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Infinite-dimensional Euler-Lagrange and Port Hamiltonian Systems

DISSERTATION

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Kurzfassung

Die Bestimmung eines mathematischen Modells stellt einen sehr wichtigen Schritt in der Analyse und Regelung von physikalischen Systemen dar. Die erhaltene mathematische Beschreibung liefert die Grundlage für die Simulation, den Reglerentwurf, die Stabilitätsanalyse oder aber auch für einen Neuentwurf der Anlage. Im Rahmen dieser Arbeit werden primär zwei mathematische Modellierungswerkzeuge behandelt, welche auf eine besonders strukturierte Systemdarstellungen führen. Die behandelten Systemklassen – welche im Folgenden als Euler-Lagrange und Port Hamilton'sche Systeme bezeichnet werden – sind bereits seit langem in der Analyse und Regelung von finit-dimensionalen Systemen in Verwendung. Diese Arbeit untersucht nun deren Anwendbarkeit auf infinit-dimensionale Systeme, wobei auf die geometrische Struktur der Gleichungen und Randbedingungen besonderer Wert gelegt wird.

Im ersten, einführenden Kapitel werden die Variationsrechnung und geometrische Strukturen auf Mannigfaltigkeiten untersucht, um einige Fragestellungen aufzuwerfen. Die Beantwortung dieser Fragen geschieht in den nachfolgenden Kapiteln.

Ein robustes, mathematisches Regelwerk ist unverzichtbar für die Untersuchung von infinit-dimensionalen Systemen. Im Abschnitt I sind einige Kapitel der Einführung des Eckpfeilers der nachfolgenden Untersuchungen – der Theorie von Jet Bündeln – gewidmet.

Nach diesem eher formalen Teil wird das vorgestellte Regelwerk bei der Analyse von Euler-Lagrange Systemen angewendet. Die Bewegungsgleichungen dieser Systemklasse folgen aus einem Variationsprinzip – dem Prinzip der kleinsten Wirkung. Die Formulierung dieses Prinzips auf der Basis von Jet Bündeln wird als Ausgangspunkt für die Untersuchung des finit- und infinit-dimensionalen Falls verwendet. Es wird gezeigt, dass die Bestimmung der Randbedingungen durch die Einführung der erweiterten Cartan Form möglich ist. Einige Untersuchungen zur zeitlichen Evolution von Euler-Lagrange System und ein Anwendungsbeispiel zur erarbeiteten Theorie beschließen diesen Abschnitt.

Die Analyse von Euler-Lagrange Systemen macht eine bestimmte Struktur in den Gleichungen sichtbar, deren Verallgemeinerung zur Klasse der „Port Hamiltonian Systems with Dissipation" (kurz pHd Systeme) führt. Im Abschnitt III wird die geometrische Darstellung von finit-dimensionalen pHd Systemen eingeführt und verwendet, um eine entsprechende infinit-dimensionale Version zu erarbeiten. Wie im Fall der Euler-Lagrange Systeme wird die geometrische Struktur und das Auftreten der Randbedingungen, oder vielmehr der Randtore (boundary ports), untersucht. Danach wird das Verhalten der Systembeschreibung im Hinblick auf Zusammenschaltungen untersucht.

Der letzte Abschnitt dieser Arbeit ist der Regelung von infinit-dimensionalen Systemen gewidmet. Zuerst wird die Stabilität von finit-dimensionalen Systemen im Sinne von Lyapunov wiederholt. Diese wohlbekannten Resultate werden im Weiteren verwendet um eine erweiterte Version der Stabilitätsdefinition im Sinne von Lyapunov für infinit-dimensionale Systeme zu erhalten. Einige allgemeine Bemerkungen bzgl. Reglerentwurf beenden diesen Abschnitt.

Der Anhang enthält eine Sammlung von mathematischen Definitionen, die Bestimmung der Bewegungsgleichungen der Kirchhoff-Platte unter Verwendung von partieller Integration und eine Beschreibung des Maple Packets „JetVariationalCalculus", welche eine Implementierung der Algorithmen aus Abschnitt II entspricht.

Abstract

Modeling is an essential, or rather the most important step in the analysis and control of physical systems. The derived mathematical description of a plant serves as a basis for simulation, controller design, stability analysis, and even redesign of the whole structure. This thesis treats mainly two mathematical modeling tools, which supply system representations with particular rich structure. These system classes – referred to as Euler-Lagrange and port Hamiltonian systems – are well established for the analysis and control of finite-dimensional systems. This thesis will investigate their applicability for infinite-dimensional systems. Special attention is spent on the geometric structure and the determination of the boundary conditions.

The first, introductory chapter is dedicated to a short review on calculus of variations and structures on manifolds. This review is used to state several questions whose answering is the content of the main chapters of this thesis.

A robust mathematical framework is indispensable for the treatment of infinite-dimensional systems. In part I several chapters are spent on the introduction of the cornerstone of the subsequent analysis – the theory of jet bundles. These chapters are additionally dedicated to the definition of the used notation, which will turn out to be a crucial point of the subsequent derivations.

After this rather formal part the introduced mathematical framework is applied to the analysis of Euler-Lagrange systems. These systems are characterized by the fact, that their equations of motion could be extracted from a variational principle – Hamilton's principle. The formulation of this principle in the language of jet bundles serves as the point of departure for the analysis of the finite- and infinite-dimension case. It will be shown, that the derivation of the boundary conditions of such systems could be done by the introduction of the extended Cartan form. Some considerations on the time evolution of Euler-Lagrange systems and an application close this part.

The presented analysis of Euler-Lagrange systems makes a certain structure visible, whose generalization leads to the class of port Hamiltonian systems with dissipation – or pHd systems for short. In part III the geometric representation of finite-dimensional pHd systems is introduced and used for the definition of a corresponding infinite-dimensional version. Again the geometric construction of the boundary conditions, or rather boundary ports, is investigated. After that it is possible to analyze the behavior of infinite-dimensional pHd systems with respect to interconnection. An example closes this part.

The final part of this thesis is dedicated to the control of infinite-dimensional systems. At first the stability of the finite-dimensional systems in the sense of Lyapunov is recalled. These well known results are used to introduce an extended version for the stability of infinite-dimensional systems. Finally this criterion based on Sobolev norms is applied to an I-pHd system. Some remarks on infinite-dimensional controller design close this part.

The appendix includes a rather dense collection of mathematical definitions, the determination of the equations of motion for the Kirchhoff-plate using the integration by parts technique, and the description of the computer algebra package "JetVariationalCalculus", which corresponds to the results presented in part II.

To Katharina and Jakob

… to understand is all I ever wanted - hopefully I did!

Preface

Antoine de Saint-Exupéry presents in his book “The Little Prince” his painting depicted in figure 1.

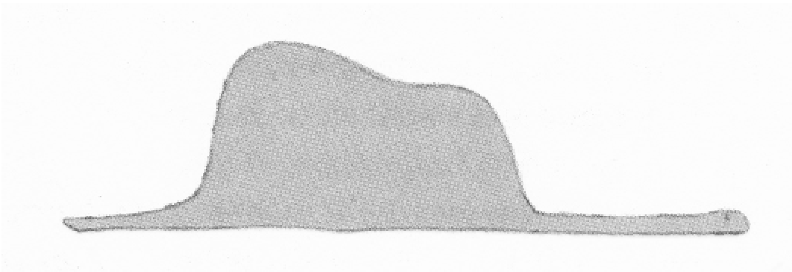

Figure 1: The hat

He further explains, that all people in his surrounding misinterpreted the contents of this painting as a simple hat. But in fact he painted a snake, that had swallowed an elephant.

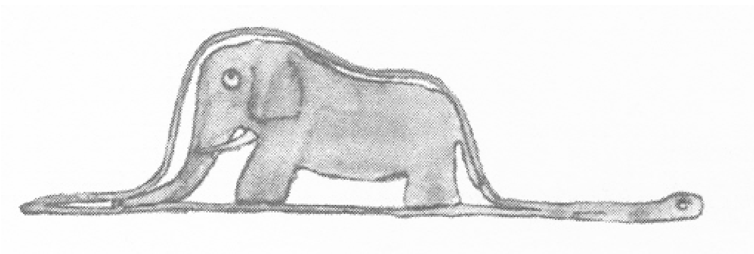

Figure 2: The snake with elephant

These pictures and the experience, the six years old Antoine made, give me the perfect opportunity to explain the emotions I had and impressions I got throughout the preparation of this thesis.

Prof. Kurt Schlacher invited me to participate as a Ph.D. student in the European sponsored project GeoPlex (www.geoplex.cc) in the year 2002. During my work at the Institute of Automatic Control and Control Systems Technology at the JK university Linz, he introduced me to the theory of the calculus of variations and port Hamiltonian systems. He had derived a pomising algorithm for the determination of the Euler-Lagrange equations and I started to implement these ideas within a computer algebra package. This situation represents more or less the “hat” of the story. Unfortunately I discovered the “snake with the swallowed elephant” in the form of unsatisfactory boundary conditions resulting from the derived theory for higher order problems in the year 2003.

An unbelievable high number of wrong ideas for the solution of this problems caused the same kind of uncertainty, that must have been familiar to the snake with this huge elephant in its stomach and could be summarized by a simple question: “wasn’t it to big for me?”. In fact the used mathematical framework is far beyond that, what is taught during the education of technical engineers. Additionally the investigated mathematical problems in their highest generality are not directly linked to physical or even engineering problems that are known to me.

Fortunately all these discouraging circumstances were balanced by small successes that appeared from time to time during my calculations. Additionally the fruitful discussions with almost every colleague at the Institute of Automatic Control and Control Systems Technology supplied indispensable support and encouraged me to continue my investigations.

I would like to thank Richard Stadlmayr and Martin Staudecker for reminding me of the relationship between down-to-earth problems and jet theory. From a scientific point of view Markus Schöberl, Bernhard Roider, Gernot Grabmair, and Kurt Zehetleitner have owned special regards. Beside the already mentioned colleagues I would like to thank Hannes Seyrkammer, Johann Holl, Stefan Fuchshumer, Harald Pachler, and Brigitta Peitl for the marvellous working atmosphere. Last but not least I would like to thank Prof. Dr. Kurt Schlacher for his never ending support and his clever advices.

The scientific work in the last four years convinced me in the fact that science is not a „linear working area“. One has to accept, that the increase of working power, or rather personal sacrifice does not necessarily lead to better scientific output. Cognitions are the result of hard work, a lot of discussions, and in the end luck. Especially people responsible for the funding of scientific research should always be aware of this fact and accept that not every grain of seed will yield a large crop.

Finally I hope that the subsequent results, whose generation was sometimes painful, gruelling, but also exciting and thrilling, are correct and useful in the scientific evolution of the theory of automatic control.

Linz, 14. Februar 2006 Helmut Ennsbrunner

Contents

Chapter 1

Introduction

The symbiotic coexistence of mathematics, physics, and engineering has a long tradition. Problems of all three disciplines caused the discovery of cognitions and solutions in each other scientific area. This thesis will treat a physical principle, its mathematical formulation and solution, and finally the implementation of the derived algorithms in computer algebra. The generated software is intended to be used in the modeling of physical systems by engineers. Thus all three areas are linked in the upcoming investigations.

Control theory supplies information about the properties of dynamic systems by means of the analysis of the corresponding mathematical models. Thus the mathematical description of systems is a crucial step in the derivation of control structures and stability investigations.

The derivation of mathematical models – known as modeling – is a procedure that differs drastically between different scientific areas as, e.g., economics, biology, mechanics. Throughout this thesis we will confine ourselves to the case of physical models, whose derivation is based on physical principles. In the following a short summary of well known physical principles, that are used by engineers, is given.

1.1 Physical Principles

Every engineering field is equipped with certain fundamental principles and equations. In the following enumeration several examples from mechanics, electrodynamics and thermodynamics are listed.

- conservation of mass
- conservation of momentum
- conservation of angular momentum
- conservation of energy
- Kirchhoff's current and voltage law
- Maxwell's equations
- Hamilton's principle

■

A very useful and efficient principle is the *Hamilton's principle*, which states, that a special functional $\mathfrak{L}$ is minimized by the actual motion of a system [Olver, 1986]. This principle can be evaluated by the introduction of the calculus of variations and enables the determination of the so called Euler-Lagrange equations. These equations represent only a necessary condition for the functional $\mathfrak{L}$ to be minimal.

Remark 1.1 *Variational principles like, e.g., the* Hamilton's principles*, open up new possibilities in mathematical physics or control, since they offer more insight into the structure of a problem. Therefore, it is often meaningful to study the Euler-Lagrange equations even if their solutions do not coincide with the minimal solution of the variational problem belonging to them.*

The Euler-Lagrange equations equal the equations of motion of certain systems, that bring along a variational principle like, e.g., Hamilton's principle of least action in mechanics [Gelfand, S.V. Fomin, 2000]. Thus the mathematical solution of Hamilton's principle in the framework of the calculus of variations equals the determination of the equations of motions and is consequently of particular interest.

The following section is dedicated to the introduction of the calculus of variations as it is widely accepted in engineering disciplines and defined in, e.g., [Gelfand, S.V. Fomin, 2000]. This presentation will serve as a basis for several questions, whose answering will be the content of part II of this thesis.

1.2 Calculus of Variations

Already the antique scientists studied problems to find a minimal solution like the isoperimetric problem, etc. Great interest in extremal problems has lead to the development of different variational problems in mathematics, physics or control in the last centuries. The classical calculus of variations studies the so called variational functional, like the simple example

$$\mathfrak{L} = \int_{t_1}^{t_2} l\left(t, x\left(t\right), \dot{x}\left(t\right)\right) \mathrm{d}t \ ,$$

where the boundary points $t_1, t_2 \in \mathbb{R}$, $t_1 < t_2$ are fixed and the integrand depends on the independent variable t, on a sufficiently smooth function x, as well as its first time derivative. The function l denotes the Lagrangian density. The problem is to find a function $x\left(t\right)$ such that the functional is minimal.

1.2.1 Classical Approach

The calculus of variations, also denoted as variational calculus, is a mathematical methodology that investigates minimal (or at least extremal) points of variational functionals. Subsequently we will consider the functional defined by

$$\mathfrak{L}\left(x\right) = \int_{\mathcal{D}} l\left(X, x^{(n)}\left(X\right)\right) \mathrm{d}X^1 \ldots \mathrm{d}X^r \ . \tag{1.1}$$

Here the Lagrangian density l is a sufficiently smooth function of the independent coordinates $X^i, i = 1, \ldots, r$, the n^{th} order derivatives $x^{(n)}$ of the dependent coordinates x, and the dependent coordinates x^α, $\alpha = 1, \ldots, s$ on the domain $\mathcal{D}$.

A minimal point of the functional $\mathfrak{L}$ equals an element $f : X^i \to f^\alpha (X^i)$ of a certain normed linear function space $F_{(n)}$ such that

$$\mathfrak{L}(x) \geq \mathfrak{L}(f) \quad \forall x \in F_{(n)}$$

is met. There exist several methods to determine the extremal points, i.e. the functions $f \in F_{(n)}$ as, e.g., minimizing sequences or the Ritz method (see [Gelfand, S.V. Fomin, 2000]). Here we will make use of a method which is due to Euler. This method takes into account, that the variational functional is a continuous mapping

$$\begin{aligned} \mathfrak{L}(\cdot) : F_{(n)} &\to \mathbb{R} \\ x &\to \mathfrak{L}(x) \ . \end{aligned}$$

Consequently the functional supplies for every $\varepsilon > 0$ a $\delta > 0$, such that from

$$|\mathfrak{L}(x) - \mathfrak{L}(f)| < \varepsilon$$

results

$$\|x - f\| < \delta \ .$$

Here, the norm

$$\|x\| = \sum_{r=0}^{n} \sum_{\alpha=1}^{s} \max_{X \in \mathcal{D}} \left| (x^\alpha)^{(r)} (X) \right|$$

is used. Thus one is able to derive a necessary condition, which must be met by all local extrema.

An at least local extremum (minimum resp. maximum) $f \in F_{(n)}$ is characterized by

$$\mathfrak{L}\left(f + f^{'}\right) \geq \mathfrak{L}(f) \ \text{resp.} \ \mathfrak{L}\left(f + f^{'}\right) \leq \mathfrak{L}(f)$$

for admissible functions $f^{'} \in F_{(n)}$.

This enables the definition of a new, but in general nonlinear functional for a fixed function $x(X)$ given by

$$\triangle\mathfrak{L}\left(f^{'}\right) = \mathfrak{L}\left(x + f^{'}\right) - \mathfrak{L}(x) \quad , \ x \in F_{(n)}.$$

The linear part $\delta\mathfrak{L}(f')$ of this functional

$$\triangle\mathfrak{L}\left(f^{'}\right) = \delta\mathfrak{L}(f') + \varepsilon \left\|f^{'}\right\|$$

is called the variation of the functional $\mathfrak{L}(x)$. The quantity $\varepsilon \left\|f^{'}\right\|$ incorporates higher order terms and meets $\varepsilon \to 0$ as $\left\|f^{'}\right\| \to 0$.

Theorem 1.2 *A necessary condition for the differentiable functional to have an extremum at f is that its variation vanishes for $x = f$, i.e. that*

$$\delta\mathfrak{L}(f') = 0 \tag{1.2}$$

for all admissible f'. (see [Gelfand, S.V. Fomin, 2000])

This relation marks all points $f \in F_{(n)}$ having a “horizontal tangent” over the infinite dimensional function space $F_{(n)}$ by means of partial differential equations.

1.2.2 Questions

The previous considerations lead to the following questions.

- What are the topological properties met by the domain $\mathcal{D}$?
- How could the arbitrary increments $f^{'}$ be represented in more general spaces?
- How could one derive correct boundary conditions from equation (1.2)?
- How do external inputs come into play?
- Is it possible to provide additional information about the evolution of such Euler-Lagrange systems?

All these questions will be treated in the chapters 5 to 8.

Chapter 9 is dedicated to the investigation of the time evolution of Euler-Lagrange systems. It will turn out that the derived equations are equipped with a particular rich structure.

In part III we will consider more general systems equipped with a certain structure. Additionally, we will not require that the structure results from a variational principle. In the following we introduce some well known structures on manifolds. The definitions of the used mathematical objects can be found in part I and appendix A.

1.3 Structures on Manifolds

A structure on a manifold can be seen as an additional underlying property, which could be very useful in the analysis of, e.g., dynamic systems on manifolds. Both subsequently discussed structures are well known for finite dimensional systems.

1.3.1 Poission Structure

The general definition of a Poisson structure

Definition 1.3 (Poisson structure) *Let $\mathcal{M}$ be a manifold and let $C^\infty(\mathcal{M})$ denote all smooth real functions on $\mathcal{M}$. A Poisson structure on $\mathcal{M}$ is a bilinear map – called the Poisson bracket – given by*

$$\begin{array}{rccc} \{F,G\}: & C^\infty(\mathcal{M}) \times C^\infty(\mathcal{M}) & \longrightarrow & C^\infty(\mathcal{M}) \\ & (F,G) & \longrightarrow & \{F,G\} \end{array}$$

which satisfies for $F, G, H \in C^\infty(\mathcal{M})$

- $\{F,G\} = -\{G,F\}$ *(skew symmetry)*
- $\{F,\{G,H\}\} + \{G,\{H,F\}\} + \{H,\{F,G\}\} = 0$ *(Jakobi identity)*
- $\{F,GH\} = \{F,G\}H + G\{F,H\}$ *(Leibniz rule)*

(see, e.g., [Nijmeijer, A.J. van der Schaft, 1990])

leads to the introduction of the Hamilton vector field v_F of a Hamilton function F by

$$\{F, G\} = \mathrm{L}_{v_F}(G) = v_F \rfloor \mathrm{d}G \ ,$$

where the Lie derivative L and the exterior derivative d are used.

Let $\mathcal{M}$ be a Poisson manifold with local coordinates $(x^1, \ldots, x^r)$. Then there exist locally smooth functions $w^{ij}(x)\,,\ i,j \in 1 \ldots r$, such that the Poisson bracket is given by

$$\{F, G\}(x) = \sum_{i,j=1}^{r} w^{ij}(x) \frac{\partial F}{\partial x^i}(x) \frac{\partial G}{\partial x^j}(x) \ . \tag{1.3}$$

Since

$$\begin{aligned} \{F, G\}(x) &= \mathrm{L}_{v_F}(G) = \mathrm{d}G(v_F)(x) = \frac{\partial G}{\partial x^j} v_F^j(x) \\ \{F, G\}(x) &= -\{G, F\}(x) = -\mathrm{L}_{v_G}(F)(x) = -\mathrm{d}F(v_G)(x) = -\frac{\partial F}{\partial x^i} v_G^i(x) \end{aligned}$$

implies $\frac{\partial G}{\partial x^j} v_F^j(x) = -\frac{\partial F}{\partial x^i} v_G^i(x)$, this definition is only possible if additionally

$$w^{ij}(x) = -w^{ji}(x)$$

is met. By the fact that

$$w^{ij}(x) = \left\{x^i, x^j\right\}$$

and that the Jacobi identity $\left\{x^i, \left\{x^j, x^k\right\}\right\} + \left\{x^j, \left\{x^k, x^i\right\}\right\} + \left\{x^k, \left\{x^i, x^j\right\}\right\} = 0$ must be met, the functions $w^{ij}(x)$ are also restricted to

$$\sum_{l=1}^{r} \left(w^{lj} \frac{\partial w^{ik}}{\partial x^l} + w^{li} \frac{\partial w^{kj}}{\partial x^l} + w^{lk} \frac{\partial w^{ji}}{\partial x^l} \right) = 0. \tag{1.4}$$

It is worth mentioning that this restriction represents an integrability condition.

One could extract from (1.3) a map

$$\begin{aligned} W(x): \ \mathcal{T}^*(\mathcal{M}) &\longrightarrow \mathcal{T}(\mathcal{M}) \\ \mathrm{d}F &\longrightarrow v_F = w^{ij}(x) \tfrac{\partial F}{\partial x^i}(x)\, \partial_j \end{aligned} \tag{1.5}$$

in order to derive the Poisson bracket $\{F, G\} = v_F \rfloor \mathrm{d}G$. We conclude, that the Poisson bracket is determined by a skew symmetric tensor field $W(x) = w^{ij}(x)\, \partial_i \otimes \partial_j$.

$$\begin{tikzcd} \mathcal{T}(\mathcal{M}) \arrow[dr, "\tau_{\mathcal{M}}"'] & & \mathcal{T}^*(\mathcal{M}) \arrow[ll, "W"'] \arrow[dl, "\bar{\tau}_{\mathcal{M}}"] \\ & \mathcal{M} & C^\infty(\mathcal{M}) \arrow[u, "\mathrm{d}"'] \end{tikzcd}$$

If the rank of the Poisson bracket, which equals the rank of the structure matrix $w^{ij}(x)$, is equal to $\dim(\mathcal{M})$ then it is called *nondegenerated*.

1.3.2 Symplectic Structure

In contrary to the Poisson structure the symplectic structure is directly defined by a tensor field with special properties.

Definition 1.4 *A* 2*-form* $\omega = \omega_{ij} \mathrm{d}x^i \wedge \mathrm{d}x^j \in \bigwedge^2 T^* (\mathcal{M})$ *on an even dimensional manifold* $\mathcal{M}$ *of dimension* $2n$ *is called symplectic (and then* $\mathcal{M}$ *is called symplectic manifold), iff it satisfies*

- $\mathrm{d}\omega = 0$ *(closed)*
- ω *is nondegenerate; that is, the linear transformation associating to a vector* $v = v^i \partial_i$ *the 1-form* $v \rfloor \omega$ *is nonsingular. In local coordinates* x*, since* $v \rfloor \omega = v^i \omega_{ij} \mathrm{d}x^j$*, this merely says* $\det (\omega_{ij}) \neq 0$

(see, e.g., [Frankel, 1997])

Actually a symplectic 2-form is a map

$$\begin{array}{rccc} \omega: & \mathcal{T}(\mathcal{M}) & \longrightarrow & \mathcal{T}^*(\mathcal{M}) \\ & v_H & \longrightarrow & v_H \rfloor \omega \end{array} \tag{1.6}$$

visualized in

$$\begin{tikzcd} \mathcal{T}(\mathcal{M}) \arrow[rr, "\omega"] \arrow[dr, "\tau_{\mathcal{M}}"'] & & \mathcal{T}^*(\mathcal{M}) \arrow[dl, "\bar{\tau}_{\mathcal{M}}"] \\ & \mathcal{M} & C^\infty(\mathcal{M}) \arrow[u, "\mathrm{d}"'] \end{tikzcd}$$

and consequently the dual mapping to (1.5).

It is now remarkable, that we are able to identify – in the case of a nondegenerated Poisson bracket on an even dimensional manifold – the skew symmetric tensor field W with a symplectic 2-form. From

$$\{F, G\} = \omega (v_F, v_G) = v_G \rfloor v_F \rfloor \omega$$

it follows that

$$v_G \rfloor v_F \rfloor \omega = -v_G \rfloor \mathrm{d}F \quad \Rightarrow \quad v_F \rfloor \omega = -\mathrm{d}F$$

and in local coordinates we get the correspondence

$$\omega_{ij} = - [W]^{-1}_{ij} \, .$$

Both, the Poisson and symplectic structure represent a map between the tangent and the cotangent bundle. Additionally these maps have the property that the image of the map applied on its source, by means of the interior product, always vanishes, i.e. the image is restricted to be an annihilator of the source. This consideration could be extended and summarized in the notion of a Dirac structure, whereby all elements of tangent and cotangent bundle are marked, that – beside other properties – annihilate each other (see [van der Schaft, 2000]). Consequently Poisson and symplectic structures are dirac structures.

From the modeling point of view all these structures become of interest, if the equations of motion determine such a structure. For example, if the equations of motion represent a Hamilton vector field it is possible to make use of the corresponding Poisson structure.

These considerations lead now again to several questions, that are of interest.

1.3.3 Questions

- How could dissipative elements be taken into account?
- How could this approach be extended to the case of infinite-dimensional systems?
- Which geometric objects could be used for representation?
- How do in- and outputs – or more generally speaking – ports appear in this setting and what is meant by collocation?
- How are boundary conditions, or rather boundary ports taken into account?

In part III we will consider finite- and infinite-dimensional systems, that are equipped with a Poisson structure and try to answer the stated questions.

1.4 Control

The last part of this thesis is dedicated to topics that are related to the design of controllers for infinite-dimensional systems. In fact no methodology for a certain design will be derived, but several remarks and general questions concerning this task are stated. The central question will be of course the stability of infinite-dimensional systems, as this is inseparably linked to the design of a controller. It will be shown, how the stability in the sense of Lyapunov could be used to determine certain stability criteria based on Sobolev norms.

Part I

The Mathematical Framework

Every challenge demands the introduction of appropriate tools!

This rather general statement applies in many cases. Even in everyday life it is common to adjust ones tool box before manipulating, e.g., the electrical installation. Also in computer science the implementation of complex software is done by the definition of a sophisticated class structure. Thereby programmers divide the problem is several subtasks, focus on their solution and generate frameworks. Such frameworks enable other coders to make use of techniques who are far beyond their own knowledge.

All these examples have in common that the described tools are designed by experts. These experts generate tools, structures, and mechanisms that ease the solution of problems by users. Clearly these users must be instructed in the application of such tools in order to achieve the results of interest.

Control engineering represents another example of such a development. Several questions that occurred in control theory where answered by the use of theorems, definitions, proofs etc. stated by mathematicians. Mostly their intension was not to solve certain control problems rather than the generation of a robust framework for mathematical manipulations.

This thesis will also follow this procedure. Its first part is dedicated to the introduction of the applied mathematical framework. It is intended to summarize the mathematical objects used in the subsequent investigations. Additionally, the applied notation is fixed, definitions are stated, and references to the corresponding literature are given. In order to develop a more comprehensive picture a collection of additional definitions is provided in appendix A.

The nonlinear control theory as presented in, e.g., [Isidori, 1995], [Nijmeijer, A.J. van der Schaft, 1990], [Sastry, 1999] gave rise to the use of *differential geometry*. Roughly speaking this framework generalizes the theory of surfaces in $\mathbb{R}^n$ to more complex spaces. Topology, algebra and functional analysis are united to enable the introduction of coordinate independent representations. At first glance might be astonishing that a theory on geometry could have any impact on the theory of dynamic systems. In fact the notion of coordinate transformation links both disciplines and enables engineers to interpret, e.g., finite-dimensional dynamic systems as vector fields on manifolds. These identifications enabled a completely new and joyful point of view and controller design methods like, e.g., the input-state or input-output linearization were developed. In the more recent years these investigations have been extended to infinite-dimensional systems [van der Schaft, B.M. Maschke, 2002], [Macchelli, 2002]. This class of systems has several independent and dependent coordinates, whose evolution is defined by means of partial differential equations. Such systems are of main interest in this thesis.

In the next chapter some basics in differential geometry, which are well known from the analysis of finite-dimensional systems, are recalled. After that, an appropriate geometric object for systems with several independent and dependent coordinates is defined. Finally, these objects are extended to spaces that enable us to handle partial derivatives of dependent coordinates with respect to independent ones in a geometric fashion.

Chapter 2

Manifolds

The analysis of physical systems by means of mathematical modeling immediately leads to the circumstance that there exist different, but equivalent models for the same system. This is mainly caused by the degree of freedom in the introduction of the corresponding coordinates. Additionally, it is well known that this choice is crucial for the applicability of the derived model.

Consequently we are looking for a mathematical object which allows a coordinate invariant description of the system. Consider a plant parametrized in four different ways by the coordinates (x_a^1, x_a^2), (x_b^1, x_b^2), (x_c^1, x_c^2), (x_d^1, x_d^2) as visualized in Fig. 2.1. All these coordinates partly

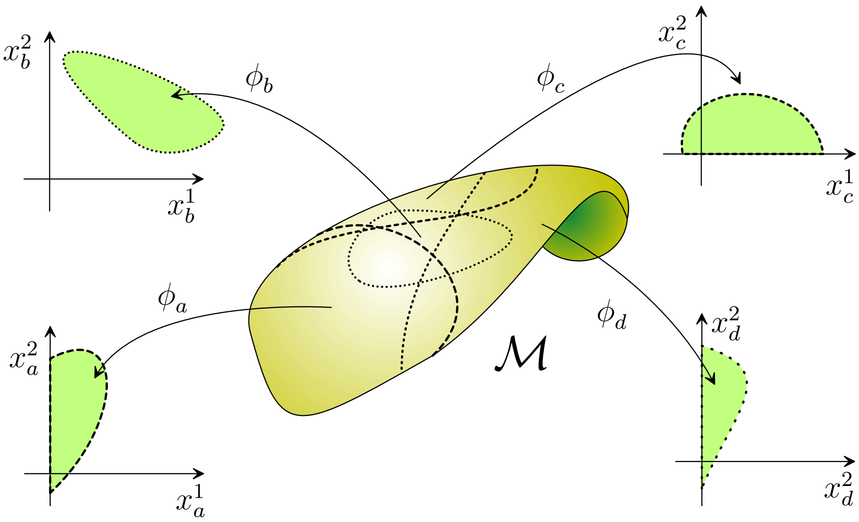

Figure 2.1: The idea of a manifold.

describe the real mathematical representative of the plant – the *manifold* $\mathcal{M}$ (Def. A.38).

As shown later, the modeling of infinite-dimensional systems is mostly related to bounded domains, what leads to the slightly more general definition 2.1.

Definition 2.1 (manifold with boundary) *An n-dimensional (topological) manifold $\mathcal{M}$ with boundary is a Hausdorff topological space such that every point has a neighborhood homeomorphic to the Euclidean half-space $\mathbb{H}^n$ ($\mathbb{H}^n = \{(x^1, \ldots, x^n) | \, x^n \geq 0\}$, $n \in \mathbb{N}$). (see, e.g., [Choquet-Bruhat, Cecile DeWitt-Morette, 1982], [Munkres, 1984])*

Here the definitions (Def. A.25), (Def. A.16), and (Def. A.31) are taken into account. All local coordinate systems are referred to as *charts* and defined by

Definition 2.2 (chart) *A chart (U, ϕ) of a manifold (with boundary) $\mathcal{M}$ is an open set U of $\mathcal{M}$, called the domain of the chart, together with a homeomorphism $\phi : U \to V$ of U onto an open set V in $\mathbb{R}^n$ ($\mathbb{H}^n$). (see, e.g., [Choquet-Bruhat, Cecile DeWitt-Morette, 1982])*

A chart is equipped with adapted *local coordinates*.

Definition 2.3 (local coordinates) *The coordinates $(x^1, \ldots, x^n)$ of the image $\phi(x) \in \mathbb{R}^n$ of the point $x \in U \subset \mathcal{M}$ are called the coordinates of x in the chart (U, ϕ). (see, e.g., [Choquet-Bruhat, Cecile DeWitt-Morette, 1982])*

A collection of compatible charts, which fully describes (i.e. covers) the manifold is named *atlas*.

Definition 2.4 (atlas) *An atlas of class C^k on a manifold $\mathcal{M}$ is a set $\{(U_a, \phi_a)\}$ of charts of $\mathcal{M}$ such that the domains $\{U_a\}$ cover $\mathcal{M}$ and the homeomorphisms $\{\phi_a\}$ enable the formulation of class C^k maps $\phi_b \circ \phi_a^{-1} : \phi_a(U_a \cap U_b) \to \phi_b(U_a \cap U_b)$. (see, e.g., [Choquet-Bruhat, Cecile DeWitt-Morette, 1982])*

If the maps between all charts of an atlas are smooth – of class C^∞ – we denote such manifolds *smooth* (Def. A.39). Unless otherwise stated we will assume all manifolds to be smooth in the subsequent investigations.

The definition of an atlas corresponding to a manifold enables the definition of additional topological properties.

Definition 2.5 (orientable manifold) *A differentiable manifold is said to be orientable if there exists an atlas such that on the overlap $U_a \cap U_b$ of any two charts (U_a, ϕ_a) and (U_b, ϕ_b) the Jacobian determinant of the map $\varphi_{ab} = \phi_b \circ \phi_a^{-1}$ is positive.*

Consequently an orientable manifold enables the introduction of an orientation and we get the following definition.

Definition 2.6 (oriented manifold) *An oriented manifold is an orientable manifold with fixed orientation in a certain coordinate chart (U_a, ϕ_a) of an atlas with positive Jacobian determinants.*

The introduction of a C^∞-atlas corresponding to a certain manifold intrinsically supplies smooth maps $\phi_b \circ \phi_a^{-1}$ between different charts on their overlap $U_a \cap U_b$, as stated by definition (Def. 2.4). Thus different parametrizations (i.e. local coordinates of a physical problem) become equivalent and it is possible to switch from one description to another. These considerations lead to the fact that the notion of manifolds frees modeling from artefacts caused by a certain choice of coordinates.

In order to be able to handle relations of different manifolds, we introduce smooth maps between manifolds.

Definition 2.7 (map between manifolds) *A map $f : \mathcal{M} \to \mathcal{N}$ between the manifolds $\mathcal{M}$ and $\mathcal{N}$ is said to be smooth, iff its local representation*

$$f_{loc} = \varphi_b \circ f \circ \phi_a^{-1} : \phi_a(U_a) \to \varphi_b(V_b)$$

is smooth for every coordinate chart (U_a, ϕ_a) on $\mathcal{M}$ and (V_b, φ_b) on $\mathcal{N}$.

In the subsequent treatment of differential geometric objects, the explicit declaration of the used coordinate charts will be suppressed. This implies that the map $f : \mathcal{M} \to \mathcal{N}$ is synonymously used to its local representation f_{loc}.

Throughout this thesis we will make use of the summation convention (Def. 2.8), which enables a compact notation.

Definition 2.8 (summation convention) *If in a product a letter figures twice, once as superscript and once as subscript, summation must be carried out with respect to this letters. The summation sign $\sum$ will be omitted. (see [Kreyszig, 1991])*

Chapter 3

Bundles

Actually the introduced coordinate invariant representation of a physical system does not allow to distinguish dependent from independent coordinates. From a differential geometric point of view this can be solved by the introduction of fibred manifolds.

Definition 3.1 (fibred manifold, fibre) *A fibred manifold is a triple $(\mathcal{E}, \pi, \mathcal{B})$ where $\mathcal{E}$ and $\mathcal{B}$ are manifolds and $\pi : \mathcal{E} \to \mathcal{B}$ is a surjective submersion. $\mathcal{E}$ is called the total space, π the projection, and $\mathcal{B}$ the base space. For each point $p \in \mathcal{B}$ the subset $\pi^{-1}(p)$ of $\mathcal{E}$ is called the fibre over p and is usually denoted $\mathcal{E}_p$. (see [Saunders, 1989])*

Thus the fibre $\mathcal{E}_p$ represents the space of dependent coordinates at a certain point $p \in \mathcal{B}$ of the independent coordinates. As a shortcut for the fibred manifold $(\mathcal{E}, \pi, \mathcal{B})$ we will use its projection π in the upcoming investigations. If there exists a local trivialisation (Def. A.42) at every point $p \in \mathcal{B}$ (see Def.3.2) the fibred manifold is referred to as bundle.

Definition 3.2 (bundle) *If $(\mathcal{E}, \pi, \mathcal{B})$ is a fibred manifold and $p \in \mathcal{B}$ then a local trivialisation of π around p is a triple $(W_p, \mathcal{F}_p, t_p)$ where W_p is a neighborhood of p, $\mathcal{F}_p$ is a manifold and $t_p : \pi^{-1}(W_p) \to W_p \times \mathcal{F}_p$ is a diffeomorphism satisfying the condition*

$$pr_1 \circ t_p = \pi|_{\pi^{-1}(W_p)} \ .$$

A fibred manifold which has at least one local trivialisation around each point of its base space is called locally trivial and is known as a bundle. (see [Saunders, 1989])

$$\begin{tikzcd}
\mathcal{E} \supset \pi^{-1}(W_p) \arrow[r, "t_p"] \arrow[d, "\pi|_{\pi^{-1}(W_p)}"'] & W_p \times F_p \arrow[d, "\mathrm{pr}_1"] \\
\mathcal{B} \supset \ W_p \arrow[r, "\mathrm{id}"'] & W_p
\end{tikzcd}$$

All manifolds $\mathcal{F}_p$ specified in a local trivialisation are related by Lemma 3.3, whereby the typical fibre $\mathcal{F}$ is introduced.

Lemma 3.3 (typical fibre) *If $(\mathcal{E}, \pi, \mathcal{B})$ is a bundle then there is a manifold $\mathcal{F}$ such that, for each local trivialisation $(W_p, \mathcal{F}_p, t_p)$ of π, the manifolds $\mathcal{F}$ and $\mathcal{F}_p$ are diffeomorphic. (see [Saunders, 1989])*

An assignment of the dependent coordinates by means of a function of the independent coordinates is of particular interest in the context of physical systems, as their solution represents such a map.

