\begin{aligned}(\partial_0 + \partial_0 s^\alpha \partial_\alpha) \rfloor \mathrm{d}(m\,(v \rfloor v \rfloor g) \circ j^1(s)) &= (\partial_0 + \partial_0 s^\alpha \partial_\alpha) \rfloor \partial_0 \left(m\,(v \rfloor v \rfloor g) \circ j^1(s)\right) \mathrm{d}t^0 \\ &= m \partial_0 \left(v \rfloor v \rfloor g \circ j^1(s)\right)\end{aligned}$$

and for the right hand side we use (3.31) and (3.32) together with $p = m(v \rfloor g)$ to obtain successively

$$v_s \rfloor \left(\nabla^\Lambda (v \circ j^1(s)) \rfloor p + v \rfloor \nabla^{\Lambda^*} (p \circ j^1(s))\right) \circ j^1(s) = \left(\frac{1}{m} \left(\hat{g} \rfloor - \mathrm{d}V\right) \rfloor p + v \rfloor \left(\gamma_c \rfloor - \mathrm{d}V\right)\right) \circ j^1(s)$$

and

$$v_s \rfloor \left(\nabla^\Lambda (v \circ j^1(s)) \rfloor p + v \rfloor \nabla^{\Lambda^*} (p \circ j^1(s))\right) \circ j^1(s) = (-v \rfloor \mathrm{d}V \ - v \rfloor \mathrm{d}V) \circ j^1(s)$$

with

$$\gamma_c = (\mathrm{d}q^\beta - \gamma^\beta_0 \mathrm{d}t^0) \otimes \partial_\beta \ .$$

A.6 Lagrangian Mechanics

A.6.1 Variational Derivative in Point Mechanics

We want to show the relation

$$h_0\left(j^1(\eta)\rfloor(\mathrm{d}\rho_L)\right) = \eta^\alpha(\delta_\alpha\mathcal{L})\mathrm{d}t^0$$

treated in section (3.3), where we already introduced the horizontal projection h_0. Therefore we start with

$$\mathrm{d}\rho_L = \mathrm{d}\left(\mathcal{L}\mathrm{d}t^0 + \partial^0_\alpha\mathcal{L}(\mathrm{d}q^\alpha - q^\alpha_0\mathrm{d}t^0)\right)$$

$$\begin{aligned}
\mathrm{d}\rho_L &= \mathrm{d}\mathcal{L}\wedge\mathrm{d}t^0 + \mathrm{d}(\partial^0_\alpha\mathcal{L})\wedge(\mathrm{d}q^\alpha - q^\alpha_0\mathrm{d}t^0) - \partial^0_\alpha\mathcal{L}\mathrm{d}q^\alpha_0\wedge\mathrm{d}t^0\\
&= \partial_\alpha\mathcal{L}\mathrm{d}q^\alpha\wedge\mathrm{d}t^0 + \partial^0_\alpha\mathcal{L}\mathrm{d}q^\alpha_0\wedge\mathrm{d}t^0 + \partial_0\partial^0_\alpha\mathcal{L}\mathrm{d}t^0\wedge\mathrm{d}q^\alpha + \partial_\beta\partial^0_\alpha\mathcal{L}\mathrm{d}q^\beta\wedge(\mathrm{d}q^\alpha - q^\alpha_0\mathrm{d}t^0)\\
&\quad +\partial^0_\beta\partial^0_\alpha\mathcal{L}\mathrm{d}q^\beta_0\wedge(\mathrm{d}q^\alpha - q^\alpha_0\mathrm{d}t^0) - \partial^0_\alpha\mathcal{L}\mathrm{d}q^\alpha_0\wedge\mathrm{d}t^0\\
&= \partial_\alpha\mathcal{L}\mathrm{d}q^\alpha\wedge\mathrm{d}t^0 + \partial_0\partial^0_\alpha\mathcal{L}\mathrm{d}t^0\wedge\mathrm{d}q^\alpha + \partial_\beta\partial^0_\alpha\mathcal{L}\mathrm{d}q^\beta\wedge(\mathrm{d}q^\alpha - q^\alpha_0\mathrm{d}t^0)\\
&\quad +\partial^0_\beta\partial^0_\alpha\mathcal{L}\mathrm{d}q^\beta_0\wedge(\mathrm{d}q^\alpha - q^\alpha_0\mathrm{d}t^0)
\end{aligned}$$

and compute

$$j^1(\eta)\rfloor(\mathrm{d}\rho_L)\;.$$

We use

$$j^1(\eta) = \eta^\alpha\partial_\alpha + \eta^\alpha_0\partial^0_\alpha$$

and consequently we obtain

$$\begin{aligned}
j^1(\eta)\rfloor(\mathrm{d}\rho_L) &= \eta^\alpha(\partial_\alpha\mathcal{L}\mathrm{d}t^0 - \partial_0\partial^0_\alpha\mathcal{L}\mathrm{d}t^0 - \partial_\beta\partial^0_\alpha\mathcal{L}\mathrm{d}q^\beta + \partial_\alpha\partial^0_\beta\mathcal{L}(\mathrm{d}q^\beta - q^\beta_0\mathrm{d}t^0) - \partial^0_\beta\partial^0_\alpha\mathcal{L}\mathrm{d}q^\beta_0)\\
&\quad +\eta^\alpha_0\partial^0_\alpha\partial^0_\beta\mathcal{L}(\mathrm{d}q^\beta - q^\beta_0\mathrm{d}t^0)\;.
\end{aligned}$$

Now we apply the horizontal projection

$$\mathrm{d}t^0\longmapsto\mathrm{d}t^0\;,\quad \mathrm{d}q^\alpha\longmapsto q^\alpha_0\mathrm{d}t^0\;,\quad \mathrm{d}q^\alpha_0\longmapsto q^\alpha_{00}\mathrm{d}t^0$$

and obtain

$$\begin{aligned}
h_0(j^1(\eta)\rfloor(\mathrm{d}\rho_L)) &= \eta^\alpha(\partial_\alpha\mathcal{L}\mathrm{d}t^0 - \partial_0\partial^0_\alpha\mathcal{L}\mathrm{d}t^0 - \left(\partial_\beta\partial^0_\alpha\mathcal{L}\right)q^\beta_0\mathrm{d}t^0 - \left(\partial^0_\beta\partial^0_\alpha\mathcal{L}\right)q^\beta_{00}\mathrm{d}t^0)\\
&= \eta^\alpha(\partial_\alpha\mathcal{L} - \partial_0\partial^0_\alpha\mathcal{L} - \left(\partial_\beta\partial^0_\alpha\mathcal{L}\right)q^\beta_0 - \left(\partial^0_\beta\partial^0_\alpha\mathcal{L}\right)q^\beta_{00})\mathrm{d}t^0\\
&= \eta^\alpha(\partial_\alpha\mathcal{L} - d_0\partial^0_\alpha\mathcal{L})\mathrm{d}t^0
\end{aligned}$$

which is the desired result.

A.6.2 Covariant Lagrangian Equations

We start with

$$\mathcal{L} = \frac{1}{2}g_{\alpha\beta}(q^\alpha_0 - \gamma^\alpha_0)(q^\beta_0 - \gamma^\beta_0)$$

and consequently we evaluate

$$-\delta_\alpha \mathcal{L} = d_0\left(\partial^0_\nu \mathcal{L}\right) - \partial_\nu \mathcal{L}$$

therefore we obtain successively

$$\begin{aligned}\partial_\nu \mathcal{L} &= \frac{1}{2}(\partial_\nu g_{\alpha\beta})(q^\alpha_0-\gamma^\alpha_0)(q^\beta_0-\gamma^\beta_0)+\frac{1}{2}g_{\alpha\beta}\Big((-\partial_\nu\gamma^\alpha_0)(q^\beta_0-\gamma^\beta_0)+(q^\alpha_0-\gamma^\alpha_0)(-\partial_\nu\gamma^\beta_0)\Big)\\ &= \frac{1}{2}(\partial_\nu g_{\alpha\beta})(q^\alpha_0-\gamma^\alpha_0)(q^\beta_0-\gamma^\beta_0)-g_{\alpha\beta}(\partial_\nu\gamma^\alpha_0)(q^\beta_0-\gamma^\beta_0)\end{aligned}$$

as well as

$$\partial^0_\nu \mathcal{L} = g_{\nu\alpha}(q^\alpha_0-\gamma^\alpha_0)\ .$$

Now we evaluate

$$d_0\left(\partial^0_\nu \mathcal{L}\right) = \partial_0\partial^0_\nu\mathcal{L} + \left(\partial_\sigma\partial^0_\nu\mathcal{L}\right)q^\sigma_0 + \left(\partial^0_\tau\partial^0_\nu\mathcal{L}\right)q^\tau_{00}$$

with

$$\begin{aligned}\partial_0\partial^0_\nu\mathcal{L} &= (\partial_0 g_{\nu\alpha})(q^\alpha_0-\gamma^\alpha_0)-g_{\nu\alpha}\partial_0\gamma^\alpha_0\\ \partial_\sigma\partial^0_\nu\mathcal{L} &= (\partial_\sigma g_{\nu\alpha})(q^\alpha_0-\gamma^\alpha_0)-g_{\nu\alpha}\partial_\sigma\gamma^\alpha_0\\ \partial^0_\tau\partial^0_\nu\mathcal{L} &= g_{\nu\tau}\ .\end{aligned}$$

Then this follows to

$$\begin{aligned}0 &= -\delta_\alpha\mathcal{L}\\ &= (\partial_0 g_{\alpha\nu})(q^\alpha_0-\gamma^\alpha_0)-g_{\nu\alpha}\partial_0\gamma^\alpha_0+((\partial_\sigma g_{\nu\alpha})(q^\alpha_0-\gamma^\alpha_0)-g_{\nu\alpha}\partial_\sigma\gamma^\alpha_0)q^\sigma_0+g_{\nu\tau}q^\tau_{00}\\ &\quad -\frac{1}{2}(\partial_\nu g_{\alpha\beta})(q^\alpha_0-\gamma^\alpha_0)(q^\beta_0-\gamma^\beta_0)+g_{\alpha\beta}(\partial_\nu\gamma^\alpha_0)(q^\beta_0-\gamma^\beta_0)\ .\end{aligned}$$

From section 3.2 we have the relation

$$\partial_0 g_{\alpha\nu} = -(\partial_\alpha\gamma^\rho_0)g_{\rho\nu}-(\partial_\nu\gamma^\rho_0)g_{\rho\alpha}-(\partial_\rho g_{\alpha\nu})\,\gamma^\rho_0$$

and consequently we obtain

$$\begin{aligned}0 &= (-(\partial_\alpha\gamma^\rho_0)g_{\rho\nu}-(\partial_\nu\gamma^\rho_0)g_{\rho\alpha}-(\partial_\rho g_{\alpha\nu})\,\gamma^\rho_0)(q^\alpha_0-\gamma^\alpha_0)-g_{\nu\alpha}\partial_0\gamma^\alpha_0\\ &\quad +((\partial_\sigma g_{\nu\alpha})(q^\alpha_0-\gamma^\alpha_0)-g_{\nu\alpha}\partial_\sigma\gamma^\alpha_0)q^\sigma_0+g_{\nu\tau}q^\tau_{00}\\ &\quad -\frac{1}{2}(\partial_\nu g_{\alpha\beta})(q^\alpha_0-\gamma^\alpha_0)(q^\beta_0-\gamma^\beta_0)+g_{\alpha\beta}(\partial_\nu\gamma^\alpha_0)(q^\beta_0-\gamma^\beta_0)\ .\end{aligned}$$

Using the inverse of the metric we have

$$\begin{aligned}0 &= -(\partial_\alpha\gamma^\phi_0)(q^\alpha_0-\gamma^\alpha_0)-\hat{g}^{\phi v}(\partial_\nu\gamma^\rho_0)g_{\rho\alpha}(q^\alpha_0-\gamma^\alpha_0)-\hat{g}^{\phi v}\,(\partial_\rho g_{\alpha\nu})\,\gamma^\rho_0(q^\alpha_0-\gamma^\alpha_0)-\partial_0\gamma^\phi_0\\ &\quad +\hat{g}^{\phi v}(\partial_\sigma g_{\nu\alpha})(q^\alpha_0-\gamma^\alpha_0)q^\sigma_0-\partial_\sigma\gamma^\phi_0 q^\sigma_0+q^\phi_{00}\\ &\quad -\hat{g}^{\phi v}\frac{1}{2}(\partial_\nu g_{\alpha\beta})(q^\alpha_0-\gamma^\alpha_0)(q^\beta_0-\gamma^\beta_0)+\hat{g}^{\phi v}g_{\alpha\beta}(\partial_\nu\gamma^\alpha_0)(q^\beta_0-\gamma^\beta_0)\end{aligned}$$

and

$$\begin{aligned}
0 &= q^\phi_{00} - \partial_0\gamma^\phi_0 - \partial_\sigma\gamma^\phi_0 q^\sigma_0 - (\partial_\alpha\gamma^\phi_0)(q^\alpha_0 - \gamma^\alpha_0) \\
&\quad -g^{\phi v}\frac{1}{2}(\partial_\nu g_{\alpha\beta})(q^\alpha_0 - \gamma^\alpha_0)(q^\beta_0 - \gamma^\beta_0) + g^{\phi v}(\partial_\rho g_{\nu\alpha})(q^\alpha_0 - \gamma^\alpha_0)(q^\rho_0 - \gamma^\rho_0)\ .
\end{aligned}$$

This finally leads to

$$\begin{aligned}
0 &= q^\phi_{00} - \partial_0\gamma^\phi_0 - \partial_\sigma\gamma^\phi_0 q^\sigma_0 - (\partial_\alpha\gamma^\phi_0)(q^\alpha_0 - \gamma^\alpha_0) - \hat{g}^{\phi v}\frac{1}{2}(\partial_\nu g_{\alpha\beta})(q^\alpha_0 - \gamma^\alpha_0)(q^\beta_0 - \gamma^\beta_0) \\
&\quad +\frac{1}{2}\hat{g}^{\phi v}(\partial_\rho g_{\nu\alpha})(q^\alpha_0 - \gamma^\alpha_0)(q^\rho_0 - \gamma^\rho_0) + \frac{1}{2}\hat{g}^{\phi v}(\partial_\alpha g_{\nu\rho})(q^\alpha_0 - \gamma^\alpha_0)(q^\rho_0 - \gamma^\rho_0)
\end{aligned}$$

and

$$\begin{aligned}
0 &= q^\phi_{00} - \partial_0\gamma^\phi_0 - \partial_\sigma\gamma^\phi_0 q^\sigma_0 - (\partial_\alpha\gamma^\phi_0)(q^\alpha_0 - \gamma^\alpha_0) \\
&\quad +\frac{1}{2}\hat{g}^{\phi v}\left(\partial_\rho g_{\nu\alpha} + \partial_\alpha g_{\rho\nu} - \partial_\nu g_{\alpha\rho}\right)(q^\alpha_0 - \gamma^\alpha_0)(q^\rho_0 - \gamma^\rho_0)\ .
\end{aligned}$$

The relation

$$2\Lambda^\kappa_{\alpha\rho} = -\hat{g}^{\kappa\varepsilon}\left(\partial_\alpha g_{\rho\varepsilon} + \partial_\rho g_{\varepsilon\alpha} - \partial_\varepsilon g_{\alpha\rho}\right)$$

then gives the desired result

$$\begin{aligned}
0 &= q^\phi_{00} - \partial_0\gamma^\phi_0 - \partial_\sigma\gamma^\phi_0 q^\sigma_0 - (\partial_\alpha\gamma^\phi_0)(q^\alpha_0 - \gamma^\alpha_0) - \Lambda^\phi_{\alpha\rho}(q^\alpha_0 - \gamma^\alpha_0)(q^\rho_0 - \gamma^\rho_0) \\
&= q^\phi_{00} - d_0(\gamma^\phi_0) - (\partial_\alpha\gamma^\phi_0)(q^\alpha_0 - \gamma^\alpha_0) - \Lambda^\phi_{\alpha\rho}(q^\alpha_0 - \gamma^\alpha_0)(q^\rho_0 - \gamma^\rho_0)\ .
\end{aligned}$$