Definition 3.4 (section) *A map $\sigma : \mathcal{B} \to \mathcal{E}$ is called a section of π if it satisfies the condition $\pi \circ \sigma = id_{\mathcal{B}}$. The set of all sections of π will be denoted $\Gamma(\pi)$. (see [Saunders, 1989])*

These definitions are depicted in figure 3.1.

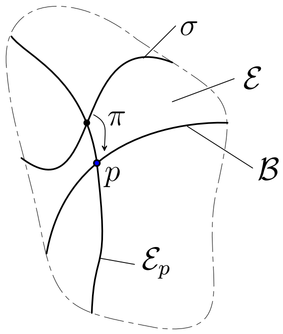

Figure 3.1: A bundle with fibre $\mathcal{E}_p$ and section σ.

3.1 Bundle maps

A map $f : \mathcal{E} \to \mathcal{H}$ between the total manifolds $\mathcal{E}, \mathcal{H}$ of certain bundles does not preserve the bundle structure in general. From a modeling point of view this implies that a local decomposition on the image of the map between dependent and independent coordinates is lost. One is able to preserve the bundle structure by a restriction of the map f to the class of *bundle morphisms*.

Definition 3.5 (bundle morphism) *If $(\mathcal{E}, \pi, \mathcal{B})$ and $(\mathcal{H}, \rho, \mathcal{N})$ are bundles then a bundle morphism from π to ρ is a pair $\left(f, \bar{f}\right)$ where $f : \mathcal{E} \to \mathcal{H}$, $\bar{f} : \mathcal{B} \to \mathcal{N}$ and $\rho \circ f = \bar{f} \circ \pi$. The map $\bar{f}$ is called the projection of f.*

$$\begin{tikzcd}
\mathcal{E} \arrow[r, "f"] \arrow[d, "\pi"'] & \mathcal{H} \arrow[d, "\rho"] \\
\mathcal{B} \arrow[r, "\bar{f}"'] & \mathcal{N}
\end{tikzcd}$$

(see [Saunders, 1989])

The image of the bundle morphism $\left(f, \bar{f}\right)$ satisfies the bundle structure of ρ. Thus a section σ on π defines also a section γ on ρ by

$$f \circ \sigma = \gamma \circ \bar{f} \ .$$

This relation could be used to determine γ by means of the inverse map $\bar{f}^{-1}$. Clearly the inverse map must exist, which is guarantied in the case of a diffeomorphic map $\bar{f}$.

Definition 3.6 (transformation of sections) *A bundle morphism $(f,\bar{f})$ induces a transformation of sections σ on $(\mathcal{E},\pi,\mathcal{B})$ to sections γ on $(\mathcal{H},\rho,\mathcal{N})$ if $\bar{f}^{-1}$ exists. With local coordinates X^i resp. Y^j on the base manifold $\mathcal{B}$ resp. $\mathcal{N}$ we get*

$$\begin{aligned}\gamma:\mathcal{N} &\rightarrow \mathcal{H}\\ Y^j &\rightarrow Y^j, f\circ\sigma\circ\bar{f}^{-1}\left(Y^j\right)\ .\end{aligned}$$

In contrary to ordinary manifolds, there exist several methods to construct new bundles from given ones. The most general way is the construction of a *product bundle* $\pi\times\rho$ of two bundles π and ρ.

Definition 3.7 (product bundle) *If $(\mathcal{E},\pi,\mathcal{B})$ and $(\mathcal{H},\rho,\mathcal{N})$ are bundles then the product bundle is the triple $(\mathcal{E}\times\mathcal{H},\pi\times\rho,\mathcal{B}\times\mathcal{N})$. (see [Saunders, 1989])*

Indeed one could consider two bundles π and ρ whose base manifolds coincide. This leads to the notion of a *fibred product bundle*.

Definition 3.8 (fibred product bundle) *If $(\mathcal{E},\pi,\mathcal{B})$ and $(\mathcal{H},\rho,\mathcal{B})$ are bundles over the same base space $\mathcal{B}$ then the fibred product bundle is the triple $(\mathcal{E}\times_{\mathcal{B}}\mathcal{H},\pi\times_{\mathcal{B}}\rho,\mathcal{B})$, where the total space is defined to equal*

$$\{(a,b)\in\mathcal{E}\times\mathcal{H}:\pi(a)=\rho(b)\}$$

and the projections map is defined by

$$(\pi\times_{\mathcal{B}}\rho)(a,b)=\pi(a)=\rho(b)\ .$$

This configuration can be visualized with

$$\begin{tikzcd}
\mathcal{E}\times_{\mathcal{B}}\mathcal{H} \arrow[r, "\pi^*(\rho)"] \arrow[d, "\rho^*(\pi)"'] \arrow[dr, "\pi\times_{\mathcal{B}}\rho"] & \mathcal{E} \arrow[d, "\pi"] \\
\mathcal{H} \arrow[r, "\rho"'] & \mathcal{B}\,.
\end{tikzcd}$$

(see [Saunders, 1989])

A more general case is given if ρ is not a bundle, but simply a smooth map $\rho:\mathcal{H}\rightarrow\mathcal{B}$. This construction is called *pull-back bundle*.

Definition 3.9 (pull-back bundle) *If $(\mathcal{E},\pi,\mathcal{B})$ is a bundle and $\rho:\mathcal{H}\rightarrow\mathcal{B}$ is a map then the pull-back of π by ρ is the bundle $(\rho^*(\mathcal{E}),\rho^*(\pi),\mathcal{H})$, where the total space $\rho^*(\mathcal{E})$ is defined to equal*

$$\{(a,b)\in\mathcal{E}\times\mathcal{H}:\pi(a)=\rho(b)\}$$

and the projection is defined by

$$\rho^*(\pi)(a,b)=b\ .$$

This construction leads to

$$\begin{array}{ccc} \rho^*(\mathcal{E}) & & \mathcal{E} \\ \Big\downarrow{\scriptstyle \rho^*(\pi)} & & \Big\downarrow{\scriptstyle \pi} \\ \mathcal{H} & \xrightarrow[\rho]{} & \mathcal{B}\,. \end{array}$$

(see [Saunders, 1989])

In the particular case when the pull-back map ρ is an embedding (Def. A.41) $\rho = \varepsilon_W : W \rightarrow \mathcal{B}$ then the pull-back $\varepsilon_W^*(\pi)$ is called the restricted bundle. As the total space $\varepsilon_W^*(\mathcal{E})$ is diffeomorphic to the submanifold $\pi^*(W)$ of $\mathcal{E}$, the restricted bundle can be regarded as *sub-bundle*.

Definition 3.10 (sub-bundle) *If $(\mathcal{E}, \pi, \mathcal{B})$ is a bundle and $\mathcal{E}' \subset \mathcal{E}$ is a submanifold such that the triple $(\mathcal{E}', \pi|_{\mathcal{E}'}, \pi(\mathcal{E}'))$ is itself a bundle, the bundle $\pi|_{\mathcal{E}'}$ is called a sub-bundle of π. A sub-bundle with the particular property $\pi^{-1}(\pi(\mathcal{E}')) = \mathcal{E}'$ is referred to as restricted bundle. (see [Saunders, 1989])*

3.2 Linear bundles

As it will be shown later, one of the most important bundle structures is characterized by a typical fibre to be a vector space (Def. A.13). Such bundles are referred to as *vector bundles*.

Definition 3.11 (vector bundle) *A vector bundle is a quintuple $(\mathcal{E}, \pi, \mathcal{B}, \sigma, \mu)$ where*

- *$(\mathcal{E}, \pi, \mathcal{B})$ is a bundle;*
- *$\sigma : \mathcal{E} \times_{\mathcal{B}} \mathcal{E} \rightarrow \mathcal{E}$ satisfies, for each $p \in \mathcal{B}$, $\sigma(\mathcal{E}_p \times \mathcal{E}_p) \subset \mathcal{E}_p$*
  $\mu : \mathbb{R} \times \mathcal{E} \rightarrow \mathcal{E}$ satisfies, for each $p \in \mathcal{B}$, $\mu(\mathbb{R} \times \mathcal{E}_p) \subset \mathcal{E}_p$
  for each $p \in \mathcal{B}$, $\left(\mathcal{E}_p, \sigma|_{\mathcal{E}_p \times \mathcal{E}_p}, \mu|_{\mathbb{R} \times \mathcal{E}_p}\right)$ is a real vector space;
- *for each $p \in \mathcal{B}$ there is a local trivialisation (W_p, R^n, t_p) called a linear local trivialisation, satisfying the condition that, for $q \in W_p$, the composite of*

  $$t_p|_{\mathcal{E}_q} : \mathcal{E}_q \rightarrow \{q\} \times \mathbb{R}^n$$

  with $pr_2 : \{q\} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a linear isomorphism. (see [Saunders, 1989])

A more general notion of linear bundles is given in the case of a typical fibre being an affine space. Similarly this leads to the notion of affine bundles.

Definition 3.12 (affine bundle) *Let $(\mathcal{E}, \pi, \mathcal{B})$ be a vector bundle. An affine bundle modeled on π is a quintuple $(\mathcal{A}, \rho, \mathcal{B}, \alpha)$ where*

- *$(\mathcal{A}, \rho, \mathcal{B})$ is a bundle;*

- *$\alpha : \mathcal{A} \times_{\mathcal{B}} \mathcal{E} \rightarrow \mathcal{A}$ satisfies, for each $p \in \mathcal{B}$, $\alpha(\mathcal{A}_p \times \mathcal{E}_p) \subset \mathcal{A}_p$ for each $p \in \mathcal{B}$, $\left(\mathcal{A}_p, \mathcal{E}_p, \sigma|_{\mathcal{A}_p \times \mathcal{E}_p}\right)$ is an affine space;*
- *for each $p \in \mathcal{B}$ there is a local trivialisation (W_p, R^n, t_p) called an affine local trivialisation, satisfying the condition that, for $q \in W_p$, the composite of*

$$t_p|_{\mathcal{A}_q} : \mathcal{A}_q \rightarrow \{q\} \times \mathbb{R}^n$$

with $pr_2 : \{q\} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is an affine isomorphism. (see [Saunders, 1989])

In the following we will define some very important vector bundles.

3.2.1 Tangent and cotangent bundle

The rather simple observation that it is possible to assign a linear tangent space to a sufficiently smooth surface at a point p, as visualized in figure 3.2, leads directly to the definition of the

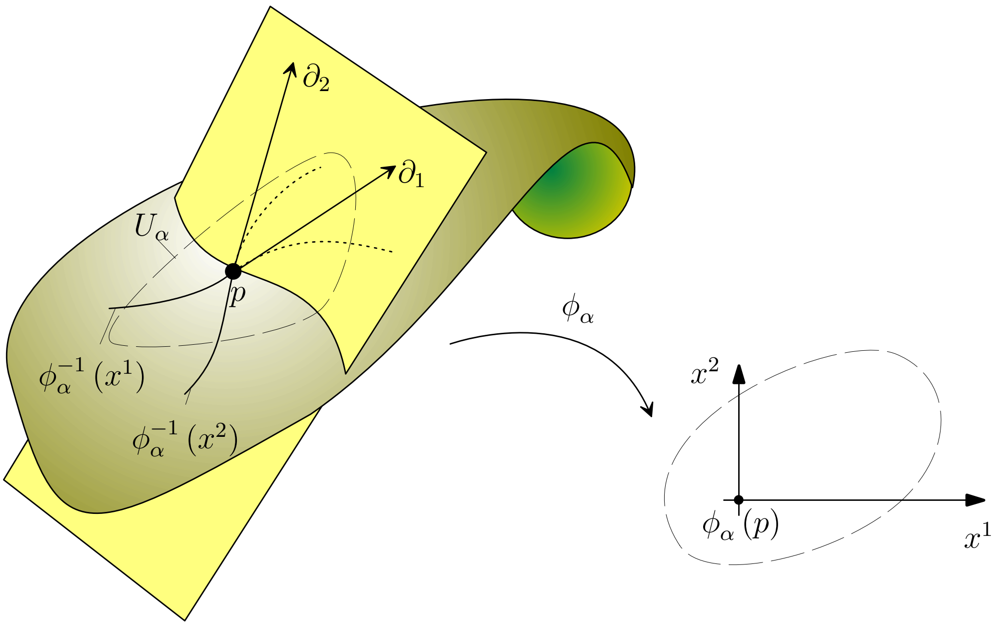

Figure 3.2: The idea of the tangent space.

tangent space.

Definition 3.13 (tangent space) *A tangent space $\mathcal{T}_p(\mathcal{M})$ to the n-dimensional manifold $\mathcal{M}$ at p is the set of all mappings $v_p, w_p : C^\infty(p) \rightarrow \mathbb{R}$ satisfying for all $\alpha, \beta \in \mathbb{R}$ and $f, g \in C^\infty(p)$ the two conditions*

$$\begin{aligned} v_p(\alpha f + \beta g) &= \alpha(v_p(f)) + \beta(v_p(g)) \\ v_p(fg) &= v_p(f)\ g(p) + v_p(g)\ f(p) \end{aligned}$$

with the vector space operations in $\mathcal{T}_p(\mathcal{M})$ defined by

$$\begin{aligned} (v_p + w_p)(f) &= v_p(f) + w_p(f) \\ (\alpha v_p)(f) &= \alpha\, v_p(f)\ . \end{aligned}$$

Thus $\mathcal{T}_p(\mathcal{M})$ is an n-dimensional vector space. A tangent vector to $\mathcal{M}$ at p is any $v_p \in \mathcal{T}_p(\mathcal{M})$. (see [Boothby, 1986])

Obviously, it is possible to assign a linear tangent space to every point of the manifold and thus we are able to define the tangent bundle.

Definition 3.14 (tangent bundle) *The tangent bundle $(\mathcal{T}(\mathcal{M}), \tau_{\mathcal{M}}, \mathcal{M})$ of a manifold $\mathcal{M}$ consists of the total manifold*

$$\mathcal{T}(\mathcal{M}) = \bigcup\nolimits_{p\in\mathcal{M}} \mathcal{T}_p(\mathcal{M})$$

and the natural projection

$$\begin{aligned} \tau_{\mathcal{M}} : \mathcal{T}(\mathcal{M}) &\rightarrow \mathcal{M} \\ \mathcal{T}_p(\mathcal{M}) &\rightarrow p\,. \end{aligned}$$

A section on the tangent bundle assigns to each point p of the manifold $\mathcal{M}$ an element of the linear tangent space $\mathcal{T}_p(\mathcal{M})$, i.e. a vector.

Definition 3.15 (vector field) *A vector field v of class C^r on $\mathcal{M}$ is a function assigning to each point $p \in \mathcal{M}$ a vector whose components in the frames of any local coordinates (U_α, ϕ_α) are functions of class C^r on the domain U_α of the coordinates. (see [Boothby, 1986]). Consequently v is a section of $\tau_{\mathcal{M}}$.*

Using local coordinates, we represent a vector field on an m-dimensional manifold with local coordinates $x^i, i = 1, \ldots, m$ by

$$v^i(x)\frac{\partial}{\partial x^i} = v^i(x)\,\partial_i \in \Gamma(\tau_{\mathcal{M}}) \;, \quad i = 1, \ldots, m\,.$$

We are now able to mark vector fields on the total manifold of a bundle $(\mathcal{E}, \pi, \mathcal{B})$ with special properties. A vector field $v \in \Gamma(\tau_{\mathcal{E}})$ is said to be π-projectable, iff there exists a field $w \in \Gamma(\tau_{\mathcal{B}})$ such that

$$\pi_* \circ v = \pi \circ w$$

$$\begin{tikzcd}
\mathcal{E} \arrow[r, shift left, "v"] \arrow[d, "\pi"'] & \mathcal{T}(\mathcal{E}) \arrow[l, shift left, "\tau_{\mathcal{E}}"] \arrow[d, "\pi_*"] \\
\mathcal{B} \arrow[r, shift right, "w"'] & \mathcal{T}(\mathcal{B}) \arrow[l, shift right, "\tau_{\mathcal{M}}"']
\end{tikzcd}$$

is met. We say v is π-vertical in the case $\pi_* \circ v = 0$. It is easy to show that the set of all π-vertical vectors $V\pi$ form a sub-bundle of $\tau_{\mathcal{E}}$.

Definition 3.16 (vertical bundle) *If $(\mathcal{E}, \pi, \mathcal{B})$ is a bundle, then the vertical bundle to π is the vector sub-bundle $(V\pi, \tau_{\mathcal{E}}|_{V\pi}, \mathcal{E})$ of the tangent bundle $\tau_{\mathcal{E}}$ whose total space $V\pi$ is defined by*

$$V\pi = \{\zeta \in \mathcal{T}(\mathcal{E}) : \pi_*(\zeta) = 0\}\,.$$

(see [Saunders, 1989])

Thus we are able to define a vertical vector field.

Definition 3.17 (vertical vector field) *A vertical vector field v is a section on the vertical bundle $(V\pi, \tau_{\mathcal{E}}|_{V\pi}, \mathcal{E})$, i.e. $v \in \Gamma(\tau_{\mathcal{E}}|_{V\pi})$. (see [Saunders, 1989])*

A π-projectable vector field generates locally the 1-parameter transformation group with parameter ε, which is also a bundle automorphism

$$\left(f, \bar{f}\right) : \pi \to \pi$$

with $\bar{f} = \exp(\varepsilon\pi_* \circ v)$, $f = \exp(\varepsilon v)$. For $v \in \Gamma(\tau_{\mathcal{E}}|_{V\pi})$ one gets the fibre preserving automorphism, i.e. $\bar{f} = \mathrm{id}_{\mathcal{B}}$, $f = \exp(\varepsilon v)$.

The dual object to the tangent space is the cotangent space and defined by

Definition 3.18 (cotangent space) *The dual space to the tangent space $\mathcal{T}_p(\mathcal{M})$ is the space of linear forms on $\mathcal{M}$. It is an n-dimensional vector space called the cotangent space $\mathcal{T}_p^*(\mathcal{M})$ to $p \in \mathcal{M}$. A cotangent vector or 1-form is any $\omega_p \in \mathcal{T}_p^*(\mathcal{M})$.*

Similarly to the definition of the tangent bundle we are able to construct the cotangent bundle.

Definition 3.19 (cotangent bundle) *The cotangent bundle $(\mathcal{T}^*(\mathcal{M}), \bar{\tau}_{\mathcal{M}}, \mathcal{M})$ of a manifold $\mathcal{M}$ consists of the total manifold*

$$\mathcal{T}^*(\mathcal{M}) = \bigcup\nolimits_{p \in \mathcal{M}} \mathcal{T}_p^*(\mathcal{M})$$

and the natural projection

$$\begin{array}{rcl} \bar{\tau}_{\mathcal{M}} : \mathcal{T}^*(\mathcal{M}) & \to & \mathcal{M} \\ \mathcal{T}_p^*(\mathcal{M}) & \to & p\,. \end{array}$$

The sections of the cotangent bundle are denoted covector fields or 1-forms. Using local coordinates, we represent a covector field on an m-dimensional manifold with local coordinates $x^i, i = 1, \ldots, m$ by

$$\omega_i(x)\,\mathrm{d}x^i \in \Gamma(\bar{\tau}_{\mathcal{M}}) \ , \quad i = 1, \ldots, m\,.$$

3.2.2 Tensor Bundles, Exterior Bundles and Algebra

It is possible to extend the idea of dual spaces to more general spaces, whose elements are multi-linear maps and denoted as tensors. Tensor fields and a very important sub-class – the exterior forms – are defined in the chapter A.3 of the appendix.

Chapter 4

Jet Theory

Having partial differential equations at ones disposal, the mathematical framework introduced so far is not sufficient. In fact, one has to handle higher-order partial derivatives of the dependent coordinates x^α with respect to the independent ones X^i. From a differential geometric point of view this can be achieved by the use of jet theory.

4.1 Notation

Let γ be a smooth section of a bundle $(\mathcal{E}, \pi, \mathcal{B})$ with adapted coordinates (X^i, x^α), $i = 1, \ldots, r$, $\alpha = 1, \ldots, s$. The k^{th} order partial derivatives of γ^α will be denoted by

$$\frac{\partial^k}{(\partial X^1)^{j_1} \cdots (\partial X^r)^{j_r}} \gamma^\alpha = \partial_{[J]} \gamma^\alpha = \gamma^\alpha_{[J]} \ ,$$

with the multi-index J, $\#J = k = \sum\limits_{i=1}^{r} j_i$.

Definition 4.1 (multi-index) *An ordered multi-index $J = j_1 \ldots j_r$, has the length $\#J = \sum_{i=1}^r j_i$. The special index $J = j_1, \ldots, j_r$, $j_i = \delta_{il}$, $i = 1, \ldots, r$, $l \in \{1, \ldots, r\}$ will be denoted by 1_l and $J + 1_l$ is a shortcut for $j_i + \delta_{il}$ with the Kronecker symbol δ_{il}. (see [Pommaret, 2001])*

In the construction of the Cartan form as presented in part II, we have to choose a multi-index from a set of multi-indices with equal length. This process can be made unique by the introduction of a multi-index order.

Definition 4.2 (multi-index order) *Let $J_a = j_{a1} \ldots j_{ar}$ and $J_b = j_{b1} \ldots j_{br}$ be two multi-indices. We say $J_a > J_b$ if in the difference $J_a - J_b$ the right-most nonzero entry is positive.*

Remark 4.3 *The introduced multi-index order is motivated by the inverse lexicographic order as defined in, e.g., [Cox, J. Little, D. O'Shea, 1992].*

4.2 First order jet bundles

As mentioned before, in the treatment of partial differential equations one has to deal with partial derivatives of dependent coordinates with respect to independent coordinates. To be more precise, one does not consider derivatives of coordinates rather than the derivatives of sections. This immediately leads to the 1-jet of a section.

Definition 4.4 (1-jet of a section) *Let $(\mathcal{E}, \pi, \mathcal{B})$ be a bundle and let $p \in \mathcal{B}$. Define the sections $\phi, \psi \in \Gamma(\pi)$ to be 1-equivalent at p if $\phi(p) = \psi(p)$ and if, in some adapted coordinate system (X^i, x^α), $i = 1 \ldots r$, $\alpha = 1 \ldots s$ around $\phi(p)$*

$$\left.\frac{\partial \phi^\alpha}{\partial X^i}\right|_p = \left.\frac{\partial \psi^\alpha}{\partial X^i}\right|_p .$$

The equivalence class containing ϕ is called the 1-jet of ϕ at p and is denoted $j^1_p \phi$.

One can provide the set of all 1-jets of sections $\Gamma(\pi)$ with the structure of a differentiable manifold.

Definition 4.5 (first jet manifold) *The first jet manifold of $(\mathcal{E}, \pi, \mathcal{B})$ is the set*

$$\left\{ j^1_p \phi : p \in \mathcal{B},\ \phi \in \Gamma(\pi) \right\}$$

and is denoted $J^1\pi$. The functions π^1 and π^1_0, called the source and target projection respectively, are defined by

$$\begin{array}{rcl} \pi^1 : J^1\pi & \to & \mathcal{M} \\ j^1_p\phi & \to & p \end{array} \quad \textit{and} \quad \begin{array}{rcl} \pi^1_0 : J^1\pi & \to & \mathcal{E} \\ j^1_p\phi & \to & \phi(p) \end{array}$$

and visualized in

$$\begin{tikzcd} J^1\pi \arrow[r, "\pi^1_0"] \arrow[d, "\pi^1"'] & \mathcal{E} \arrow[d, "\pi"] \\ \mathcal{B} \arrow[r, "\mathrm{id}"'] & \mathcal{B} . \end{tikzcd}$$

The functions π^1 and π^1_0 are surjective submersions.

An adapted coordinate system of π induces an adapted system on $J^1\pi$, which is denoted by $\left(X^i, x^\alpha, x^\alpha_{[1_i]}\right)$ with the $r \cdot s$ new coordinates $x^\alpha_{[1_i]}$. Both projections π^1, π^1_0 allow the definition of bundles as stated by proposition 4.6 .

Proposition 4.6 *If $(\mathcal{E}, \pi, \mathcal{B})$ is a bundle then $(J^1\pi, \pi^1, \mathcal{B})$ and $(J^1\pi, \pi^1_0, \mathcal{E})$ are bundles. (see [Saunders, 1989])*

By construction we can prolong a section of π to π^1 and get the definition.

Definition 4.7 (1^{st} prolongation of a section) *If $(\mathcal{E}, \pi, \mathcal{B})$ is a bundle then the first prolongation of a section $\phi \in \Gamma(\pi)$ is the section $j^1\phi \in \Gamma(\pi^1)$ defined by*

$$j^1\phi(p) = j^1_p\phi$$

for $p \in \mathcal{B}$. (see [Saunders, 1989])

It is worth mentioning that a section γ of the bundle $(J^1\pi, \pi^1, \mathcal{B})$ is not necessarily the first jet $j^1(\sigma)$ of a section σ of π, since those sections must meet the relations $\partial_i\gamma^\alpha - \gamma^\alpha_{[1_i]} = 0$. Even more important is the possibility to prolong a given bundle morphism to the first jet.

Definition 4.8 (1^{st} prolongation of a morphism) *Let $(\mathcal{E}, \pi, \mathcal{B})$ and $(\mathcal{H}, \rho, \mathcal{N})$ be bundles, and let $(f, \bar{f})$ be a bundle morphism, where $\bar{f}$ is a diffeomorphism. The first prolongation of $(f, \bar{f})$ is the map defined by*

$$j^1(f, \bar{f})(j^1_p\phi) = j^1_{\bar{f}(p)}(f \circ \phi \circ \bar{f}^{-1}) \ .$$

This could be visualized in the following diagram.

$$\begin{tikzcd}
J^1\pi \arrow[r, "\mathrm{id}"] \arrow[dd, shift right, "\pi^1"'] & J^1\pi \arrow[r, "j^1(f,\bar{f})"] \arrow[d, "\pi^1_0"'] & J^1\rho \arrow[d, "\rho^1_0"] & J^1\rho \arrow[l, "\mathrm{id}"'] \arrow[dd, shift right, "\rho^1"'] \\
 & \mathcal{E} \arrow[r, "f"] \arrow[d, shift right, "\pi"'] & \mathcal{H} \arrow[d, shift right, "\rho"'] & \\
\mathcal{B} \arrow[r, "\mathrm{id}"'] \arrow[uu, shift right, "j^1_p\phi"'] & \mathcal{B} \arrow[r, "\bar{f}"'] \arrow[u, shift right, "\phi"'] & \mathcal{N} \arrow[u, shift right, "(f\circ\phi\circ\bar{f}^{-1})"'] & \mathcal{N} \arrow[l, "\mathrm{id}"] \arrow[uu, shift right, "j^1_{\bar{f}(p)}(f\circ\phi\circ\bar{f}^{-1})"']
\end{tikzcd}$$

If no confusion is possible, the notation j^1f will be used rather than $j^1(f, \bar{f})$. With local coordinates (X^i, x^α) resp. (Y^j, y^β) for π resp. ρ the first prolongation results in

$$\begin{aligned}
j^1f \quad &: \quad J^1\pi \to J^1\rho \\
X^i, x^\alpha, x^\alpha_{[1_i]} \quad &\to \quad Y^j = \bar{f}^j(X^i), y^\beta = f^\beta(X^i, x^\alpha), y^\alpha_{[1_j]} = j^1f(X^i, x^\alpha, x^\alpha_{[1_i]})
\end{aligned}$$

where

$$\begin{aligned}
y^\beta_{[1_j]} &= j^1f^\beta(X^i, x^\alpha, x^\alpha_{[1_i]}) \\
&= d_j \rfloor \mathrm{d}y^\beta = \left(\frac{\partial y^\beta}{\partial x^\alpha} d_i \rfloor \mathrm{d}x^\alpha + \frac{\partial y^\beta}{\partial X^i}\right)\left(\frac{\partial X^i}{\partial Y^j} \circ \bar{f}(X^i)\right) \\
&= d_i \rfloor \mathrm{d}y^\beta \left(\frac{\partial X^i}{\partial Y^j} \circ \bar{f}(X^i)\right)
\end{aligned}$$

is used. The vector fields d_j resp. d_i are referred as total derivative (see Def. 4.11) with respect to Y^j resp. X^i. (see [Saunders, 1989])

4.3 n^{th} order jet bundles

Analogously to the first jet of a section γ, we define the n^{th}-jet $j^n\gamma$ of γ by $j^n\gamma = (x^i, \gamma^\alpha(x), \partial_{[J]}\gamma^\alpha(x))$, $\#J = 1, \ldots, n$. The n^{th}-jet manifold $J^n\pi$ of π may be considered as a container for n^{th}-jets of sections of π. Furthermore, an adapted coordinate system of $\mathcal{E}$ induces an adapted system on $J^n\pi$ with $\left(X^i, x^\alpha_{[J]}\right)$, $\alpha = 1, \ldots, q$, $\#J = 0, \ldots, n$. The natural projections and the corresponding bundles are given by

$$\begin{aligned} \pi^n &: J^n\pi \rightarrow \mathcal{B}\,, \quad (J^n\pi, \pi^n, \mathcal{B}) \text{ and} \\ \pi^n_m &: J^n\pi \rightarrow J^m\pi\,, \quad (J^n\pi, \pi^n_m, J^m\pi)\,, \quad m < n \end{aligned} \tag{4.1}$$

for $m = 1, \ldots, n-1$ with $\pi^n(j^n(\gamma(x))) = x$ and $\pi^n_m(j^n(\gamma(x))) = j^m(\gamma(x))$.

All these bundle lead to the following definition.

Definition 4.9 (jet framework) *The n^{th}-order jet framework Π^n of a bundle π is defined as the collection of all jet bundles*

$$\pi^k\,,\ \pi^k_w\,,\quad w < k\,,\quad k = 1, \ldots, n\,,\quad w = 0, \ldots, n-1\,.$$

In fact all bundles of a jet framework are related. It is obvious that it is not possible to define arbitrary sections on a jet bundle π^n_m without violating the jet framework, i.e. such a section is not necessarily the prolongation of a section on π. These relations can be summarized in the so called contact structure.

Definition 4.10 (contact structure) *A contact structure on an n^{th} order jet bundle $J^n\pi$ is given in local coordinates $\left(X^i, x^\alpha, x^\alpha_{[J]}\right)$ by two vector valued one forms*

$$h = \mathrm{d}X^i \otimes \left(\partial_i + x^\alpha_{[1_i]}\partial_\alpha + \cdots + x^\alpha_{[J+I_1]}\partial^{[J]}_\alpha\right) \tag{4.2}$$

$$v = \left(\mathrm{d}x^\alpha_{[J]} - x^\alpha_{[J+I_1]}\mathrm{d}X^i\right) \otimes \partial^{[J]}_\alpha\,, \tag{4.3}$$

where

$$I_1 = 1_i, \#J = 0, \ldots, n\,,\quad \partial^{[J]}_\alpha = \frac{\partial}{\partial x^\alpha_{[J]}}\,.$$

(see [Saunders, 1989])

The contact structure incorporates two very important objects – the total derivatives

Definition 4.11 (total derivative) *The field $d_i \in \Gamma\left((\pi^{n+1}_n)^*(\tau_{J^n\pi})\right)$*

$$d_i = \partial_i + x^\alpha_{[1_i]}\partial_\alpha + \cdots + x^\alpha_{[J+I_1]}\partial^{[J]}_\alpha\,,\quad I_1 = 1_i\,,\quad \#J = 1, \ldots, n\,,\quad \partial^{[J]}_\alpha = \frac{\partial}{\partial x^\alpha_{[J]}}$$

is referred to as n^{th} order total derivative with respect to the independent coordinate X^i. It is the unique operator that meets

$$\left(j^{n+1}\sigma\right)^*(d_i f) = \partial_i\left((j^n\sigma)^* f\right)\,,\quad \partial_i = \frac{\partial}{\partial X^i}$$

for all functions $f \in C^\infty(J^n\pi)$ and sections $\sigma \in \Gamma(\pi)$. (see [Saunders, 1989]) and the contact forms

Definition 4.12 (contact form) *The form $\omega^\alpha \in \Gamma\left(\left(\pi_n^{n+1}\right)^*\left(\bar{\tau}_{J^n\pi}\right)\right)$*

$$\omega^\alpha_{[J]} = \mathrm{d}x^\alpha_{[J]} - x^\alpha_{[J+I_1]}\mathrm{d}X^i\ , \quad I_1 = 1_i\ ,\ \#J = 1,\ldots,n$$

is referred to as contact form on the n^{th} jet manifold and meets

$$\left(j^{n+1}\sigma\right)^*\omega^\alpha_{[J]} = 0$$

for all sections $\sigma \in \Gamma(\pi)$. (see [Saunders, 1989])

It is now remarkable that the contact forms allow the introduction of an exterior ideal (Def. A.58) denoted as contact ideal.

Definition 4.13 (contact ideal) *The contact forms $\omega^\alpha_{[J]}$ on the n^{th} jet manifold generate an ideal I_n over the exterior algebra $\bigwedge J^n\pi$ on the jet manifold $J^n\pi$. Such an ideal will be denoted contact ideal.*

Every element of the contact ideal $\alpha \in I_n$ annihilates the distribution of n^{th} order total derivatives $\Delta = \mathrm{span}\,(d_i)$ and defines thereby a dual distribution $\Delta^\perp$.

An exterior ideal is said to be closed with respect to the exterior differentiation, if the exterior derivative of an element of the ideal is again an element of the ideal, i.e.

$$\alpha \in I_n \longmapsto \mathrm{d}\alpha \in I_n\ .$$

Such an ideal is called a differential ideal (see, e.g., [Sastry, 1999]). The investigation of the closeness of the contact ideal leads immediately to the following theorem.

Theorem 4.14 *The contact ideal I_n on $\bigwedge J^n\pi$ is not closed on $\bigwedge J^n\pi$ with respect to the exterior derivative.*

Proof. It is obvious that the exterior derivative of a contact form $\omega^\alpha_{[J]} \in I_n$ given by

$$\mathrm{d}\left(\mathrm{d}x^\alpha_{[J]} - x^\alpha_{[J+I_1]}\mathrm{d}X^i\right) = \mathrm{d}x^\alpha_{[J+I_1]} \wedge \mathrm{d}X^i\ , \quad I_1 = 1_i\ , \quad J = n$$

cannot be an element of the ideal I_n. ■

This theorem is closely related to the non-involutivity of the Cartan distribution as stated in, e.g., [Saunders, 1989] and [Giachetta, G. Sardanashvily, L. Mangiarotti, 1994].

In part II we will apply the "inverted" procedure used in the non-closedness proof, i.e. we will asks for elements of the contact ideal whose exterior derivatives lead to a certain form

$$\lambda_0 = \mathrm{d}x^\alpha_{[J]} \wedge \mathrm{dX}\ ,$$

where dX represents the volume form $\mathrm{dX} = \mathrm{d}X^1 \wedge \ldots \wedge \mathrm{d}X^r$. This results in $m \geq 1$ elements of the contact ideal $\omega^\alpha_{[J-I_1]} \wedge \partial_{I_1} \rfloor \mathrm{dX}$ that meet

$$\begin{aligned}
\mathrm{d}\left(\omega^\alpha_{[J-I_1]} \wedge \partial_{I_1}\rfloor\mathrm{dX}\right) &= \mathrm{d}\left(\underbrace{\mathrm{d}x^\alpha_{[J-I_1]} \wedge \partial_{I_1}\rfloor\mathrm{dX}}_{\lambda_{I_1}} - x^\alpha_{[J]}\mathrm{dX}\right)\ , \quad I_1 = 1_i \\
&= -\mathrm{d}x^\alpha_{[J]} \wedge \mathrm{dX}\ .
\end{aligned}$$

Consequently this is in general not a unique procedure. It is now remarkable that the m obtained forms

$$\lambda_{I_1} = \mathrm{d}x^\alpha_{[J-I_1]} \wedge \partial_{I_1} \rfloor \mathrm{dX}$$

are pairwise contained in the exterior derivative of $\binom{m}{2}$ different, lower-indexed contact elements $\omega^\alpha_{[J-I_2]} \wedge \partial_{I_2} \rfloor \mathrm{dX}$ given by

$$\mathrm{d}\left(\underbrace{\left(\mathrm{d}x^\alpha_{[J-I_2]} - x^\alpha_{[J-I_2+1_i]}\mathrm{d}X^i - x^\alpha_{[J-I_2+1_k]}\mathrm{d}X^k\right)}_{\omega^\alpha_{[J-I_2]}} \wedge \partial_{I_2} \rfloor \mathrm{dX}\right) \ , \quad I_2 = 1_i + 1_k \ , \ \#I_2 = 2 \ , \ i < k \ .$$

Surprisingly three forms out of the $\binom{m}{2}$ similarly obtained forms

$$\lambda_{I_2} = \mathrm{d}x^\alpha_{[J-I_2]} \wedge \partial_{I_2} \rfloor \mathrm{dX}$$

are once again contained in the exterior derivatives of the $\binom{m}{3}$ different, lower-indexed contact elements $\omega^\alpha_{[J-I_3]} \wedge \partial_{I_3} \rfloor \mathrm{dX}$ given by

$$\mathrm{d}\left(\left(\mathrm{d}x^\alpha_{[J-I_3]} - x^\alpha_{[J-I_3+1_i]}\mathrm{d}X^i - x^\alpha_{[J-I_3+1_k]}\mathrm{d}X^k - x^\alpha_{[J-I_3+1_r]}\mathrm{d}X^r\right) \wedge \partial_{I_3} \rfloor \mathrm{dX}\right) \ , \quad \begin{array}{l} I_3 = 1_i + 1_k + 1_r, \\ i < k < r \end{array} \ .$$

This sequence of contact elements related under the exterior derivative of lower-indexed ones can be continued and finally ends up in one single, i.e. $\binom{m}{m}$, contact element $\omega^\alpha_{J-I_m} \wedge \partial_{I_m} \rfloor \mathrm{dX}$

$$\mathrm{d}\left(\left(\mathrm{d}x^\alpha_{[J-I_m]} - x^\alpha_{[J-I_m+1q]}\mathrm{d}x^q\right) \wedge \partial_{I_m} \rfloor \mathrm{dX}\right) \ , \quad \#I = m \ .$$

This investigation shows that a form $\mathrm{d}x^\alpha_{[J]} \wedge \mathrm{dX}$ is related to a unique, lower indexed contact element $\omega^\alpha_{[J-I_m]} \wedge \partial_{I_m} \rfloor \mathrm{dX}$, $\#I_m = m$. This observation is visualized in the following example.