A.7 Hamiltonian Mechanics

A.7.1 Splittings - The Autonomous Case

We want to show the relation (3.42)

$$v_H(H\mathrm{d}t^0) = v_{H,\mathcal{H}}(H)\mathrm{d}t^0$$

and we start with

$$v_H(H\mathrm{d}t^0) = \mathrm{d}(v_H\rfloor H\mathrm{d}t^0) + v_H\rfloor\mathrm{d}(H\mathrm{d}t^0)\ ,$$

where we used Cartan's magic formula, see [Frankel, 2nd ed. 2004], which immediately gives

$$v_H(H\mathrm{d}t^0) = \mathrm{d}(H) + v_H\rfloor(\mathrm{d}H\wedge\mathrm{d}t^0)$$

with the relation (3.38) which was given as

$$v_H = \partial_0 + \dot{\partial}^\alpha(H + p_\beta\gamma^\beta_0)\partial_\alpha - \partial_\alpha(H + p_\beta\gamma^\beta_0)\dot{\partial}^\alpha\ .$$

Consequently we derive

$$v_H(H\mathrm{d}t^0) = \partial_0 H\mathrm{d}t^0 + \partial_\beta H\mathrm{d}q^\beta + \dot{\partial}^\alpha H\mathrm{d}p_\alpha + v_H\rfloor(\partial_\alpha H\mathrm{d}q^\alpha\wedge\mathrm{d}t^0 + \dot{\partial}^\alpha H\mathrm{d}p_\alpha\wedge\mathrm{d}t^0)$$

and from

$$\begin{aligned} v_H \rfloor (\partial_\alpha H \mathrm{d}q^\alpha \wedge \mathrm{d}t^0 + \dot{\partial}^\alpha H \mathrm{d}p_\alpha \wedge \mathrm{d}t^0) &= -\partial_\alpha H \mathrm{d}q^\alpha - \dot{\partial}^\alpha H \mathrm{d}p_\alpha \\ &\quad + \partial_\alpha H \dot{\partial}^\alpha (H + p_\beta \gamma_0^\beta) \mathrm{d}t^0 - \dot{\partial}^\alpha H \partial_\alpha (H + p_\beta \gamma_0^\beta) \mathrm{d}t^0 \end{aligned}$$

it follows that the relation

$$v_H(H \mathrm{d}t^0) = \Big(\partial_0 H + \partial_\alpha H \dot{\partial}^\alpha (H + p_\beta \gamma_0^\beta) - \dot{\partial}^\alpha H \partial_\alpha (H + p_\beta \gamma_0^\beta)\Big)\, \mathrm{d}t^0$$

is met. This simplifies to

$$\begin{aligned} v_H(H \mathrm{d}t^0) &= \Big(\partial_0 H + \partial_\alpha H \dot{\partial}^\alpha (p_\beta \gamma_0^\beta) - \dot{\partial}^\alpha H \partial_\alpha (p_\beta \gamma_0^\beta)\Big)\, \mathrm{d}t^0 \\ &= \Big(\partial_0 H + \partial_\alpha H \gamma_0^\alpha - p_\beta \dot{\partial}^\alpha H \partial_\alpha \gamma_0^\beta\Big)\, \mathrm{d}t^0 \end{aligned}$$

and this is

$$v_H(H \mathrm{d}t^0) = \Big(\partial_0 + \gamma_0^\alpha \partial_\alpha - p_\beta \partial_\alpha \gamma_0^\beta \dot{\partial}^\alpha\Big)\, (H) \mathrm{d}t^0 = v_{H,\mathcal{H}}(H) \mathrm{d}t^0$$

which completes the proof.

A.7.2 Splittings - The Case of Inputs

Let us consider the case of the extended Hamiltonian

$$H = H_0 - H_\rho u^\rho\,, \quad H_0,\ H_\rho \in C^\infty(\mathcal{V}^*(\mathcal{Q}))$$

with the input functions $u^\rho \in \mathcal{C}^\infty(\mathcal{B})$ and we want to show that

$$v_H(H_0 \mathrm{d}t^0) = \left(v_{H,\mathcal{H}}(H_0) + v_{H,\mathcal{V}}(H_\varepsilon) u^\varepsilon\right) \mathrm{d}t^0$$

is met. From the same considerations as above we successively have

$$v_H(H_0 \mathrm{d}t^0) = \partial_0 H_0 \mathrm{d}t^0 + \partial_\beta H_0 \mathrm{d}q^\beta + \dot{\partial}^\alpha H_0 \mathrm{d}p_\alpha + v_H \rfloor (\partial_\alpha H_0 \mathrm{d}q^\alpha \wedge \mathrm{d}t^0 + \dot{\partial}^\alpha H_0 \mathrm{d}p_\alpha \wedge \mathrm{d}t^0)$$

and the expression

$$v_H \rfloor (\partial_\alpha H_0 \mathrm{d}q^\alpha \wedge \mathrm{d}t^0 + \dot{\partial}^\alpha H_0 \mathrm{d}p_\alpha \wedge \mathrm{d}t^0)$$

can be rewritten as

$$-\partial_\alpha H_0 \mathrm{d}q^\alpha - \dot{\partial}^\alpha H_0 \mathrm{d}p_\alpha + \partial_\alpha H_0 \dot{\partial}^\alpha (H_0 - H_\rho u^\rho + p_\beta \gamma_0^\beta) \mathrm{d}t^0 - \dot{\partial}^\alpha H_0 \partial_\alpha (H_0 - H_\rho u^\rho + p_\beta \gamma_0^\beta) \mathrm{d}t^0$$

with

$$v_H = \partial_0 + \dot{\partial}^\alpha (H_0 - H_\rho u^\rho + p_\beta \gamma_0^\beta) \partial_\alpha - \partial_\alpha (H_0 - H_\rho u^\rho + p_\beta \gamma_0^\beta) \dot{\partial}^\alpha\ .$$

Consequently we have

$$v_H(H_0 \mathrm{d}t^0) = \Big(\partial_0 H_0 + \partial_\alpha H_0 \dot{\partial}^\alpha (H_0 - H_\rho u^\rho + p_\beta \gamma_0^\beta) - \dot{\partial}^\alpha H_0 \partial_\alpha (H_0 - H_\rho u^\rho + p_\beta \gamma_0^\beta)\Big)\, \mathrm{d}t^0$$

which gives

$$v_H(H_0 \mathrm{d}t^0) = \Big(\partial_0 H_0 + \partial_\alpha H_0 \dot{\partial}^\alpha(-H_\rho u^\rho + p_\beta \gamma_0^\beta) - \dot{\partial}^\alpha H_0 \partial_\alpha(-H_\rho u^\rho + p_\beta \gamma_0^\beta)\Big)\,\mathrm{d}t^0 \ .$$

Finally we obtain

$$v_H(H_0 \mathrm{d}t^0) = \Big(\partial_0 H_0 + \gamma_0^\alpha \partial_\alpha H_0 - p_\beta \dot{\partial}^\alpha H_0 \partial_\alpha \gamma_0^\beta + \Big(\dot{\partial}^\alpha H_0 \partial_\alpha H_\rho - \partial_\alpha H_0 \dot{\partial}^\alpha H_\rho\Big)\, u^\rho\Big)\,\mathrm{d}t^0 \ .$$

Since

$$\begin{aligned} v_{H,\mathcal{V}} &= \dot{\partial}^\alpha H \partial_\alpha - \partial_\alpha H \dot{\partial}^\alpha \\ &= \dot{\partial}^\alpha (H_0 - H_\rho u^\rho)\partial_\alpha - \partial_\alpha (H_0 - H_\rho u^\rho)\dot{\partial}^\alpha \end{aligned}$$

we have

$$\begin{aligned} v_{H,\mathcal{V}}(H_\rho) &= \dot{\partial}^\alpha (H_0 - H_\tau u^\tau)\partial_\alpha H_\rho - \partial_\alpha (H_0 - H_\tau u^\tau)\dot{\partial}^\alpha H_\rho \\ &= \dot{\partial}^\alpha H_0 \partial_\alpha H_\rho - \dot{\partial}^\alpha H_\tau \partial_\alpha H_\rho u^\tau - \partial_\alpha H_0 \dot{\partial}^\alpha H_\rho + \partial_\alpha H_\tau \dot{\partial}^\alpha H_\rho u^\tau \\ &= \dot{\partial}^\alpha H_0 \partial_\alpha H_\rho - \partial_\alpha H_0 \dot{\partial}^\alpha H_\rho + \Big(\partial_\alpha H_\tau \dot{\partial}^\alpha H_\rho - \dot{\partial}^\alpha H_\tau \partial_\alpha H_\rho\Big)\, u^\tau \end{aligned}$$

and

$$\begin{aligned} v_{H,\mathcal{V}}(H_\rho)u^\rho &= \Big(\dot{\partial}^\alpha H_0 \partial_\alpha H_\rho - \partial_\alpha H_0 \dot{\partial}^\alpha H_\rho\Big)\, u^\rho + \Big(\partial_\alpha H_\tau \dot{\partial}^\alpha H_\rho - \dot{\partial}^\alpha H_\tau \partial_\alpha H_\rho\Big)\, u^\tau u^\rho \\ v_{H,\mathcal{V}}(H_\rho)u^\rho &= \Big(\dot{\partial}^\alpha H_0 \partial_\alpha H_\rho - \partial_\alpha H_0 \dot{\partial}^\alpha H_\rho\Big)\, u^\rho \end{aligned}$$

and therefore the final result follows to

$$v_H(H_0 \mathrm{d}t^0) = (v_{H,\mathcal{H}}(H_0) + v_{H,\mathcal{V}}(H_\rho)u^\rho)\,\mathrm{d}t^0 \ .$$

A.8 Continuum Mechanics

A.8.1 Mass Balance Eulerian Picture

We consider the field

$$v_\phi = \partial_0 + v_\phi^\alpha \partial_\alpha$$

and compute

$$v_\phi(\mathrm{vol} \wedge \mathrm{d}t^0) \ .$$

The following abbreviation will simplify the calculation. We use the volume form

$$\mathrm{vol} \wedge \mathrm{d}t^0 = \sqrt{|\det(g_{\alpha\beta})|}\mathrm{d}q^1 \wedge ... \wedge \mathrm{d}q^n \wedge \mathrm{d}t^0$$

and define

$$\mathrm{vol}^0 = \mathrm{d}q^1 \wedge ... \wedge \mathrm{d}q^n \wedge \mathrm{d}t^0$$

consequently we use

$$\mathrm{vol} \wedge \mathrm{d}t^0 {=} \sqrt{|\det(g_{\alpha\beta})|}\mathrm{vol}^0 \ .$$

Successively we obtain

$$\begin{aligned} v_\phi(\mathrm{vol} \wedge \mathrm{d}t^0) &= \left(\partial_0 \sqrt{|\det(g_{\alpha\beta})|} + v_\phi^\alpha \partial_\alpha \sqrt{|\det(g_{\alpha\beta})|}\right) \mathrm{vol}^0 \\ &\quad + \sqrt{|\det(g_{\alpha\beta})|} \mathrm{d}(\left(\partial_0 + v_\phi^\alpha \partial_\alpha\right) \rfloor \mathrm{vol}^0) \\ &= \left(\partial_0 \sqrt{|\det(g_{\alpha\beta})|} + v_\phi^\alpha \partial_\alpha \sqrt{|\det(g_{\alpha\beta})|}\right) \mathrm{vol}^0 + \sqrt{|\det(g_{\alpha\beta})|} \mathrm{d}v_\phi^\alpha \wedge \partial_\alpha \rfloor \mathrm{vol}^0 \\ &= \left(\partial_0 \sqrt{|\det(g_{\alpha\beta})|}) + \partial_\alpha \left(v_\phi^\alpha \sqrt{|\det(g_{\alpha\beta})|}\right)\right) \mathrm{vol}^0 \end{aligned}$$

$$v_\phi(\mathrm{vol} \wedge \mathrm{d}t^0) = \left(\partial_0 \sqrt{|\det(g_{\alpha\beta})|}) + \partial_\alpha \left(v_\phi^\alpha \sqrt{|\det(g_{\alpha\beta})|}\right)\right) \mathrm{vol}^0$$

and finally the result follows to

$$v_\phi(\mathrm{vol} \wedge \mathrm{d}t^0) = \mathrm{div}(v_\phi) \left(\mathrm{vol} \wedge \mathrm{d}t^0\right)$$

with

$$\mathrm{div}(v_\phi) = \frac{1}{\sqrt{|\det(g_{\alpha\beta})|}} \left(\partial_0 \sqrt{|\det(g_{\alpha\beta})|} + \partial_\alpha (v_\phi^\alpha \sqrt{|\det(g_{\alpha\beta})|})\right) \ .$$

A.8.2 Energy Principles

This part gives coordinate proofs of the relations

$$\hat{\otimes} \left(\sigma \rfloor \nabla^{\Lambda^*}(v \rfloor g)\right) \wedge \mathrm{d}t^0 = \hat{\otimes} \left(\sigma \rfloor \left(\frac{1}{2} v_\phi(g)\right)\right) \wedge \mathrm{d}t^0 \tag{A.12}$$

and

$$\left(v_\phi \rfloor \nabla^\Lambda(v)\right) \rfloor v \rfloor g = v_\phi \left(\frac{1}{2} v \rfloor v \rfloor g\right) = v_\phi \left(e_k\right) \ . \tag{A.13}$$

We start with the proof of relation (A.12) and observe that the left hand side can be rewritten as

$$\hat{\otimes} \left(\sigma \rfloor \nabla^{\Lambda^*}(v \rfloor g)\right) \wedge \mathrm{d}t^0 = \hat{\otimes} \left(\sigma \rfloor \nabla^\Lambda(v) \rfloor g\right) \wedge \mathrm{d}t^0$$

with the help of the relation (3.26). This yields

$$\begin{aligned} \hat{\otimes} \left(\sigma \rfloor \nabla^\Lambda(v) \rfloor g\right) \wedge \mathrm{d}t^0 &= \hat{\otimes} \left(\sigma \rfloor ((\partial_0 v^\rho - \Lambda_{0\alpha}^\rho v^\alpha) \, \mathrm{d}t^0 \otimes \partial_\rho) \rfloor g\right) \wedge \mathrm{d}t^0 \\ &\quad + \hat{\otimes} \left(\sigma \rfloor ((\partial_\alpha v^\rho - \Lambda_{\alpha\beta}^\rho v^\beta) \mathrm{d}q^\alpha \otimes \partial_\rho) \rfloor g\right) \wedge \mathrm{d}t^0 \\ &= \hat{\otimes} \left(\sigma^{\alpha\nu} \partial_\nu \rfloor \mathrm{vol} \otimes \partial_\alpha \rfloor ((\partial_\alpha v^\rho - \Lambda_{\alpha\beta}^\rho v^\beta) \mathrm{d}q^\alpha) g_{\tau\rho} \otimes \mathrm{d}q^\tau\right) \wedge \mathrm{d}t^0 \\ &= \sigma^{\alpha\nu} g_{\nu\rho} (\partial_\alpha v^\rho - \Lambda_{\alpha\beta}^\rho v^\beta) \mathrm{vol} \wedge \mathrm{d}t^0 \ . \end{aligned}$$

Next we inspect the right side of the equation (A.12) and compute

$$\begin{aligned} v_\phi(g) &= v_\phi(g_{\alpha\beta})(\mathrm{d}q^\alpha - \gamma_0^\alpha \mathrm{d}t^0) \otimes (\mathrm{d}q^\beta - \gamma_0^\beta \mathrm{d}t^0) \\ &\quad + g_{\alpha\beta}\left(v_\phi(\mathrm{d}q^\alpha - \gamma_0^\alpha \mathrm{d}t^0) \otimes (\mathrm{d}q^\beta - \gamma_0^\beta \mathrm{d}t^0) + (\mathrm{d}q^\alpha - \gamma_0^\alpha \mathrm{d}t^0) \otimes v_\phi(\mathrm{d}q^\beta - \gamma_0^\beta \mathrm{d}t^0)\right) \end{aligned}$$