Example 4.15 *Let us consider the form*

$$\lambda_{000} = \mathrm{d}x^\alpha_{[J]} \wedge \mathrm{dX} = \mathrm{d}x^1_{[121]} \wedge \mathrm{d}X^1 \wedge \mathrm{d}X^2 \wedge \mathrm{d}X^3 \ .$$

Obviously we obtain three elements of the contact ideal, i.e. $m = 3$*, whose exterior derivative supplies* λ_{000}

$$\begin{array}{l}
\left(\mathrm{d}x^1_{[021]} - x^1_{[121]}\mathrm{d}X^1 - x^1_{[031]}\mathrm{d}X^2 - x^1_{[022]}\mathrm{d}X^3\right) \wedge \partial_1 \rfloor \mathrm{d}X^1 \wedge \mathrm{d}X^2 \wedge \mathrm{d}X^3 \\
\left(\mathrm{d}x^1_{[111]} - x^1_{[211]}\mathrm{d}X^1 - x^1_{[121]}\mathrm{d}X^2 - x^1_{[112]}\mathrm{d}X^3\right) \wedge \partial_2 \rfloor \mathrm{d}X^1 \wedge \mathrm{d}X^2 \wedge \mathrm{d}X^3 \\
\left(\mathrm{d}x^1_{[120]} - x^1_{[220]}\mathrm{d}X^1 - x^1_{[130]}\mathrm{d}X^2 - x^1_{[121]}\mathrm{d}X^3\right) \wedge \partial_3 \rfloor \mathrm{d}X^1 \wedge \mathrm{d}X^2 \wedge \mathrm{d}X^3 \ .
\end{array}$$

The forms λ_{I_1} *are in this case given by*

$$\begin{array}{rcl}
\lambda_{100} & = & \mathrm{d}x^1_{[021]} \wedge \partial_1 \rfloor \mathrm{d}X^1 \wedge \mathrm{d}X^2 \wedge \mathrm{d}X^3 \\
\lambda_{010} & = & \mathrm{d}x^1_{[111]} \wedge \partial_2 \rfloor \mathrm{d}X^1 \wedge \mathrm{d}X^2 \wedge \mathrm{d}X^3 \\
\lambda_{001} & = & \mathrm{d}x^1_{[120]} \wedge \partial_3 \rfloor \mathrm{d}X^1 \wedge \mathrm{d}X^2 \wedge \mathrm{d}X^3 \ .
\end{array}$$

The three elements of the contact ideal, i.e. $\binom{3}{2} = 3$*, whose exterior derivative supplies* λ_{000} *are*

$$\begin{array}{l}
\left(\mathrm{d}x^1_{[011]} - x^1_{[111]}\mathrm{d}X^1 - x^1_{[021]}\mathrm{d}X^2 - x^1_{[012]}\mathrm{d}X^3\right) \wedge \partial_2 \rfloor \partial_1 \rfloor \mathrm{d}X^1 \wedge \mathrm{d}X^2 \wedge \mathrm{d}X^3 \\
\left(\mathrm{d}x^1_{[110]} - x^1_{[210]}\mathrm{d}X^1 - x^1_{[120]}\mathrm{d}X^2 - x^1_{[111]}\mathrm{d}X^3\right) \wedge \partial_3 \rfloor \partial_2 \rfloor \mathrm{d}X^1 \wedge \mathrm{d}X^2 \wedge \mathrm{d}X^3 \\
\left(\mathrm{d}x^1_{[020]} - x^1_{[120]}\mathrm{d}X^1 - x^1_{[030]}\mathrm{d}X^2 - x^1_{[021]}\mathrm{d}X^3\right) \wedge \partial_1 \rfloor \partial_3 \rfloor \mathrm{d}X^1 \wedge \mathrm{d}X^2 \wedge \mathrm{d}X^3 \ .
\end{array}$$

The forms $\omega^{\alpha}_{J-I_2} \wedge \partial_{I_2} \rfloor \mathrm{dX}$ contain pairwise the forms λ_{I_1}, i.e. λ_{100}, λ_{010} and λ_{001}.The forms λ_{I_2} are consequently given by

$$
\begin{aligned}
\lambda_{110} &= \mathrm{d}x^1_{[011]} \wedge \partial_2 \rfloor \partial_1 \rfloor \mathrm{d}X^1 \wedge \mathrm{d}X^2 \wedge \mathrm{d}X^3 \\
\lambda_{011} &= \mathrm{d}x^1_{[110]} \wedge \partial_3 \rfloor \partial_2 \rfloor \mathrm{d}X^1 \wedge \mathrm{d}X^2 \wedge \mathrm{d}X^3 \\
\lambda_{101} &= \mathrm{d}x^1_{[020]} \wedge \partial_1 \rfloor \partial_3 \rfloor \mathrm{d}X^1 \wedge \mathrm{d}X^2 \wedge \mathrm{d}X^3 \,.
\end{aligned}
$$

Finally the single, i.e. $\binom{3}{3}$ contact element $\omega^{\alpha}_{J-I_3} \wedge \partial_{I_3} \rfloor \mathrm{dX}$ is give by

$$
\mathrm{d}x^1_{[010]} - x^1_{[110]}\mathrm{d}X^1 - x^1_{[020]}\mathrm{d}X^2 - x^1_{[011]}\mathrm{d}X^3 \,,
$$

and $\lambda_{111} = \mathrm{d}x^1_{[010]}$. The exterior derivative of the form $\omega^{\alpha}_{J-I_3} \wedge \partial_{I_3} \rfloor \mathrm{dX}$ contains all three λ_{I_2}. This construction is visualized in the following diagram.

$$
\begin{tikzcd}
 & \lambda_{000} \arrow[dl] \arrow[d] \arrow[dr] & \\
\lambda_{100} \arrow[d] \arrow[dr] & \lambda_{010} \arrow[dl] \arrow[dr] & \lambda_{001} \arrow[dl] \arrow[d] \\
\lambda_{110} \arrow[dr] & \lambda_{101} \arrow[d] & \lambda_{011} \arrow[dl] \\
 & \lambda_{111} &
\end{tikzcd}
$$

The contact structure could be used to introduce two differentials – the horizontal d_h and the vertical differential d_v. These differentials represent a decomposition of the exterior derivative

$$
\mathrm{d} = \mathrm{d}_h + \mathrm{d}_v
$$

(see, e.g., [Giachetta, G. Sardanashvily, L. Mangiarotti, 1994]) and are defined as follows.

Definition 4.16 (horizontal differential) *The horizontal differential of a form $\omega \in \bigwedge J^n\pi$ is defined by*

$$
\begin{aligned}
\mathrm{d}_h(\omega) &= \mathrm{asym}\left(h \rfloor \mathrm{d}(\omega)\right) + \mathrm{d}\left(\mathrm{asym}\left(h \rfloor (\omega)\right)\right) \\
&= \mathrm{d}X^i \wedge \mathrm{L}_{d_i}(\omega) \;,
\end{aligned}
$$

where the contact structure of Def.4.10 is used. (see, e.g., [Saunders, 1989] or [Giachetta, G. Sardanashvily, L. Mangiarotti, 1994])

Definition 4.17 (vertical differential) *The vertical differential of a form $\omega \in \bigwedge J^n\pi$ is defined by*

$$
\begin{aligned}
\mathrm{d}_v(\omega) &= \mathrm{asym}\left(v \rfloor \mathrm{d}(\omega)\right) - \mathrm{d}\left(\mathrm{asym}\left(v \rfloor (\omega)\right)\right) \\
&= \mathrm{d}(\omega) - \mathrm{d}_h(\omega) \;,
\end{aligned}
$$

where the contact structure of Def.4.10 is used. (see, e.g., [Saunders, 1989] or [Giachetta, G. Sardanashvily, L. Mangiarotti, 1994])

Here the alternating map $\mathrm{asym}(\cdot)$ as defined in (Def.A.50) is used.

These differentials meet the homology properties

$$\mathrm{d}_h\mathrm{d}_h = 0\ ,\ \ \mathrm{d}_v\mathrm{d}_v = 0\ ,\ \ \mathrm{d}_v\mathrm{d}_h + \mathrm{d}_h\mathrm{d}_v = 0\ .$$

The following lemma provides a connection between Stokes's theorem (Def. A.70) and the horizontal differential.

Lemma 4.18 *For every $\sigma \in \Gamma(\pi)$ the following relation holds*

$$\mathrm{d} \circ (j^n\sigma)^* = \left(j^{n+1}\sigma\right)^* \circ \mathrm{d}_h\ .$$

Proof. As the exterior derivative commutes with the pull-back it follows that

$$\mathrm{d} \circ (j^n\sigma)^* = (j^n\sigma)^* \circ \mathrm{d} = (j^n\sigma)^* \circ (\mathrm{d}_h + \mathrm{d}_v) = \left(j^{n+1}\sigma\right)^* \circ \mathrm{d}_h\ ,$$

where $(j^{n+1}\sigma)^* \circ \mathrm{d}_v = 0$ is used. (see, e.g., [Saunders, 1989]) ■

Consequently we get for the integral of the horizontal differential of a form $\omega \in \bigwedge J^n\pi$ over the bounded base manifold $\mathcal{D}$ of the bundle $(\mathcal{E}, \pi, \mathcal{D})$

$$\int_{\mathcal{D}} \left(j^{n+1}\sigma\right)^* (\mathrm{d}_h\omega) = \int_{\mathcal{D}} \mathrm{d}\left((j^n\sigma)^*\omega\right) = \int_{\partial\mathcal{D}} \iota^*\left((j^n\sigma)^*\omega\right)\ .$$

Thus the relation of the horizontal differential and the boundary operator ∂ is visualized.

Remark 4.19 *The application of the horizontal differential d_h onto an $(r-1)$-form $\omega \in (\pi^n)^* \wedge^{r-1} \mathcal{B}$ equals the total divergence as defined in [Olver, 1986].*

The introduction of a bundle morphism between two jet bundles $(\Psi^n, \psi) : J^n\rho \to J^n\pi$ enables one to pull-back elements of $\wedge J^n\pi$ onto $\wedge J^n\rho$. With regards to the corresponding contact structures, it is possible to mark a special kind of bundle morphisms - the contact bundle morphisms.

Definition 4.20 (contact bundle morphism) *A bundle morphism $(\Psi^n, \psi) : \rho^n \to \pi^n$ qualifies as contact bundle morphism if*

$$\begin{array}{rcl} (\Psi^n)^* : I_{\pi^n} & \to & I_{\rho^n} \\ \Theta & \to & (\Psi^n)^*\,\Theta \end{array}$$

maps elements Θ of the contact ideal I_{π^n} on π^n onto elements $(\Psi^n)^*\,\Theta$ of the contact ideal I_{ρ^n} on ρ^n.

It is worth mentioning that we do not require Ψ to be a diffeomorphism. Another, even more important property of a contact bundle morphism $(\Psi^n, \psi) : \rho^n \to \pi^n$ is given by

$$\psi^* \circ (j^n\sigma)^* = (j^n\gamma)^* \circ (\Psi^n)^*$$

where the prolongations of the sections $\sigma \in \Gamma(\pi)$, $\gamma \in \Gamma(\rho)$ are used. This is nothing else than the basic property of a bundle morphism. In the upcoming investigation of boundary conditions, where the map ψ is given by the inclusion map ι (see section 8.1) we will make heavy use of this property.

Finally it is left to discuss the prolongation of a bundle morphism to the n^{th} jet.

Definition 4.21 (the n^{th} prolongation of a morphism) *The second prolongation of a bundle morphism leads to*

$$\begin{aligned} y^{\beta}_{[1_j+1_k]} &= d_k \rfloor \mathrm{d}y^{\beta}_{[1_j]} = \left(\frac{\partial y^{\beta}_{[1_j]}}{\partial x^{\alpha}_{[1_i]}} d_w \rfloor \mathrm{d}x^{\alpha}_{[1_i]} + \frac{\partial y^{\beta}_{[1_j]}}{\partial x^{\alpha}} d_w \rfloor \mathrm{d}x^{\alpha} + \frac{\partial y^{\beta}_{[1_j]}}{\partial X^w}\right) \left(\frac{\partial X^w}{\partial Y^k} \circ \bar{f}\left(Y^j\right)\right) \\ & d_w \left(y^{\beta}_{[1_j]}\right) \left(\frac{\partial Y^w}{\partial X^k} \circ \bar{f}\left(Y^j\right)\right) . \end{aligned}$$

Finally the n^{th} prolongation is defined recursively by

$$y^{\beta}_{[J+1_k]} = d_w \left(y^{\beta}_{[J]}\right) \left(\frac{\partial X^w}{\partial Y^k} \circ \bar{f}\left(Y^j\right)\right)$$

as shown in [Giachetta, G. Sardanashvily, L. Mangiarotti, 1994].

It is obvious that these calculations represent a costly job. If the bundle morphism is induced by a 1-parameter transformation group $\bar{f} = \exp\left(\varepsilon \pi_* \circ v\right), f = \exp\left(\varepsilon v\right)$, we are able to consider its infinitesimal generator v instead. Fortunately one obtains a much simpler relation given by

Definition 4.22 (n^{th} prolongation of a vector field) *Let us consider a vector field*

$$v = \dot{X}^i \partial_i + \dot{x}^{\alpha} \partial_{\alpha} \in \Gamma\left(\tau_{\mathcal{E}}\right)$$

where $\dot{X}^i = \dot{X}^i\left(X^i, x^{\alpha}\right)$ and $\dot{x}^{\alpha} = \dot{x}^{\alpha}\left(X^i, x^{\alpha}\right)$ is used. The n^{th} prolongation of this vector field is given by

$$j^n v = \dot{X}^i \partial_i + \dot{x}^{\alpha} \partial_{\alpha} + \left(\mathrm{L}_{d_J}\left(\dot{x}^{\alpha} - \dot{X}^i \partial_i x^{\alpha}\right) + \dot{X}^i \partial_i x^{\alpha}_{[J]}\right) \partial^{[J]}_{\alpha} \in \Gamma\left(\tau_{J^n \pi}\right) , \quad 1 \leq \#J \leq n$$

where $d_J = (d_1)^{j_1} \circ \cdots \circ (d_p)^{j_p}$ and $\partial^{[J]}_{\alpha} = \frac{\partial}{\partial x^{\alpha}_{[J]}}$ is used. (see, e.g., [Olver, 1986])

and in the case of a vertical vector field $v = v^{\alpha} \partial_{\alpha} \in \Gamma\left(\tau_{\mathcal{E}}|_{V\pi}\right)$ we obtain an even simpler version given by

$$j^n\left(v\right) = v + \mathrm{L}_{d_J}\left(v^{\alpha}\right) \partial^{[J]}_{\alpha} , \quad 1 \leq \#J \leq n . \tag{4.4}$$

Part II

Euler-Lagrange Systems

Physical systems supply more than just a bunch of differential equations!

This introductory statement tries to summarize the fact that physical systems provide more information than just their equations of motion. This additional information can be extracted from, e.g., energy considerations or structural properties of the equations of motion.

Here we will consider Euler-Lagrange equations that are equipped with a particular rich structure. These equations are defined to be the necessary condition met by certain functions $x^\alpha\left(X^i\right)$ such that the functional

$$\mathfrak{L} = \int_{\mathcal{D}} l\left(X^i, x^\alpha, x^{(n)}\right) \mathrm{d}X^1 \cdots \mathrm{d}X^r \ ,$$

is minimal. Here we have introduced $r \geq 1$ independent variables (X^i), $i = 1, \ldots, r$ and $s \geq 1$ dependent variables (x^α), $\alpha = 1, \ldots, s$. The Lagrangian density l may also depend on the partial derivatives $x^{(n)}$ of the dependent variables with respect to the independent ones up to the order $n > 0$. Furthermore, the function l is assumed to be smooth on the domain $\mathcal{D}$ of integration. The methodology to determine the Euler-Lagrange equations is referred to as calculus of variations.

The following chapter is dedicated to the introduction of the calculus of variations within the language of jet manifolds. The general problem, the solution using integration by parts, and the subsequently applied solution based on the so called Cartan form is discussed. After that, the variational formula is used for finite-dimensional and infinite-dimensional Euler-Lagrange systems in chapter 6 and 7. Special attention is paid on infinite-dimensional Euler-Lagrange systems and the non-uniqueness of the Cartan form. The achieved results give rise to an extension of the approach, which leads finally to the so called extended Cartan form presented in chapter 8. This mathematical object allows the unique derivation of the domain and boundary conditions of an infinite-dimensional Euler-Lagrange system of arbitrary order.

This part is closed by chapter 9 which is dedicated to the analysis of the time evolution of Euler-Lagrange systems. Again we treat the finite- and the infinite-dimensional case separately. It will turn out that there exists a certain function h, which is invariant under the motion of a finite-dimensional Euler-Lagrange system in the time-invariant scenario. In the infinite-dimensional, time-invariant case this object is replaced by an invariant functional.

Finally we conclude that the Euler-Lagrange systems are equipped with a Poisson structure and directly supply the Hamilton operator v_h.

Chapter 5

Calculus of Variations

Particular instances of problems involving the concept of a functional were considered more than three hundred years ago, and in fact, the first important results are due to Euler (1707-1783). The most developed branch of the calculus of functionals is concerned with finding the maxima and minima of functionals and called the calculus of variations.

In the introduction the classical approach based on function spaces is sketched and several questions are stated. Here we reformulate this task in the context of differential geometry in order to answer the stated questions.

5.1 The Variational Formula

The representation of the calculus of variations on jet bundles requires in a first step the precise definition of the used spaces.

5.1.1 Local coordinates

We assume a plant described by r independent and s dependent coordinates. As already mentioned in Chap. 3 the appropriate mathematical representative of such plants is a bundle. Consequently we consider the bundle $(\mathcal{E}, \pi, \mathcal{D})$ with $\dim(\mathcal{E}) = r + s$, $\dim(\mathcal{D}) = r$ and choose the local coordinates (X^i, x^α), $i = 1, \ldots, r$, $\alpha = 1, \ldots, s$ according to $\mathcal{E}$ and (X^i) according to $\mathcal{D}$. Additionally, it is assumed that the base manifold $\mathcal{D}$ is a bounded, oriented manifold with coherently oriented boundary $\partial\mathcal{D}$. Furthermore we introduce the n^{th} order domain jet framework Π^n as the collection of all corresponding jet bundles up to the n^{th} order – see Def. 4.9.

Remark 5.1 *We have provided the domain $\mathcal{D}$ with the topological properties of a bounded, oriented manifold and thus answered the first introductory question.*

Now we are able to introduce the Lagrangian functional in the language of jet manifolds.

5.1.2 Lagrangian functional

The Lagrangian functional $\mathfrak{L}$ represents a map

$$\begin{aligned} \mathfrak{L} : \Gamma(\pi) &\rightarrow \mathbb{R} \\ \sigma &\rightarrow \mathfrak{L}(\sigma) \end{aligned}$$

given by

$$\mathfrak{L}(\sigma) = \int_{\mathcal{D}} (j^n\sigma)^* \left(l\left(X^i, x^\alpha, x^\alpha_{[J]}\right) \mathrm{dX}\right) \ , \quad \mathrm{dX} = \mathrm{d}X^1 \wedge \cdots \wedge \mathrm{d}X^r, \#J = 1, \ldots, n \tag{5.1}$$

where dX denotes the volume form on $\mathcal{D}$. The r-form $l\ \mathrm{dX}$ is called the Lagrangian of the variational problem, where the Lagrangian density $l \in C^\infty(J^n\pi)$ is used. Thus the Lagrangian is a section of the pull-back bundle $(\pi^n)^*(\tau_{\wedge^r\mathcal{D}})$, i.e. $l\ \mathrm{dX} \in \Gamma\left((\pi^n)^*(\tau_{\wedge^r\mathcal{D}})\right)$. Additionally, we make use of the n^{th} prolongation of the section σ

$$\begin{aligned} \sigma : \mathcal{D} &\rightarrow \mathcal{E} \\ X^i &\rightarrow \left(X^i, x^\alpha = \sigma^\alpha\left(X^i\right)\right) \end{aligned}$$

to pull-back the Lagrangian to $\tau_{\wedge^r\mathcal{D}}$, i.e.

$$(j^n\sigma)^*(l\ \mathrm{dX}) \in \Gamma(\tau_{\wedge^r\mathcal{D}}) \ .$$

This construction can be visualize by the following commutative diagram

$$\begin{tikzcd}
(\pi^n)^*(\wedge^r\mathcal{D}) \arrow[d, shift right, "(\pi^n)^*(\tau_{\wedge^r\mathcal{D}})"'] & \wedge^r\mathcal{D} \arrow[d, "\tau_{\wedge^r\mathcal{D}}"] \\
J^n\pi \arrow[u, shift right, "l\mathrm{dX}"'] \arrow[r, shift right, "\pi^n"'] & \mathcal{D}. \arrow[l, shift right, "j^n\sigma"']
\end{tikzcd}$$

Now we are able to introduce the variation of a section $\sigma^\alpha(X^i)$.

5.1.3 Variation

We introduce the variation of the section $\sigma \in \Gamma(\pi)$ as a fibre preserving bundle automorphism $(\exp(\varepsilon v), \mathrm{id}_{\mathcal{D}})$ generated by a π-vertical field $v \in \Gamma\left(\tau_{\mathcal{E}}|_{V\pi}\right)$. A section $\sigma \in \Gamma(\pi)$ is called extremal, iff it meets the condition

$$\begin{aligned} \left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathfrak{L}(\exp(\varepsilon v)(\sigma))\right|_{\varepsilon=0} &= \left.\int_{\mathcal{D}} \frac{\mathrm{d}}{\mathrm{d}\varepsilon}(j^n(\exp(\varepsilon v)(\sigma)))^*(l\ \mathrm{dX})\right|_{\varepsilon=0} \\ &= \int_{\mathcal{D}} (j^n(\mathrm{L}_v(\sigma)))^*(l\ \mathrm{dX}) \\ &= \int_{\mathcal{D}} j^n(\sigma)^* \mathrm{L}_{j^n v}(l\ \mathrm{dX}) \\ &= 0 \end{aligned}$$

for any admissible variational field $v \in \Gamma\left(\tau_{\mathcal{E}}|_{V\pi}\right)$ (see, e.g., [Saunders, 1989]). From this representation we want to derive the necessary condition for extremal points – the Euler-Lagrange equations.

Remark 5.2 *The Lie derivative of the Lagrangian along the vertical vector field v is the differential geometric analogon to the linear part $\delta\mathfrak{L}\left(f'\left(X\right)\right)$ of the functional $\triangle\mathfrak{L}\left(f'\left(X\right)\right)$ defined in the introduction. Consequently the second question is answered.*

5.2 Euler-Lagrange equations

To derive the Euler-Lagrange equations for (5.1), we apply Cartan's formula (see Def. A.64) given by

$$\mathrm{L}_{j^n(v)}\left(\omega\right) = j^n\left(v\right)\rfloor\mathrm{d}\omega + \mathrm{d}\left(j^n\left(v\right)\rfloor\omega\right)$$

and get

$$\mathfrak{L}_1\left(\sigma\right) + \mathfrak{L}_2\left(\sigma\right) = 0\ , \tag{5.2}$$

with

$$\mathfrak{L}_1\left(\sigma\right) = \int_{\mathcal{D}} \left(j^n\sigma\right)^*\left(j^n v\rfloor\mathrm{d}l\wedge\mathrm{dX}\right)\ ,\quad \mathfrak{L}_2\left(\sigma\right) = \int_{\partial\mathcal{D}} \iota^*\left(\left(j^n\sigma\right)^*\left(j^n v\rfloor\left(l\mathrm{dX}\right)\right)\right)\ , \tag{5.3}$$

where Stokes's theorem (see Def. A.70) with the inclusion mapping $\iota : \partial\mathcal{D} \to \mathcal{D}$ is used. Since the functional $\mathfrak{L}_1\left(\sigma\right)$ of equation (5.2) depends on the interior $\mathring{\mathcal{D}}$ of $\mathcal{D}$ and the functional $\mathfrak{L}_2\left(\sigma\right)$ on the boundary $\partial\mathcal{D}$, both of them must vanish. In the present form, the functional $\mathfrak{L}_2\left(\sigma\right)$ vanishes for every arbitrary section and $\mathfrak{L}_1\left(\sigma\right)$ leads to conditions for every jet coordinate $x^\alpha_{[J]}$, the Lagrangian density depends on.

In fact this trivial solution is much too restrictive, as the used variational vector field on $J^n\pi$ is obtained by the prolongation of a vertical vector field on $\mathcal{E}$ – denoted by $j^n v$.

In the subsequent investigations, we will discuss two different ways to incorporate this additional information. The first possibility evaluates the functional $\mathfrak{L}_1\left(\sigma\right)$ of equation (5.3), where the prolongation of the vertical vector field is carried out using equation (4.4). Consequently the obtained integrand contains total derivatives that could be used to derive a minimal amount of domain conditions[1] by means of integration by parts. Obviously the application of integration by parts also introduces additional boundary conditions.

The second possibility, which is due to Cartan, introduces a modification of the initial Lagrangian functional (5.1) in order to prevent the necessity of a prolonged variational vector field. The modification of the integrand by means of elements of a special contact ideal leads to the minimal amount of domain and boundary conditions without the use of integration by parts.

5.3 Integration by parts solution

A common strategy to derive the domain conditions is the integration by parts technique applied to $\mathfrak{L}_1\left(\sigma\right)$, e.g., see [Olver, 1986]. A straightforward application of this method is possible if all boundary terms vanish. Additionally, one has to keep in mind that classical integration by parts is only valid in $\mathbb{R}^1$([Zeidler, 1990]). Thus it is not the appropriate tool to handle integrals over higher dimensional domains. In appendix B the rectangular Kirchhoff plate is investigated using classical notation as introduced in, e.g., [Gelfand, S.V. Fomin, 2000] and the classical integration by parts technique is applied to determine the domain and boundary

[1]The domain conditions must be met only on the interior $\mathring{\mathcal{D}}$ of the domain $\mathcal{D}$, as the boundary $\partial\mathcal{D}$ is of measure zero in the functional $\mathfrak{L}_1\left(\sigma\right)$.

conditions. In fact the same results are obtained as given by Ritz in [Ritz, 1909]. Finally it is shown that the stated condition on the edges is a result of the wrong application of the integration by parts technique.

Fortunately there exists another solution that allows the straight forward derivation of the domain and boundary conditions – the Cartan form solution. Additionally, it is remarkable that this method is more suitable for computer algebra systems (see appendix C).

5.4 Cartan form solution

The upcoming modifications of the Lagrangian functional in equation (5.1) is based on the following theorem.

Theorem 5.3 *The Lagrangian functional is invariant under any additional element of the contact ideal $\omega \in I_n$ (see Def. 4.13), i.e.*

$$\int_{\mathcal{D}} (j^n\sigma)^* (l\ \mathrm{dX}) = \int_{\mathcal{D}} (j^n\sigma)^* (l\ \mathrm{dX} + \omega)\ .$$

Proof. This theorem is a trivial consequence of the property of elements of the contact ideal, i.e.

$$(j^n\sigma)^* \omega = 0\ .$$

■

Consequently we may add any term ω to $l\mathrm{dX}$ in (5.1) and try to construct a new functional $\mathfrak{C} : \Gamma(\pi) \to \mathbb{R}$,

$$\mathfrak{C}(\sigma) = \int_{\mathcal{D}} \left(j^{2n-1}\sigma\right)^* (c) = \mathfrak{L}(\sigma)$$

with the so called Cartan form $c = l\mathrm{dX} + \omega$, such that

$$\int_{\mathcal{D}} \left(j^{2n-1}\sigma\right)^* \left(j^{2n-1}v \rfloor \mathrm{d}c\right) = \int_{\mathcal{D}} \left(j^{2n-1}\sigma\right)^* (v \rfloor \mathrm{d}c) \tag{5.4}$$

is met.

Remark 5.4 *The prolongation order of the section σ must be increased, i.e.*

$$j^n\sigma \to j^{2n-1}\sigma,$$

due to the special construction of the Cartan form c as shown in section 6.2.

If the stated suppression of the prolongation of v in (5.4) is possible, then we can pick the domain condition directly from the expression

$$\int_{\mathcal{D}} \left(j^{2n-1}\sigma\right)^* (v \rfloor \mathrm{d}c) = 0\ . \tag{5.5}$$

The boundary conditions for first order Lagrangians $l \in C^\infty(J^1\pi)$ result simply from the expression

$$\int_{\partial\mathcal{D}} \iota^* \left(\left(j^1\sigma\right)^* \left(j^1 v \rfloor c\right)\right) = \int_{\partial\mathcal{D}} \left(j^1\bar{\sigma}\right)^* \left(v_\partial \rfloor \left(\Psi^1\right)^* c\right) = 0\ . \tag{5.6}$$

Here we have introduced the boundary section $\bar{\sigma} : \partial\mathcal{D} \to \bar{\mathcal{E}}$ and the contact bundle morphism (Ψ^1, ι) (see section 8.1.2) and the variational vector field restricted to the boundary denoted v_∂.

Remark 5.5 *In the case of higher order Lagrangians the Cartan form construction is not sufficient anymore and we have to introduce – as shown in chapter 8 – the extended Cartan form.*

Since the construction of $\mathfrak{C}$ is quite different for the finite-dimensional case[2], i.e. $r = 1$, $n \geq 1$, the first order infinite-dimensional case[3], i.e. $r > 1$, $n = 1$, and the n^{th} order infinite-dimensional case, i.e. $r > 1$, $n > 1$, we discuss them in detail in the chapters 6, 7, and 8.

[2]The Euler-Lagrange equations become ordinary differential equations in the finite-dimensional case.
[3]The Euler-Lagrange equations become partial differential equations in the infinite-dimensional case.

Chapter 6

Finite-dimensional Euler-Lagrange Systems (F-EL Systems)

In the finite-dimensional case we consider the bundle $(\mathcal{E}, \pi, \mathcal{D})$ with $\dim(\mathcal{E}) = s+1$, $\dim(\mathcal{D}) = 1$ and the adapted coordinates (X^1, x^α), $\alpha = 1, \ldots, s$. The volume form simplifies to $\mathrm{dX} = \mathrm{d}X^1$ and the domain of integration is the one dimensional bounded manifold $\mathcal{D}$ with the boundary points $a, b \in \mathbb{R}$, $a < b$ in the local coordinate X^1. Thus the zero dimensional boundary manifold $\partial\mathcal{D}$ consists of the two points a, b.

6.1 Structure of the Cartan form

Since every contact form $\omega^\alpha_{[J-1_1]} = \left(\mathrm{d}x^\alpha_{[J-1_1]} - x^\alpha_{[J]}\mathrm{d}X^1\right)$, $\#J < n$ meets $(j^n\sigma)^*\left(\omega^\alpha_{[J-1_1]}\right) = 0$ for an arbitrary section $\sigma \in \Gamma(\pi)$, we choose

$$c = l\mathrm{d}X + p^{[J]}_\alpha\left(\mathrm{d}x^\alpha_{[J-1_1]} - x^\alpha_{[J]}\mathrm{d}X^1\right)\ , \quad \#J = J = 1, \ldots, n\ , \quad x^\alpha_{[0]} = x^\alpha \tag{6.1}$$

as Cartan form c with suitable functions $p^{[J]}_\alpha \in C^\infty\left(J^{2n-1}\pi\right)$. Thus the determination of the functions $p^{[J]}_\alpha$ is now of interest.

6.2 Derivation of the Cartan form

The Euler-Lagrange equations are the necessary conditions that must be met by a section σ, such that

$$\int_{\mathcal{D}} (j^n\sigma)^*\,(l\ \mathrm{dX})$$

becomes extremal. Up to now we have determine the necessary condition

$$\left(j^{2n-1}\sigma\right)^*(v\rfloor\mathrm{d}c) = 0\ ,$$

which does not directly lead to the Euler-Lagrange equations, as it depends on the unknown section σ. Fortunately we are able to determine the Euler-Lagrange from the horizontal part of $v\rfloor\mathrm{d}c$. This is a simple consequence of the fact, that the vertical part of $v\rfloor\mathrm{d}c$ vanishes automatically under the pull-back $\left(j^{2n-1}\sigma\right)^*$.

Consequently we introduce the horizontal projection (see [Giachetta, G. Sardanashvily, L. Mangiarotti, 1994]) in equation (5.5) and obtain

$$\left(j^{2n-1}\sigma\right)^{*}(v\rfloor \mathrm{d}c) = \left(j^{2n}\sigma\right)^{*}(d_1\rfloor (v\rfloor \mathrm{d}c)) \wedge \mathrm{d}X^1 \tag{6.2}$$

with the vector field of the total derivative

$$d_1 = \partial_1 + x^{\alpha}_{[1_1]}\partial_\alpha + \cdots + x^{\alpha}_{[J+1_1]}\partial^{[J]}_\alpha + \cdots \ , \quad \partial^{[J]}_\alpha = \frac{\partial}{\partial x^\alpha_{[J]}}$$

(see Def. 4.11). Additionally, we are now able to determine the missing functions $p^{[J]}_\alpha$ from the following expression

$$\begin{aligned}
&(d_1\rfloor (j^n v\rfloor \mathrm{d}c)) \wedge \mathrm{d}X^1 = \\
&\quad = \ j^n v\rfloor \left((-d_1\rfloor \mathrm{d}c) \wedge \mathrm{d}X^1\right) \\
&\quad = \ -j^n v\rfloor \left(\left(d_1\rfloor \left(\mathrm{d}l \wedge \mathrm{d}X^1 + \mathrm{d}p^{[J]}_\alpha \wedge \left(\mathrm{d}x^\alpha_{[J-1_1]} - x^\alpha_{[J]}\mathrm{d}X^1\right) - p^{[J]}_\alpha \mathrm{d}x^\alpha_{[J]} \wedge \mathrm{d}X^1\right)\right) \wedge \mathrm{d}X^1\right) \\
&\quad = \ j^n v\rfloor \left(\mathrm{d}l - \left(d_1\rfloor \mathrm{d}p^{[J]}_\alpha\right) \wedge \left(\mathrm{d}x^\alpha_{[J-1_1]} - x^\alpha_{[J]}\mathrm{d}X^1\right) - p^{[J]}_\alpha \mathrm{d}x^\alpha_{[J]}\right) \wedge \mathrm{d}X^1 \\
&\quad = \ j^n v\rfloor \left(\mathrm{d}l \wedge \mathrm{d}X^1 - \left(d_1\rfloor \mathrm{d}p^{[J]}_\alpha\right) \wedge \mathrm{d}x^\alpha_{[J-1_1]} \wedge \mathrm{d}X^1 - p^{[J]}_\alpha \mathrm{d}x^\alpha_{[J]} \wedge \mathrm{d}X^1\right) \ ,
\end{aligned}$$

which results in

$$(d_1\rfloor (j^n v\rfloor \mathrm{d}c)) \wedge \mathrm{d}X^1 = j^n v\rfloor \left(\mathrm{d}l \wedge \mathrm{d}X^1 - \mathrm{L}_{d_1}\left(p^{[J]}_\alpha\right) \wedge \mathrm{d}x^\alpha_{[J-1_1]} \wedge \mathrm{d}X^1 - p^{[J]}_\alpha \mathrm{d}x^\alpha_{[J]} \wedge \mathrm{d}X^1\right) \ .$$

We are able to achieve the suppression of the prolongation of the vertical vector v (or simply we can replace $j^n v$ by v), iff we choose the functions $p^{[J]}_\alpha$ according to

$$\begin{array}{lcll}
p^{[J]}_\alpha & = & \partial^{[J]}_\alpha l & \#J = J = n \\
p^{[J]}_\alpha & = & \partial^{[J]}_\alpha l - \mathrm{L}_{d_1}\left(p^{[J+1_1]}_\alpha\right) = \partial^{[J]}_\alpha l + (-1)\,\mathrm{L}_{d_1}\left(\partial^{[J+1_1]}_\alpha l\right) & \#J = J = n-1 \\
\vdots & = & \vdots & \\
p^{[J]}_\alpha & = & \partial^{[J]}_\alpha l - \mathrm{L}_{d_1}\left(p^{[J+1_1]}_\alpha\right) = \partial^{[J]}_\alpha l + \sum_{\#I=1}^{n-\#J}(-1)^{\#I}\,\mathrm{L}_{d_I}\left(\partial^{[J+I]}_\alpha l\right) & \mathrm{L}_{d_I} = \underbrace{\mathrm{L}_{d_1}\circ\cdots\circ \mathrm{L}_{d_1}}_{\#I-\text{times}} \\
\vdots & = & \vdots & \\
p^{[J]}_\alpha & = & \partial^{[J]}_\alpha l - \mathrm{L}_{d_1}\left(p^{[J+1_1]}_\alpha\right) = \partial^{[1]}_\alpha l + \sum_{\#I=1}^{n-1}(-1)^{\#I}\,\mathrm{L}_{d_I}\left(\partial^{[1+I]}_\alpha l\right) & \#J = J = 1 \ .
\end{array} \tag{6.3}$$

Combining the relations (6.1), (6.3), we can rewrite the exterior derivative of the Cartan form as

$$
\begin{aligned}
\mathrm{d}c &= \mathrm{d}l \wedge \mathrm{d}X^1 + \mathrm{d}p_\alpha^{[J]} \wedge \left(\mathrm{d}x_{[J-1_1]}^\alpha - x_{[J]}^\alpha \mathrm{d}X^1\right) - p_\alpha^{[J]} \mathrm{d}x_{[J]}^\alpha \wedge \mathrm{d}X^1 \\
&= \mathrm{d}l \wedge \mathrm{d}X^1 + \mathrm{d}\left(\partial_\alpha^{[J]} l + \sum_{\#I=1}^{n-\#J} (-1)^{\#I} \mathrm{L}_{d_I}\left(\partial_\alpha^{[J+I]} l\right)\right) \wedge \left(\mathrm{d}x_{[J-1_1]}^\alpha - x_{[J]}^\alpha \mathrm{d}X^1\right) \\
&\quad - \left(\partial_\alpha^{[J]} l + \sum_{\#I=1}^{n-\#J} (-1)^{\#I} \mathrm{L}_{d_I}\left(\partial_\alpha^{[J+I]} l\right)\right) \mathrm{d}x_{[J]}^\alpha \wedge \mathrm{d}X^1 \\
&= \mathrm{d}l \wedge \mathrm{d}X^1 + (\mathrm{d}_h + \mathrm{d}_v)\left(\partial_\alpha^{[J]} l + \sum_{\#I=1}^{n-\#J} (-1)^{\#I} \mathrm{L}_{d_I}\left(\partial_\alpha^{[J+I]} l\right)\right) \wedge \left(\mathrm{d}x_{[J-1_1]}^\alpha - x_{[J]}^\alpha \mathrm{d}X^1\right) \\
&\quad - \left(\partial_\alpha^{[J]} l + \sum_{\#I=1}^{n-\#J} (-1)^{\#I} \mathrm{L}_{d_I}\left(\partial_\alpha^{[J+I]} l\right)\right) \mathrm{d}x_{[J]}^\alpha \wedge \mathrm{d}X^1 \\
&= \mathrm{d}l \wedge \mathrm{d}X^1 + \mathrm{d}_h\left(\partial_\alpha^{[J]} l + \sum_{\#I=1}^{n-\#J} (-1)^{\#I} \mathrm{L}_{d_I}\left(\partial_\alpha^{[J+I]} l\right)\right) \wedge \mathrm{d}x_{[J-1_1]}^\alpha \\
&\quad + \mathrm{d}_v\left(\partial_\alpha^{[J]} l + \sum_{\#I=1}^{n-\#J} (-1)^{\#I} \mathrm{L}_{d_I}\left(\partial_\alpha^{[J+I]} l\right)\right) \wedge \left(\mathrm{d}x_{[J-1_1]}^\alpha - x_{[J]}^\alpha \mathrm{d}X^1\right) \\
&\quad - \left(\partial_\alpha^{[J]} l + \sum_{\#I=1}^{n-\#J} (-1)^{\#I} \mathrm{L}_{d_I}\left(\partial_\alpha^{[J+I]} l\right)\right) \mathrm{d}x_{[J]}^\alpha \wedge \mathrm{d}X^1 \\
&= \partial_\alpha l \mathrm{d}x^\alpha \wedge \mathrm{d}X^1 + \sum_{i=1}^{n} (-1)^i \mathrm{L}_{d_i}\left(\partial_\alpha^i l\right) \mathrm{d}x^\alpha \wedge \mathrm{d}X^1 \\
&\quad + \mathrm{d}_v\left(\partial_\alpha^{[J]} l + \sum_{\#I=1}^{n-\#J} (-1)^{\#I} \mathrm{L}_{d_I}\left(\partial_\alpha^{[J+I]} l\right)\right) \wedge \left(\mathrm{d}x_{[J-1_1]}^\alpha - x_{[J]}^\alpha \mathrm{d}X^1\right) \\
&= \delta_\alpha(l)\, \mathrm{d}x^\alpha \wedge \mathrm{d}X^1 + \underbrace{\mathrm{d}_v\left(\partial_\alpha^{[J]} l + \sum_{\#I=1}^{n-\#J} (-1)^{\#I} \mathrm{L}_{d_I}\left(\partial_\alpha^{[J+I]} l\right)\right) \wedge \left(\mathrm{d}x_{[J-1_1]}^\alpha - x_{[J]}^\alpha \mathrm{d}X^1\right)}_{\theta}
\end{aligned}
$$

with the Euler-Lagrange operator δ_α, which is given by

$$
\delta_\alpha(\cdot) = \partial_\alpha(\cdot) + \sum_{i=1}^{n} (-1)^i (\mathrm{L}_{d_1})^i \left(\partial_\alpha^{[i]}(\cdot)\right), \quad (\mathrm{L}_{d_1})^i = \underbrace{\mathrm{L}_{d_1} \circ \cdots \circ \mathrm{L}_{d_1}}_{i-\text{times}}, \ \partial_\alpha^{[i]} = \frac{\partial}{\partial x_{[i]}^\alpha} \tag{6.4}
$$

in finite-dimensional case. It is remarkable that the 2-form θ meets

$$
d_1 \rfloor \theta = 0 \ \text{ or equivalently } \ \left(j^{2n}\sigma\right)^*(\theta) = 0
$$

and consequently the prolongation of the variational vector field becomes unnecessary.