with

$$v_\phi = \partial_0 + v_\phi^\rho \partial_\rho$$

and

$$\begin{aligned} v_\phi(\mathrm{d}q^\alpha - \gamma_0^\alpha \mathrm{d}t^0) &= v_\phi \rfloor \left(-\mathrm{d}\gamma_0^\alpha \wedge \mathrm{d}t^0\right) + \mathrm{d}\left(v_\phi \rfloor (\mathrm{d}q^\alpha - \gamma_0^\alpha \mathrm{d}t^0)\right) \\ &= v_\phi \rfloor \left(-\partial_\beta \gamma_0^\alpha \mathrm{d}q^\beta \wedge \mathrm{d}t^0\right) + \mathrm{d}(v_\phi^\alpha - \gamma_0^\alpha) \\ &= \partial_\beta \gamma_0^\alpha \mathrm{d}q^\beta - v_\phi^\beta \partial_\beta \gamma_0^\alpha \mathrm{d}t^0 + \mathrm{d}(v_\phi^\alpha - \gamma_0^\alpha) \\ &= \left(\partial_0(v_\phi^\alpha - \gamma_0^\alpha) - v_\phi^\delta \partial_\delta \gamma_0^\alpha\right) \mathrm{d}t^0 + \partial_\beta v_\phi^\alpha \mathrm{d}q^\beta \end{aligned}$$

we end up with

$$\begin{aligned} v_\phi(g) &= \left(\partial_0 g_{\alpha\beta} + v_\phi^\rho \partial_\rho g_{\alpha\beta}\right)(\mathrm{d}q^\alpha - \gamma_0^\alpha \mathrm{d}t^0) \otimes (\mathrm{d}q^\beta - \gamma_0^\beta \mathrm{d}t^0) \\ &\quad + g_{\alpha\beta}\left(\left(\partial_0(v_\phi^\alpha - \gamma_0^\alpha) - v_\phi^\delta \partial_\delta \gamma_0^\alpha\right)\mathrm{d}t^0 + \partial_\varepsilon v_\phi^\alpha \mathrm{d}q^\varepsilon\right) \otimes (\mathrm{d}q^\beta - \gamma_0^\beta \mathrm{d}t^0) \\ &\quad + g_{\alpha\beta}(\mathrm{d}q^\alpha - \gamma_0^\alpha \mathrm{d}t^0) \otimes \left(\left(\partial_0(v_\phi^\beta - \gamma_0^\beta) - v_\phi^\nu \partial_\nu \gamma_0^\beta\right)\mathrm{d}t^0 + \partial_\rho v_\phi^\beta \mathrm{d}q^\rho\right) \ . \end{aligned}$$

Now we use equation (3.24) and derive

$$\begin{aligned} v_\phi(g) &= \left(-\partial_\alpha \gamma_0^\tau g_{\tau\beta} - \partial_\beta \gamma_0^\tau g_{\tau\alpha} + (v_\phi^\rho - \gamma_0^\rho)\partial_\rho g_{\alpha\beta}\right)(\mathrm{d}q^\alpha - \gamma_0^\alpha \mathrm{d}t^0) \otimes (\mathrm{d}q^\beta - \gamma_0^\beta \mathrm{d}t^0) \\ &\quad + g_{\alpha\beta}\left(\left(\partial_0(v_\phi^\alpha - \gamma_0^\alpha) - v_\phi^\delta \partial_\delta \gamma_0^\alpha\right)\mathrm{d}t^0 + \partial_\varepsilon v_\phi^\alpha \mathrm{d}q^\varepsilon\right) \otimes (\mathrm{d}q^\beta - \gamma_0^\beta \mathrm{d}t^0) \\ &\quad + g_{\alpha\beta}(\mathrm{d}q^\alpha - \gamma_0^\alpha \mathrm{d}t^0) \otimes \left(\left(\partial_0(v_\phi^\beta - \gamma_0^\beta) - v_\phi^\nu \partial_\nu \gamma_0^\beta\right)\mathrm{d}t^0 + \partial_\rho v_\phi^\beta \mathrm{d}q^\rho\right) \ . \end{aligned}$$

From

$$\begin{aligned} \left(\sigma \rfloor \left(\frac{1}{2} v_\phi(g)\right)\right) \wedge \mathrm{d}t^0 &= \frac{1}{2}\sigma^{\alpha\nu}\partial_\nu \rfloor \mathrm{vol} \otimes \partial_\alpha \rfloor \left(-\partial_\alpha \gamma_0^\tau g_{\tau\beta} - \partial_\beta \gamma_0^\tau g_{\tau\alpha}\right) \mathrm{d}q^\alpha \otimes \mathrm{d}q^\beta \wedge \mathrm{d}t^0 \\ &\quad + \frac{1}{2}\sigma^{\alpha\nu}\partial_\nu \rfloor \mathrm{vol} \otimes \partial_\alpha \rfloor \left((v_\phi^\rho - \gamma_0^\rho)\partial_\rho g_{\alpha\beta}\right) \mathrm{d}q^\alpha \otimes \mathrm{d}q^\beta \wedge \mathrm{d}t^0 \\ &\quad + \frac{1}{2}\sigma^{\alpha\nu}\partial_\nu \rfloor \mathrm{vol} \otimes \partial_\alpha \rfloor \left(g_{\kappa\beta}\partial_\alpha v_\phi^\kappa + g_{\alpha\tau}\partial_\beta v_\phi^\tau\right) \mathrm{d}q^\alpha \otimes \mathrm{d}q^\beta \wedge \mathrm{d}t^0 \end{aligned}$$

we derive

$$\hat{\otimes}\left(\sigma \rfloor \left(\frac{1}{2} v_\phi(g)\right)\right) \wedge \mathrm{d}t^0 = \frac{1}{2}\sigma^{\alpha\beta}\left((v_\phi^\rho - \gamma_0^\rho)\partial_\rho g_{\alpha\beta} + g_{\varepsilon\beta}\partial_\alpha v^\varepsilon + g_{\alpha\varepsilon}\partial_\beta v^\varepsilon\right) \mathrm{vol} \wedge \mathrm{d}t^0 \ .$$

With the equation (3.17) which reads as

$$(\partial_\rho g_{\alpha\varepsilon}) = -g_{\kappa\varepsilon}\Lambda^\kappa_{\alpha\rho} - g_{\alpha\kappa}\Lambda^\kappa_{\rho\varepsilon}$$

the result follows to

$$\hat{\otimes}\left(\sigma\rfloor\left(\frac{1}{2}v_\phi(g)\right)\right)\wedge \mathrm{d}t^0 = g_{\beta\eta}\sigma^{\varepsilon\eta}\left(\partial_\varepsilon v^\beta - \Lambda^\beta_{\varepsilon\tau}v^\tau\right)\mathrm{vol}\wedge\mathrm{d}t^0\ ,$$

where it is worth mentioning that we have

$$v^\rho = v^\rho_\phi - \gamma^\rho_0$$

and this proves the relation (A.12).

Remark A.1 *The tensor d which meets*

$$d\wedge\mathrm{d}t^0 = \frac{1}{2}v_\phi(g)\wedge\mathrm{d}t^0$$

which in coordinates gives

$$d\wedge\mathrm{d}t^0 = \frac{1}{2}\left(g_{\varepsilon\beta}\left(\partial_\alpha v^\varepsilon - \Lambda^\varepsilon_{\alpha\rho}v^\rho\right) + g_{\alpha\varepsilon}\left(\partial_\beta v^\varepsilon - \Lambda^\varepsilon_{\rho\beta}v^\rho\right)\right)\mathrm{d}q^\alpha\otimes\mathrm{d}q^\beta\wedge\mathrm{d}t^0$$

is called the rate of the deformation and has the components

$$d_{\alpha\beta} = \frac{1}{2}\left(g_{\varepsilon\beta}\left(\partial_\alpha v^\varepsilon - \Lambda^\varepsilon_{\alpha\rho}v^\rho\right) + g_{\alpha\varepsilon}\left(\partial_\beta v^\varepsilon - \Lambda^\varepsilon_{\rho\beta}v^\rho\right)\right)\ .$$

Therefore we conclude that we have

$$\hat{\otimes}\left(\sigma\rfloor\left(\frac{1}{2}v_\phi(g)\right)\right)\wedge\mathrm{d}t^0 = \sigma^{\alpha\beta}d_{\alpha\beta}\mathrm{vol}\wedge\mathrm{d}t^0\ .$$