Remark 6.1 *It is obvious that the constructed Cartan form qualifies as a Lepagian form (see [Giachetta, G. Sardanashvily, L. Mangiarotti, 1994]).*

6.2.1 Extracting the Euler-Lagrange equations

Obviously, the Euler-Lagrange equations can be extracted from

$$
(-d_1 \rfloor \mathrm{d}c) \wedge \mathrm{d}X^1 = 0 \tag{6.5}
$$

or equally

$$
\delta_\alpha(l)\, \mathrm{d}x^\alpha \wedge \mathrm{d}X^1 = 0\,. \tag{6.6}
$$

The application of the horizontal projection in (6.2) leads obviously to an algebraic relation, i.e. the Euler-Lagrange equations $\delta_\alpha(l) = 0$, on $J^{2n}\pi$.

Additionally, we get the boundary conditions

$$j^n v_\partial \rfloor \left(\Psi^1\right)^* c = j^n v_\partial \rfloor \left(\left(p_\alpha^{[J]} \circ \Psi^1\right) \mathrm{d}x^\alpha_{[J-1_1]}\right) = 0 \; . \tag{6.7}$$

This condition has to be met at the “time”-boundary points (a, b). In the case of Euler-Lagrange systems, it is assumed that the variational vector field v vanishes at the “time”-boundary. Consequently no boundary conditions appear for finite-dimensional Euler-Lagrange systems, as the equation 6.7 is always met.

Remark 6.2 *In the theory of calculus of variations all independent coordinates are treated equivalently, i.e. they are assumed to be local coordinates of the manifold $\mathcal{D}$. In fact, the additional assumption in the context of Euler-Lagrange systems that the variational vector field vanishes on “time”-boundaries is the first and the only distinction of an independent coordinate X^1 – the time t.*

Chapter 7

Infinite-dimensional Euler-Lagrange Systems (I-EL Systems)

In the case of infinite-dimensional EL systems we make use of a bundle $(\mathcal{E}, \pi, \mathcal{D})$ with the total manifold $\mathcal{E}$, $\dim(\mathcal{E}) = r + s$ and the bounded base manifold $\mathcal{D}$, $\dim(\mathcal{D}) = r > 1$. Additionally, we choose the local coordinates (X^i), $i = 1, \ldots, r$ according to $\mathcal{D}$ and (X^i, x^α), $\alpha = 1, \ldots, s$ corresponding to $\mathcal{E}$. Before we actually start to analyze the infinite-dimensional case we discuss the considered domains.

7.0.2 Considered Domains

The switch from the finite-dimensional to the infinite-dimensional case equals the consideration of domains of integration with possibly higher topological complexity. We will confine ourselves to the case of a base manifold (or domain of integration) $\mathcal{D}$ homeomorphic to an r-dimensional unit sphere. We will consider base manifolds $\mathcal{D}$ without boundary, i.e.

$$\partial\mathcal{D} = 0$$

as shown in Fig.7.1. It is obvious that the derivation of the boundary conditions for infinite-

Figure 7.1: Domains without boundary.

dimensional EL-systems becomes trivial, if no boundary exists. Thus we consider also domains $\mathcal{D}$ with boundary, i.e.

$$\partial\mathcal{D} \neq 0$$

as shown in the figures 7.2 and 7.3. Here we distinguish between smooth and non-smooth boundaries. It will turn out that from the differential geometric point of view the distinction

Figure 7.2: Domains with smooth boundary.

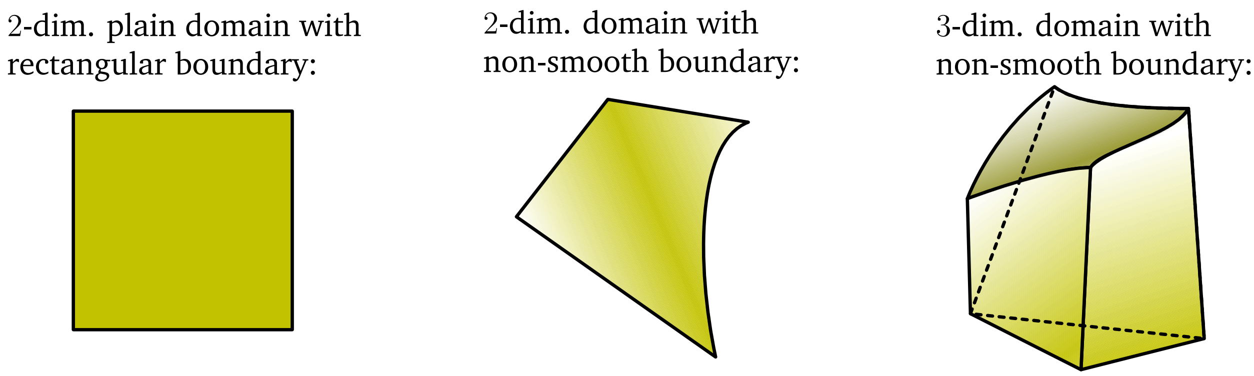

Figure 7.3: Domains with non-smooth boundary.

between smooth and non-smooth domains is not of main importance, as long as we confine ourselves to the case of non-smooth boundaries that are built up by a finite amount of bounded smooth manifolds. As the construction of the extended Cartan form is a local procedure, it will not differ for smooth and non-smooth boundaries (see section 8.4).

Subsequently we have divided the investigation of the I-EL systems in two sections. At first, we consider the case of first order Lagrangians $l \in C^\infty(J^1\pi)$. After that, the more general case of n^{th} order Lagrangians $l \in C^\infty(J^n\pi)$ is treated.

7.1 Systems with 1^{st} order Lagrangian

The preliminary work of section 5.1 enables us to establish the Cartan form solution in a straight forward manner. At first let us define the structure of the Cartan form c.

7.1.1 Structure of the Cartan Form

Following the considerations from above, one sees that a possible choice for the Cartan form is given by

$$c = l\mathrm{dX} + p_\alpha^{[1_j]}\left(\mathrm{d}x^\alpha - x^\alpha_{[1_i]}\mathrm{d}X^i\right) \wedge (\partial_j \rfloor \mathrm{dX}) \ , \quad j,i = 1,\dots,r \ , \tag{7.1}$$

with suitable functions $p_\alpha^{[1_j]} \in C^\infty(J^1\pi)$.

In the next step we have to determine these functions such that equation (5.4) is met.

7.1.2 Derivation of the Cartan Form

We are again able to apply the horizontal projection in the form of

$$\left(j^1\sigma\right)^* \left(j^1 v \rfloor (\mathrm{d}c)\right) = \left(j^2\sigma\right)^* \left(D\rfloor \left(j^1 v\rfloor (\mathrm{d}c)\right) \wedge \mathrm{dX}\right) \ , \quad D = d_r\rfloor \cdots \rfloor d_1 \ , \tag{7.2}$$

in order to obtain the Euler-Lagrange equations as algebraic relations on $J^2\pi$. Here the total derivatives d_i of definition 4.11 are used again.

Having the horizontal projection at ones disposal, we determine the expression

$$\begin{aligned}
&D\rfloor \left(j^1 v\rfloor (\mathrm{d}c)\right) \wedge \mathrm{dX} = \\
&\quad = \ j^1 v\rfloor \left((-1)^r D\rfloor (\mathrm{d}c)\right) \wedge \mathrm{dX} \\
&\quad = \ (-1)^r \ j^1 v\rfloor \left(D\rfloor \left(\mathrm{d}\left(l \,\mathrm{dX} + p_\alpha^{[1_j]} \left(\mathrm{d}x^\alpha \wedge \partial_j\rfloor \mathrm{dX} - x_{[1_j]}^\alpha \mathrm{dX}\right)\right)\right)\right) \wedge \mathrm{dX} \\
&\quad = \ (-1)^r \ j^1 v\rfloor \left(D\rfloor \left(\mathrm{d}l \wedge \mathrm{dX} + \mathrm{d}p_\alpha^{[1_j]} \wedge \left(\mathrm{d}x^\alpha \wedge \partial_j\rfloor \mathrm{dX} - x_{[1_j]}^\alpha \mathrm{dX}\right) - p_\alpha^{[1_j]} \mathrm{d}x_{[1_j]}^\alpha \wedge \mathrm{dX}\right)\right) \wedge \mathrm{dX} \\
&\quad = \ j^1 v\rfloor \left(\mathrm{d}l \wedge \mathrm{dX} - \left(d_j\rfloor \mathrm{d}p_\alpha^{[1_j]}\right) \mathrm{d}x^\alpha \wedge \mathrm{dX} - p_\alpha^{[1_j]} \mathrm{d}x_{[1_j]}^\alpha \wedge \mathrm{dX}\right) \ .
\end{aligned}$$

This relation becomes independent of the prolongation of the vertical vector field $j^1 v$, i.e. no $\mathrm{L}_{d_i}\left(v^\alpha\right)$ will appear, iff the functions $p_\alpha^{[1_j]}$ are determined by

$$p_\alpha^{[J]} \ = \ \partial_\alpha^{[J]} l \quad , \quad \#J = 1 \ , \ J = 1_j \ . \tag{7.3}$$

Remark 7.1 *There exists only a single contact form that can be used to cancel out a certain* $\mathrm{d}x_{[1_j]}^\alpha$ *form entry of* $\mathrm{d}c$*. Thus the proposed Cartan form is unique.*

The combination of (7.1) and (7.3) leads to

$$\begin{aligned}
\mathrm{d}c \ &= \ \mathrm{d}l \wedge \mathrm{dX} + \mathrm{d}p_\alpha^{[1_j]} \wedge \left(\mathrm{d}x^\alpha \wedge \partial_j\rfloor \mathrm{dX} - x_{[1_j]}^\alpha \mathrm{dX}\right) - p_\alpha^{[1_j]} \mathrm{d}x_{[1_j]}^\alpha \wedge \mathrm{dX} \\
&= \ \mathrm{d}l \wedge \mathrm{dX} + \left(\mathrm{d}_h\left(\partial_\alpha^{[1_j]} l\right) + \mathrm{d}_v\left(\partial_\alpha^{[1_j]} l\right)\right) \wedge \left(\mathrm{d}x^\alpha \wedge \partial_j\rfloor \mathrm{dX} - x_{[1_j]}^\alpha \mathrm{dX}\right) - \partial_\alpha^{[1_j]} l \ \mathrm{d}x_{[1_j]}^\alpha \wedge \mathrm{dX} \\
&= \ \partial_\alpha l \mathrm{d}x^\alpha \wedge \mathrm{dX} + \partial_\alpha^{[1_j]} l \ \mathrm{d}x_{[1_j]}^\alpha \wedge \mathrm{dX} + \mathrm{L}_{d_j}\left(\partial_\alpha^{[1_j]} l\right) \mathrm{d}X^j \wedge \mathrm{d}x^\alpha \wedge \partial_j\rfloor \mathrm{dX} \\
&\quad + \mathrm{d}_v\left(\partial_\alpha^{[1_j]} l\right) \wedge \left(\mathrm{d}x^\alpha \wedge \partial_j\rfloor \mathrm{dX} - x_{[1_j]}^\alpha \mathrm{dX}\right) - \partial_\alpha^{[1_j]} l \ \mathrm{d}x_{[1_j]}^\alpha \wedge \mathrm{dX} \\
&= \ \delta_\alpha(l) \, \mathrm{d}x^\alpha \wedge \mathrm{dX} + \underbrace{\mathrm{d}_v\left(\partial_\alpha^{[1_j]} l\right) \wedge \left(\mathrm{d}x^\alpha \wedge \partial_j\rfloor \mathrm{dX} - x_{[1_j]}^\alpha \mathrm{dX}\right)}_{\theta}
\end{aligned}$$

with the Euler-Lagrange operator δ_α,

$$\delta_\alpha(\cdot) = \partial_\alpha(\cdot) + \sum\nolimits_{i=1}^{n} (-1) \, \mathrm{L}_{d_i}\left(\partial_\alpha^{[1_i]}(\cdot)\right) \tag{7.4}$$

and a $(p+1)$-form θ, which meets $D\rfloor\theta = 0$. Consequently we are able to determine the domain and boundary conditions.

It is worth mentioning, that the Cartan form qualifies also in this case as a Lepagian form (see [Giachetta, G. Sardanashvily, L. Mangiarotti, 1994]).

7.1.3 Extraction of the Euler-Lagrange Equations

From the functional (5.5) and the considerations in equation (7.2), we are able to determine the Euler-Lagrange equations as the partial differential equations

$$\left((-1)^r D\rfloor \mathrm{d}c\right) \wedge \mathrm{dX} = 0$$

or equivalently

$$\delta_\alpha (l)\, \mathrm{d}x^\alpha \wedge \mathrm{dX} = 0\ .$$

It is obvious that these partial differential equations represent the algebraic relations on $J^2\pi$ of interest.

7.1.4 Extraction of the Boundary Conditions

The integral of equation (5.6) leads in this case to the expression

$$\begin{aligned}
\iota^*\left(\left(j^1\sigma\right)^*(v\rfloor c)\right) &= \left(j^1\bar\sigma\right)^*\left(\left(\Psi^1\right)^*(v\rfloor c)\right) \\
&= \left(j^1\bar\sigma\right)^*\left(\left(\Psi^1\right)^*\left(v^\alpha\partial_\alpha\rfloor\left(l\mathrm{dX} + p_\alpha^{[1_j]}\left(\mathrm{d}x^\alpha - x^\alpha_{[1_i]}\mathrm{d}X^i\right)\wedge\left(\partial_j\rfloor\mathrm{dX}\right)\right)\right)\right) \\
&= \left(j^1\bar\sigma\right)^*\left(\left(\Psi^1\right)^*\left(v^\alpha\partial_\alpha\rfloor\left(p_\alpha^{[1_j]}\mathrm{d}x^\alpha\wedge\left(\partial_j\rfloor\mathrm{dX}\right)\right)\right)\right) \\
&= \left(j^1\bar\sigma\right)^*\left(\left(v^\alpha\circ\Psi^1\right)\partial_\alpha\rfloor\left(\Psi^1\right)^*\left(p_\alpha^{[1_j]}\mathrm{d}x^\alpha\wedge\left(\partial_j\rfloor\mathrm{dX}\right)\right)\right) \\
&= \left(j^1\bar\sigma\right)^*\left(v^\alpha_\partial\partial_\alpha\rfloor\left(\left(p_\alpha^{[1_r]}\circ\Psi^1\right)\mathrm{d}x^\alpha\wedge\mathrm{d\bar X}\right)\right) \\
&= \left(j^1\bar\sigma\right)^*\left(v_\partial\rfloor\left(\left(\delta^{[1_r]}_{\partial\alpha}(l)\right)\mathrm{d}x^\alpha\wedge\mathrm{d\bar X}\right)\right)\ ,
\end{aligned} \tag{7.5}$$

which must vanish on the boundary for any admissible variational field v. Here we have introduced the boundary Euler-Lagrange operator $\delta^{[1_r]}_{\partial\alpha}(l) = p_\alpha^{[1_r]}\circ\Psi^1$. Thus we get the boundary condition

$$\delta^{[1_r]}_{\partial\alpha}(l)\,\mathrm{d}x^\alpha\wedge\mathrm{d\bar X} = 0\ , \tag{7.6}$$

with is again an algebraic relation. Additionally, we made use of the boundary volumes form $\mathrm{d\bar X}$ and assumed that $\partial_r\rfloor\mathrm{dX} = \mathrm{d\bar X}$. This is always possible by the introduction of a special change of coordinates and explained in more detail in section 8.1.1.

Now we are able to switch to a more general case of infinite-dimensional Euler-Lagrange systems.

7.2 Systems with n^{th} order Lagrangians

7.2.1 General Problem

Unfortunately the case $l \in C^\infty(J^n\pi)$, $n > 1$ is substantially more difficult than the first order case. The main reason for this fact is that in the determination process of the Cartan form, one can find several contact forms $\omega^\alpha_{[J-I_1]}$, $\#I_1 = 1$ that allow the cancellation of a certain form part $\mathrm{d}x^\alpha_{[J]}$ in $\mathrm{d}c$ (see theorem 4.14). The only exceptions are the previously treated finite-dimensional case $r = 1$ and the infinite-dimensional case $r > 1$ with first order Lagrangian $n = 1$. In both configurations, there exists only one unique index I_1 and thus the corresponding Cartan form is unique.

In the next step we will analyze the impact of this non-uniqueness on the domain and boundary conditions.

7.2.2 Construction of the Cartan Form

In the upcoming investigations we will use contact forms in the following representation

$$\omega^{\alpha}_{[J-I_1]} = \left(\mathrm{d}x^{\alpha}_{[J-I_1]} - x^{\alpha}_{[J-I_1+1_i]}\mathrm{d}X^i\right) , \quad I_1 = 1_{k_l} , \quad k_l \in \{1,\ldots,r\} , \quad l = 1,\ldots,m_J .$$

Remark 7.2 *The used notation $\omega^{\alpha}_{[J-I_1]}$ implies that the corresponding contact form exists only for non-negative entries in the multi-index $[J-I_1]$.*

We assume that there exist $1 \leq m_J \leq r$ contact forms for a given multi-index J, $\#J > 0$. Furthermore let us apply all possible contact forms $\omega^{\alpha}_{[J-I_1]}$ in the construction of the Cartan form, i.e.

$$c = l\,\mathrm{dX} + \sum_{l=1}^{m_J} p_{\alpha}^{[J-I_1,J]}\left(\mathrm{d}x^{\alpha}_{[J-I_1]} - x^{\alpha}_{[J]}\mathrm{d}X^{k_l}\right) \wedge \partial_{k_l} \rfloor \mathrm{dX} , \quad \#J = 1,\ldots,n , \quad I_1 = 1_{k_l} \tag{7.7}$$

to realize $p_{\alpha}^{[J-I_1,J]}\ x^{\alpha}_{[J]}\ \mathrm{dX}$ entries in c. It is necessary to introduce the double multi-indexed functions $p_{\alpha}^{[J-I_1,J]}$ in order to incorporate the non-uniqueness of the correspondence between $x^{\alpha}_{[J]}$ and $\omega^{\alpha}_{[J-I_1]}$. Again we require the Euler-Lagrange equations to be algebraic relation on a certain jet manifold and consequently we introduce the horizontal projection

$$\left(j^{2n-1}\sigma\right)^{*}\left(j^{2n-1}v\rfloor(\mathrm{d}c)\right) = \left(j^{2n}\sigma\right)^{*}\left(D\rfloor\left(j^{2n-1}v\rfloor(\mathrm{d}c)\right)\wedge\mathrm{dX}\right) , \quad D = d_r\rfloor\cdots\rfloor d_1 .$$

Finally we obtain

$$\begin{aligned}
&\left(D\rfloor j^{2n-1}v\rfloor \mathrm{d}c\right)\wedge\mathrm{dX} = \\
&= j^{2n}v\rfloor\left((-1)^r D\rfloor\mathrm{d}c\right)\wedge\mathrm{dX} \\
&= (-1)^r\left(j^{2n}v\rfloor D\rfloor\mathrm{d}\left(l\mathrm{dX} + \sum_{l=1}^{m_J} p_{\alpha}^{[J-I_1,J]}\left(\mathrm{d}x^{\alpha}_{[J-I_1]}\wedge\partial_{k_l}\rfloor\mathrm{dX} - x^{\alpha}_{[J]}\mathrm{dX}\right)\right)\right)\wedge\mathrm{dX} \\
&= (-1)^r\left(j^{2n}v\rfloor D\rfloor\left(\mathrm{d}l\wedge\mathrm{dX}\right.\right. \\
&\qquad \left.\left. + \sum_{l=1}^{m_J}\left(\mathrm{d}p_{\alpha}^{[J-I_1,J]}\wedge\left(\mathrm{d}x^{\alpha}_{[J-I_1]}\wedge\partial_{k_l}\rfloor\mathrm{dX} - x^{\alpha}_{[J]}\mathrm{dX}\right) - p_{\alpha}^{[J-I_1,J]}\mathrm{d}x^{\alpha}_{[J]}\wedge\mathrm{dX}\right)\right)\right)\wedge\mathrm{dX} \\
&= j^{2n}v\rfloor\left(\mathrm{d}l\wedge\mathrm{dX} - \sum_{l=1}^{m_J}\left(d_{k_l}\rfloor\mathrm{d}p_{\alpha}^{[J-I_1,J]}\wedge\mathrm{d}x^{\alpha}_{[J-I_1]}\wedge\mathrm{dX} + p_{\alpha}^{[J-I_1,J]}\mathrm{d}x^{\alpha}_{[J]}\wedge\mathrm{dX}\right)\right) \\
&= j^{2n}v\rfloor\left(\mathrm{d}l\wedge\mathrm{dX} - \sum_{l=1}^{m_J}\left(\mathrm{L}_{d_{k_l}}\left(p_{\alpha}^{[J-I_1,J]}\right)\wedge\mathrm{d}x^{\alpha}_{[J-I_1]}\wedge\mathrm{dX} + p_{\alpha}^{[J-I_1,J]}\mathrm{d}x^{\alpha}_{[J]}\wedge\mathrm{dX}\right)\right) .
\end{aligned} \tag{7.8}$$

Thus all $\mathrm{d}x^\alpha_{[J]}$ form parts disappear in $\mathrm{d}c$ or equivalently the prolongation of the vertical vector field v becomes unnecessary, see again (5.4), iff we choose

$$\begin{aligned}
\sum_{l=1}^{m_J} p_\alpha^{[J-I_1,J]} &= \partial_\alpha^{[J]} l && \#J=n\ ,\ I_1=1_{k_l} \\
\sum_{l=1}^{m_J} p_\alpha^{[J-I_1,J]} &= \partial_\alpha^{[J]} l - \sum_{i=1}^{r} \mathrm{L}_{d_i}\left(p_\alpha^{[J,J+1_i]}\right) && \#J=n-1\ ,\ I_1=1_{k_l} \\
\vdots\quad &= \quad\vdots \\
p_\alpha^{[0,J]} &= \partial_\alpha^{\left[1_{k_1}\right]} l - \sum_{i=1}^{r} \mathrm{L}_{d_i}\left(p_\alpha^{\left[1_{k_1},1_{k_1}+1_i\right]}\right) && \#J=1\ ,\ J=0+1_{k_1}\ ,\ m_J=1\ .
\end{aligned} \tag{7.9}$$

This construction does obviously not determine the Cartan form uniquely. Actually it determines a family of forms. In the next step, we investigate the domain conditions that can be determined from every element of this family.

7.2.3 Conditions on the domain

If we consider the equation (7.8) we see that all $p_\alpha^{[J,J+1_i]}$ are canceled out by lower indexed $p_\alpha^{[J-I_1,J]}$ except for the case $\#J=0$. This is a consequence of the fact that a jet variable $x^\alpha_{[J]}$ with multi-index $J=1_{k_1},\ \#J=1$ leads to the introduction of a unique function $p_\alpha^{\left[0,1_{k_1}\right]}$.

Remark 7.3 *This fact was responsible for the uniqueness of the Cartan form in the case of first order Lagrangians.*

Thus we have to focus on $p_\alpha^{\left[0,1_{k_1}\right]}$ in equation (7.8) and get

$$\begin{aligned}
\sum_{k_1=1}^{r} \mathrm{L}_{d_{k_1}}\left(p_\alpha^{\left[0,1_{k_1}\right]}\right) &= \sum_{k_1=1}^{r} \mathrm{L}_{d_{k_1}}\left(\partial_\alpha^{\left[1_{k_1}\right]} l - \sum_{i=1}^{r} \mathrm{L}_{d_i}\left(p_\alpha^{\left[1_{k_1},1_{k_1}+1_i\right]}\right)\right) \\
&= \sum_{k_1=1}^{r} \mathrm{L}_{d_{k_1}} \partial_\alpha^{\left[1_{k_1}\right]} l - \sum_{k_1=1}^{r} \mathrm{L}_{d_{k_1}} \sum_{i=1}^{r} \mathrm{L}_{d_i}\left(p_\alpha^{\left[1_{k_1},1_{k_1}+1_i\right]}\right) \\
&= \sum_{k_1=1}^{r} \mathrm{L}_{d_{k_1}} \partial_\alpha^{\left[1_{k_1}\right]} l - \sum_{\#J=2} \mathrm{L}_{d_J}\left(\partial_\alpha^{[J]} l - \sum_{j=1}^{r} \mathrm{L}_{d_i}\left(p_\alpha^{[J,J+1_j]}\right)\right) \\
&= \sum_{k_1=1}^{r} \mathrm{L}_{d_{k_1}} \partial_\alpha^{\left[1_{k_1}\right]} l - \sum_{\#J=2} \mathrm{L}_{d_J} \partial_\alpha^{[J]} l \\
&\quad + \sum_{\#J=2} \mathrm{L}_{d_J} \sum_{j=1}^{r} \mathrm{L}_{d_i}\left(p_\alpha^{[J,J+1_j]}\right) \\
&= \sum_{k_1=1}^{r} \mathrm{L}_{d_{k_1}} \partial_\alpha^{\left[1_{k_1}\right]} l - \sum_{\#J=2} \mathrm{L}_{d_J} \partial_\alpha^{[J]} l \\
&\quad + \sum_{\#J=3} \mathrm{L}_{d_J}\left(\partial_\alpha^{[J]} l - \sum_{j=1}^{r} \mathrm{L}_{d_i}\left(p_\alpha^{[J,J+1_j]}\right)\right) \\
&= \dots\ ,
\end{aligned}$$

where we have made extensive use of

$$\mathrm{L}_{d_i}\mathrm{L}_{d_j}(f) = \mathrm{L}_{d_j}\mathrm{L}_{d_i}(f) \ , \quad f \in C^\infty(J^n\pi) \ , \quad i,j = 1,\ldots,r$$

and

$$\mathrm{L}_{d_J} = \underbrace{(\mathrm{L}_{d_1})^{j_1} \circ \cdots \circ (\mathrm{L}_{d_r})^{j_r}}_{\#J\text{-times}} \ , \quad [J] = [j_1 \ldots j_r] \ .$$

We see that all $p_\alpha^{[J-I_1,J]}$, $\#J > 0$ can be replaced by the assignments of equation (7.9) and consequently it follows that

$$\begin{aligned}
\mathrm{d}l \wedge \mathrm{dX} - \sum_{l=1}^{m_J} \left(\mathrm{L}_{d_{k_l}}\left(p_\alpha^{[J-I_1,J]}\right) \wedge \mathrm{d}x^\alpha_{[J-I_1]} \wedge \mathrm{dX} + p_\alpha^{[J-I_1,J]}\mathrm{d}x^\alpha_{[J]} \wedge \mathrm{dX}\right) &= \\
= \left(\partial_\alpha l + \sum_{\#I=1}^{n} (-1)^{\#I}\,\mathrm{L}_{d_I}\left(\partial_\alpha^{[I]} l\right)\right) \mathrm{d}x^\alpha \wedge \mathrm{dX} & \\
= \delta_\alpha(l)\ \mathrm{d}x^\alpha \wedge \mathrm{dX}\ , &
\end{aligned}$$

with the Euler-Lagrange operator δ_α given by

$$\delta_\alpha(l) = \partial_\alpha l + \sum_{\#I=1}^{n} (-1)^{\#I}\,\mathrm{L}_{d_I}\left(\partial_\alpha^{[I]} l\right) \ , \quad \mathrm{L}_{d_I} = \underbrace{(\mathrm{L}_{d_1})^{i_1} \circ \cdots \circ (\mathrm{L}_{d_r})^{i_r}}_{\#I-\text{times}} \ . \tag{7.10}$$

This derivation shows that every element of the determined family leads to the same domain conditions, i.e. partial differential equations. On the one hand this result is rather satisfactory, because the determination of the domain conditions is solved. On the other hand this implies that there exists no distinguished Cartan form using the presented construction. Thus it is necessary to add additional restrictions if one wants to obtain a unique Cartan form. This can be achieved by the use of an additional index ordering as presented in section 7.2.5.

In the next step we will investigate the consequences of the non-uniqueness of the Cartan form on the determination of the boundary conditions.

7.2.4 Conditions on the boundary

The condition on the boundary results in

$$\begin{aligned}
\left(\Psi^{2n-1}\right)^*\left(j^{2n-1}v\rfloor c\right) &= j^{2n-1}v_\partial \rfloor \left(\left(\Psi^{2n-1}\right)^*\left(l\mathrm{dX} + p_\alpha^{[J-I_1,J]}\left(\mathrm{d}x^\alpha_{[J-I_1]} - x^\alpha_{[J]}\mathrm{d}X^{k_l}\right)\wedge \partial_{k_l}\rfloor\mathrm{dX}\right)\right) \\
&= j^{2n-1}v_\partial \rfloor \left(\left(p_\alpha^{[J-1_r,J]} \circ \Psi^{2n-1}\right)\mathrm{d}x^\alpha_{[J-1_r]} \wedge \mathrm{d}\bar{\mathrm{X}}\right) \\
&= j^{2n-1}v_\partial \rfloor \left(p_\alpha^{[J-1_r,J]} \circ \Psi^{2n-1}\right)\mathrm{d}x^\alpha_{[J-1_r]} \wedge \mathrm{d}\bar{\mathrm{X}} \\
&= 0
\end{aligned} \tag{7.11}$$

with

$$I_1 = 1_{k_l} \ , \quad k_l \in \{1,\ldots,r\} \ , \quad l = 1,\ldots,m_J \ , \quad \#J = 1,\ldots,n \ .$$

Additionally, we have again assumed that the boundary volume form is given by $\mathrm{d}\bar{\mathrm{X}} = \partial_r\rfloor\mathrm{dX}$. Unfortunately the boundary conditions are not uniquely determined, which is visualized by the still unknown functions $p_\alpha^{[J-1_r,J]}$ in (7.11).

To overcome this non-uniqueness problem, we will make use of an additional index order (see Def. 4.2) in 7.2.5 and discuss the resulting partial differential equations and boundary conditions. It will turn out that the chosen ordering does also not allow a correct determination of the boundary conditions.

7.2.5 Unique Cartan Form using Index Ordering

The introduction of an index order (see Def. 4.2) enables us to identify uniquely, which contact form $\omega^{\alpha}_{[J-I_1]}$, $\#I_1 = 1$ has to be used in the construction process of the Cartan form, i.e., which $\omega^{\alpha}_{[J-I_1]}$ has to be used to cancel out a $\mathrm{d}x^{\alpha}_{[J]}$ entry in $\mathrm{d}c$. This can be done by choosing the minimal index $(J-I_1)_{\min}$ in the determination process of $\omega^{\alpha}_{[J-I_1]}$.

Having this ordering at ones disposal, we are able to define the Cartan form to be of the following structure

$$c = l\mathrm{dX} + p^{[J]}_{\alpha}\left(\mathrm{d}x^{\alpha}_{[J-I_1]} \wedge \partial_k \rfloor \mathrm{dX} - x^{\alpha}_{[J]}\mathrm{dX}\right)\;, \quad \#J = 1,\ldots,n\;, \quad I_1 = 1_k. \tag{7.12}$$

Here the uniquely determined functions $p^{[J]}_{\alpha}$ result from the index ordering and thus they are no more equipped with a double multi-index.

Remark 7.4 *The used index order is not geometrically motivated and consequently the determined Cartan form is not invariant under coordinate transformations. It will be shown that this is the major drawback of this approach in the determination of the boundary conditions.*

Following the considerations from above, we see that the expression

$$\begin{aligned}
\left(D\rfloor j^{2n-1}v\rfloor \mathrm{d}c\right)\wedge \mathrm{dX} &= j^{2n}v\rfloor\left(\mathrm{d}l\wedge\mathrm{dX} - d_k\rfloor \mathrm{d}p^{[J]}_{\alpha}\wedge \mathrm{d}x^{\alpha}_{[J-I_1]}\wedge\mathrm{dX} - p^{[J]}_{\alpha}\mathrm{d}x^{\alpha}_{[J]}\wedge\mathrm{dX}\right)\\
&= j^{2n}v\rfloor\left(\mathrm{d}l\wedge\mathrm{dX} - \left(\mathrm{L}_{d_k}\left(p^{[J]}_{\alpha}\right)\wedge \mathrm{d}x^{\alpha}_{[J-I_1]}\wedge\mathrm{dX} + p^{[J]}_{\alpha}\mathrm{d}x^{\alpha}_{[J]}\wedge\mathrm{dX}\right)\right)
\end{aligned}$$

does not require the prolongation of the variational vector field anymore, iff we choose

$$\begin{aligned}
p^{[J]}_{\alpha} &= \partial^{[J]}_{\alpha}l && \#J = n,\ I_1 = 1_k\\
p^{[J]}_{\alpha} &= \partial^{[J]}_{\alpha}l - \mathrm{L}_{d_i}\left(p^{[J+1_i]}_{\alpha}\right) = \partial^{[J]}_{\alpha}l + (-1)\,\mathrm{L}_{d_i}\left(\partial^{[J+1_i]}_{\alpha}l\right) && \#J = n-1,\ I_1 = 1_k\\
\vdots &= \vdots\\
p^{[J]}_{\alpha} &= \partial^{[J]}_{\alpha}l - \mathrm{L}_{d_i}\left(p^{[J+1_i]}_{\alpha}\right) = \partial^{[J]}_{\alpha}l + \textstyle\sum_{\#I=1}^{n-\#J}(-1)^{\#I}\,\mathrm{L}_{d_I}\left(\partial^{[J+I]}_{\alpha}l\right) && n > \#J > 1,\ I_1 = 1_k\\
\vdots &= \vdots\\
p^{[J]}_{\alpha} &= \partial^{[J]}_{\alpha}l - \mathrm{L}_{d_i}\left(p^{[J+1_i]}_{\alpha}\right) = \partial^{[J]}_{\alpha}l + \textstyle\sum_{\#I=1}^{n-1}(-1)^{\#I}\,\mathrm{L}_{d_I}\left(\partial^{[J+I]}_{\alpha}l\right) && \#J = 1\,.
\end{aligned} \tag{7.13}$$

Like before we are now able to derive the domain conditions.

7.2.6 Conditions on the domain

Combining the relations (7.12) and (7.13), we can rewrite $\mathrm{d}c$ as

$$\begin{aligned}
\mathrm{d}c &= \mathrm{d}l\wedge\mathrm{dX} + \mathrm{d}p^{[J]}_{\alpha}\wedge\left(\mathrm{d}x^{\alpha}_{[J-I_1]}\wedge\partial_k\rfloor\mathrm{dX} - x^{\alpha}_{[J]}\mathrm{dX}\right) - p^{[J]}_{\alpha}\mathrm{d}x^{\alpha}_{[J]}\wedge\mathrm{dX}\\
&= \mathrm{d}l\wedge\mathrm{dX} + \left(\mathrm{d}_h p^{[J]}_{\alpha} + \mathrm{d}_v p^{[J]}_{\alpha}\right)\wedge\left(\mathrm{d}x^{\alpha}_{[J-I_1]}\wedge\partial_k\rfloor\mathrm{dX} - x^{\alpha}_{[J]}\mathrm{dX}\right) - p^{[J]}_{\alpha}\mathrm{d}x^{\alpha}_{[J]}\wedge\mathrm{dX}\\
&= \delta_{\alpha}(l)\,\mathrm{d}u^{\alpha}\wedge\mathrm{dX} + \underbrace{\mathrm{d}_v\left(p^{[J]}_{\alpha}\right)\wedge\left(\mathrm{d}x^{\alpha}_{[J-I_1]}\wedge\partial_k\rfloor\mathrm{dX} - x^{\alpha}_{[J]}\mathrm{dX}\right)}_{\theta}
\end{aligned}$$

with the Euler-Lagrange operator $\delta_\alpha(\cdot)$ defined by equation (7.10) and a $(p+1)$-form θ, which meets $D\rfloor\theta = 0$. The system of partial differential equations follows again from

$$((-1)^r D\rfloor \mathrm{d}c) \wedge \mathrm{dX} = 0 \;, \tag{7.14}$$

which is of course equivalent to $\delta_\alpha(l) = 0$.

7.2.7 Conditions on the boundary

Again, one gets the conditions on the boundary from the expression

$$\begin{aligned} \left(\Psi^{2n-1}\right)^* \left(j^{2n-1}v\rfloor c\right) &= j^{2n-1}v_\partial \rfloor \left(\Psi^{2n-1}\right)^* \left(l\mathrm{dX} + p_\alpha^{[J]}\left(\mathrm{d}x^\alpha_{[J-1_k]} - x^\alpha_{[J]}\mathrm{d}X^k\right) \wedge \partial_k \rfloor \mathrm{dX}\right) \\ &= j^{2n-1}v_\partial \rfloor \left(\left(p_\alpha^{[J]} \circ \Psi^{2n-1}\right) \mathrm{d}x^\alpha_{[J-1_r]} \wedge \mathrm{d}\bar{\mathrm{X}}\right) \;, \end{aligned} \tag{7.15}$$

which must vanish for any admissible variational field $v = v^\alpha \partial_\alpha$. Unfortunately the derived conditions are not the boundary conditions of interest. In fact the coordinate dependents of the index ordering implies that the same system would have different boundary conditions in different coordinate systems. Consequently this approach does not allow to derive the boundary conditions of infinite-dimensional Euler-Lagrange systems.

In the next chapter we will discuss an extension of this approach that enables the derivation of the domain and boundary conditions using the so called extended Cartan form. It is worth mentioning that the extended Cartan form approach makes use of certain index ordering principles that are geometrically motivated.

Chapter 8

The Extended Cartan Form

Up to now all investigations were based on the ideas presented in section 5.1. The determination of the Cartan form was discussed in the finite- and infinite-dimensional case. It was shown that the non-uniqueness of the Cartan form in the infinite-dimensional case with n^{th} order Lagrangian density does not influence the domain conditions and that this does not apply for the boundary conditions.

Before we actually start to extend the Cartan form approach, some general remarks on the geometric meaning of boundary conditions must be given.

8.1 Pull-back of the Jet Framework

The oriented base manifold $\mathcal{D}$ is assumed to be bounded by the coherently oriented boundary manifold $\partial\mathcal{D}$. The inclusion mapping $\iota : \partial\mathcal{D} \to \mathcal{D}$ describes the relation of both manifolds and enables us to pull-back the jet framework as shown in

$$\begin{tikzcd}
\left(X^i\right) & \left(X^i, x^\alpha\right) & & \left(X^i, x^\alpha, x^\alpha_{[J]}\right) \\
\mathcal{D} & \mathcal{E} \arrow[l, "\pi"'] & \dots \arrow[l, "\pi^1_0"'] & J^n\pi \arrow[l, "\pi^n_{n-1}"'] \\
\partial\mathcal{D} \arrow[u, "\iota"] & \iota^*\mathcal{E} \arrow[l, "\iota^*\pi"] & \dots \arrow[l, "\iota^*\left(\pi^1_0\right)"] & \iota^* J^n\pi \arrow[l, "\iota^*\left(\pi^n_{n-1}\right)"] \\
\left(X^j_\partial\right) & \left(X^j_\partial, x^\alpha\right) & & \left(X^j_\partial, x^\alpha, x^\alpha_{[J]}\right) .
\end{tikzcd}$$

The achieved bundle structure on the boundary is obviously no more a jet framework. Roughly speaking there are too less independent coordinates available to qualify as a jet framework. On the other hand the additional undetermined coordinates can be seen as a degree of freedom and this gives a first idea of what boundary conditions are all about.

In order to make this idea more visible, we introduce a special class of coordinate transformations on the domain $\mathcal{D}$.

8.1.1 Coordinate Transformation

We introduce special local coordinates (Y^i) on the domain $\mathcal{D}$, such that the inclusion mapping $\iota : \partial\mathcal{D} \to \mathcal{D}$ becomes

$$\begin{aligned} \iota :\ \partial\mathcal{D} &\rightarrow \mathcal{D} \\ \bar{Y}^i &\rightarrow Y^i = \bar{Y}^i,\ Y^r = Y^r_\partial = \text{const.}\ , \quad i = 1, \ldots, (r-1)\ . \end{aligned} \tag{8.1}$$

Here we have introduced new adapted coordinates $\left(\bar{Y}^i\right)$ on the boundary manifold $\partial\mathcal{D}$. Additionally, we require that points of the domain $p \in \mathcal{D}$ meet in this new local coordinates $p^r \leq Y^r_\partial$, i.e. the coordinate Y^r has to point out of the domain (see Fig. 8.1).

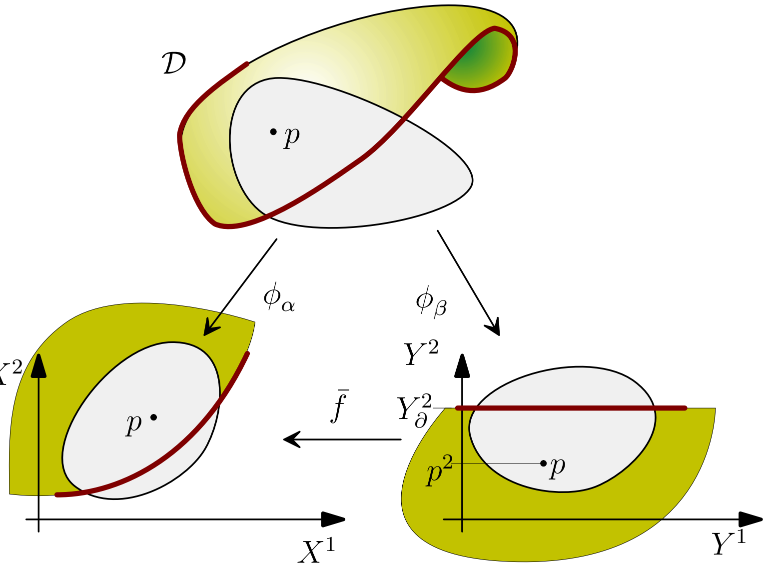

Figure 8.1: Change of coordinates on the domain.

An appropriate bundle morphism (see Def. 3.5) for the coordinate transformation is given by

$$\begin{aligned} \bar{f} : \mathcal{D} &\rightarrow \mathcal{D} \\ Y^j &\rightarrow X^i\left(Y^j\right)\ , \quad j, i = 1, \ldots, r \\ f : \mathcal{E} &\rightarrow \mathcal{E} \\ Y^j, y^\alpha &\rightarrow X^i\left(Y^j\right)\ , \ x^\alpha = y^\alpha\ , \quad \alpha = 1, \ldots, s\ . \end{aligned}$$

The prolongation of this bundle morphism to the n^{th} jet (see Def. 4.21) enables us to pull-back the Lagrangian and to obtain

$$l'\left(Y^i, y^\alpha, y^\alpha_{[J]}\right) \mathrm{dY} = \left(j^n f\right)^* \left(l\left(X^i, x^\alpha, x^\alpha_{[J]}\right) \mathrm{dX}\right)\ ,$$

whereby the transformed Lagrangian density l' and the transformed volume form dY are introduced. Subsequently we will suppress the superscript, i.e. we will use $l\ \mathrm{dY}$ instead of $l'\mathrm{dY}$ to keep a concise notation. Of course this construction does not lead to a uniquely determined bundle morphism as illustrated in figure 8.2. This non-uniqueness is an immediate consequence of the non-uniqueness of the map $\bar{f}$.

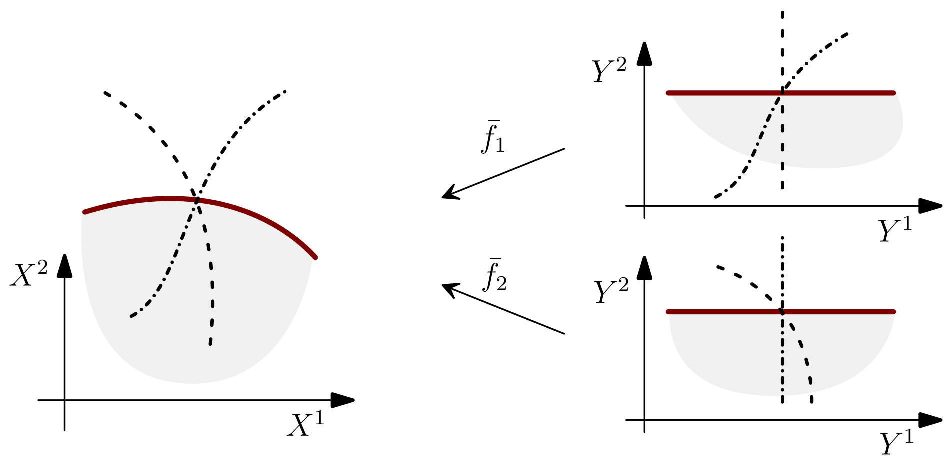

Figure 8.2: Non uniqueness of $\bar{f}$.

8.1.2 Boundary Jet Framework

Having the special local coordinates (Y^i) at ones disposal, we are able to enlighten the relation of the inclusion map ι and a section on the domain $\sigma \in \Gamma(\pi)$. In order to motivate the upcoming modifications we consider the following example.

Example 8.1 *Let us consider a bundle π with local coordinates (Y^1, Y^2, y^1) and its first and second jet $J^1\pi$ resp. $J^2\pi$ with local coordinates $\left(Y^1, Y^2, y^1, y^1_{[10]}, y^1_{[01]}\right)$ resp. $\left(Y^1, Y^2, y^1, y^1_{[10]}, y^1_{[01]}, y^1_{[20]}, y^1_{[11]}, y^1_{[02]}\right)$. The pull-back of this structure along ι results in $\iota^*\pi$ with local coordinates $(\bar{Y}^1, y^1)$, $\iota^* J^1\pi$ with $\left(\bar{Y}^1, y^1, y^1_{[10]}, y^1_{[01]}\right)$, and $\iota^* J^1\pi$ with $\left(\bar{Y}^1, y^1, y^1_{[10]}, y^1_{[01]}, y^1_{[20]}, y^1_{[11]}, y^1_{[02]}\right)$. Now we are able to introduce a section of the pull-back bundle structure*

$$\begin{aligned} \sigma_0 : \partial\mathcal{D} &\rightarrow \iota^*\mathcal{E} \\ \bar{Y}^1 &\rightarrow \bar{Y}^1\,,\ y^1 = \sigma^1_{\partial 0}\left(\bar{Y}^1\right)\ . \end{aligned}$$

This section is related to a section σ of π by $\sigma \circ \iota = \sigma^1_{\partial 0}$. Obviously an arbitrary choice of $\sigma^1_{\partial 0}$ does not violate the domain jet structure. Due to the special choice of the independent coordinates, we are able to derive a section σ_1 on $\iota^* J^1\pi$ from σ_0 by

$$\begin{aligned} \sigma_1 : \partial\mathcal{D} &\rightarrow \iota^* J^1\pi \\ \bar{Y}^1 &\rightarrow \bar{Y}^1\,,\ y^1 = \sigma^1_{\partial 0}\left(\bar{Y}^1\right)\,,\ y^1_{[10]} = \frac{\partial\sigma^1_{\partial 0}\left(\bar{Y}^1\right)}{\partial\bar{Y}^1}\,,\ y^1_{[01]} = \sigma^1_{\partial 1}\left(\bar{Y}^1\right)\ . \end{aligned}$$

Obviously one can again choose an arbitrary function $\sigma^1_{\partial 1}$ without violating the domain jet structure. Thereby the degree of freedom obtained by the pull-back of the jet framework is incorporated. Consequently we are able to introduce a section σ_2 on $\iota^* J^2\pi$ by

$$\begin{aligned} \sigma_2 : \partial\mathcal{D} &\rightarrow \iota^* J^2\pi \\ \bar{Y}^1 &\rightarrow \begin{array}{l} \bar{Y}^1,\ y^1 = \sigma^1_{\partial 0}\left(\bar{Y}^1\right)\,,\ y^1_{[10]} = \frac{\partial\sigma^1_{\partial 0}\left(\bar{Y}^1\right)}{\partial\bar{Y}^1}\,,\ y^1_{[01]} = \sigma^1_{\partial 1}\left(\bar{Y}^1\right) \\ y^1_{[20]} = \frac{\partial^2\sigma^1_{\partial 0}\left(\bar{Y}^1\right)}{\partial\left(\bar{Y}^1\right)^2}\,,\ y^1_{[11]} = \frac{\partial\sigma^1_{\partial 1}\left(\bar{Y}^1\right)}{\partial\bar{Y}^1}\,,\ y^1_{[02]} = \sigma^1_{\partial 2}\left(\bar{Y}^1\right) \end{array} \ . \end{aligned}$$

Again the function $\sigma^1_{\partial 2}\left(\bar{Y}^1\right)$ incorporates the gained degree of freedom. It is worth mentioning that σ_2 and σ_1 meet by construction all integrability conditions, i.e. they are compatible with σ_0 resp. σ_0 and σ_1.

This example illustrates that every section $\sigma_k : \partial\mathcal{D} \to \iota^* J^k\mathcal{E},\ k = 0,\dots,n$ is equipped with a function $\sigma^\alpha_{\partial k}$ that could be freely chosen without violating the domain jet structure. We conclude that all local coordinates $y^\alpha_{[0\dots 0 j_r]}$ can be assigned freely on the pull-back structure in accordance to the domain jet framework. From a mathematical point of view, this effect represents the loss of the contact structure on the pull-back bundles $\iota^*\pi$.

Fortunately it is possible to introduce a jet bundle structure on the boundary. This enables us to introduce the bundle morphisms $(\Psi^n, \iota) : \bar{\pi}^n \to \pi^n$ that qualify as contact bundle morphisms, i.e.

$$(\Psi^n)^* : I_n \to I_{n\partial}.$$

All elements of the domain contact ideal I_n are mapped onto elements of the boundary contact ideal $I_{n\partial}$ by means of the pull-back $(\Psi^n)^*$. Here we have already used the boundary bundle $\bar{\pi}$

Definition 8.2 (boundary bundle) *The boundary bundle $\left(\bar{\mathcal{E}}, \bar{\pi}, \partial\mathcal{D}\right)$ according to a domain bundle π consists of the $(r-1)$-dimensional base manifold $\partial\mathcal{D}$, the boundary projection $\bar{\pi}$ and the $(s\cdot k_r)$-dimensional total manifold $\bar{\mathcal{E}}$. We equip $\bar{\mathcal{E}}$ with the local coordinates $\Big(\bar{Y}^j, y^\alpha_{[0\dots 0;0]}, y^\alpha_{[0\dots 0;1]}, y^\alpha_{[0\dots 0;2]}, \dots, y^\alpha_{[0\dots 0;k_r-1]}\Big)$ or in a shorter notation $\left(\bar{Y}^j, y^\alpha_{[0\dots 0;j_r]}\right)$, $j_r = 0,\dots,k_r-1$.*
The dimension of $\bar{\mathcal{E}}$ is determined by the quantity $0 \le k_r \le 2n$.

Thus we are able to define the *boundary jet framework* Π^n_∂ consisting of the bundles

$$\bar{\pi}^j\ ,\ \bar{\pi}^j_w\ ,\quad w < j\ ,\quad j = 1,\dots,n\ ,\quad w = 0,\dots,n-1$$

according to the boundary bundle $\bar{\pi}$.

Additionally, we obtain a boundary contact structure (see Def. 4.10) corresponding to Π^n_∂ in local coordinates $\left(\bar{Y}^j, y^\alpha_{[0\dots 0;j_r]}, y^\alpha_{\left[\bar{J};j_r\right]}\right)$ by two vector valued one forms

$$\begin{aligned} h_\partial &= \mathrm{d}\bar{Y}^i \otimes \left(\bar{\partial}_i + y^\alpha_{[1_i;j_r]}\partial_\alpha^{[0\dots 0;j_r]} + \dots + y^\alpha_{\left[\bar{J}+I_1;j_r\right]}\partial_\alpha^{\left[\bar{J};j_r\right]}\right)\ ,\quad \bar{\partial}_i = \frac{\partial}{\partial \bar{Y}^i} \\ v_\partial &= \left(\mathrm{d}y^\alpha_{\left[\bar{J};j_r\right]} - y^\alpha_{\left[\bar{J}+I_1;j_r\right]}\mathrm{d}\bar{Y}^i\right) \otimes \partial_\alpha^{\left[\bar{J};j_r\right]} \end{aligned}$$

where

$$i = 1,\dots,r-1\ ,\quad j_r = 0,\dots,k_r-1\ ,\quad I_1 = 1_i\ ,\quad \#\bar{J} = 0,\dots,n$$

is used. Here we have introduced the boundary multi-index $\left[\bar{J}\right] = [j_1 \dots j_{r-1}]$.

The boundary contact structure leads to the definition of the boundary total derivatives

$$d_{\partial i} = \bar{\partial}_i + y^\alpha_{[1_i;j_r]}\partial_\alpha^{[0\dots 0;j_r]} + \dots + y^\alpha_{\left[\bar{J}+I_1;j_r\right]}\partial_\alpha^{\left[\bar{J};j_r\right]}\ ,\quad i = 1,\dots,r-1\ .$$

Furthermore the boundary contact forms are given by

$$\omega^\alpha_{\partial\left[\bar{J};j_r\right]} = \mathrm{d}y^\alpha_{\left[\bar{J};j_r\right]} - y^\alpha_{\left[\bar{J}+I_1;j_r\right]}\mathrm{d}\bar{Y}^i\ ,\quad i = 1,\dots,r-1\ ,\quad j_r = 0,\dots,k_r-1\ .$$

The definition of boundary contact forms forces the introduction of a boundary contact ideal.

Definition 8.3 *The boundary contact forms $\omega^{\alpha}_{\partial\left[\bar{J};j_r\right]}$ of the n^{th} jet manifold $J^n\bar{\pi}$ form an ideal $I_{\partial n}$ over the exterior algebra $\bigwedge J^n\left(\bar{\mathcal{E}}\right)$. Such an ideal will be denoted boundary contact ideal.*

All elements ω_∂ of this boundary contact ideal $I_{\partial n}$ meet the following relation

$$\left(j^n\bar{\sigma}\right)^*\left(\omega_\partial\right) = 0\,,$$

where the prolongation of the boundary section

$$\begin{array}{rcl} \bar{\sigma}:\partial\mathcal{D} & \rightarrow & \bar{\mathcal{E}} \\ \bar{Y}^i & \rightarrow & \left(\bar{Y}^i, y^{\alpha}_{[0\ldots0;0]} = \sigma^{\alpha}_{\partial 0}\left(\bar{Y}^i\right), y^{\alpha}_{[0\ldots0;1]} = \sigma^{\alpha}_{\partial 1}\left(\bar{Y}^i\right), \ldots, y^{\alpha}_{[0\ldots0;k]} = \sigma^{\alpha}_{\partial k}\left(\bar{Y}^i\right)\right)\,, \end{array}$$

is used. The functions $\sigma^{\alpha}_{\partial j_r}$ correspond to the used notation used in example 8.1.

The definition of the boundary bundle enables now the introduction of the subsequently used contact bundle morphisms.

Definition 8.4 *Boundary contact bundle morphisms are contact bundle morphisms, whose projection ψ is given by the inclusion map $\iota:\partial\mathcal{D}\rightarrow\mathcal{D}$. The boundary contact bundle morphisms*

$$\left(\Psi^n,\iota\right):\bar{\pi}^n\rightarrow\pi^n$$

are given in local coordinates $\left(Y^j, y^{\alpha}_{[J]}\right)$ resp. $\left(\bar{Y}^i, y^{\alpha}_{\left[\bar{J};j_r\right]}\right)$ by

$$\left(\bar{Y}^i, y^{\alpha}_{\left[\bar{J};j_r\right]}\right)\rightarrow\left(Y^i=\bar{Y}^i, Y^r=Y^r_\partial=\mathrm{const.},\ y^{\alpha}_{[J]}=y^{\alpha}_{\left[\bar{J};j_r\right]}\right)\,,\qquad [J]=\left[\bar{J}j_r\right]\,.$$

Thus the pull-back of the domain contact forms along the map Ψ^n

$$\left(\Psi^n\right)^*\left(\mathrm{d}y^{\alpha}_{[J]} - y^{\alpha}_{[J+1_j]}\mathrm{d}Y^j\right) = \mathrm{d}y^{\alpha}_{\left[\bar{J};j_r\right]} - y^{\alpha}_{\left[\bar{J}+1_i;j_r\right]}\mathrm{d}\bar{Y}^i = \,,\quad \#J=n-1,\ j=1,\ldots,r$$

results in boundary contact forms. These bundle morphisms are visualized in the following commutative diagram

$$\begin{tikzcd}
\left(Y^j\right) & \left(Y^j,y^{\alpha}\right) & & \left(Y^j,y^{\alpha},y^{\alpha}_{[J]}\right) \\
\mathcal{D} & \mathcal{E} \arrow[l, "\pi"'] & \ldots \arrow[l, "\pi^1_0"'] & J^n\pi \arrow[l, "\pi^n_{n-1}"'] \\
\partial\mathcal{D} \arrow[u, "\iota"] & \bar{\mathcal{E}} \arrow[l, "\bar{\pi}"] \arrow[u, "\Psi"] & \ldots \arrow[l, "\bar{\pi}^1_0"] & J^n\bar{\pi} \arrow[l, "\bar{\pi}^n_{n-1}"] \arrow[u, "\Psi^n"] \\
\left(\bar{Y}^i\right) & \left(\bar{Y}^i,y^{\alpha}_{[0\ldots0;j_r]}\right) & & \left(\bar{Y}^i,y^{\alpha}_{[0\ldots0;j_r]},y^{\alpha}_{\left[\bar{J};j_r\right]}\right).
\end{tikzcd}$$

Before we are able to present the structure of the extended Cartan form, we have to investigate some relations between domain and boundary contact ideal.

Corollary 8.5 *Every element $\lambda \in \Gamma\left(\wedge J^n\pi\right)$ that meets*

$$\left(\Psi^n\right)^* \lambda = \lambda_\partial \in I_{\partial n}$$

vanishes under the concatenation of the pull-back along a domain section σ and the inclusion mapping ι, i.e.

$$\iota^*\left(\left(j^n\sigma\right)^* \lambda\right) = 0 \,.$$

Proof. It is obvious that λ must be of the form

$$\lambda = \left(\mathrm{d}y^\alpha_{[J]} - y^\alpha_{[J+1_j]}\mathrm{d}Y^j + g\mathrm{d}Y^r\right) \wedge f \,, \quad g \in C^\infty\left(J^n\pi\right) \,,\ f \in \wedge\left(J^n\pi\right) \,,\ j = 1,\ldots,r-1$$

due to $\left(\Psi^n\right)^* \lambda \in I_{\partial n}$. Thus we have to prove that

$$\iota^*\left(\left(j^n\sigma\right)^*\left(\left(\mathrm{d}y^\alpha_{[J]} - y^\alpha_{[J+1_j]}\mathrm{d}Y^j + g\mathrm{d}Y^r\right) \wedge f\right)\right) = 0 \,.$$

The pull-back along the prolongation of a domain section leads to

$$\left(j^n\sigma\right)^*\left(\left(\mathrm{d}y^\alpha_{[J]} - y^\alpha_{[J+1_j]}\mathrm{d}Y^j + g\mathrm{d}Y^r\right) \wedge f\right) = \left(\partial_r\left(\partial_{[J]}\sigma\right) + \left(g \circ j^n\sigma\right)\right)\mathrm{d}Y^r \wedge \left(j^n\sigma\right)^* f$$

and by the application of

$$\iota^*\left(\left(\partial_r\left(\partial_{[J]}\sigma\right) + \left(g \circ j^n\sigma\right)\right)\mathrm{d}Y^r\right) \wedge \iota^*\left(j^n\sigma\right)^* f = 0$$

we have proved the corollary. ■

It is worth mentioning that every element of the domain contact ideal I_n is an element of this class.

This preliminary work enables us now to define the structure of the extended Cartan form.

8.2 Structure of the Extended Cartan Form

The definition of the domain and boundary contact ideals are the cornerstone of the following definition.

Definition 8.6 (Extended Cartan Form) *The Extended Cartan Form – or ECF for short – is given by*

$$c_{ext} = l\ \mathrm{dY} + \omega + \mathrm{d}\omega_\partial = c + \mathrm{d}\omega_\partial$$

with the elements of the domain contact ideal $\omega \in I_n$ and the boundary contact ideal $\left(\Psi^n\right)^* \omega_\partial \in I_{\partial n}$.

This definition is mainly forced by the following theorem.

Theorem 8.7 *The Lagrangian functional is invariant under any additional elements of the domain contact ideal $\omega \in I_n$ and the exterior derivative of any elements of the boundary contact ideal $\left(\Psi^n\right)^* \omega_\partial \in I_{\partial n}$, i.e.*

$$\int_{\mathcal{D}} \left(j^n\sigma\right)^*\left(l\ \mathrm{dY}\right) = \int_{\mathcal{D}} \left(j^n\sigma\right)^*\left(l\ \mathrm{dY} + \omega + \mathrm{d}\omega_\partial\right) \,.$$

Proof. Similarly to theorem 5.3 we make essential use of the property of elements of contact ideals on the domain

$$(j^n\sigma)^*\,\omega = 0$$

and on the boundary

$$\iota^*\left((j^n\sigma)^*\,\omega_\partial\right) = 0\;.$$

These properties can be used in

$$\begin{aligned}\int_{\mathcal{D}} (j^n\sigma)^*\left(l\;\mathrm{dY} + \omega + \mathrm{d}\omega_\partial\right) &= \int_{\mathcal{D}} (j^n\sigma)^*\left(l\;\mathrm{dY} + \omega\right) + \int_{\mathcal{D}} \mathrm{d}\,(j^n\sigma)^*\,\omega_\partial \\ &= \int_{\mathcal{D}} (j^n\sigma)^*\left(l\;\mathrm{dY} + \omega\right) + \int_{\partial\mathcal{D}} \iota^*\left((j^n\sigma)^*\,\omega_\partial\right) \\ &= \int_{\mathcal{D}} (j^n\sigma)^*\left(l\;\mathrm{dY}\right)\end{aligned}$$

to prove the theorem. ■

The impact of the extension of the Cartan form on the domain conditions is treated in corollary 8.8.

Corollary 8.8 *The extended Cartan form c_{ext} and the original Cartan form c lead to the same domain conditions, i.e.*

$$\left((-1)^r\,D\rfloor\mathrm{d}c\right)\wedge\mathrm{dY} = \left((-1)^r\,D\rfloor\mathrm{d}c_{ext}\right)\wedge\mathrm{dY}\;.$$

Proof. This is a trivial consequence of the identity

$$\mathrm{d}c_{ext} = \mathrm{d}c + \mathrm{d}\left(\mathrm{d}\omega_\partial\right) = \mathrm{d}c\;,$$

which has to be used in

$$\left((-1)^r\,D\rfloor\mathrm{d}c_{ext}\right)\wedge\mathrm{dY} = 0$$

to determine the domain conditions. ■

This result confirms the applied modification.

Remark 8.9 *The fact that an additional exact form does not influence the domain conditions in the calculus of variations is well known. Such an exact form generates a null Lagrangian as defined in [Olver, 1986].*

The introduced extensions will provide the necessary degrees of freedom to determine the boundary conditions uniquely. It is remarkable that the consideration of any further extension by means of an additional exact form, i.e.

$$l\;\mathrm{dY} + \omega + \mathrm{d}\left(\omega_\partial + \mathrm{d}\omega_{\partial\partial}\right)$$

or equally by

$$\int_{\mathcal{D}} (j^n\sigma)^*\left(l\;\mathrm{dY} + \omega\right) + \int_{\partial\mathcal{D}} (j^n\bar{\sigma})^*\,\omega_\partial + \int_{\partial\partial\mathcal{D}} \left(j^n\bar{\bar{\sigma}}\right)^*\,\omega_{\partial\partial}$$

does not make any sense for the given Lagrangian functionals of the form

$$\int_{\mathcal{D}} l\;\mathrm{dY}\;,$$

because of the closeness of the exterior derivative, i.e. $\mathrm{dd}\left(\cdot\right) = 0$. This could be equally explained by the fact that the boundary manifold has itself no boundary, i.e. it is closed.

Remark 8.10 *These considerations do not apply for Euler-Lagrange systems whose Lagrangian functional is given by, e.g.,*

$$\int_{\mathcal{D}_1} l_1 \, \mathrm{dY}_1 + \int_{\mathcal{D}_2} l_2 \, \mathrm{dY}_2 + \int_{\mathcal{D}_3} l_3 \, \mathrm{dY}_3$$

where $\mathcal{D}_2 \subset \partial\mathcal{D}_1$, $\mathcal{D}_3 \subset \partial\mathcal{D}_2$. Such problems of coupled systems of different base-dimension will not be treated in the Euler-Lagrange part of this thesis.

In the next section we will present a procedure to construct the extended Cartan form.

8.3 Construction of the Extended Cartan Form

8.3.1 Coordinate Expression of the ECF

The introduced bundle morphism $(f, \bar{f})$ and its prolongations enable us to pull-back the Lagrangian to $l\mathrm{dY}$, $l \in C^\infty(J^n\pi)$ with the adapted local coordinates $\left(Y^j, y^\beta, y^\beta_{[J]}\right)$.

We define the extended Cartan form in local coordinates to be given by

$$c_{ext} = l\mathrm{dY} + p_\alpha^{[J-I_1,J]}\omega^\alpha_{[J-I_1]} \wedge \partial_{[I_1]} \rfloor \mathrm{dY} + \mathrm{d}\left(p_\alpha^{\left[\bar{J}-I_2,\bar{J};j_r\right]}\omega^\alpha_{\left[\bar{J}-I_2;j_r\right]} \wedge \partial_{[I_2]} \rfloor \partial_{[1_r]} \rfloor \mathrm{dY}\right)$$

where

$$\begin{aligned}
\#J &= 1,\ldots,n \\
\#I_1 &= 1\,, \quad I_1 = 1_j\,, \quad j = 1,\ldots,r \\
\#\bar{J} &= 1,\ldots,n\,, \quad j_r = 0,\ldots,\frac{k_r}{2}-1 \\
\#I_2 &= 1\,, \quad I_2 = 1_i\,, \quad i = 1,\ldots,r-1
\end{aligned}$$

is used. The pull-back of the form $\omega^\alpha_{\left[\bar{J}-I_2;j_r\right]}$ meets

$$(\Psi^n)^*\,\omega^\alpha_{\left[\bar{J}-I_2;j_r\right]} = \omega^\alpha_{\partial\left[\bar{J}-I_2;j_r\right]},$$

i.e. is a contact form on the boundary jet bundle.

The still unknown functions $p_\alpha^{[J-I_1,J]} \in C^\infty(J^{2n-1}\mathcal{E})$, $p_\alpha^{\left[\bar{J}-I_2,\bar{J};j_r\right]} \in C^\infty(J^{2n-1}\mathcal{E})$ represent the necessary degree of freedom for the determination of the domain and boundary conditions.

Obviously all $p_\alpha^{[J-I_1,J]}\omega^\alpha_{[J-I_1]} \wedge \partial_{I_1} \rfloor \mathrm{dY}$ are elements of the domain contact ideal I_n and similarly all $(\Psi^n)^*\left(p_\alpha^{\left[\bar{J}-I_2,\bar{J};j_r\right]}\omega^\alpha_{\left[\bar{J}-I_2;j_r\right]} \wedge \partial_{[I_2]} \rfloor \partial_{[1_r]} \rfloor \mathrm{dY}\right)$ are elements of the boundary contact ideal $I_{\partial n}$. Thus the stated requirements to qualify as extended Cartan form are met.

8.3.2 Derivation of the ECF

The multi-index ordering of definition 4.2 is an arbitrary choice and no geometric information is used in its construction. Having the local coordinates $\left(Y^j, y^\beta, y^\beta_{[J]}\right)$ that incorporate some additional information about the boundary shape at ones disposal, we are able to introduce a geometrically motivated partial ordering. This is mainly caused by the existence of a distinguished coordinate Y^r and consequently of a distinguished index j_r.

Definition 8.11 (partial ordering) *In the local coordinates* $\left(Y^j, y^\beta, y^\beta_{[J]}\right)$ *we define the following partial ordering. Let* $J_a = j_{a1} \dots j_{ar}$ *and* $J_b = j_{b1} \dots j_{br}$ *be two multi-indices. We say* $J_a >_{part} J_b$ *if in the difference* $J_a - J_b$ *the* r^{th} *entry is positive.*

This partial ordering is compatible with Def. 4.2, i.e.

$$J_a >_{part} J_b \quad \text{implies} \quad J_a > J_b \, .$$

In order to obtain a minimal amount of boundary conditions, it is necessary to apply the partial ordering in the determination process of the functions $p_\alpha^{[J-I_1,J]}$. Consequently we will use the contact forms $\omega^\alpha_{[J-I_1]}$ with the smallest indices $[J-I_1]$ with respect to $>_{part}$ in the determination process of the functions $p_\alpha^{[J-I_1,J]}$. Due to the compatibility with the general multi-index ordering, one can also use the general ordering $>$. This leads precisely to the results as presented in section 7.2.5, if the local coordinates $\left(Y^i, y^\alpha, y^\alpha_{[J]}\right)$ are used.

From the general boundary condition (5.6) we know that

$$\iota^* \left(\left(j^{2n-1}\sigma\right)^* \left(j^n v \rfloor c_{ext}\right)\right) = 0$$

must be satisfied. Fortunately we are able to reformulate this equation

$$\begin{aligned}
\iota^* \left(\left(j^{2n-1}\sigma\right)^* \left(j^n v \rfloor c_{ext}\right)\right) &= \left(j^{2n-1}\bar{\sigma}\right)^* \left(\Psi^{2n-1}\right)^* \left(j^n v \rfloor c_{ext}\right) \\
&= \left(j^{2n-1}\bar{\sigma}\right)^* \left(j^n v_\partial \rfloor \left(\Psi^{2n-1}\right)^* c_{ext}\right) = 0
\end{aligned}$$

where we have used the boundary section $\bar{\sigma} : \partial\mathcal{D} \to \bar{\mathcal{E}}$, the properties of the boundary contact bundle morphism, and the prolonged variational vector field $j^n v$ restricted to the boundary[1] denoted by $j^n v_\partial$. The pull-back of the extended Cartan form to the boundary $\left(\Psi^{2n}\right)^* c_{ext}$ is in the local coordinates $\left(Y^j, y^\beta, y^\beta_{[J]}\right)$ a trivial operation and given by

$$\begin{aligned}
\left(\Psi^{2n}\right)^* c_{ext} &= \left(p_\alpha^{[J-I_1,J]} \circ \Psi^{2n-1}\right) \mathrm{d}y^\alpha_{[\bar{J};j_r]} \wedge \mathrm{d}\bar{\mathrm{Y}} + \mathrm{d}\left(p_\alpha^{[\bar{J}-I_2,\bar{J};j_r]} \omega^\alpha_{\partial[\bar{J}-I_2;j_r]} \wedge \partial_{1_i} \rfloor \mathrm{d}\bar{\mathrm{Y}}\right) \\
&= \left(p_\alpha^{[J-I_1,J]} \circ \Psi^{2n-1}\right) \mathrm{d}y^\alpha_{[\bar{J};j_r]} \wedge \mathrm{d}\bar{\mathrm{Y}} + \mathrm{d}p_\alpha^{[\bar{J}-1_i,\bar{J};j_r]} \wedge \omega^\alpha_{\partial[\bar{J}-1_i;j_r]} \wedge \partial_{1_i} \rfloor \mathrm{d}\bar{\mathrm{Y}} \\
&\quad - p_\alpha^{[\bar{J}-1_i,\bar{J};j_r]} \mathrm{d}y^\alpha_{[\bar{J};j_r]} \wedge \mathrm{d}\bar{\mathrm{Y}} \, ,
\end{aligned}$$

where

$$J = \left[\bar{J}; j_r + 1_r\right] \, , \quad j_r = 0, \dots, \frac{k_r}{2} - 1 \, , \quad I_2 = 1_i \, , \quad i = 1, \dots, r-1$$

is used. In order to extract the boundary conditions we have to consider the horizontal projection

$$\begin{aligned}
\left(j^{2n-1}\bar{\sigma}\right)^* \left(j^n v_\partial \rfloor \left(\Psi^{2n-1}\right)^* c_{ext}\right) &= \left(j^{2n}\bar{\sigma}\right)^* \left(\left(D_\partial \rfloor j^n v_\partial \rfloor \left(\Psi^{2n-1}\right)^* c_{ext}\right) \wedge \mathrm{d}\bar{\mathrm{Y}}\right) \\
&= (-1)^{r-1} \left(j^{2n}\bar{\sigma}\right)^* \left(j^n v_\partial \rfloor \left(D_\partial \rfloor \left(\Psi^{2n-1}\right)^* c_{ext}\right) \wedge \mathrm{d}\bar{\mathrm{Y}}\right) = 0
\end{aligned}$$

where $D_\partial = d_{\partial r-1} \rfloor \cdots \rfloor d_{\partial 1}$ is used.

[1]As the variational vector field is a vertical vector field, this restriction is always possible.

Remark 8.12 *All boundary total derivatives $d_{\partial 1}, \cdots, d_{\partial r-1}$ used in D_∂ annihilate the introduced boundary contact forms $\omega^\alpha_{\partial\left[\bar{J}-I_2;j_r\right]}$.*

We obtain finally

$$\begin{aligned}
&\left(j^{2n}\bar{\sigma}\right)^* \left(j^n v_\partial \rfloor \left((-1)^{r-1} D_\partial\rfloor \left(\Psi^{2n-1}\right)^* c_{ext}\right) \wedge \mathrm{d}\bar{\mathrm{Y}}\right) = \qquad (8.2)\\
&= \left(j^{2n}\bar{\sigma}\right)^* \left(j^n v_\partial \rfloor \left(\left(p_\alpha^{[J-1_r,J]} \circ \Psi^{2n-1}\right) \mathrm{d}y^\alpha_{\left[\bar{J};j_r\right]} \wedge \mathrm{d}\bar{\mathrm{Y}} - d_i \rfloor \mathrm{d}p_\alpha^{\left[\bar{J}-1_i,\bar{J};j_r\right]} \mathrm{d}y^\alpha_{\left[\bar{J}-1_i;j_r\right]} \wedge \mathrm{d}\bar{\mathrm{Y}} \right.\right.\\
&\quad \left.\left. -p_\alpha^{\left[\bar{J}-1_i,\bar{J};j_r\right]} \mathrm{d}y^\alpha_{\left[\bar{J};j_r\right]} \wedge \mathrm{d}\bar{\mathrm{Y}}\right)\right)\\
&= 0\,.
\end{aligned}$$

This calculation shows that it is possible to cancel out a certain $\left(p_\alpha^{[J-1_r,J]} \circ \Psi^{2n-1}\right) \mathrm{d}y^\alpha_{[J-1_r]} \wedge \mathrm{d}\bar{\mathrm{Y}}$ form part entirely by means of an appropriate choice of the function $p_\alpha^{\left[\bar{J}-1_i,\bar{J};j_r\right]}$ in the term $-p_\alpha^{\left[\bar{J}-1_i,\bar{J};j_r\right]} \mathrm{d}y^\alpha_{\left[\bar{J};j_r\right]} \wedge \mathrm{d}\bar{\mathrm{Y}}$, iff the multi-index of $\mathrm{d}y^\alpha_{[J-1_r]}$ is not of the form

$$[J-1_r] = J_r = [0..0j_r]\ .$$

Every such cancellation operation introduces a lower-index form $d_i\rfloor \mathrm{d}p_\alpha^{\left[\bar{J}-1_i,\bar{J};j_r\right]} \mathrm{d}y^\alpha_{\left[\bar{J}-1_i;j_r\right]} \wedge \mathrm{d}\bar{\mathrm{Y}}$, $i = 1, \ldots, r-1$ and thus we obtain precisely the same elimination process for every dependent coordinate $y^\alpha_{[0\ldots0;j_r]}$ of the total boundary manifold $\bar{\mathcal{E}}$ as we had used for the dependent coordinates y^α on the total domain manifold $\mathcal{E}$.