Let us now proceed with the equation (A.13). The left hand side in coordinates gives

$$\begin{aligned}\left(v_\phi\rfloor\nabla^\Lambda(v)\right)\rfloor v\rfloor g &= \left((\partial_0 v^\rho - \Lambda^\rho_{0\alpha}v^\alpha) + v^\alpha_\phi(\partial_\alpha v^\rho - \Lambda^\rho_{\alpha\beta}v^\beta)\right)v^\kappa g_{\rho\kappa}\\ &= \left(\partial_0 v^\rho + v^\alpha_\phi\partial_\alpha v^\rho - (\partial_\alpha\gamma^\rho_0)v^\alpha - \Lambda^\rho_{\alpha\beta}v^\beta v^\alpha)\right)v^\kappa g_{\rho\kappa}\end{aligned}$$

where essential use of the relation (3.25)

$$\Lambda^\rho_{0\alpha} = (\partial_\alpha\gamma^\rho_0) - \Lambda^\rho_{\beta\alpha}\gamma^\beta_0$$

was made. The right hand side of the equation (A.13) is more crucial and we have

$$v_\phi\left(\frac{1}{2}v\rfloor v\rfloor g\right) = \partial_0(\frac{1}{2}v^\rho v^\tau g_{\rho\tau}) + v^\alpha_\phi\partial_\alpha(\frac{1}{2}v^\rho v^\tau g_{\rho\tau})$$

which leads to

$$\begin{aligned}v_\phi\left(\frac{1}{2}v\rfloor v\rfloor g\right) &= \frac{1}{2}\left(\partial_0 v^\rho\right)v^\tau g_{\rho\tau} + \frac{1}{2}\left(\partial_0 v^\tau\right)v^\rho g_{\rho\tau} + \frac{1}{2}v^\rho v^\tau\left(\partial_0 g_{\rho\tau}\right)\\ &+\frac{1}{2}(v^\alpha_\phi\partial_\alpha v^\rho)v^\tau g_{\rho\tau} + \frac{1}{2}(v^\alpha_\phi\partial_\alpha v^\tau)v^\rho g_{\rho\tau} + \frac{1}{2}v^\alpha_\phi v^\rho v^\tau\left(\partial_\alpha g_{\rho\tau}\right)\ .\end{aligned}$$

The relations (3.17)

$$(\partial_\rho g_{\alpha\varepsilon}) = -g_{\kappa\varepsilon}\Lambda^\kappa_{\alpha\rho} - g_{\alpha\kappa}\Lambda^\kappa_{\rho\varepsilon}$$

and (3.24)

$$\partial_0 g_{\alpha\nu} = -(\partial_\alpha \gamma_0^\rho) g_{\rho\nu} - (\partial_\nu \gamma_0^\rho) g_{\rho\alpha} - (\partial_\rho g_{\alpha\nu})\,\gamma_0^\rho$$

are again used and we successively obtain

$$\begin{aligned} v_\phi\left(\frac{1}{2}v\rfloor v\rfloor g\right) &= (\partial_0 v^\rho)\, v^\tau g_{\rho\tau} + \frac{1}{2}v^\rho v^\tau(-\partial_\tau\gamma_0^\kappa g_{\kappa\rho} - \partial_\rho\gamma_0^\kappa g_{\kappa\tau} - (\partial_\kappa g_{\rho\tau})\,\gamma_0^\kappa) \\ &\quad +(v_\phi^\alpha\partial_\alpha v^\tau) v^\rho g_{\rho\tau} + \frac{1}{2}v_\phi^\alpha v^\rho v^\tau(-g_{\kappa\rho}\Lambda^\kappa_{\alpha\tau} - g_{\tau\kappa}\Lambda^\kappa_{\alpha\rho}) \end{aligned}$$

and this this is

$$\begin{aligned} v_\phi\left(\frac{1}{2}v\rfloor v\rfloor g\right) &= (\partial_0 v^\rho)\, v^\tau g_{\rho\tau} - v^\rho v^\tau \partial_\tau\gamma_0^\kappa g_{\kappa\rho} - \frac{1}{2}v^\rho v^\tau\,(\partial_\kappa g_{\rho\tau})\,\gamma_0^\kappa \\ &\quad +(v_\phi^\alpha\partial_\alpha v^\tau) v^\rho g_{\rho\tau} - v_\phi^\alpha v^\rho v^\tau g_{\kappa\rho}\Lambda^\kappa_{\alpha\tau} \end{aligned}$$

Again using (3.17) we end up with

$$\begin{aligned} v_\phi\left(\frac{1}{2}v\rfloor v\rfloor g\right) &= (\partial_0 v^\rho)\, v^\tau g_{\rho\tau} - \partial_\tau\gamma_0^\kappa g_{\kappa\rho} v^\rho v^\tau + v^\rho v^\tau\,(g_{\varepsilon\rho}\Lambda^\varepsilon_{\kappa\tau})\,\gamma_0^\kappa) \\ &\quad +(v_\phi^\alpha\partial_\alpha v^\rho) v^\tau g_{\rho\tau} - v_\phi^\alpha v^\rho v^\tau (g_{\tau\kappa}\Lambda^\kappa_{\alpha\rho}) \end{aligned}$$

and finally

$$v_\phi\left(\frac{1}{2}v\rfloor v\rfloor g\right) = \left(\partial_0 v^\rho - v^\alpha\partial_\alpha\gamma_0^\rho + v_\phi^\alpha\partial_\alpha v^\rho - v^\alpha v^\beta\Lambda^\rho_{\alpha\beta}\right) v^\tau g_{\rho\tau}$$

this proofs that

$$v_\phi\rfloor\nabla^\Lambda(v)\rfloor v\rfloor g = v_\phi\left(\frac{1}{2}v\rfloor v\rfloor g\right)$$

is met.

A.8.3 Piola Transformation Relations

In the following we show a relation involving the 2nd Piola tensor and the Cauchy Green tensor discussed in section (4.2.1). The expression to be shown is

$$\hat{\otimes}\left(S\rfloor\nabla_\Phi^{\Lambda^*}(V\rfloor g)\right)\wedge \mathrm{d}t^0 = \hat{\otimes}\left(S\rfloor\frac{1}{2}\,(\partial_0 C)\right)\wedge \mathrm{d}t^0\,.$$

We start with

$$\hat{\otimes}\left(S\rfloor\nabla_\Phi^{\Lambda^*}(V\rfloor g)\right)\wedge \mathrm{d}t^0 = \hat{\otimes}\left(S\rfloor\nabla_\Phi^{\Lambda}(V)\rfloor g\right)\wedge \mathrm{d}t^0$$

and obtain

$$\begin{aligned} \hat{\otimes}\left(S\rfloor\nabla_\Phi^{\Lambda}(V)\rfloor g\right)\wedge \mathrm{d}t^0 &= \hat{\otimes}\left(S^{ij}\partial_i\rfloor\mathrm{VOL}\otimes\left((\partial_i\,(V^\rho) - \Lambda^\rho_{\alpha\beta}F_i^\alpha V^\beta)\mathrm{d}X^i\right) g_{\rho\tau}F_j^\tau\right)\wedge \mathrm{d}t^0 \\ &= S^{ij}(\partial_i\,(V^\rho) - \Lambda^\rho_{\alpha\beta}F_i^\alpha V^\beta) g_{\rho\tau}F_j^\tau \mathrm{VOL}\wedge \mathrm{d}t^0\,. \end{aligned}$$

The next step is to consider

$$\begin{aligned}\partial_0(C) &= \partial_0\left(\left(g_{\alpha\beta}\circ\Phi\right)F_i^\alpha F_j^\beta\right)(\mathrm{d}X^i-\Gamma_0^i\mathrm{d}t^0)\otimes(\mathrm{d}X^j-\Gamma_0^j\mathrm{d}t^0)\\ &\quad -g_{\alpha\beta}F_i^\alpha F_j^\beta\partial_0\Gamma_0^i\mathrm{d}t^0\otimes(\mathrm{d}X^j-\Gamma_0^j\mathrm{d}t^0)-g_{\alpha\beta}F_i^\alpha F_j^\beta\partial_0\Gamma_0^j(\mathrm{d}X^i-\Gamma_0^i\mathrm{d}t^0)\otimes\mathrm{d}t^0\end{aligned}$$

since

$$\partial_0(\mathrm{d}X^i-\Gamma_0^i\mathrm{d}t^0)=-\partial_0\Gamma_0^i\mathrm{d}t^0$$

and therefore

$$\hat{\otimes}\left(S\rfloor\frac{1}{2}\left(\partial_0C\right)\right)\wedge\mathrm{d}t^0=\hat{\otimes}\left(S^{ij}\partial_i\rfloor\mathrm{VOL}\otimes\partial_j\rfloor\frac{1}{2}\partial_0\left(\left(g_{\alpha\beta}\circ\Phi\right)F_i^\alpha F_j^\beta\right)\mathrm{d}X^i\otimes\mathrm{d}X^j\right)\wedge\mathrm{d}t^0\ .$$