Remark 8.13 *The procedure used in the derivation of the domain condition results that the Cartan form c was built, such that no*

$$g\ \mathrm{d}y^\alpha_{[J]} \wedge \mathrm{dX}\ , \quad g \in C^\infty\left(J^n\pi\right)\ , \quad [J] \neq [0\ldots0]$$

forms appear in $\left((-1)^r D\rfloor \mathrm{d}c\right) \wedge \mathrm{dX}$. Another consequence of this construction is that no prolongation of the variational vector field v is necessary anymore.

Consequently we are able to define all $p_\alpha^{\left[\bar{J}-1_i,\bar{J};j_r\right]}$ functions in the extended Cartan form, such that no

$$g\ \mathrm{d}y^\alpha_{\left[\bar{J};j_r\right]} \wedge \mathrm{d}\bar{\mathrm{Y}}\ , \quad g \in C^\infty\left(J^n\left(\bar{\mathcal{E}}\right)\right)\ , \quad \left[\bar{J}\right] \neq [0\ldots0]\ , \quad j_r = 0, \ldots, \frac{k_r}{2} - 1$$

forms appear in $\left((-1)^{r-1} D_\partial\rfloor \left(\Psi^{2n-1}\right)^* c_{ext}\right) \wedge \mathrm{d}\bar{\mathrm{Y}}$. It is remarkable that we can replace the prolongation of the variational field on the boundary $j^n v_\partial$ simply by

$$v_\partial = v_\partial{}^\alpha_{[0\ldots0;jr]} \frac{\partial}{\partial y^\alpha_{[0\ldots0;jr]}}\ , \quad j_r = 0, \ldots, \frac{k_r}{2} - 1\ ,$$

i.e. the boundary variational vector field, if the functions $p_\alpha^{\left[\bar{J}-1_i,\bar{J};j_r\right]}$ are chosen properly.

The determination of the relations that must be fulfilled by the functions $p_\alpha^{[\bar{J}-1_i,\bar{J};j_r]}$ are formulated in the following listing (8.3)

$$
\begin{array}{lcll}
\sum\limits_{l=1}^{m_{\partial J}} p_\alpha^{[\bar{J}-1_{w_l},\bar{J};j_r]} & = & p_\alpha^{[J-1_r,J]} \circ \Psi^{2n-1} & \#J=n\ ,\ [J-1_r]=[\bar{J}\ j_r]\ , \\ & & & w_l\in\{1,\ldots,r-1\} \\
\sum\limits_{l=1}^{m_{\partial J}} p_\alpha^{[\bar{J}-1_{w_l},\bar{J};j_r]} & = & \left(p_\alpha^{[J-1_r,J]} \circ \Psi^{2n-1}\right) - \sum\limits_{j=1}^{r-1} \mathrm{L}_{d_{\partial j}} \left(p_\alpha^{[\bar{J},\bar{J}+1_j;j_r]}\right) & \#J=n-1\ , \\ & & & [J-1_r]=[\bar{J}\ j_r]\ , \\ & & & w_l\in\{1,\ldots,r-1\} \\
\vdots & = & \vdots & \\
p_\alpha^{[0\ldots0,1_{w_l};j_r]} & = & \left(p_\alpha^{[J_r-1_r,J]}\circ\Psi^{2n-1}\right) - \sum\limits_{j=1}^{r-1} \mathrm{L}_{d_{\partial j}} \left(p_\alpha^{[1_{w_l},1_{w_l}+1_j;j_r]}\right) & \#J=j_r+2\ , \\ & & & [J-1_r]=1_{w_l}+J_r\ , \\ & & & m_{\partial J}=1, w_l\in\{1,\ldots,r-1\}\,.
\end{array}
\tag{8.3}
$$

Obviously we have determined a family of functions according to a certain index j_r or equally to every dependent coordinate $y^\alpha_{[0\ldots0;j_r]}$ in $\bar{\mathcal{E}}$. Now it is left to show that the derived boundary conditions are invariant under a certain choice of the functions $p_\alpha^{[\bar{J}-1_{w_l},\bar{J};j_r]}$ out of the determined family. In order to prove this, we have to consider equation (8.2). Similarly to the construction on the domain, we can start on the bottom index $J_r=[0\ldots0;j_r]$ and step upwards in the scheme (8.3). This results in

$$
\begin{aligned}
&\left(p_\alpha^{[J_r,J_r+1_r]}\circ\Psi^{2n-1}\right)\mathrm{d}y^\alpha_{[J_r]}\wedge\mathrm{d}\bar{\mathrm{Y}} - \sum_{w_l=1}^{r-1} d_{w_l}\rfloor\mathrm{d}p_\alpha^{[0\ldots0,1_{w_l};j_r]}\mathrm{d}y^\alpha_{[J_r]}\wedge\mathrm{d}\bar{\mathrm{Y}} \\
&= \left(p_\alpha^{[J_r,J_r+1_r]}\circ\Psi^{2n-1}\right)\mathrm{d}y^\alpha_{[J_r]}\wedge\mathrm{d}\bar{\mathrm{Y}} - \sum_{w_l=1}^{r-1}\mathrm{L}_{d_{\partial w_l}}\left(\left(p_\alpha^{[J_r+1_{w_l},J_r+1_r+1_{w_l}]}\circ\Psi^{2n-1}\right)\right. \\
&\quad \left. - \sum_{j=1}^{r-1}\mathrm{L}_{d_{\partial j}}\left(p_\alpha^{[1_{w_l},1_{w_l}+1_j;j_r]}\right)\right)\mathrm{d}y^\alpha_{[J_r]}\wedge\mathrm{d}\bar{\mathrm{Y}} \\
&= \left(p_\alpha^{[J_r,J_r+1_r]}\circ\Psi^{2n-1} - \sum_{w_l=1}^{r-1}\mathrm{L}_{d_{\partial w_l}}p_\alpha^{[J_r+1_{w_l},J_r+1_r+1_{w_l}]}\circ\Psi^{2n-1}\right)\mathrm{d}y^\alpha_{[J_r]}\wedge\mathrm{d}\bar{\mathrm{Y}} \\
&\quad + \sum_{w_l=1}^{r-1}\sum_{j=1}^{r-1}\mathrm{L}_{d_{\partial w_l}}\mathrm{L}_{d_{\partial j}}\left(p_\alpha^{[1_{w_l},1_{w_l}+1_j;j_r]}\right)\mathrm{d}y^\alpha_{[J_r]}\wedge\mathrm{d}\bar{\mathrm{Y}} \\
&= \left(p_\alpha^{[J_r,J_r+1_r]}\circ\Psi^{2n-1} - \sum_{w_l=1}^{r-1}\mathrm{L}_{d_{\partial w_l}}p_\alpha^{[J_r+1_{w_l},J_r+1_r+1_{w_l}]}\circ\Psi^{2n-1}\right)\mathrm{d}y^\alpha_{[J_r]}\wedge\mathrm{d}\bar{\mathrm{Y}} \\
&\quad + \sum_{\#\bar{J}=2}\mathrm{L}_{d_{\bar{J}}}\left(p_\alpha^{[J_r+\bar{J},J_r+\bar{J}+1_r]}\circ\Psi^{2n-1} - \sum_{j=1}^{r-1}\mathrm{L}_{d_j}\left(p_\alpha^{[\bar{J},\bar{J}+1_j;j_r]}\right)\right)\mathrm{d}y^\alpha_{[J_r]}\wedge\mathrm{d}\bar{\mathrm{Y}} \\
&= \cdots \\
&= \delta_{\partial\alpha}^{[J_r]}(l)\,\mathrm{d}y^\alpha_{[J_r]}\wedge\mathrm{d}\bar{\mathrm{Y}} \\
&= 0\ ,
\end{aligned}
$$

where we have made again extensive use of

$$\mathrm{L}_{d_{\partial i}}\mathrm{L}_{d_{\partial j}}(f) = \mathrm{L}_{d_{\partial j}}\mathrm{L}_{d_{\partial i}}(f) \ , \quad f \in C^{\infty}\left(J^n\left(\bar{\mathcal{E}}\right)\right) \ , \quad i,j = 1,\ldots,r-1$$

and

$$\mathrm{L}_{d_{\bar{J}}} = \underbrace{\left(\mathrm{L}_{d_{\partial 1}}\right)^{j_1} \circ \cdots \circ \left(\mathrm{L}_{d_{\partial r-1}}\right)^{j_{r-1}}}_{\#\bar{J}\text{-times}} \ , \quad \left[\bar{J}\right] = \left[j_1 \ldots j_{r-1}\right] \ .$$

Thus we see from (8.2) that the derived boundary conditions are independent of the functions $p_{\alpha}^{\left[\bar{J}-I_2,\bar{J};j_r\right]}$ iff (8.3) is met. Furthermore the derived boundary conditions are independent of the used $p_{\alpha}^{[J-I_1,J]}$ in the construction of c_{ext}. This is caused by the fact that all appearing functions $p_{\alpha}^{[J_r,J_r+1_r]}$, $p_{\alpha}^{\left[J_r+1_{w_l},J_r+1_r+1_{w_l}\right]}$, ... , $p_{\alpha}^{\left[J_r+\bar{J},J_r+\bar{J}+1_r\right]}$ are unique due to the introduced partial ordering. In other words for every multi-index $J = J_r + \bar{J} + 1_r$ there exists a unique minimal multi-index $J_r + \bar{J}$ with respect to the introduced partial multi-index ordering $>_{part}$ (see Def. 8.11).

8.3.3 Condition on the domain

As shown by corollary 8.8 we are able to extract the domain condition similarly to the original Cartan form domain condition, i.e.

$$\left(\left(-1\right)^r D\rfloor \mathrm{d}c_{ext}\right) \wedge \mathrm{dY} = 0$$

or equally $\delta_{\alpha}(l)\ \mathrm{d}x^{\alpha} \wedge \mathrm{dX} = 0$.

8.3.4 Condition on the boundary

The boundary conditions for an infinite-dimensional Euler-Lagrange system with n^{th} order Lagrangian are given by

$$\left(\left(-1\right)^{r-1} D_{\partial}\rfloor \left(\Psi^{2n-1}\right)^{*} c_{ext}\right) \wedge \mathrm{d}\bar{\mathrm{Y}} = 0$$

or equally $\delta_{\partial\alpha}^{[J_r]}(l)\ \mathrm{d}y_{[J_r]}^{\alpha} \wedge \mathrm{d}\bar{\mathrm{Y}} = 0$. It is obvious that these conditions are analogous to the domain conditions.

Finally we are able to state the following theorem.

Theorem 8.14 *The domain and boundary conditions extracted from the extended Cartan form are unique.*

Proof. This theorem is a immediate consequence of the investigations shown in 7.2.3 and 8.3.2. ■

8.4 Domain with Non-smooth Boundary

Here we will consider domains $\mathcal{D}$ with non-smooth boundary manifold $\partial\mathcal{D}$. In Fig. 8.3 a comparison of a domain with smooth and non-smooth boundary is given. This situation implies that every point of the manifold $\mathcal{D}$ is only at least homeomorphic to the half plane (see 2.1).

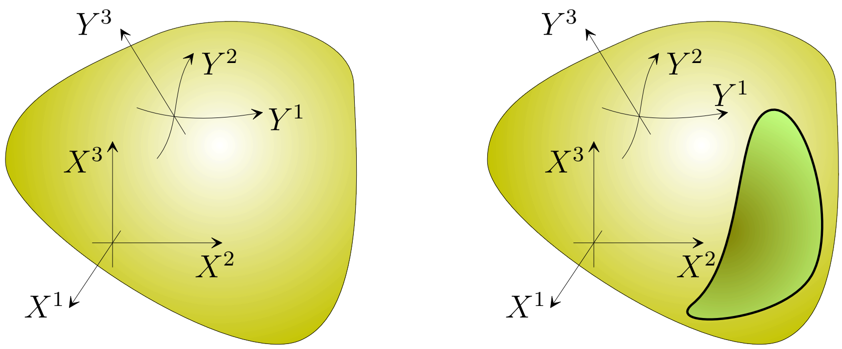

Figure 8.3: Comparison between smooth and non-smooth boundary.

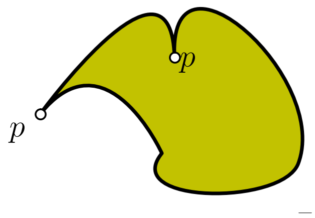

Figure 8.4: Domain $\mathcal{D}$ that does not allow a coherent orientation of the boundary $\partial\mathcal{D}$ at the points p.

We have already assumed that the orientable boundary manifold $\partial\mathcal{D}$ is coherently oriented to the oriented domain manifold $\mathcal{D}$. This prevents domains that do not allow a coherent orientation like, e.g., Fig. 8.4. Furthermore we require $\partial\mathcal{D}$ to be built up by a finite amount of bounded smooth manifolds $\partial\mathcal{D}_i$, i.e.

$$\partial\mathcal{D} = \bigcup_{i=1}^{k} \partial\mathcal{D}_i \ , \ k < \infty \ .$$

Such a domain is depicted in Fig. 8.5. These prerequisites enable us to rewrite the Lagrangian

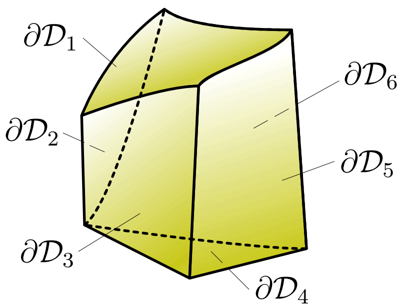

Figure 8.5: Domain $\mathcal{D}$ with boundary built up by smooth manifolds $\partial\mathcal{D}_i$.

functional as used in proof 8.2 in the form of

$$\begin{aligned}
\int_{\mathcal{D}} (j^n\sigma)^* \left(l \ \mathrm{dY} + \omega + \mathrm{d}\omega_\partial\right) &= \int_{\mathcal{D}} (j^n\sigma)^* \left(l \ \mathrm{dY} + \omega\right) + \int_{\partial\mathcal{D}} \iota^* \left((j^n\sigma)^* \omega_\partial\right) \\
&= \int_{\mathcal{D}} (j^n\sigma)^* \left(l \ \mathrm{dY} + \omega\right) + \sum_{i=1}^{k} \int_{\partial\mathcal{D}_i} \iota_i^* \left((j^n\sigma)^* \omega_\partial\right) \\
&= \int_{\mathcal{D}} (j^n\sigma)^* \left(l \ \mathrm{dY}\right) \ .
\end{aligned}$$

It is remarkable that all boundaries of the manifolds $\partial\mathcal{D}_i$, i.e. $\partial\partial\mathcal{D}_i$ are by construction of measure zero in the Lie derivative of the Lagrangian functional. Thus we can exclude these domains and the construction presented for a smooth boundary applies also for all boundary parts $\partial\mathcal{D}_i$.

Remark 8.15 *These investigations imply that Hamilton's principle applied to a certain Lagrangian $l\,\mathrm{dY}$ on an r-dimensional domain $\mathcal{D}$ could not result in any condition on a domain of dimension less than $r-1$. Consequently the analysis of the 2-dim. rectangular Kirchhoff plate (see e.g. section 8.6) could not result in any condition on the edges of the plate, as they are 0-dim. domains. This implies that the edge conditions presented in [Ritz, 1909] must be too restrictive. In appendix B this effect is discussed using classical notation.*

The analysis of physical systems from control point of view is undoubtedly interested in a correct modeling of external inputs.

8.5 Systems with Inputs

The introduction of domain and boundary inputs into the theory of Euler-Lagrange systems from a geometric point of view is the aim of this section.

In order to achieve this, we recall the domain

$$\left((-1)^r\, D\rfloor \mathrm{d}c_{ext}\right)\wedge \mathrm{dY}=0$$

and boundary condition

$$\left((-1)^{r-1}\, D_\partial\rfloor \left(\Psi^{2n-1}\right)^* c_{ext}\right)\wedge \mathrm{d}\bar{\mathrm{Y}}=0$$

developed in the recent chapters. An evaluation of these conditions leads to the following representation

$$f_\beta\left(Y^i,y^\alpha,y^\alpha_{[J]}\right)\mathrm{d}y^\beta\wedge \mathrm{dY}=0 \tag{8.4}$$

respectively

$$f_\beta^{[0\ldots 0;j_r]}\left(\bar{Y}^i,y^\alpha_{\left[\bar{J};\bar{j}_r\right]}\right)\mathrm{d}y^\beta_{[0\ldots 0;j_r]}\wedge \mathrm{d}\bar{\mathrm{Y}}=0\ ,\quad \bar{j}_r=0,\ldots,\frac{k_r}{2}\ ,\ j_r=0,\ldots,\frac{k_r}{2}-1. \tag{8.5}$$

In the case of a free system, we are able to identify the domain and boundary conditions by

$$f_\beta\left(Y^j,y^\alpha,y^\alpha_{[J]}\right)=0 \tag{8.6}$$

respectively

$$f_\beta^{[0\ldots 0;j_r]}\left(\bar{Y}^i,y^\alpha_{\left[\bar{J};\bar{j}_r\right]}\right)=0\ . \tag{8.7}$$

Thereby we get in general a system of partial differential equations on the domain and on the boundary.

These equations do not represent all conditions that can be extracted from the equations (8.4) and (8.5). For example both equations are met, as soon as all dependent variables y^β are determined by means of external setting as functions of the independent coordinates, i.e. $y^\beta\left(Y^j\right)$. This equals the plug in of a section $\sigma:\mathcal{D}\to\mathcal{E},\ y^\beta\to y^\beta\left(Y^j\right)$ and implies that

$$\mathrm{d}y^\beta\wedge \mathrm{dY}=0\ \ ,\ \ \mathrm{d}y^\beta_{[0\ldots 0;j_r]}\wedge \mathrm{d}\bar{\mathrm{Y}}=0\ . \tag{8.8}$$

Consequently equation (8.4) and (8.5) are met. Another scenario would be the assignment of the dependent boundary coordinates, i.e.

$$y^{\beta}_{[0\ldots0;j_r]} \rightarrow y^{\beta}_{[0\ldots0;j_r]}\left(\bar{Y}^i\right) \tag{8.9}$$

by means of external restrictions. Thereby the corresponding equation (8.5) is also met. This behavior of boundary conditions is well known in mechanics and referred to as dynamic boundary condition in the case of equation (8.7) and static boundary condition in the case of equation (8.9).

This observations illustrate that any external setting of the dependent variables on the domain or boundary causes the disappearance of the corresponding dynamic conditions. Consequently the set of equations of motion is reduced and the remaining ones automatically satisfy the introduced statical condition.

These preliminary considerations can now be used to introduce the notion of external inputs to the language of Euler-Lagrange systems. The external assignment of the dependent coordinates must be equal to the choice of an appropriate input. Thus we extend for example the domain condition

$$f_\beta\left(Y^j, y^\alpha, y^\alpha_{[J]}\right)\mathrm{d}y^\beta \wedge \mathrm{dY} + u_\beta \mathrm{d}y^\beta \wedge \mathrm{dY} = 0\;,$$

whereby no modification of the static domain condition appears. On the other hand we are also able to meet this relation by plugging the external setting $y^\alpha\left(Y^j\right)$ into f_β and choosing

$$u_\beta = -\, f_\beta\left(Y^j, y^\alpha, y^\alpha_{[J]}\right)\Big|_{y^\alpha(Y^j)}\;.$$

This equation determines precisely the external input u_β, which is necessary to force the solution on the externally assigned function $y^\beta\left(Y^j\right)$.

An even more important consequence of these considerations is the information about the correct geometrical introduction of external inputs to the extended Cartan form. From the form representation on the domain

$$f_\beta\left(Y^i, y^\alpha, y^\alpha_{[J]}\right)\mathrm{d}y^\beta \wedge \mathrm{dY} + u_\beta \mathrm{d}y^\beta \wedge \mathrm{dY} = 0$$

respectively on the boundary

$$f^{[0\ldots0;j_r]}_\beta\left(\bar{Y}^i, y^\alpha_{\left[\bar{J};\bar{j}_r\right]}\right)\mathrm{d}y^\beta_{[0\ldots0;j_r]} \wedge \mathrm{d\bar{Y}} + \bar{u}^{[0\ldots0;j_r]}_\beta \mathrm{d}y^\beta_{[0\ldots0;j_r]} \wedge \mathrm{d\bar{Y}} = 0$$

we obtain directly the extended domain condition

$$\left(\left(-1\right)^r D\rfloor\left(\mathrm{d}c_{ext}\right) + u_\beta \mathrm{d}y^\beta\right) \wedge \mathrm{dY} = 0$$

and the extended boundary condition

$$\left(\left(-1\right)^{r-1} D_\partial\rfloor\left(\left(\Psi^n\right)^* c_{ext}\right) + \bar{u}^{[0\ldots0;j_r]}_\beta \mathrm{d}y^\beta_{[0\ldots0;j_r]}\right) \wedge \mathrm{d\bar{Y}} = 0\;.$$

If the inputs are assumed to be functions of the independent coordinates only, i.e.

$$u_\beta = u_\beta\left(Y^j\right) \quad \text{respectively } \bar{u}^{[0\ldots0;j_r]}_\beta = \bar{u}^{[0\ldots0;j_r]}_\beta\left(\bar{Y}^i\right)\;,$$

then it is possible to incorporate the external inputs directly within the extended Cartan form by

$$\begin{aligned} c_{ext} &= l\mathrm{dY} + p_\alpha^{[J-I_1,J]}\omega^\alpha_{[J-I_1]} \wedge \partial_{[I_1]}\rfloor\mathrm{dY} + \mathrm{d}\left(p_\alpha^{\left[\bar{J}-I_2,\bar{J};j_r\right]}\omega^\alpha_{\left[\bar{J}-I_2;j_r\right]} \wedge \partial_{[I_2]}\rfloor\partial_{[1_r]}\rfloor\mathrm{dY}\right) \\ &\quad + u_\beta\, y^\beta\mathrm{dY} + \mathrm{d}\left(\bar{u}_\beta^{[0\ldots0;j_r]}\; y^\beta_{[0\ldots0;j_r]}\partial_{[1_r]}\rfloor\mathrm{dY}\right) \\ &= \left(l + u_\beta\, y^\beta\right)\mathrm{dY} + p_\alpha^{[J-I_1,J]}\omega^\alpha_{[J-I_1]} \wedge \partial_{I_1}\rfloor\mathrm{dY} \\ &\quad + \mathrm{d}\left(p_\alpha^{\left[\bar{J}-I_2,\bar{J};j_r\right]}\omega^\alpha_{\left[\bar{J}-I_2;j_r\right]} \wedge \partial_{[I_2]}\rfloor\partial_{[1_r]}\rfloor\mathrm{dY} + \bar{u}_\beta^{[0\ldots0;j_r]}\; y^\beta_{[0\ldots0;j_r]}\partial_{[1_r]}\rfloor\mathrm{dY}\right) \ . \end{aligned}$$

The extension of the Lagrangian density by

$$l + u_\beta\, y^\beta$$

in order to take external inputs into account, is also known from mechanics and the term $u_\beta\, y^\beta$ is denoted as *viral*. It is remarkable that the construction of the viral is not applicable for the analysis of the closed loop behavior generated by a control law of the form

$$u_\beta\left(Y^j, y^\alpha, y^\alpha_{[J]}\right) \ .$$

8.6 Application: The Kirchhoff plate

Here we will consider the rectangular Kirchhoff plate as depicted in figure 8.6, with distributed linear damping force f on the spatial domain $\mathcal{D}_S$ and a damping torque M along the border $Y^3 = Y^3_\partial = \mathrm{const.}$.

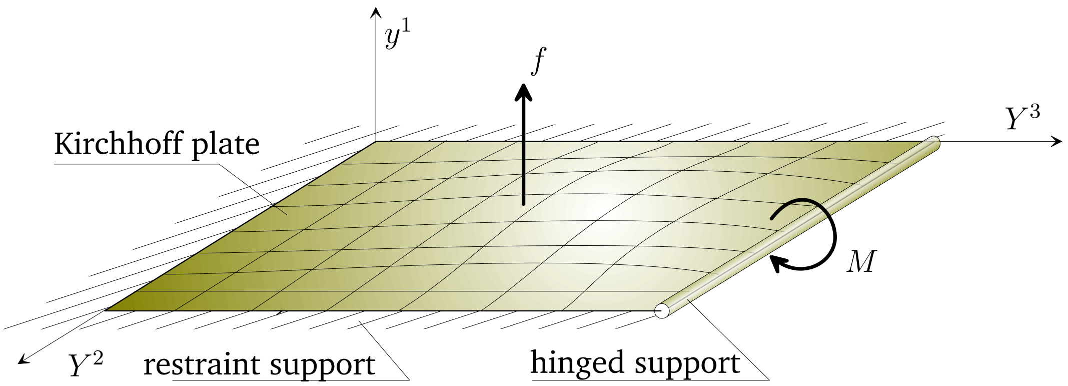

Figure 8.6: The rectangular Kirchhoff plate with damping force f and daming torque M.

8.6.1 Local coordinate representation

We introduce the independent coordinates $Y^1 = t, Y^2, Y^3$ on the domain $\mathcal{D}$. The motion of the plate is described by the dependent coordinate y^1. Here we are interested in the domain and boundary conditions and consequently it is necessary to define a suitable inclusion mapping. We will derive the boundary conditions on the border $Y^3 = Y^3_\partial = \mathrm{const.}$ and thus we have to make use of the following inclusion map

$$\begin{array}{rcl} \iota_1 : \partial\mathcal{D}_1 & \to & \mathcal{D} \\ \left(\bar{Y}^1 = t,\ \bar{Y}^2\right) & \to & \left(Y^1 = \bar{Y}^1 = t,\ Y^2 = \bar{Y}^2,\ Y^3 = Y^3_\partial = \text{const.}\right) \ , \end{array}$$

where $\partial\mathcal{D}_1 \subset \partial\mathcal{D}$. This configuration is visualized in figure 8.7. Now we are able to start the

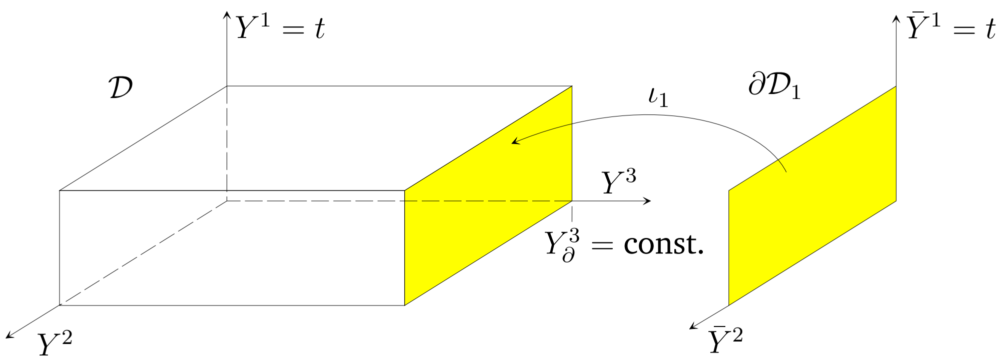

Figure 8.7: The used inclusion map ι_1.

construction of the extended Cartan form.

8.6.2 Construction of the extended Cartan form

The Lagrangian density is given by (see [Ritz, 1909])

$$l = \frac{1}{2}\rho\Lambda\left(y^1_{[100]}\right)^2 - \frac{1}{2}\varsigma\left(\left(y^1_{[020]}\right)^2 + \left(y^1_{[002]}\right)^2 + 2\nu y^1_{[002]}y^1_{[020]} + 2\left(1-\nu\right)\left(y^1_{[011]}\right)^2\right)$$

with $\rho, \Lambda, \varsigma, \nu \in \mathbb{R}^+$. Consequently we are ready to determine the functions $p_\alpha^{[J-I_1,J]}$ and $p_\alpha^{\left[\bar{J}-1_{w_l},\bar{J};j_r\right]}$.

Determination of the functions $p_\alpha^{[J-I_1,J]}$

In a first step we derive all functions according to multi-indices meeting $\#J = 2$. We get

$$\begin{aligned}
p_1^{[010,020]} &= \partial_1^{[020]}l = -\varsigma\left(y^1_{[020]} + \nu y^1_{[002]}\right)\\
p_1^{[001,002]} &= \partial_1^{[002]}l = -\varsigma\left(y^1_{[002]} + \nu y^1_{[020]}\right)\\
p_1^{[010,011]} + p_1^{[001,011]} &= \partial_1^{[011]}l = -2\varsigma\left(\left(1-\nu\right)y^1_{[011]}\right) \ .
\end{aligned}$$

Here we recognize the non-uniqueness of the functions for the multi-index $J = [011]$. In fact we have to make use of the partial ordering and get

$$p_1^{[010,011]} = \partial_1^{[011]}l = -2\varsigma\left(\left(1-\nu\right)y^1_{[011]}\right) \ .$$

In the next step we consider multi-indices J whose length is $\#J = 1$. We obtain

$$\begin{aligned}
p_1^{[000,100]} &= \partial_1^{[100]}l - \sum_{i=1}^{r}\mathrm{L}_{d_i}\left(p_1^{[100,100+1_i]}\right) = \rho\Lambda y^1_{[100]}\\
p_1^{[000,010]} &= \partial_1^{[010]}l - \sum_{i=1}^{r}\mathrm{L}_{d_i}\left(p_1^{[010,010+1_i]}\right) =\\
&= \partial_1^{[010]}l - \mathrm{L}_{d_2}\left(p_1^{[010,020]}\right) - \mathrm{L}_{d_3}\left(p_1^{[010,011]}\right) = +\varsigma\left(y^1_{[030]} + \nu y^1_{[012]}\right) + 2\varsigma\left(1-\nu\right)y^1_{[012]}\\
p_1^{[000,001]} &= \partial_1^{[001]}l - \sum_{i=1}^{r}\mathrm{L}_{d_i}\left(p_1^{[001,001+1_i]}\right) = -\mathrm{L}_{d_3}\left(p_1^{[001,002]}\right) = \varsigma\left(y^1_{[003]} + \nu y^1_{[021]}\right) \ .
\end{aligned}$$

Consequently it is left to derive the functions $p_\alpha^{\left[\bar{J}-1_{w_l},\bar{J};j_r\right]}$.

Determination of the functions $p_\alpha^{\left[\bar{J}-1_{w_l},\bar{J};j_r\right]}$

Here we obtain only a single function from the multi-index $[J-1_r] = \left[\bar{J}\ j_r\right]$ to be given by

$$p_1^{[00,01;0]} = p_1^{[010,011]} = -2\varsigma\left(\left(1-\nu\right)y^1_{[011]}\right) \ ,$$

where obviously $j_r = 0$ and $\left[\bar{J}\right] = [01]$ is used.

Finally we are able to formulate the extended Cartan form

$$\begin{aligned}
c_{ext} \;=\; & \frac{1}{2}\rho\Lambda\left(y^1_{[100]}\right)^2 \mathrm{d}t\wedge \mathrm{d}Y^2\wedge \mathrm{d}Y^3 \\
& -\frac{1}{2}\varsigma\left(\left(y^1_{[020]}\right)^2+\left(y^1_{[002]}\right)^2+2\nu y^1_{[002]}y^1_{[020]}+2\left(1-\nu\right)\left(y^1_{[011]}\right)^2\right)\mathrm{d}t\wedge \mathrm{d}Y^2\wedge \mathrm{d}Y^3 \\
& +p_1^{[010,020]}\left(\mathrm{d}y^1_{[010]}-y^1_{[020]}\mathrm{d}Y^2\right)\wedge\partial_2\rfloor\left(\mathrm{d}t\wedge \mathrm{d}Y^2\wedge \mathrm{d}Y^3\right) \\
& +p_1^{[001,002]}\left(\mathrm{d}y^1_{[001]}-y^1_{[002]}\mathrm{d}Y^3\right)\wedge\partial_3\rfloor\left(\mathrm{d}t\wedge \mathrm{d}Y^2\wedge \mathrm{d}Y^3\right) \\
& +p_1^{[010,011]}\left(\mathrm{d}y^1_{[010]}-y^1_{[011]}\mathrm{d}Y^3\right)\wedge\partial_3\rfloor\left(\mathrm{d}t\wedge \mathrm{d}Y^2\wedge \mathrm{d}Y^3\right) \\
& +p_1^{[000,100]}\left(\mathrm{d}y^1_{[000]}-y^1_{[100]}\mathrm{d}t\right)\wedge\partial_1\rfloor\left(\mathrm{d}t\wedge \mathrm{d}Y^2\wedge \mathrm{d}Y^3\right) \\
& +p_1^{[000,010]}\left(\mathrm{d}y^1_{[000]}-y^1_{[010]}\mathrm{d}Y^2\right)\wedge\partial_2\rfloor\left(\mathrm{d}t\wedge \mathrm{d}Y^2\wedge \mathrm{d}Y^3\right) \\
& +p_1^{[000,001]}\left(\mathrm{d}y^1_{[000]}-y^1_{[001]}\mathrm{d}Y^3\right)\wedge\partial_3\rfloor\left(\mathrm{d}t\wedge \mathrm{d}Y^2\wedge \mathrm{d}Y^3\right) \\
& +\mathrm{d}\left(p_1^{[00,01;0]}\left(\mathrm{d}y^1_{[000]}-y^1_{[010]}\mathrm{d}Y^2\right)\wedge\partial_2\rfloor\left(\mathrm{d}t\wedge \mathrm{d}Y^2\right)\right)\ .
\end{aligned}$$

8.6.3 The domain condition

From the general domain condition

$$\left(\left(-1\right)^r D\rfloor\left(\mathrm{d}c_{ext}\right)-Ry^1_{[100]}\mathrm{d}y^1_{[000]}\right)\wedge \mathrm{dY}=0\ ,$$

which is extended by the distributed damping force $u=f=-Ry^1_{[100]}$, we are able to extract the domain conditions. Here this results in

$$\left(-d_1\rfloor \mathrm{d}p_1^{[000,100]}-d_2\rfloor \mathrm{d}p_1^{[000,010]}-d_3\rfloor \mathrm{d}p_1^{[000,001]}-Ry^1_{[100]}\right)\mathrm{d}y^1_{[000]}\wedge \mathrm{d}t\wedge \mathrm{d}X^2\wedge \mathrm{d}X^3=0$$

and finally the equation of motion is obtained by

$$\begin{aligned}
&\rho\Lambda y^1_{[200]}+\varsigma\left(y^1_{[040]}+\nu y^1_{[022]}\right)+2\varsigma\left(\left(1-\nu\right)y^1_{[022]}\right)+\varsigma\left(y^1_{[004]}+\nu y^1_{[022]}\right)+Ry^1_{[100]} \\
&\quad=\ \rho\Lambda y^1_{[200]}+\varsigma y^1_{[040]}+\varsigma\nu y^1_{[022]}+2\varsigma\left(1-\nu\right)y^1_{[022]}+\varsigma y^1_{[004]}+\varsigma\nu y^1_{[022]}+Ry^1_{[100]} \\
&\quad=\ \rho\Lambda y^1_{[200]}+\varsigma y^1_{[040]}+\varsigma y^1_{[004]}+2\varsigma y^1_{[022]}+Ry^1_{[100]}=0\ .
\end{aligned}$$

8.6.4 The boundary condition on $\partial\mathcal{D}_1$

We consider the general boundary condition

$$\left(\left(-1\right)^{r-1} D_\partial\rfloor\left(\left(\Psi^n\right)^* c_{ext}\right)-R_\partial y^1_{[10;1]}\mathrm{d}y^1_{[00;1]}\right)\wedge \mathrm{d}\bar{\mathrm{Y}}=0\ .$$

where the pull-back of the extended Cartan form along Ψ^n, i.e.

$$\begin{aligned}
\left(\Psi^n\right)^* c_{ext} \;=\; & \left(\Psi^n\right)^* p_1^{[001,002]}\mathrm{d}y^1_{[00;1]}\wedge\partial_3\rfloor \mathrm{d}t\wedge \mathrm{d}Y^2\wedge \mathrm{d}Y^3 \\
& +\left(\Psi^n\right)^* p_1^{[010,011]}\mathrm{d}y^1_{[01;0]}\wedge\partial_3\rfloor \mathrm{d}t\wedge \mathrm{d}Y^2\wedge \mathrm{d}Y^3 \\
& +\left(\Psi^n\right)^* p_1^{[000,001]}\mathrm{d}y^1_{[00;0]}\wedge\partial_3\rfloor \mathrm{d}t\wedge \mathrm{d}Y^2\wedge \mathrm{d}Y^3 \\
& +\mathrm{d}\left(p_1^{[00,01;0]}\left(\mathrm{d}y^1_{[00;0]}-y^1_{[01;0]}\mathrm{d}Y^2\right)\wedge\partial_2\rfloor \mathrm{d}t\wedge \mathrm{d}Y^2\right)
\end{aligned}$$

has to be used. Additionally, we make use of the damping torque $u_\partial = M = -R_\partial y^1_{[10;1]}$ on $\partial\mathcal{D}_1$. Finally we end up with the form

$$\left(\left(\Psi^n\right)^* p_1^{[001,002]} - R_\partial y^1_{[10;1]}\right) \mathrm{d}y^1_{[00;1]} \wedge \mathrm{d}t \wedge \mathrm{d}Y^2 - d_{\partial 2} \rfloor \mathrm{d}p_1^{[00,01;0]} \mathrm{d}y^1_{[00;0]} \wedge \mathrm{d}t \wedge \mathrm{d}Y^2 + \\ + \left(\Psi^n\right)^* p_1^{[000,001]} \mathrm{d}y^1_{[00;0]} \wedge \mathrm{d}t \wedge \mathrm{d}Y^2 = 0 \, .$$

Consequently we obtain the boundary conditions on $\partial\mathcal{D}_1$

$$\begin{aligned} y^1_{[00;0]} &= 0 \\ \left(\Psi^n\right)^* p_1^{[001,002]} - R_\partial y^1_{[10;1]} &= -\varsigma\left(y^1_{[00;2]} + \nu y^1_{[02;0]}\right) - R_\partial y^1_{[10;1]} = 0 \, . \end{aligned}$$

The boundary conditions on $\partial\mathcal{D} - \partial\mathcal{D}_1$ are given by

$$\begin{aligned} y^1_{[00;0]} &= 0 \\ y^1_{[00;1]} &= 0 \, , \end{aligned}$$

where the restraint support is taken into account.

Chapter 9

The Evolution of Euler-Lagrange Systems

The base manifold of the bundle π is given by the domain of integration $\mathcal{D}$. We have introduced local coordinates X^i resp. Y^i corresponding to this domain. The representation using the coordinates Y^i enables us to determine all boundary conditions by distinguishing the coordinate Y^r. Having physical systems at ones disposal, we have to consider an additional distinguished coordinate – the time coordinate t.

Remark 9.1 *The time coordinate t was already distinguished in the determination of the boundary conditions of Euler-Lagrange systems. It is assumed that no variation takes place at the time-boundary of the EL system.*

In order to incorporate this additional information in the presented framework, we mark

$$Y^1 = X^1 = t$$

as the independent coordinate representing the time. In figure 9.1 the domain $\mathcal{D}$ is depicted

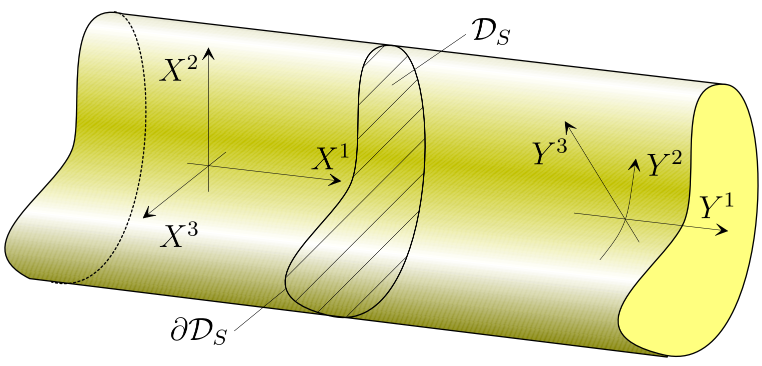

Figure 9.1: The space-time zylinder.

as a space-time cylinder, whereby the separation of time domain and spatial domain $\mathcal{D}_S$ is visualized. The time coordinate t is of special interest, as it allows to analyze the evolution of certain functionals on $\mathcal{D}_S$ along the solution σ of the EL system.

The extraction of the evolution information differs significantly for the finite-dimensional, infinite-dimensional of 1st order, and the infinite-dimensional n^{th} order case. For this reason we discuss them separately.

9.1 The finite-dimensional case

The extended Cartan form is in the finite-dimensional case given by

$$c_{ext} = c = l\mathrm{d}Y^1 + p_\alpha^{[J]}\left(\mathrm{d}y^\alpha_{[J-1_1]} - y^\alpha_{[J]}\mathrm{d}Y^1\right) \ , \quad \#J = 1,\ldots,n.$$

The single independent coordinate Y^1 equals the time coordinate t and consequently no spatial domain $\mathcal{D}_S$ exists.

In order to obtain the evolution information of interest, we reformulate the extended Cartan form in the following way

$$\begin{aligned} c_{ext} &= -\left(p_\alpha^{[J]}y^\alpha_{[J]} - l\right)\mathrm{d}Y^1 + p_\alpha^{[J]}\mathrm{d}y^\alpha_{[J-1_1]} \\ &= -h\mathrm{d}Y^1 + p_\alpha^{[J]}\mathrm{d}y^\alpha_{[J-1_1]} \ , \end{aligned}$$

where we have introduced the function $h = \left(p_\alpha^{[J]}y^\alpha_{[J]} - l\right) \in C^\infty\left(J^{2n-1}\mathcal{E}\right)$. It is remarkable that this representation hides in some sense the application of the elements of the contact ideal $p_\alpha^{[J]}\left(\mathrm{d}y^\alpha_{[J-1_1]} - y^\alpha_{[J]}\mathrm{d}Y^1\right)$ that incorporate a $\mathrm{d}Y^1$ term. This procedure will be also applied in the infinite-dimensional case.

From the domain condition

$$(-1)^r\left(D\rfloor\mathrm{d}c_{ext}\right)\wedge\mathrm{dY} = (-1)\left(d_1\rfloor\mathrm{d}c_{ext}\right)\wedge\mathrm{d}Y^1 = 0$$

we get

$$\begin{aligned} &(-1)\left(d_1\rfloor\left(-\mathrm{d}h\wedge\mathrm{d}Y^1 + \mathrm{d}p_\alpha^{[J]}\wedge\mathrm{d}y^\alpha_{[J-1_1]}\right)\right)\wedge\mathrm{d}Y^1 = \\ &\quad = -\mathrm{d}h\wedge\mathrm{d}Y^1 - d_1\rfloor\mathrm{d}p_\alpha^{[J]}\mathrm{d}y^\alpha_{[J-1_1]}\wedge\mathrm{d}Y^1 + d_1\rfloor\mathrm{d}y^\alpha_{[J-1_1]}\mathrm{d}p_\alpha^{[J]}\wedge\mathrm{d}Y^1 = 0 \ . \end{aligned}$$

The exterior derivative of the function h results in

$$\mathrm{d}h\wedge\mathrm{d}Y^1 = \frac{\partial h}{\partial y^\alpha}\mathrm{d}y^\alpha\wedge\mathrm{d}Y^1 + \frac{\partial h}{\partial p_\alpha^{[J]}}\mathrm{d}p_\alpha^{[J]}\wedge\mathrm{d}Y^1$$

by construction. Consequently the domain condition is met iff

$$\begin{aligned} d_1\rfloor\mathrm{d}y^\alpha_{[J-1_1]} &= \frac{\partial h}{\partial p_\alpha^{[J]}} \\ d_1\rfloor\mathrm{d}p_\alpha^{[1_1]} &= -\frac{\partial h}{\partial y^\alpha} \\ d_1\rfloor\mathrm{d}p_\alpha^{[J]} &= 0 \ , \quad \#J > 1 \ . \end{aligned} \tag{9.1}$$

This representation states the equivalence of certain total derivatives and functions on the jet manifolds. Consequently we obtained the evolution information of interest.

The first equation of (9.1) represents simply

$$d_1\rfloor\mathrm{d}y^\alpha_{[J-1_1]} = \frac{\partial h}{\partial p_\alpha^{[J]}} = y^\alpha_{[J]} \ .$$

Roughly speaking all information about the evolution of the lumped parameter EL-system is contained in the equations of (9.1) that determine the quantities $d_1\rfloor\mathrm{d}p_\alpha^{[J]}$, $\#J = 0,\ldots,n$.

Having the evolutionary system representation (9.1) at ones disposal, it is easy to derive the total time derivative of certain functions f whose exterior derivative is given by

$$\mathrm{d}f = \frac{\partial f}{\partial Y^1}\mathrm{d}Y^1 + \frac{\partial f}{\partial y^\alpha_{[J-1_1]}}\mathrm{d}y^\alpha_{[J-1_1]} + \frac{\partial f}{\partial p_\alpha^{[1_1]}}\mathrm{d}p_\alpha^{[1_1]} + \frac{\partial f}{\partial p_\alpha^{[J]}}\mathrm{d}p_\alpha^{[J]}\ , \quad \#J > 1\ .$$

It is obvious that the time derivative of such functions along the solution of the system[1] is determined by

$$\begin{aligned} d_1\rfloor\mathrm{d}f &= \frac{\partial f}{\partial Y^1}d_1\rfloor\mathrm{d}Y^1 + \frac{\partial f}{\partial y^\alpha_{[J-1_1]}}d_1\rfloor\mathrm{d}y^\alpha_{[J-1_1]} + \frac{\partial f}{\partial p_\alpha^{[1_1]}}d_1\rfloor\mathrm{d}p_\alpha^{[1_1]} + \frac{\partial f}{\partial p_\alpha^{[J]}}d_1\rfloor\mathrm{d}p_\alpha^{[J]} \\ &= \frac{\partial f}{\partial Y^1} + \frac{\partial f}{\partial y^\alpha_{[J-1_1]}}\frac{\partial h}{\partial p_\alpha^{[J]}} - \frac{\partial f}{\partial p_\alpha^{[1_1]}}\frac{\partial h}{\partial y^\alpha}\ . \end{aligned}$$

It is remarkable that the previously introduced function $h \in C^\infty\left(J^{2n-1}\pi\right)$ is precisely of this class and owns a time derivative given by

$$\begin{aligned} \mathrm{L}_{d_1}(h) &= d_1\rfloor\mathrm{d}h = \frac{\partial h}{\partial Y^1} + \frac{\partial h}{\partial y^\alpha}d_1\rfloor\mathrm{d}y^\alpha + \frac{\partial h}{\partial p_\alpha^{[1_1]}}d_1\rfloor\mathrm{d}p_\alpha^{[1_1]} + \frac{\partial h}{\partial p_\alpha^{[J]}}d_1\rfloor\mathrm{d}p_\alpha^{[J]} \\ &= \frac{\partial h}{\partial Y^1}\ , \end{aligned}$$

which is nothing else, than its partial derivative with respect to time t. In the case of $\frac{\partial h}{\partial Y^1} = 0$ we see that the function h is invariant under the motion of the n^{th} order finite-dimensional Euler-Lagrange system.

Remark 9.2 *This result confirms Noether's theorem as the group of time translation becomes a symmetry group of the variational problem if* $\mathrm{L}_{\partial_1}(l) = 0$ *is met. (see, e.g., [Olver, 1986])*

9.2 The infinite-dimensional case

9.2.1 Systems with 1^{st} order Lagrangian

In the case of 1^{st} order Lagrangians the extended Cartan form coincides again with the ordinary Cartan form and is given by

$$c_{ext} = c = l\mathrm{dY} + p_\alpha^{[1_j]}\left(\mathrm{d}y^\alpha\partial_j\rfloor\mathrm{dY} - y^\alpha_{[1_j]}\mathrm{dY}\right)\ , \quad j = 1, \ldots, r\ .$$

Now we want to reformulate the domain condition

$$(-1)^r\left(D\rfloor\mathrm{d}c_{ext}\right)\wedge\mathrm{dY} = 0$$

and the corresponding boundary condition

$$(-1)^{r-1}\left(D_\partial\rfloor\left(\Psi^2\right)^* c_{ext}\right)\wedge\mathrm{d}\bar{\mathrm{Y}} = 0$$

[1]Here we assume the existence of a solution of the system.

in order to make the time evolution of the I-EL system visible. This could be obtained by choosing the following representation of the extended Cartan form

$$c_{ext} = l\mathrm{dY} + p_\alpha^{[1_1]}\left(\mathrm{d}y^\alpha \wedge \partial_1 \rfloor \mathrm{dY} - y^\alpha_{[1_1]}\mathrm{dY}\right) + p_\alpha^{[1_i]}\left(\mathrm{d}y^\alpha \wedge \partial_i \rfloor \mathrm{dY} - y^\alpha_{[1_i]}\mathrm{dY}\right) \ , \quad i = 2,\ldots,r \ ,$$

whereby the contact form corresponding to the time coordinate, i.e. $p_\alpha^{[1_1]}\left(\mathrm{d}y^\alpha \wedge \partial_1 \rfloor \mathrm{dY} - y^\alpha_{[1_1]}\mathrm{dY}\right)$ is separated. Consequently we are again able to introduce a function $h \in C^\infty\left(J^1\pi\right)$ such that

$$\begin{aligned} c_{ext} &= -\left(p_\alpha^{[1_1]}y^\alpha_{[1_1]} - l\right)\mathrm{dY} + p_\alpha^{[1_1]}\mathrm{d}y^\alpha \wedge \partial_1 \rfloor \mathrm{dY} + p_\alpha^{[1_i]}\left(\mathrm{d}y^\alpha \wedge \partial_i \rfloor \mathrm{dY} - y^\alpha_{[1_i]}\mathrm{dY}\right) \\ &= -h\mathrm{dY} + p_\alpha^{[1_1]}\mathrm{d}y^\alpha \wedge \partial_1 \rfloor \mathrm{dY} + p_\alpha^{[1_i]}\left(\mathrm{d}y^\alpha \wedge \partial_i \rfloor \mathrm{dY} - y^\alpha_{[1_i]}\mathrm{dY}\right) \ . \end{aligned}$$

Remark 9.3 *It is obvious that this representation "hides" again the contact form corresponding to the time coordinate.*

In order to derive the domain conditions we determine the exterior derivative

$$\begin{aligned} \mathrm{d}c_{ext} &= -\mathrm{d}h \wedge \mathrm{dY} + \mathrm{d}p_\alpha^{[1_1]} \wedge \mathrm{d}y^\alpha \wedge \partial_1 \rfloor \mathrm{dY} \\ &\quad + \mathrm{d}p_\alpha^{[1_i]} \wedge \left(\mathrm{d}y^\alpha \wedge \partial_i \rfloor \mathrm{dY} - y^\alpha_{[1_i]}\mathrm{dY}\right) - p_\alpha^{[1_i]}\mathrm{d}y^\alpha_{[1_i]} \wedge \mathrm{dY} \end{aligned}$$

and consequently we get

$$\begin{aligned} (-1)^r \left(D \rfloor \left(\mathrm{d}c_{ext}\right)\right) \wedge \mathrm{dY} &= -\mathrm{d}h \wedge \mathrm{dY} - p_\alpha^{[1_i]}\mathrm{d}y^\alpha_{[1_i]} \wedge \mathrm{dY} - \left(d_i \rfloor \mathrm{d}p_\alpha^{[1_i]}\right) \wedge \mathrm{d}y^\alpha \wedge \mathrm{dY} \\ &\quad - \left(d_1 \rfloor \mathrm{d}p_\alpha^{[1_1]}\right)\mathrm{d}y^\alpha \wedge \mathrm{dY} + \left(d_1 \rfloor \mathrm{d}y^\alpha\right)\mathrm{d}p_\alpha^{[1_1]} \wedge \mathrm{dY} \\ &= -\frac{\partial h}{\partial p_\alpha^{[1_1]}}\mathrm{d}p_\alpha^{[1_1]} \wedge \mathrm{dY} - \frac{\partial h}{\partial y^\alpha}\mathrm{d}y^\alpha \wedge \mathrm{dY} - \left(d_i \rfloor \mathrm{d}p_\alpha^{[1_i]}\right)\mathrm{d}y^\alpha \wedge \mathrm{dY} \\ &\quad - \left(d_1 \rfloor \mathrm{d}p_\alpha^{[1_1]}\right)\mathrm{d}y^\alpha \wedge \mathrm{dY} + \left(d_1 \rfloor \mathrm{d}y^\alpha\right)\mathrm{d}p_\alpha^{[1_1]} \wedge \mathrm{dY} = 0 \ . \end{aligned}$$

Thus the domain conditions are given by

$$\begin{aligned} d_1 \rfloor \mathrm{d}y^\alpha &= \frac{\partial h}{\partial p_\alpha^{[1_1]}} = \delta^\alpha(h) \\ d_1 \rfloor \mathrm{d}p_\alpha^{[1_1]} &= -\frac{\partial h}{\partial y^\alpha} - d_i \rfloor \mathrm{d}p_\alpha^{[1_i]} = -\left(\frac{\partial h}{\partial y^\alpha} - \mathrm{L}_{d_i}\frac{\partial h}{\partial y^\alpha_{[1_i]}}\right) = -\delta_\alpha(h) \ , \end{aligned} \tag{9.2}$$

in this representation. Here we have applied that

$$-\frac{\partial h}{\partial y^\alpha_{[1_i]}} = \frac{\partial l}{\partial y^\alpha_{[1_i]}} = p_\alpha^{[1_i]} \ , \quad i = 2,\ldots,r \ .$$

The boundary conditions follow from

$$(-1)^{r-1}\left(D_\partial \rfloor \left(p_\alpha^{[1_r]} \circ \Psi^2\right)\left(\mathrm{d}y^\alpha_{[0\ldots0;0]} \wedge \mathrm{d}\bar{\mathrm{Y}}\right)\right) \wedge \mathrm{d}\bar{\mathrm{Y}} = 0$$

to be given by

$$\left(p_\alpha^{[1_r]} \circ \Psi^2\right)\mathrm{d}y^\alpha_{[0\ldots0;0]} \wedge \mathrm{d}\bar{\mathrm{Y}} = 0$$

or similarly $\left(p_\alpha^{[1_r]} \circ \Psi^2\right) = 0$ for a free boundary.

This representation can now be used to derive the formal time derivative of a functional

$$\int_{\mathcal{D}_S} \left(j^1\sigma\right)^* \left(f\ \partial_1 \rfloor \mathrm{dY}\right)\ , \quad f \in C^\infty\left(J^1\pi\right)\ .$$

Additionally, we assume that the exterior derivative of f is given by

$$\mathrm{d}f = \frac{\partial f}{\partial Y^i}\mathrm{d}Y^i + \frac{\partial f}{\partial y^\alpha}\mathrm{d}y^a + \frac{\partial f}{\partial y^\alpha_{[1_i]}}\mathrm{d}y^\alpha_{[1_i]} + \frac{\partial f}{\partial p^{[1_1]}_\alpha}\mathrm{d}p^{[1_1]}_\alpha\ , \quad i = 2, \ldots, r\ . \tag{9.3}$$

Remark 9.4 *The spatial domain $\mathcal{D}_S$ is characterized by the fact that $Y^1 = t = \mathrm{const.}$ and consequently all form parts $\mathrm{d}Y^1 = 0$ vanish in the corresponding functional.*

Similar to the considerations presented in section 5.1 the simple approach

$$\begin{aligned}
\mathrm{L}_{\partial_1}\int_{\mathcal{D}_S}\left(j^1\sigma\right)^*\left(f\ \partial_1\rfloor\mathrm{dY}\right) &= \int_{\mathcal{D}_S}\left(j^2\sigma\right)^*\left(\mathrm{L}_{d_1}\left(f\ \partial_1\rfloor\mathrm{dY}\right)\right)\\
&= \int_{\mathcal{D}_S}\left(j^2\sigma\right)^*\left(d_1\rfloor\mathrm{d}f\wedge\partial_1\rfloor\mathrm{dY} + \mathrm{d}\underbrace{\left(d_1\rfloor f\ \partial_1\rfloor\mathrm{dY}\right)}_{=0}\right)\\
&= \int_{\mathcal{D}_S}\left(j^2\sigma\right)^*\left(d_1\rfloor\mathrm{d}f\wedge\mathrm{dY}\right)
\end{aligned}$$

does not supply a separation of domain and boundary part of the time derivative of the functional! Thus we extend the form $f\ \partial_1\rfloor\mathrm{dY}$ by elements of the contact ideal I_n restricted to domains with constant Y^1 coordinate in the following way

$$\begin{aligned}
&j^1\sigma^*\mathrm{L}_{d_1}\left(f\partial_1\rfloor\mathrm{dY} + \mathfrak{p}^{[1_i]}_\alpha\left(\mathrm{d}y^a - y^\alpha_{[1_i]}\mathrm{d}Y^i\right)\wedge\partial_i\rfloor\partial_1\rfloor\mathrm{dY}\right) =\\
&= j^1\sigma^*\left(d_1\rfloor\left(\mathrm{d}\left(f\partial_1\rfloor\mathrm{dY} + \mathfrak{p}^{[1_i]}_\alpha\left(\mathrm{d}y^a - y^\alpha_{[1_i]}\mathrm{d}Y^i\right)\wedge\partial_i\rfloor\partial_1\rfloor\mathrm{dY}\right)\right)\right.\\
&\quad\left. + \mathrm{d}\left(d_1\rfloor\left(f\partial_1\rfloor\mathrm{dY} + \mathfrak{p}^{[1_i]}_\alpha\left(\mathrm{d}y^a - y^\alpha_{[1_i]}\mathrm{d}Y^i\right)\wedge\partial_i\rfloor\partial_1\rfloor\mathrm{dY}\right)\right)\right)\\
&= j^1\sigma^*\left(d_1\rfloor\left(\mathrm{d}f\wedge\partial_1\rfloor\mathrm{dY} + \mathrm{d}\mathfrak{p}^{[1_i]}_\alpha\wedge\left(\mathrm{d}y^a - y^\alpha_{[1_i]}\mathrm{d}Y^i\right)\wedge\partial_i\rfloor\partial_1\rfloor\mathrm{dY} - \mathfrak{p}^{[1_i]}_\alpha\mathrm{d}y^\alpha_{[1_i]}\wedge\partial_1\rfloor\mathrm{dY}\right)\right.\\
&\quad\left. + \mathrm{d}\left(\mathfrak{p}^{[1_i]}_\alpha d_1\rfloor\mathrm{d}y^a\wedge\partial_i\rfloor\partial_1\rfloor\mathrm{dY}\right)\right)\\
&= j^2\sigma^*\left(d_1\rfloor\left(\mathrm{d}f\wedge\partial_1\rfloor\mathrm{dY} + \mathrm{d}\mathfrak{p}^{[1_i]}_\alpha\wedge\left(\mathrm{d}y^a - y^\alpha_{[1_i]}\mathrm{d}Y^i\right)\wedge\partial_i\rfloor\partial_1\rfloor\mathrm{dY} - \mathfrak{p}^{[1_i]}_\alpha\mathrm{d}y^\alpha_{[1_i]}\wedge\partial_1\rfloor\mathrm{dY}\right)\right.\\
&\quad\left. + \mathrm{d}\left(\mathfrak{p}^{[1_i]}_\alpha d_1\rfloor\mathrm{d}y^a\ \partial_i\rfloor\partial_1\rfloor\mathrm{dY}\right)\right)\ ,
\end{aligned}$$

where

$$i = 2, \ldots, r$$

is used. Additionally, we choose

$$\mathfrak{p}^{[1_i]}_\alpha = \frac{\partial f}{\partial y^\alpha_{[1_i]}}$$

and make use of $D_t = d_r\rfloor\cdots\rfloor d_2$ in the horizontal projection of the domain part

$$\begin{aligned}
&\left(j^2\sigma\right)^* D_t\rfloor d_1\rfloor\left(\mathrm{d}f\wedge\partial_1\rfloor\mathrm{dY} + \mathrm{d}\mathfrak{p}^{[1_i]}_\alpha\wedge\left(\mathrm{d}y^a - y^\alpha_{[1_i]}\mathrm{d}Y^i\right)\wedge\partial_i\rfloor\partial_1\rfloor\mathrm{dY}\right.\\
&\left. - \mathfrak{p}^{[1_i]}_\alpha\mathrm{d}y^\alpha_{[1_i]}\wedge\partial_1\rfloor\mathrm{dY}\right)\wedge\partial_1\rfloor\mathrm{dY} =\\
&\quad= \left(j^2\sigma\right)^*(-1)^{r-1}d_1\rfloor D_t\rfloor\left(\mathrm{d}f\wedge\partial_1\rfloor\mathrm{dY} + \mathrm{d}\mathfrak{p}^{[1_i]}_\alpha\wedge\left(\mathrm{d}y^a - y^\alpha_{[1_i]}\mathrm{d}Y^i\right)\wedge\partial_i\rfloor\partial_1\rfloor\mathrm{dY}\right.\\
&\quad\quad\left. - \mathfrak{p}^{[1_i]}_\alpha\mathrm{d}y^\alpha_{[1_i]}\wedge\partial_1\rfloor\mathrm{dY}\right)\wedge\partial_1\rfloor\mathrm{dY}\\
&\quad= \left(j^2\sigma\right)^* d_1\rfloor\left(\mathrm{d}f\wedge\partial_1\rfloor\mathrm{dY} - d_i\rfloor\mathrm{d}\mathfrak{p}^{[1_i]}_\alpha\wedge\mathrm{d}y^a\wedge\partial_1\rfloor\mathrm{dY} - \mathfrak{p}^{[1_i]}_\alpha\mathrm{d}y^\alpha_{[1_i]}\wedge\partial_1\rfloor\mathrm{dY}\right)\ .
\end{aligned}$$

This leads finally to

$$\int_{\mathcal{D}_S} \left(j^2\sigma\right)^* \left(\frac{\partial f}{\partial Y^1}\partial_1\rfloor \mathrm{dY} + \frac{\partial f}{\partial y^\alpha} d_1\rfloor \mathrm{d}y^\alpha \partial_1\rfloor \mathrm{dY} - d_i\rfloor \mathrm{d}\mathfrak{p}_\alpha^{[1_i]} d_1\rfloor \mathrm{d}y^a \partial_1\rfloor \mathrm{dY} + \frac{\partial f}{\partial p_\alpha^{[1_1]}} d_1\rfloor \mathrm{d}p_\alpha^{[1_1]}\partial_1\rfloor \mathrm{dY}\right)$$
$$= \int_{\mathcal{D}_S} \left(j^2\sigma\right)^* \left(\frac{\partial f}{\partial Y^1} + \left(\frac{\partial f}{\partial y^\alpha} - d_i\rfloor \mathrm{d}\frac{\partial f}{\partial y^\alpha_{[1_i]}}\right) d_1\rfloor \mathrm{d}y^a + \frac{\partial f}{\partial p_\alpha^{[1_1]}} d_1\rfloor \mathrm{d}p_\alpha^{[1_1]}\right)\partial_1\rfloor \mathrm{dY}\ .$$

Obviously we have obtained a structure of the functional that enables a straight forward derivation of the domain impact on the formal time derivative of the functional along the solution[2] of the EL system.

The resulting time derivative is given by

$$\mathrm{L}_{\partial_1} \int_{\mathcal{D}_S} \left(j^1\sigma\right)^* \left(f\ \partial_1\rfloor \mathrm{dY}\right) =$$
$$= \int_{\mathcal{D}_S} \left(j^2\sigma\right)^* \left(\frac{\partial f}{\partial Y^1} + \left(\frac{\partial f}{\partial y^\alpha} - d_i\rfloor \mathrm{d}\frac{\partial f}{\partial y^\alpha_{[1_i]}}\right)\delta^\alpha\left(h\right) - \frac{\partial f}{\partial p_\alpha^{[1_1]}}\delta_\alpha\left(h\right)\right)\partial_1\rfloor \mathrm{dY}$$
$$+ \int_{\partial\mathcal{D}_S} \left(j^1\bar{\sigma}\right)^* \left(\mathfrak{p}_\alpha^{[1_r]}\circ\Psi^2\right) d_1\rfloor \mathrm{d}y^a_{[0\ldots0;0]}\partial_1\rfloor \mathrm{d}\bar{\mathrm{Y}}$$

using the introduced domain and boundary condition representation of equation (9.2).

Remark 9.5 *One could also derive the time derivative of a functional of the form*

$$\int_{\mathcal{D}_S} \left(j^1\sigma\right)^* \left(f'\left(Y^i, y^\alpha, y^\alpha_{[1_i]}, y^\alpha_{[1_1]}\right)\ \partial_1\rfloor \mathrm{dY}\right)$$

if it is possible to determine $y^\alpha_{[1_1]}$ by means of $p_\alpha^{[1_1]}$, i.e.

$$y^\alpha_{[1_1]} = y^\alpha_{[1_1]}\left(Y^i, y^\alpha, y^\alpha_{[1_i]}, p_\alpha^{[1_1]}\right)\ .$$

Additionally it is remarkable that the function h meets precisely (9.3) and thus we can determine its exterior derivative to be given by

$$\mathrm{d}h = \frac{\partial h}{\partial Y^i}\mathrm{d}Y^i + \frac{\partial h}{\partial y^\alpha}\mathrm{d}y^a + \frac{\partial h}{\partial y^\alpha_{[1_i]}}\mathrm{d}y^\alpha_{[1_i]} + \frac{\partial h}{\partial p_\alpha^{[1_1]}}\mathrm{d}p_\alpha^{[1_1]}\ .$$

Consequently we are able to state

$$\mathrm{L}_{\partial_1} \int_{\mathcal{D}_S} \left(j^1\sigma\right)^* \left(h\ \partial_1\rfloor \mathrm{dY}\right) =$$
$$= \int_{\mathcal{D}_S} \left(j^2\sigma\right)^* \left(\frac{\partial h}{\partial Y^1} + \left(\frac{\partial h}{\partial y^\alpha} - d_i\rfloor \mathrm{d}\frac{\partial h}{\partial y^\alpha_{[1_i]}}\right) d_1\rfloor \mathrm{d}y^a + \frac{\partial h}{\partial p_\alpha^{[1_1]}} d_1\rfloor \mathrm{d}p_\alpha^{[1_1]}\right)\partial_1\rfloor \mathrm{dY}$$
$$+ \int_{\partial\mathcal{D}_S} \left(j^1\bar{\sigma}\right)^* \left(\mathfrak{p}_\alpha^{[1_r]}\circ\Psi^2\right) d_1\rfloor \mathrm{d}y^a\partial_1\rfloor \mathrm{d}\bar{\mathrm{Y}}$$
$$= \int_{\mathcal{D}_S} \left(j^2\sigma\right)^* \left(\frac{\partial h}{\partial Y^1} + \left(\frac{\partial h}{\partial y^\alpha} - d_i\rfloor \mathrm{d}\frac{\partial h}{\partial y^\alpha_{[1_i]}}\right)\frac{\partial h}{\partial p_\alpha^{[1_1]}} - \frac{\partial h}{\partial p_\alpha^{[1_1]}}\left(\frac{\partial h}{\partial y^\alpha} - \mathrm{L}_{d_i}\frac{\partial h}{\partial y^\alpha_{[1_i]}}\right)\right)\partial_1\rfloor \mathrm{dY}$$
$$+ \int_{\partial\mathcal{D}_S} \left(j^1\bar{\sigma}\right)^* \left(\mathfrak{p}_\alpha^{[1_r]}\circ\Psi^2\right)\ d_1\rfloor \mathrm{d}y^a\ \partial_r\rfloor\partial_1\rfloor \mathrm{d}\bar{\mathrm{Y}}\ .$$

[2] Here we assume again the existence of a solution of the infinite-dimensional system.

Using the relation $\mathfrak{p}_\alpha^{[1_r]} \circ \Psi^2 = -p_\alpha^{[1_r]} \circ \Psi^2$ and taking the domain and boundary conditions into account, we obtain

$$\mathrm{L}_{\partial_1} \int_{\mathcal{D}_S} \left(j^1\sigma\right)^* \left(h\ \partial_1 \rfloor \mathrm{dY}\right) = \int_{\mathcal{D}_S} \left(j^2\sigma\right)^* \frac{\partial h}{\partial Y^1} \partial_1 \rfloor \mathrm{dY} \ .$$

Consequently the functional $\int_{\mathcal{D}_S} j^1\sigma^* \left(h\ \partial_1 \rfloor \mathrm{dY}\right)$ is invariant under the motion of the I-EL system, if

$$\frac{\partial h}{\partial Y^1} = 0$$

is met.

Remark 9.6 *This result confirms again Noether's theorem as the group of time translation becomes a symmetry group of the variational problem if* $\mathrm{L}_{\partial_1}(l) = 0$ *is met. (see, e.g., [Olver, 1986])*

9.2.2 Systems with n^{th} order Lagrangian

In the case of n^{th} order Lagrangians the extended Cartan form is given by

$$\begin{aligned} c_{ext} \quad &= \quad l\mathrm{dY} + p_\alpha^{[J-1_j,J]} \left(\mathrm{d}y^\alpha_{[J-1_j]} \wedge \partial_j \rfloor \mathrm{dY} - y^\alpha_{[J]}\mathrm{dY}\right) \\ & \quad\quad + \mathrm{d}\left(p_\alpha^{\left[\bar{J}-1_w,\bar{J};j_r\right]} \left(\mathrm{d}y^\alpha_{\left[\bar{J}-1_w;j_r\right]} \wedge \partial_w \rfloor \mathrm{d\bar{Y}} - y^\alpha_{\left[\bar{J};j_r\right]}\mathrm{d\bar{Y}}\right)\right) \end{aligned}$$

where

$$j = 1, \ldots, r \ , \quad w = 1, \ldots, r-1 \ , \quad \#J, \#\bar{J} = 1, \ldots, n \ , \ \mathrm{d\bar{Y}} = \partial_{[1_r]} \rfloor \mathrm{dY}.$$

The domain condition

$$(-1)^r \left(D \rfloor \mathrm{d}c_{ext}\right) \wedge \mathrm{dY} = 0$$

and the corresponding boundary condition

$$(-1)^{r-1} \left(D_\partial \rfloor \left(\Psi^{2n-1}\right)^* c_{ext}\right) \wedge \mathrm{d\bar{Y}} = 0$$

have to be reformulated in order to obtain a pleasant evolution information. As already indicated by the previous results, we want to investigate the time evolution of functionals formulated on the spatial domain $\mathcal{D}_S$.

The construction of the extended Cartan form is not a unique procedure. Until now we have only used the partial ordering that corresponds to the coordinate Y^r or rather to the spatial boundary of the domain $\mathcal{D}$. The physically motivated assignment of Y^1 to belong to the system time t enables now the introduction of a second ordering.

Definition 9.7 (partial ordering 2) *In the local coordinates* $\left(Y^j, y^\beta, y^\beta_{[J]}\right)$ *we define the following partial ordering. Let* $J_a = j_{a1} \ldots j_{ar}$ *and* $J_b = j_{b1} \ldots j_{br}$ *be two multi-indices. We say* $J_a >_{part2} J_b$ *if in the difference* $J_a - J_b$ *the* 1^{st} *entry is positive.*

Consequently we confine ourselves in the determination of the contact forms in the construction of the extended Cartan form to the set of contact forms $\omega^\alpha_{[J]}$ resp. $\omega^\alpha_{\partial[J]}$ with the largest multi-indices with respect to the partial multi-index ordering $>_{part2}$. From this set of indices we determine the subset with the smallest indexes with respect to $>_{part}$ and use this in the construction of the extended Cartan form. This concatenation of orderings is compatible to the general index ordering and leads to a minimal amount of equations.

The concatenated orderings lead to the following representation of the extended Cartan form

$$\begin{aligned} c_{ext} &= l\mathrm{dY} + p_\alpha^{[J_1-1_1,J_1]}\left(\mathrm{d}y^\alpha_{[J_1-1_1]}\wedge\partial_1\rfloor\mathrm{dY} - y^\alpha_{[J_1]}\mathrm{dY}\right) \\ &\quad +p_\alpha^{[J-1_i,J]}\left(\mathrm{d}y^\alpha_{[J-1_i]}\wedge\partial_i\rfloor\mathrm{dY} - y^\alpha_{[J]}\mathrm{dY}\right) + \\ &\quad \mathrm{d}\left(p_\alpha^{\left[\bar J_1-1_1,\bar J_1;j_r\right]}\left(\mathrm{d}y^\alpha_{\left[\bar J_1-1_1;j_r\right]}\wedge\partial_1\rfloor\mathrm{d}\bar{\mathrm{Y}} - y^\alpha_{\left[\bar J_1;j_r\right]}\mathrm{d}\bar{\mathrm{Y}}\right)\right. \\ &\quad \left. + p_\alpha^{\left[\bar J-1_l,\bar J;j_r\right]}\left(\mathrm{d}y^\alpha_{\left[\bar J-1_l,j_r\right]}\wedge\partial_l\rfloor\mathrm{d}\bar{\mathrm{Y}} - y^\alpha_{\left[\bar J;j_r\right]}\mathrm{d}\bar{\mathrm{Y}}\right)\right) \end{aligned}$$

with

$$i = 2,\ldots,r\ , \quad l = 2,\ldots,r-1\ , \quad [J_1] = [j_1 0\ldots 0]\ , \quad j_1 = 0,\ldots,n-1$$

and

$$\left[\bar J_1; j_r\right] = [j_1 0\ldots 0; j_r]\ , \quad j_1 = 0,\ldots,n-1\ \#J = \#J = 1,\ldots,n.$$

Here we have again separated all elements of the contact ideals that incorporate any $\partial_1\rfloor\mathrm{dY}$ resp. $\partial_1\rfloor\mathrm{d}\bar{\mathrm{Y}}$ entry. The reason for this reformulation is that the domain resp. boundary condition will cause a total derivative d_1 to operate on such entries and thus we obtain the representation of interest.