Let us inspect the expression

$$\partial_0\left(\left(g_{\alpha\beta}\circ\Phi\right)F_i^\alpha F_j^\beta\right)$$

which follows to

$$\left(\partial_0g_{\alpha\beta}+V_0^\rho\partial_\rho g_{\alpha\beta}\right)F_i^\alpha F_j^\beta+g_{\alpha\beta}\left(\partial_{0i}\Phi^\alpha\right)F_j^\beta+g_{\alpha\beta}F_i^\alpha\left(\partial_{0j}\Phi^\beta\right)$$

and therefore we obtain

$$\partial_0\left(\left(g_{\alpha\beta}\circ\Phi\right)F_i^\alpha F_j^\beta\right)=\left(\partial_0g_{\alpha\beta}+V_0^\rho\partial_\rho g_{\alpha\beta}\right)F_i^\alpha F_j^\beta+g_{\alpha\beta}\partial_iV_0^\alpha F_j^\beta+g_{\alpha\beta}F_i^\alpha\partial_jV_0^\beta\ .$$

The relations (3.17)

$$\left(\partial_\rho g_{\alpha\beta}\right)=-g_{\kappa\beta}\Lambda_{\alpha\rho}^\kappa-g_{\alpha\kappa}\Lambda_{\rho\beta}^\kappa$$

and (3.24)

$$\partial_0g_{\alpha\beta}=-(\partial_\alpha\gamma_0^\rho)g_{\rho\beta}-(\partial_\beta\gamma_0^\rho)g_{\rho\alpha}-\left(\partial_\rho g_{\alpha\beta}\right)\gamma_0^\rho$$

are used to get

$$\begin{aligned}\partial_0\left(\left(g_{\alpha\beta}\circ\Phi\right)F_i^\alpha F_j^\beta\right) &= \left(-(\partial_\alpha\gamma_0^\rho)g_{\rho\beta}-(\partial_\beta\gamma_0^\rho)g_{\rho\alpha}+\left(V_0^\rho-\gamma_0^\rho\right)\left(-g_{\kappa\beta}\Lambda_{\alpha\rho}^\kappa-g_{\alpha\kappa}\Lambda_{\rho\beta}^\kappa\right)\right)F_i^\alpha F_j^\beta\\ &\quad +g_{\alpha\beta}\partial_iV_0^\alpha F_j^\beta+g_{\alpha\beta}F_i^\alpha\partial_jV_0^\beta\ .\end{aligned}$$

Additionally we obtain

$$\begin{aligned}\partial_0\left(\left(g_{\alpha\beta}\circ\Phi\right)F_i^\alpha F_j^\beta\right) &= \left(-(\partial_\alpha\gamma_0^\rho)g_{\rho\beta}-(\partial_\beta\gamma_0^\rho)g_{\rho\alpha}+V^\rho\left(-g_{\kappa\beta}\Lambda_{\alpha\rho}^\kappa-g_{\alpha\kappa}\Lambda_{\rho\beta}^\kappa\right)\right)F_i^\alpha F_j^\beta\\ &\quad +g_{\alpha\beta}\partial_iV_0^\alpha F_j^\beta+g_{\alpha\beta}F_i^\alpha\partial_jV_0^\beta\ .\\ &= g_{\alpha\beta}\left(\partial_iV^\alpha\right)F_j^\beta+g_{\alpha\beta}F_i^\alpha\partial_jV^\beta-V^\rho\left(g_{\kappa\beta}\Lambda_{\alpha\rho}^\kappa+g_{\alpha\kappa}\Lambda_{\rho\beta}^\kappa\right)F_i^\alpha F_j^\beta\end{aligned}$$

and

$$\partial_0\left(\left(g_{\alpha\beta}\circ\Phi\right)F_i^\alpha F_j^\beta\right)=\left(g_{\alpha\beta}\left(\partial_iV^\alpha\right)-V^\rho g_{\kappa\beta}\Lambda_{\alpha\rho}^\kappa F_i^\alpha\right)F_j^\beta+\left(g_{\alpha\beta}\partial_jV^\beta-V^\rho g_{\alpha\kappa}\Lambda_{\rho\beta}^\kappa F_j^\beta\right)F_i^\alpha\ .$$

Since S is symmetric we finally have

$$\begin{aligned}&\hat{\otimes}\left(S\rfloor\frac{1}{2}\left(\partial_0C\right)\right)\wedge\mathrm{d}t^0 =\\ &\qquad\hat{\otimes}\left(S^{ij}\partial_i\rfloor\mathrm{VOL}\otimes\partial_j\rfloor\left(\left(\partial_iV^\alpha-\Lambda_{\tau\rho}^\alpha V^\rho F_i^\tau\right)g_{\alpha\beta}F_j^\beta\right)\mathrm{d}X^i\otimes\mathrm{d}X^j\right)\wedge\mathrm{d}t^0\end{aligned}$$

and

$$\hat{\otimes}\left(S\rfloor\frac{1}{2}\left(\partial_0 C\right)\right)\wedge \mathrm{d}t^0 = S^{ij}\left(\partial_i V^\alpha - \Lambda^\alpha_{\tau\rho}V^\rho F^\tau_i\right) g_{\alpha\beta}F^\beta_j \mathrm{VOL}\wedge \mathrm{d}t^0$$

which proves the assertion.

Remark A.2 *The tensor D which meets*

$$D\wedge \mathrm{d}t^0 = \frac{1}{2}\left(\partial_0 C\right)\wedge \mathrm{d}t^0$$

and in coordinates this gives

$$D\wedge \mathrm{d}t^0 = \frac{1}{2}\left(g_{\alpha\beta}\left(\partial_i V^\alpha - V^\rho\Lambda^\alpha_{\nu\rho}F^\nu_i\right)F^\beta_j + g_{\alpha\beta}\left(\partial_j V^\beta - V^\rho\Lambda^\beta_{\rho\mu}F^\mu_j\right)F^\alpha_i\right)\mathrm{d}X^i\otimes \mathrm{d}X^j\wedge \mathrm{d}t^0$$

is called the rate of the deformation and has the components

$$D_{ij} = \frac{1}{2}g_{\alpha\beta}\left(\left(\partial_i V^\alpha - V^\rho\Lambda^\alpha_{\nu\rho}F^\nu_i\right)F^\beta_j + \left(\partial_j V^\beta - V^\rho\Lambda^\beta_{\rho\mu}F^\mu_j\right)F^\alpha_i\right)\ .$$

A.8.4 Mass Balance Lagrangian Picture

We want to show that

$$\int_{\mathcal{S}}\partial_0\left(\rho_{\mathcal{R}}\mathrm{VOL}\wedge \mathrm{d}t^0\right) = 0$$

implies the condition

$$\partial_0(\rho_{\mathcal{R}}\sqrt{|\det(G_{ij})|}) = 0$$

and therefore we start with

$$\begin{aligned}\partial_0\left(\rho_{\mathcal{R}}\mathrm{VOL}\wedge \mathrm{d}t^0\right) &= \partial_0\left(\rho_{\mathcal{R}}\sqrt{|\det(G_{ij})|}\mathrm{d}X^1\wedge\ldots\wedge \mathrm{d}X^n\wedge \mathrm{d}t^0\right)\\ &= \partial_0\left(\rho_{\mathcal{R}}\sqrt{|\det(G_{ij})|}\mathrm{VOL}^0\right)\ .\end{aligned}$$

We obtain

$$\partial_0\left(\rho_{\mathcal{R}}\mathrm{VOL}\wedge \mathrm{d}t^0\right) = \partial_0\left(\rho_{\mathcal{R}}\sqrt{|\det(G_{ij})|}\right)\mathrm{VOL}^0 + \rho_{\mathcal{R}}\sqrt{|\det(G_{ij})|}(\partial_0\rfloor \mathrm{d}(\mathrm{VOL}^0) + \mathrm{d}(\partial_0\rfloor\mathrm{VOL}^0))$$

which follows to

$$\partial_0\left(\rho_{\mathcal{R}}\mathrm{VOL}\wedge \mathrm{d}t^0\right) = \partial_0\left(\rho_{\mathcal{R}}\sqrt{|\det(G_{ij})|}\right)\mathrm{VOL}^0\ .$$

A.8.5 Variational Principles

We first want to prove the relation concerning the stored energy function

$$\delta_\alpha\left(\rho_{\mathcal{R}} E_{el}\sqrt{|\det(G_{ij})|}\right) = -\sqrt{|\det(G_{ij})|}\breve{P}^{k\tau} g_{\eta\alpha}\Lambda^{\eta *}_{\tau\beta} q^\beta_k - d_k(\sqrt{|\det(G_{ij})|}\breve{P}^{k\tau})g_{\alpha\tau}$$

with

$$2\rho_{\mathcal{R}}\frac{\partial}{\partial\breve{C}_{ij}}E_{el} = \breve{S}^{ij}$$

and

$$\breve{C}_{ij} = q^\alpha_i g_{\alpha\beta} q^\beta_j \ .$$

Consequently

$$\delta_\alpha\left(\rho_{\mathcal{R}} E_{el}\sqrt{|\det(G_{ij})|}\right) = \left(\partial_\alpha - d_i\partial^i_\alpha\right)(\rho_{\mathcal{R}} E_{el}\sqrt{|\det(G_{ij})|})$$

and the right hand side gives

$$\sqrt{|\det(G_{ij})|}\rho_{\mathcal{R}}\frac{\partial}{\partial\breve{C}_{ij}}E_{el}\partial_\alpha\breve{C}_{ij} - d_k\left(\sqrt{|\det(G_{ij})|}\rho_{\mathcal{R}}\frac{\partial}{\partial\breve{C}_{ij}}E_{el}\partial^k_\alpha\breve{C}_{ij}\right)$$

as well as

$$\sqrt{|\det(G_{ij})|}\frac{1}{2}\breve{S}^{ij}\partial_\alpha(q^\rho_i g_{\rho\tau} q^\tau_j) - d_k\left(\frac{1}{2}\sqrt{|\det(G_{ij})|}\breve{S}^{ij}\partial^k_\alpha(q^\rho_i g_{\rho\tau} q^\tau_j)\right) \ .$$

This follows to

$$\begin{aligned}\delta_\alpha\left(\rho_{\mathcal{R}} E_{el}\sqrt{|\det(G_{ij})|}\right) &= \sqrt{|\det(G_{ij})|}\frac{1}{2}\breve{S}^{ij} q^\rho_i q^\tau_j \partial_\alpha g_{\rho\tau} \\ &\quad -d_k\left(\frac{1}{2}\sqrt{|\det(G_{ij})|}(\breve{S}^{kj} g_{\alpha\tau} q^\tau_j + \breve{S}^{ik} q^\rho_i g_{\rho\alpha})\right)\end{aligned}$$

and

$$\delta_\alpha\left(\rho_{\mathcal{R}} E_{el}\sqrt{|\det(G_{ij})|}\right) = \sqrt{|\det(G_{ij})|}\frac{1}{2}\breve{S}^{ij} q^\rho_i q^\tau_j \partial_\alpha g_{\rho\tau} - d_k(\sqrt{|\det(G_{ij})|}\breve{S}^{kj} g_{\alpha\tau} q^\tau_j) \ .$$

The relations (3.17) which read as

$$(\partial_\alpha g_{\tau\rho}) = -g_{\beta\rho}\Lambda^\beta_{\tau\alpha} - g_{\tau\beta}\Lambda^\beta_{\alpha\rho}$$

lead to

$$\delta_\alpha\left(\rho_{\mathcal{R}} E_{el}\sqrt{|\det(G_{ij})|}\right) = -\sqrt{|\det(G_{ij})|}\breve{S}^{ij} q^\rho_i q^\tau_j (g_{\beta\rho}\Lambda^\beta_{\tau\alpha}) - d_k(\sqrt{|\det(G_{ij})|}\breve{S}^{kj} g_{\alpha\tau} q^\tau_j)$$

and

$$\begin{aligned}\delta_\alpha\left(\rho_{\mathcal{R}}E_{el}\sqrt{|\det(G_{ij})|}\right) &= \sqrt{|\det(G_{ij})|}\breve{P}^{j\rho}q_j^\tau g_{\beta\rho}\Lambda^{\beta *}_{\tau\alpha}\\ &\quad -d_k(\sqrt{|\det(G_{ij})|}\breve{P}^{k\tau})g_{\alpha\tau} - \sqrt{|\det(G_{ij})|}\breve{P}^{k\tau}\partial_\beta(g_{\alpha\tau})q_k^\beta\ .\end{aligned}$$

Finally we obtain

$$\begin{aligned}\delta_\alpha\left(\rho_{\mathcal{R}}E_{el}\sqrt{|\det(G_{ij})|}\right) &= \sqrt{|\det(G_{ij})|}\breve{P}^{j\rho}q_j^\tau g_{\beta\rho}\Lambda^{\beta *}_{\tau\alpha} - d_k(\sqrt{|\det(G_{ij})|}\breve{P}^{k\tau})g_{\alpha\tau}\\ &\quad -\sqrt{|\det(G_{ij})|}\breve{P}^{k\tau}(g_{\phi\alpha}\Lambda^{\phi *}_{\beta\tau} + g_{\phi\tau}\Lambda^{\phi *}_{\alpha\beta})q_k^\beta\end{aligned}$$

and

$$\delta_\alpha\left(\rho_{\mathcal{R}}E_{el}\sqrt{|\det(G_{ij})|}\right) = -\sqrt{|\det(G_{ij})|}\breve{P}^{k\tau}g_{\eta\alpha}\Lambda^{\eta *}_{\tau\beta}q_k^\beta - d_k(\sqrt{|\det(G_{ij})|}\breve{P}^{k\tau})g_{\alpha\tau}$$

which is the desired relation.

The expression involving the kinetic energy function

$$\begin{aligned}&\delta_\alpha\left(\rho_{\mathcal{R}}E_k\sqrt{|\det(G_{ij})|}\right) =\\ &-\rho_{\mathcal{R}}\sqrt{|\det(G_{ij})|}g_{\phi\alpha}\left[q_{00}^\phi - \partial_0\left(\gamma_0^\phi\circ\Phi\right) - \left(q_0^\kappa - \gamma_0^\kappa\right)\partial_\kappa\gamma_0^\phi - \Lambda^\phi_{\beta\sigma}\left(q_0^\beta - \gamma_0^\beta\right)\left(q_0^\sigma - \gamma_0^\sigma\right)\right]\end{aligned}$$

can be shown rather easily based on the calculations given in the sections A.6.2 and A.8.4 of the Appendix, since we have

$$\begin{aligned}&\delta_\alpha\left(\rho_{\mathcal{R}}\frac{1}{2}(q_0^\alpha - \gamma_0^\alpha)g_{\alpha\beta}(q_0^\beta - \gamma_0^\beta)\sqrt{|\det(G_{ij})|}\right) =\\ &\qquad\left(\partial_\alpha - d_0\partial_\alpha^0\right)\left(\rho_{\mathcal{R}}\frac{1}{2}(q_0^\alpha - \gamma_0^\alpha)g_{\alpha\beta}(q_0^\beta - \gamma_0^\beta)\sqrt{|\det(G_{ij})|}\right)\end{aligned}$$

and the observation that

$$\begin{aligned}\partial_\alpha\left(\rho_{\mathcal{R}}\sqrt{|\det(G_{ij})|}\right) &= 0\\ d_0\left(\rho_{\mathcal{R}}\sqrt{|\det(G_{ij})|}\right) &= \partial_0\left(\rho_{\mathcal{R}}\sqrt{|\det(G_{ij})|}\right) = 0\end{aligned}$$

where the second equation follows from the mass balance in the Lagrangian description.

Appendix B

Afterword

This work has been done in the context of a DOC scholarship of the Austrian Academy of Sciences (OEAW).

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Eidesstattliche Erklärung

Ich erkläre an Eides statt, dass ich die vorliegende Dissertation selbstständig und ohne fremde Hilfe verfasst, andere als die angegebenen Quellen und Hilfsmittel nicht benutzt bzw. die wörtlich oder sinngemäß entnommenen Stellen als solche kenntlich gemacht habe.

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