Now we are able to introduce again the function $h = p_\alpha^{[J_1,J_1+1_1]}y^\alpha_{[J_1]} - l \in C^\infty\left(J^{2n-1}\mathcal{E}\right)$ to the domain condition and obtain

$$\begin{aligned} c_{ext} &= -\left(p_\alpha^{[J_1,J_1+1_1]}y^\alpha_{[J_1]} - l\right)\mathrm{dY} + p_\alpha^{[J_1-1_1,J_1]}\mathrm{d}y^\alpha_{[J_1-1_1]}\wedge\partial_1\rfloor\mathrm{dY} \\ &\quad +p_\alpha^{[J-1_i,J]}\left(\mathrm{d}y^\alpha_{[J-1_i]}\wedge\partial_i\rfloor\mathrm{dY} - y^\alpha_{[J]}\mathrm{dY}\right) \\ &\quad +\mathrm{d}\left(p_\alpha^{\left[\bar J_1-1_1,\bar J_1;j_r\right]}\left(\mathrm{d}y^\alpha_{\left[\bar J_1-1_1;j_r\right]}\wedge\partial_1\rfloor\mathrm{d}\bar{\mathrm{Y}} - y^\alpha_{\left[\bar J_1;j_r\right]}\mathrm{d}\bar{\mathrm{Y}}\right)\right. \\ &\quad \left. + p_\alpha^{\left[\bar J-1_l,\bar J;j_r\right]}\left(\mathrm{d}y^\alpha_{\left[\bar J-1_l;j_r\right]}\wedge\partial_l\rfloor\mathrm{d}\bar{\mathrm{Y}} - y^\alpha_{\left[\bar J;j_r\right]}\mathrm{d}\bar{\mathrm{Y}}\right)\right) \\ &= -h\mathrm{dY} + p_\alpha^{[J_1-1_1,J_1]}\mathrm{d}y^\alpha_{[J_1-1_1]}\wedge\partial_1\rfloor\mathrm{dY} + p_\alpha^{[J-1_i,J]}\left(\mathrm{d}y^\alpha_{[J-1_i]}\wedge\partial_i\rfloor\mathrm{dY} - y^\alpha_{[J]}\mathrm{dY}\right) \\ &\quad +\mathrm{d}\left(p_\alpha^{\left[\bar J_1-1_1,\bar J_1;j_r\right]}\left(\mathrm{d}y^\alpha_{\left[\bar J_1-1_1;j_r\right]}\wedge\partial_1\rfloor\mathrm{d}\bar{\mathrm{Y}} - y^\alpha_{\left[\bar J_1;j_r\right]}\mathrm{d}\bar{\mathrm{Y}}\right)\right. \\ &\quad \left. + p_\alpha^{\left[\bar J-1_l,\bar J;j_r\right]}\left(\mathrm{d}y^\alpha_{\left[\bar J-1_l;j_r\right]}\wedge\partial_l\rfloor\mathrm{d}\bar{\mathrm{Y}} - y^\alpha_{\left[\bar J;j_r\right]}\mathrm{d}\bar{\mathrm{Y}}\right)\right)\ . \end{aligned}$$

On the boundary we make use of the function $\bar{h}_{j_r} = p_\alpha^{\left[\bar{J}_1 - 1_1, \bar{J}_1; j_r\right]} y^\alpha_{\left[\bar{J}_1; j_r\right]} \in C^\infty\left(J^{2n-1}\bar{\mathcal{E}}\right)$ and get

$$
\begin{aligned}
\left(\Psi^{2n-1}\right)^* c_{ext} &= \left(p_\alpha^{[J-1_r,J]} \circ \Psi^{2n-1}\right) \mathrm{d}y^\alpha_{\left[\bar{J};j_r-1_r\right]} \wedge \mathrm{d}\bar{\mathrm{Y}} \\
&\quad + \mathrm{d}\left(p_\alpha^{\left[\bar{J}_1-1_1,\bar{J}_1;j_r\right]}\left(\mathrm{d}y^\alpha_{\left[\bar{J}_1-1_1;j_r\right]} \wedge \partial_1\rfloor \mathrm{d}\bar{\mathrm{Y}} - y^\alpha_{\left[\bar{J}_1;j_r\right]}\mathrm{d}\bar{\mathrm{Y}}\right)\right) \\
&\quad + \mathrm{d}\left(p_\alpha^{\left[\bar{J}-1_l,\bar{J};j_r\right]}\left(\mathrm{d}y^\alpha_{\left[\bar{J}-1_l;j_r\right]} \wedge \partial_l\rfloor \mathrm{d}\bar{\mathrm{Y}} - y^\alpha_{\left[\bar{J};j_r\right]}\mathrm{d}\bar{\mathrm{Y}}\right)\right) \\
&= -\mathrm{d}\left(p_\alpha^{\left[\bar{J}_1-1_1,\bar{J}_1;j_r\right]} y^\alpha_{\left[\bar{J}_1;j_r\right]}\right) \wedge \mathrm{d}\bar{\mathrm{Y}} + \left(p_\alpha^{[J-1_r,J]} \circ \Psi^{2n-1}\right) \mathrm{d}y^\alpha_{\left[\bar{J};j_r-1_r\right]} \wedge \mathrm{d}\bar{\mathrm{Y}} \\
&\quad + \mathrm{d}p_\alpha^{\left[\bar{J}_1-1_1,\bar{J}_1;j_r\right]} \wedge \mathrm{d}y^\alpha_{\left[\bar{J}_1-1_1;j_r\right]} \wedge \partial_1\rfloor \mathrm{d}\bar{\mathrm{Y}} \\
&\quad + \mathrm{d}\left(p_\alpha^{\left[\bar{J}-1_l,\bar{J};j_r\right]}\left(\mathrm{d}y^\alpha_{\left[\bar{J}-1_l;j_r\right]} \wedge \partial_l\rfloor \mathrm{d}\bar{\mathrm{Y}} - y^\alpha_{\left[\bar{J};j_r\right]}\mathrm{d}\bar{\mathrm{Y}}\right)\right) \\
&= -\mathrm{d}\bar{h}_{j_r} \wedge \mathrm{d}\bar{\mathrm{Y}} + \left(p_\alpha^{[J-1_r,J]} \circ \Psi^{2n-1}\right) \mathrm{d}y^\alpha_{\left[\bar{J};j_r-1_r\right]} \wedge \mathrm{d}\bar{\mathrm{Y}} \\
&\quad + \mathrm{d}p_\alpha^{\left[\bar{J}_1-1_1,\bar{J}_1;j_r\right]} \wedge \mathrm{d}y^\alpha_{\left[\bar{J}_1-1_1;j_r\right]} \wedge \partial_1\rfloor \mathrm{d}\bar{\mathrm{Y}} \\
&\quad + \mathrm{d}\left(p_\alpha^{\left[\bar{J}-1_l,\bar{J};j_r\right]}\left(\mathrm{d}y^\alpha_{\left[\bar{J}-1_l;j_r\right]} \wedge \partial_l\rfloor \mathrm{d}\bar{\mathrm{Y}} - y^\alpha_{\left[\bar{J};j_r\right]}\mathrm{d}\bar{\mathrm{Y}}\right)\right) \ .
\end{aligned}
$$

In order to derive the domain conditions, we determine the exterior derivative of the introduced extended Cartan form representation

$$
\begin{aligned}
\mathrm{d}c_{ext} &= -\mathrm{d}h \wedge \mathrm{dY} - p_\alpha^{[J-1_i,J]}\mathrm{d}y^\alpha_{[J]} \wedge \mathrm{dY} + \mathrm{d}p_\alpha^{[J-1_i,J]} \wedge \left(\mathrm{d}y^\alpha_{[J-1_i]} \wedge \partial_i\rfloor \mathrm{dY} - y^\alpha_{[J]}\mathrm{dY}\right) \\
&\quad + \mathrm{d}p_\alpha^{[J_1-1_1,J_1]} \wedge \mathrm{d}y^\alpha_{[J_1-1_1]} \wedge \partial_1\rfloor \mathrm{dY} \ .
\end{aligned}
$$

Consequently we get

$$
\begin{aligned}
&(-1)^r\left(D\rfloor\left(\mathrm{d}c_{ext}\right)\right) \wedge \mathrm{dY} = \\
&= -\mathrm{d}h \wedge \mathrm{dY} - p_\alpha^{[J-1_i,J]}\mathrm{d}y^\alpha_{[J]} \wedge \mathrm{dY} - \left(d_i\rfloor \mathrm{d}p_\alpha^{[J-1_i,J]}\right) \wedge \mathrm{d}y^\alpha_{[J-1_i]} \wedge \mathrm{dY} \\
&\quad - \left(d_1\rfloor \mathrm{d}p_\alpha^{[J_1-1_1,J_1]}\right)\mathrm{d}y^\alpha_{[J_1-1_1]} \wedge \mathrm{dY} + \left(d_1\rfloor \mathrm{d}y^\alpha_{[J_1-1_1]}\right)\mathrm{d}p_\alpha^{[J_1-1_1,J_1]} \wedge \mathrm{dY} \\
&= -\frac{\partial h}{\partial p_\alpha^{[J_1-1_1,J_1]}}\mathrm{d}p_\alpha^{[J_1-1_1,J_1]} \wedge \mathrm{dY} - \frac{\partial h}{\partial y^\alpha_{[J_1-1_1]}}\mathrm{d}y^\alpha_{[J_1-1_1]} \wedge \mathrm{dY} - \left(d_i\rfloor \mathrm{d}p_\alpha^{[J-1_i,J]}\right)\mathrm{d}y^\alpha_{[J-1_i]} \wedge \mathrm{dY} \\
&\quad - \left(d_1\rfloor \mathrm{d}p_\alpha^{[J_1-1_1,J_1]}\right)\mathrm{d}y^\alpha_{[J_1-1_1]} \wedge \mathrm{dY} + \left(d_1\rfloor \mathrm{d}y^\alpha_{[J_1-1_1]}\right)\mathrm{d}p_\alpha^{[J_1-1_1,J_1]} \wedge \mathrm{dY} = 0 \ .
\end{aligned}
$$

Here we have applied that

$$
-\frac{\partial h}{\partial y^\alpha_{[J]}}\mathrm{d}y^\alpha_{[J-1_i]} = \frac{\partial l}{\partial y^\alpha_{[J]}}\mathrm{d}y^\alpha_{[J-1_i]} \ , \quad J \neq J_1 \ , \quad \#J = 1, \ldots, n \ .
$$

Thus the domain conditions are given by

$$
\begin{aligned}
d_1\rfloor \mathrm{d}y^\alpha_{[J_1-1_1]} &= \frac{\partial h}{\partial p_\alpha^{[J_1-1_1,J_1]}} \ , \quad \#J_1 = 1, \ldots, n \\
d_1\rfloor \mathrm{d}p_\alpha^{[0,1_1]} &= -\frac{\partial h}{\partial y^\alpha} - d_i\rfloor \mathrm{d}p_\alpha^{[0,1_i]} = -\delta_\alpha(h) \ , \quad i = 2, \ldots, r \\
d_1\rfloor \mathrm{d}p_\alpha^{[J_1-1_1,J_1]} &= 0 \ , \quad \#J_1 = 2, \ldots, n \ .
\end{aligned}
$$

The boundary conditions follow from

$$\begin{aligned}
&(-1)^{r-1}\left(D_\partial\rfloor\left(\Psi^{2n-1}\right)^* c_{ext}\right)\wedge \mathrm{d}\bar{\mathrm{Y}} = \\
= \;& (-1)^{r-1}\Big(D_\partial\rfloor\Big(-\mathrm{d}\bar{h}_{j_r}\wedge \mathrm{d}\bar{\mathrm{Y}} + \left(p_\alpha^{[J-1_r,J]}\circ\Psi^{2n-1}\right)\mathrm{d}y^\alpha_{\left[\bar{J};j_r-1_r\right]}\wedge\mathrm{d}\bar{\mathrm{Y}} \\
& +\mathrm{d}p_\alpha^{\left[\bar{J}_1-1_1,\bar{J}_1;j_r\right]}\wedge \mathrm{d}y^\alpha_{\left[\bar{J}_1-1_1;j_r\right]}\wedge\partial_1\rfloor\mathrm{d}\bar{\mathrm{Y}} \\
& +\mathrm{d}\left(p_\alpha^{\left[\bar{J}-1_l,\bar{J};j_r\right]}\left(\mathrm{d}y^\alpha_{\left[\bar{J}-1_l;j_r\right]}\wedge\partial_l\rfloor\mathrm{d}\bar{\mathrm{Y}} - y^\alpha_{\left[\bar{J};j_r\right]}\mathrm{d}\bar{\mathrm{Y}}\right)\right)\Big)\Big)\wedge\mathrm{d}\bar{\mathrm{Y}} \\
= \;& -\mathrm{d}\bar{h}_{j_r}\wedge\mathrm{d}\bar{\mathrm{Y}} + \left(p_\alpha^{[J-1_r,J]}\circ\Psi^{2n-1}\right)\mathrm{d}y^\alpha_{\left[\bar{J};j_r-1_r\right]}\wedge\mathrm{d}\bar{\mathrm{Y}} \\
& -d_1\rfloor\mathrm{d}p_\alpha^{\left[\bar{J}_1-1_1,\bar{J}_1;j_r\right]}\;\mathrm{d}y^\alpha_{\left[\bar{J}_1-1_1;j_r\right]}\wedge\mathrm{d}\bar{\mathrm{Y}} + d_1\rfloor\mathrm{d}y^\alpha_{\left[\bar{J}_1-1_1;j_r\right]}\;\mathrm{d}p_\alpha^{\left[\bar{J}_1-1_1,\bar{J}_1;j_r\right]}\wedge\mathrm{d}\bar{\mathrm{Y}} \\
& +d_l\rfloor\mathrm{d}p_\alpha^{\left[J_{1\partial},\bar{J}+1_l;j_r\right]}\mathrm{d}y^\alpha_{\left[\bar{J}-1_l;j_r\right]}\wedge\mathrm{d}\bar{\mathrm{Y}} - p_\alpha^{\left[\bar{J}-1_l,\bar{J};j_r\right]}\mathrm{d}y^\alpha_{\left[\bar{J};j_r\right]}\wedge\mathrm{d}\bar{\mathrm{Y}} \\
= \;& -\mathrm{d}\bar{h}_{j_r}\wedge\mathrm{d}\bar{\mathrm{Y}} + p_\alpha^{[J_1+J_r,J_1+J_r+1_r]}\mathrm{d}y^\alpha_{\left[\bar{J}_1;j_r\right]}\wedge\mathrm{d}\bar{\mathrm{Y}} \\
& -d_1\rfloor\mathrm{d}p_\alpha^{\left[\bar{J}_1-1_1,\bar{J}_1;j_r\right]}\;\mathrm{d}y^\alpha_{\left[\bar{J}_1-1_1;j_r\right]}\wedge\mathrm{d}\bar{\mathrm{Y}} + d_1\rfloor\mathrm{d}y^\alpha_{\left[\bar{J}_1-1_1;j_r\right]}\;\mathrm{d}p_\alpha^{\left[\bar{J}_1-1_1,\bar{J}_1;j_r\right]}\wedge\mathrm{d}\bar{\mathrm{Y}} \\
& +d_l\rfloor\mathrm{d}p_\alpha^{\left[\bar{J}_1-1_1,\bar{J}-1_1+1_l;j_r\right]}\mathrm{d}y^\alpha_{\left[\bar{J}_1-1_1,j_r\right]}\wedge\mathrm{d}\bar{\mathrm{Y}} \\
= \;& 0\,,
\end{aligned}$$

where we have already used the construction rule of the functions $p_\alpha^{\left[\bar{J}-1_l,\bar{J};j_r\right]}$. By means of these functions all $p_\alpha^{[J-1_r,J]}\mathrm{d}y^\alpha_{[J-1_r]}\wedge\mathrm{d}\bar{\mathrm{Y}}$ terms, where $[J-1_r]\neq\left[\bar{J}_1;j_r\right]$, are cancelled out. The definition of $\bar{h}_{j_r}$ leads to an exterior derivative of the form

$$\mathrm{d}\bar{h}_{j_r} = p_\alpha^{\left[\bar{J}_1-1_1,\bar{J}_1;j_r\right]}\mathrm{d}y^\alpha_{\left[\bar{J}_1;j_r\right]} + y^\alpha_{\left[\bar{J}_1;j_r\right]}\mathrm{d}p_\alpha^{\left[\bar{J}_1-1_1,\bar{J}_1;j_r\right]}$$

and finally we obtain the boundary conditions

$$\begin{aligned}
d_1\rfloor\mathrm{d}y^\alpha_{\left[\bar{J}_1-1_1,j_r\right]} &= \frac{\partial}{\partial p_\alpha^{\left[\bar{J}_1-1_1,\bar{J}_1;j_r\right]}}\rfloor\mathrm{d}\bar{h}_{j_r} = y^\alpha_{\left[\bar{J}_1;j_r\right]} \\
d_1\rfloor\mathrm{d}p_\alpha^{\left[\bar{J}_1-1_1,\bar{J}_1;j_r\right]} &= -\frac{\partial}{\partial y^\alpha_{\left[\bar{J}_1-1_1,j_r\right]}}\rfloor\mathrm{d}\bar{h}_{j_r} + d_l\rfloor\mathrm{d}p_\alpha^{\left[\bar{J}_1-1_1,\bar{J}_1-1_1+1_l;j_r\right]} + p_\alpha^{[J_1+J_r,J_1+J_r+1_r]}\,,
\end{aligned}$$

where $J_r = [0\ldots 0j_r]$ is used.

In this n^{th} order Lagrangian case, we will not try to derive the time derivative of an arbitrary functional. Here we will focus only on the time derivative of the functional

$$\mathrm{L}_{\partial_1}\left(\int_{\mathcal{D}_S}(j^n\sigma)^*\left(h\;\partial_1\rfloor\mathrm{dY} + \mathrm{d}\left(\bar{h}_{j_r}\;\partial_1\rfloor\mathrm{d}\bar{\mathrm{Y}}\right)\right)\right)\,.$$

The application of Cartan's idea using elements of the contact ideal $I_n, I_{\partial n}$ restricted to $\mathcal{D}_S$

leads to

$$\begin{aligned}
&\left(j^{2n}\sigma\right)^* \mathrm{L}_{d_1} \left(h\partial_1\rfloor \mathrm{dY} + \mathfrak{p}_\alpha^{[J-1_i,J]} \left(\mathrm{d}y^a_{[J-1_i]} \wedge \partial_i\rfloor\partial_1\rfloor\mathrm{dY} - y^\alpha_{[J]}\partial_1\rfloor\mathrm{dY}\right)\right.\\
&+\mathrm{d}\left(\bar{h}_{j_r}\ \partial_1\rfloor \mathrm{d}\bar{\mathrm{Y}} + \mathfrak{p}_\alpha^{\left[\bar{J}-1_l,\bar{J};j_r\right]} \left(\mathrm{d}y^a_{\left[\bar{J}-1_l;j_r\right]}\partial_l\rfloor\mathrm{d}\bar{\mathrm{Y}} - y^\alpha_{\left[\bar{J};j_r\right]}\mathrm{d}\bar{\mathrm{Y}}\right)\right)\Bigg)\\
= \;&\left(j^{2n}\sigma\right)^* d_1\rfloor \mathrm{d}\left(h\partial_1\rfloor\mathrm{dY} + \mathfrak{p}_\alpha^{[J-1_i,J]}\left(\mathrm{d}y^a_{[J-1_i]} \wedge \partial_i\rfloor\partial_1\rfloor\mathrm{dY} - y^\alpha_{[J]}\partial_1\rfloor\mathrm{dY}\right)\right)\\
&+\mathrm{d}\left(d_1\rfloor \left(h\partial_1\rfloor\mathrm{dY} + \mathfrak{p}_\alpha^{[J-1_i,J]}\left(\mathrm{d}y^a_{[J-1_i]} \wedge \partial_i\rfloor\partial_1\rfloor\mathrm{dY} - y^\alpha_{[J]}\partial_1\rfloor\mathrm{dY}\right)\right.\right.\\
&+\ \mathrm{d}\left(\bar{h}_{j_r}\ \partial_1\rfloor\mathrm{d}\bar{\mathrm{Y}} + \mathfrak{p}_\alpha^{\left[\bar{J}-1_l,\bar{J};j_r\right]}\left(\mathrm{d}y^a_{\left[\bar{J}-1_l;j_r\right]}\partial_l\rfloor\partial_1\rfloor\mathrm{d}\bar{\mathrm{Y}} - y^\alpha_{\left[\bar{J};j_r\right]}\partial_1\rfloor\mathrm{d}\bar{\mathrm{Y}}\right)\right)\Bigg)\Bigg)\ .
\end{aligned}$$

Consequently we get on the domain $\mathcal{D}_S$

$$\begin{aligned}
&d_1\rfloor D_t\rfloor\left(\mathrm{d}h\wedge\partial_1\rfloor\mathrm{dY} + \mathrm{d}\mathfrak{p}_\alpha^{[J-1_i,J]} \wedge \left(\mathrm{d}y^a_{[J-1_i]}\wedge\partial_i\rfloor\partial_1\rfloor\mathrm{dY} - y^\alpha_{[J]}\partial_1\rfloor\mathrm{dY}\right)\right.\\
&\left. - \mathfrak{p}_\alpha^{[J-1_i,J]}\mathrm{d}y^\alpha_{[J]}\wedge\partial_1\rfloor\mathrm{dY}\right)\wedge\partial_1\rfloor\mathrm{dY}\\
= \;& d_1\rfloor\left(\mathrm{d}h\wedge\partial_1\rfloor\mathrm{dY} - d_i\rfloor\mathrm{d}\mathfrak{p}_\alpha^{[J-1_i,J]}\wedge\mathrm{d}y^a_{[J-1_i]}\wedge\partial_1\rfloor\mathrm{dY} - \mathfrak{p}_\alpha^{[J-1_i,J]}\mathrm{d}y^\alpha_{[J]}\wedge\partial_1\rfloor\mathrm{dY}\right)\\
= \;& \left(\frac{\partial h}{\partial Y^1} + \frac{\partial h}{\partial y^\alpha}d_1\rfloor\mathrm{d}y^\alpha + \frac{\partial h}{\partial p_\alpha^{[0,1_1]}}d_1\rfloor\mathrm{d}p_\alpha^{[0,1_1]} - d_i\rfloor\mathrm{d}\mathfrak{p}_\alpha^{[0,1_i]}\ d_1\rfloor\mathrm{d}y^a\right)\wedge\partial_1\rfloor\mathrm{dY}\\
= \;& \frac{\partial h}{\partial Y^1} + \left(\frac{\partial h}{\partial y^\alpha} - d_i\rfloor\mathrm{d}\mathfrak{p}_\alpha^{[0,1_i]}\right)d_1\rfloor\mathrm{d}y^\alpha\wedge\partial_1\rfloor\mathrm{dY} + \frac{\partial h}{\partial p_\alpha^{[0,1_1]}}d_1\rfloor\mathrm{d}p_\alpha^{[0,1_1]}\wedge\partial_1\rfloor\mathrm{dY}\\
= \;& \frac{\partial h}{\partial Y^1}\ .
\end{aligned}$$

Here the functions $\mathfrak{p}_\alpha^{[J-1_i,J]}$ are determined in such a way that all $\mathrm{d}y^\alpha_{[J]}\wedge\partial_1\rfloor\mathrm{dY}$ entries with $\#J > 1$ are cancelled out. Due to the introduction of the partial ordering $>_{part2}$ in the determination of $p_\alpha^{[J-1_i,J]}$ and the fact that

$$-\frac{\partial h}{\partial y^\alpha_{[J]}}\mathrm{d}y^\alpha_{[J-1_i]} = \frac{\partial l}{\partial y^\alpha_{[J]}}\mathrm{d}y^\alpha_{[J-1_i]}\ , \quad J\neq J_1\ , \quad \#J = 1,\ldots,n\ ,$$

it is guarantied that

$$d_1\rfloor\mathrm{d}p_\alpha^{[0,1_1]} = -\frac{\partial h}{\partial y^\alpha} - d_i\rfloor\mathrm{d}p_\alpha^{[0,1_i]} = -\left(\frac{\partial h}{\partial y^\alpha} - d_i\rfloor\mathrm{d}\mathfrak{p}_\alpha^{[0,1_i]}\right)\ .$$

This identity was used in the simplification applied in the last line of the latter equation.

Additionally, we get on the boundary $\partial\mathcal{D}_S$

$$
\begin{aligned}
& d_1\rfloor D_{t\partial}\rfloor \left(\Psi^{2n-1}\right)^* \left(h\partial_1\rfloor \mathrm{dY} + \mathfrak{p}_\alpha^{[J-1_i,J]}\left(\mathrm{d}y^a_{[J-1_i]}\wedge\partial_i\rfloor\partial_1\rfloor\mathrm{dY} - y^\alpha_{[J]}\partial_1\rfloor\mathrm{dY}\right)\right.\\
& \left.+\,\mathrm{d}\left(\bar h_{j_r}\wedge\partial_1\rfloor\mathrm{d\bar Y} + \mathfrak{p}_\alpha^{\left[\bar J-1_l,\bar J;j_r\right]}\left(\mathrm{d}y^a_{\left[\bar J-1_l;j_r\right]}\partial_l\rfloor\partial_1\rfloor\mathrm{d\bar Y} - y^\alpha_{\left[\bar J;j_r\right]}\partial_1\rfloor\mathrm{d\bar Y}\right)\right)\right)\wedge\partial_1\rfloor\mathrm{d\bar Y}\\
= \; & d_1\rfloor D_{t\partial}\rfloor\left(-\mathfrak{p}_\alpha^{[J-1_r,J]}\mathrm{d}y^a_{[J-1_r]}\wedge\partial_1\rfloor\partial_r\rfloor\mathrm{dY} + \mathrm{d}\left(\bar h_{j_r}\wedge\partial_1\rfloor\mathrm{d\bar Y}\right)\right.\\
& \left.+\,\mathrm{d}\left(\mathfrak{p}_\alpha^{\left[\bar J-1_l,\bar J;j_r\right]}\left(\mathrm{d}y^a_{\left[\bar J-1_l;j_r\right]}\partial_l\rfloor\partial_1\rfloor\mathrm{d\bar Y} - y^\alpha_{\left[\bar J;j_r\right]}\partial_1\rfloor\mathrm{d\bar Y}\right)\right)\right)\wedge\partial_1\rfloor\mathrm{d\bar Y}\\
= \; & d_1\rfloor D_{t\partial}\rfloor\left(-\mathfrak{p}_\alpha^{[J-1_r,J]}\mathrm{d}y^a_{[J-1_r]}\wedge\partial_1\rfloor\mathrm{d\bar Y} - \mathfrak{p}_\alpha^{\left[\bar J-1_l,\bar J;j_r\right]}\mathrm{d}y^\alpha_{\left[\bar J;j_r\right]}\wedge\partial_1\rfloor\mathrm{d\bar Y}\right.\\
& +\mathrm{d}\left(\bar h_{j_r}\wedge\partial_1\rfloor\mathrm{d\bar Y}\right)\\
& \left.+\,\mathrm{d}\mathfrak{p}_\alpha^{\left[\bar J-1_l,\bar J;j_r\right]}\wedge\left(\mathrm{d}y^a_{\left[\bar J-1_l;j_r\right]}\partial_l\rfloor\partial_1\rfloor\mathrm{d\bar Y} - y^\alpha_{\left[\bar J;j_r\right]}\partial_1\rfloor\mathrm{d\bar Y}\right)\right)\wedge\partial_1\rfloor\mathrm{d\bar Y}\\
= \; & d_1\rfloor\left(-\mathfrak{p}_\alpha^{[J_1+J_r,J_1+J_r+1_r]}\mathrm{d}y^\alpha_{[J_1+J_r]} + \mathrm{d}\bar h_{j_r} - d_l\rfloor\mathrm{d}\mathfrak{p}_\alpha^{\left[\bar J-1_l,\bar J;j_r\right]}\mathrm{d}y^a_{\left[\bar J_1-1_l;j_r\right]}\right)\wedge\partial_1\rfloor\mathrm{d\bar Y}\\
= \; & -\mathfrak{p}_\alpha^{[J_1+J_r,J_1+J_r+1_r]}d_1\rfloor\mathrm{d}y^\alpha_{[J_1+J_r]} + \frac{\partial\bar h_{j_r}}{\partial p_\alpha^{\left[\bar J_1-1_1,\bar J_1;j_r\right]}}d_1\rfloor\mathrm{d}p_\alpha^{\left[\bar J_1-1_1,\bar J_1;j_r\right]}\\
& +\frac{\partial\bar h_{j_r}}{\partial y^\alpha_{\left[\bar J_1-1_1,j_r\right]}}d_1\rfloor\mathrm{d}y^\alpha_{\left[\bar J_1-1_1,j_r\right]} - d_l\rfloor\mathrm{d}\mathfrak{p}_\alpha^{\left[\bar J_1,\bar J+1_l;j_r\right]}\,d_1\rfloor\mathrm{d}y^a_{\left[\bar J_1;j_r\right]}\\
= \; & \frac{\partial\bar h_{j_r}}{\partial p_\alpha^{\left[\bar J_1-1_1,\bar J_1;j_r\right]}}d_1\rfloor\mathrm{d}p_\alpha^{\left[\bar J_1-1_1,\bar J_1;j_r\right]}\\
& +\left(\frac{\partial\bar h_{j_r}}{\partial y^\alpha_{\left[\bar J_1-1_1,j_r\right]}} - d_l\rfloor\mathrm{d}\mathfrak{p}_\alpha^{\left[\bar J_1-1_1,\bar J_1-1_1+1_l;j_r\right]} - \mathfrak{p}_\alpha^{\left[\bar J_1-1_1+J_r,\bar J_1-1_1+J_r\right]}\right)d_1\rfloor\mathrm{d}y^\alpha_{[J_{1\partial}-1_1,j_r]}\\
= \; & 0\,.
\end{aligned}
$$

Consequently the functional

$$\int_{\mathcal{D}_S}\left(j^n\sigma\right)^*\left(h\;\partial_1\rfloor\mathrm{dY} + \mathrm{d}\left(\bar h_{j_r}\;\partial_1\rfloor\mathrm{d\bar Y}\right)\right)$$

is invariant along the solution of the corresponding Euler-Lagrange system if $\frac{\partial\bar h}{\partial Y^1} = 0$.

Remark 9.8 *This result confirms again Noether's theorem as the group of time translation becomes a symmetry group of the variational problem if* $\mathrm{L}_{\partial_1}(l) = 0$ *is met. (see, e.g., [Olver, 1986])*

9.3 Application: The Kirchhoff plate

Here we will make use of the model of the damped Kirchhoff plate as derived in section 8.6. We will show, how the time evolution of the system could be described.

9.3.1 The extended Cartan form

The reformulation of the extended Cartan form leads to

$$
\begin{aligned}
c_{ext} \;=\; & \Big(\frac{1}{2}\rho\Lambda\left(y^1_{[100]}\right)^2 - p_1^{[000,100]} y^1_{[100]} \\
& -\frac{1}{2}\varsigma\left(\left(y^1_{[020]}\right)^2 + \left(y^1_{[002]}\right)^2 + 2\nu y^1_{[002]} y^1_{[020]} + 2\left(1-\nu\right)\left(y^1_{[011]}\right)^2\right)\Big)\, \mathrm{d}t \wedge \mathrm{d}Y^2 \wedge \mathrm{d}Y^3 \\
& + p_1^{[010,020]}\left(\mathrm{d}y^1_{[010]} - y^1_{[020]}\mathrm{d}Y^2\right) \wedge \partial_2 \rfloor \mathrm{d}t \wedge \mathrm{d}Y^2 \wedge \mathrm{d}Y^3 \\
& + p_1^{[001,002]}\left(\mathrm{d}y^1_{[001]} - y^1_{[002]}\mathrm{d}Y^3\right) \wedge \partial_3 \rfloor \mathrm{d}t \wedge \mathrm{d}Y^2 \wedge \mathrm{d}Y^3 \\
& + p_1^{[010,011]}\left(\mathrm{d}y^1_{[010]} - y^1_{[011]}\mathrm{d}Y^3\right) \wedge \partial_3 \rfloor \mathrm{d}t \wedge \mathrm{d}Y^2 \wedge \mathrm{d}Y^3 \\
& + p_1^{[000,100]}\mathrm{d}y^1_{[000]} \wedge \partial_1 \rfloor \mathrm{d}t \wedge \mathrm{d}Y^2 \wedge \mathrm{d}Y^3 \\
& + p_1^{[000,010]}\left(\mathrm{d}y^1_{[000]} - y^1_{[010]}\mathrm{d}Y^2\right) \wedge \partial_2 \rfloor \mathrm{d}t \wedge \mathrm{d}Y^2 \wedge \mathrm{d}Y^3 \\
& + p_1^{[000,001]}\left(\mathrm{d}y^1_{[000]} - y^1_{[001]}\mathrm{d}Y^3\right) \wedge \partial_3 \rfloor \mathrm{d}t \wedge \mathrm{d}Y^2 \wedge \mathrm{d}Y^3 \\
& + \mathrm{d}\left(p_1^{[00,01;0]}\left(\mathrm{d}y^1_{[000]} - y^1_{[010]}\mathrm{d}Y^2\right) \wedge \partial_2 \rfloor \mathrm{d}t \wedge \mathrm{d}Y^2\right) .
\end{aligned}
$$

whereby the function

$$
\begin{aligned}
-h \;=\; & \frac{1}{2}\rho\Lambda\left(y^1_{[100]}\right)^2 - p_1^{[000,100]} y^1_{[100]} \\
& -\frac{1}{2}\varsigma\left(\left(y^1_{[020]}\right)^2 + \left(y^1_{[002]}\right)^2 + 2\nu y^1_{[002]} y^1_{[020]} + 2\left(1-\nu\right)\left(y^1_{[011]}\right)^2\right)
\end{aligned}
$$

is introduced. We do not have to consider a function $\bar{h}_{j_r}$, because no $p_\alpha^{\left[\bar{J}_1 - 1_1, \bar{J}_1; j_r\right]}$ function appears in this example.

9.3.2 Domain Condition

From the general domain condition with external input

$$
\left(\left(-1\right)^r D \rfloor \mathrm{d}c_{ext} - R y^1_{[100]} \mathrm{d}y^1_{[000]}\right) \wedge \mathrm{dY} = 0 ,
$$

we obtain

$$
\begin{aligned}
d_1 \rfloor \mathrm{d}y^\alpha_{[000]} &= \frac{\partial h}{\partial p_\alpha^{[000,100]}} = y^1_{[100]} \qquad (9.4) \\
d_1 \rfloor \mathrm{d}p_\alpha^{[000,100]} &= -\frac{\partial h}{\partial y^\alpha_{[000]}} - d_2 \rfloor \mathrm{d}p_\alpha^{[000,010]} - d_3 \rfloor \mathrm{d}p_\alpha^{[000,001]} - R y^1_{[100]} \\
&= -\varsigma y^1_{[040]} - \left(2\varsigma - \varsigma\nu\right) y^1_{[022]} - \varsigma\left(y^1_{[004]} + \nu y^1_{[022]}\right) - R y^1_{[100]} \\
&= -\varsigma y^1_{[040]} - \varsigma y^1_{[004]} - 2\varsigma y^1_{[022]} - R y^1_{[100]}
\end{aligned}
$$

where

$$\begin{aligned} p_1^{[000,010]} &= +\varsigma y^1_{[030]} + \varsigma\nu y^1_{[012]} + (2\varsigma - 2\varsigma\nu)\, y^1_{[012]} = \varsigma y^1_{[030]} + (2\varsigma - \varsigma\nu)\, y^1_{[012]} \\ p_1^{[000,001]} &= \varsigma\left(y^1_{[003]} + \nu y^1_{[021]}\right) \,. \end{aligned}$$

is used. It is obvious that the obtained domain condition representation is equivalent to the one already determined in section 8.6.

9.3.3 Boundary Condition

We have already determined the conditions on $\partial\mathcal{D}_1$

$$p_1^{[001,002]} - R_\partial y^1_{[101]} = -\varsigma\left(y^1_{[002]} + \nu y^1_{[020]}\right) - R_\partial y^1_{[101]} = 0 \,.$$

9.3.4 Time evolution of a functional

We investigate now the evolution of the functional

$$\begin{aligned} &\int_{\mathcal{D}_S} \left(j^2\sigma\right)^* \left(h \mathrm{d}Y^2 \wedge \mathrm{d}Y^3\right) = \\ &= \int_{\mathcal{D}_S} j^2\sigma^* \left(\frac{1}{2}\varsigma\left(\left(y^1_{[020]}\right)^2 + \left(y^1_{[002]}\right)^2 + 2\nu y^1_{[002]} y^1_{[020]} + 2\,(1-\nu)\left(y^1_{[011]}\right)^2\right)\right. \\ &\qquad \left. -\frac{1}{2}\rho\Lambda\left(y^1_{[100]}\right)^2 + p_1^{[000,100]} y^1_{[100]}\right) \mathrm{d}Y^2 \wedge \mathrm{d}Y^3 \end{aligned}$$

along the solution σ of the system. Consequently we have to modify the functional following Cartan's idea. We choose the following elements of the contact ideal I_2 restricted to $\mathcal{D}_S$:

$$\mathfrak{p}_\alpha^{[J-I_1,J]}\left(dy^\alpha_{[J-I_1]} - y^\alpha_{[J]}\mathrm{d}Y^i\right) \wedge \partial_i \rfloor \left(\mathrm{d}Y^2 \wedge \mathrm{d}Y^3\right) \;, \quad I_1 = 1_i \;, \quad i \in \{2,3\} \;, \quad \#J \in \{1,2\}$$

Determination of the functions $\mathfrak{p}_\alpha^{[J-I_1,J]}$

In a first step we derive all functions according to multi-indices with $\#J = 2$. We get similar to the results of section 8.6

$$\begin{aligned} \mathfrak{p}_1^{[010,020]} &= \partial_1^{[020]} h = -\partial_1^{[020]} l = \varsigma\left(y^1_{[020]} + \nu y^1_{[002]}\right) = -p_1^{[010,020]} \\ \mathfrak{p}_1^{[001,002]} &= \partial_1^{[002]} h = -\partial_1^{[002]} l = \varsigma\left(y^1_{[002]} + \nu y^1_{[020]}\right) = -p_1^{[001,002]} \\ \mathfrak{p}_1^{[010,011]} &= \partial_1^{[011]} h = -\partial_1^{[011]} l = 2\varsigma\left((1-\nu)\, y^1_{[011]}\right) = -p_1^{[010,011]} \,. \end{aligned}$$

In the next step we consider multi-indices J whose length is $\#J = 1$. We obtain

$$\begin{aligned} \mathfrak{p}_1^{[000,010]} &= \partial_1^{[010]} h - \sum_{i=2}^{r} \mathrm{L}_{d_i}\left(\mathfrak{p}_1^{[010,010+1_i]}\right) = -p_1^{[000,010]} \\ &= -\mathrm{L}_{d_2}\left(\mathfrak{p}_1^{[010,020]}\right) - \mathrm{L}_{d_3}\left(\mathfrak{p}_1^{[010,011]}\right) = -\varsigma\left(y^1_{[030]} + \nu y^1_{[012]}\right) - 2\varsigma\,(1-\nu)\, y^1_{[012]} \\ \mathfrak{p}_1^{[000,001]} &= \partial_1^{[001]} h - \sum_{i=2}^{r} \mathrm{L}_{d_i}\left(\mathfrak{p}_1^{[001,001+1_i]}\right) = -p_1^{[000,001]} \\ &\quad -\mathrm{L}_{d_3}\left(\mathfrak{p}_1^{[001,002]}\right) = -\varsigma\left(y^1_{[003]} + \nu y^1_{[021]}\right) \,. \end{aligned}$$

Consequently it is only left to derive the functions $\mathfrak{p}_\alpha^{\left[\bar{J}-1_{w_l},\bar{J};j_r\right]}$.

Determination of the functions $\mathfrak{p}_\alpha^{\left[\bar{J}-1_{w_l},\bar{J};j_r\right]}$

Here we obtain a single function from $[J-1_r]=\left[\bar{J};j_r\right]$ to be given by

$$\mathfrak{p}_1^{[00,01;0]}=\mathfrak{p}_1^{[010,011]}=+2\varsigma\left((1-\nu)\,y^1_{[011]}\right)\ ,$$

where $j_r=0$ and $\#\bar{J}=1$ is used.

Finally we end up with the following functional

$$\begin{aligned}
\int_{\mathcal{D}_S}\left(j^2\sigma\right)^*\left(h\mathrm{d}Y^2\wedge\mathrm{d}Y^3\right) &= \int_{\mathcal{D}_S}\left(j^3\sigma\right)^*\breve{c}\\
&= \int_{\mathcal{D}_S}\left(j^3\sigma\right)^*\Bigg(\Bigg(-\frac{1}{2}\rho\Lambda\left(y^1_{[100]}\right)^2+p_1^{[000,100]}y^1_{[100]}\\
&\quad+\frac{1}{2}\varsigma\left(\left(y^1_{[020]}\right)^2+\left(y^1_{[002]}\right)^2+2\nu y^1_{[002]}y^1_{[020]}+2\,(1-\nu)\left(y^1_{[011]}\right)^2\right)\Bigg)\,\mathrm{d}Y^2\wedge\mathrm{d}Y^3\\
&\quad+\mathfrak{p}_1^{[010,020]}\left(\mathrm{d}y^1_{[010]}-y^1_{[020]}\mathrm{d}Y^2\right)\wedge\partial_2\rfloor\mathrm{d}Y^2\wedge\mathrm{d}Y^3\\
&\quad+\mathfrak{p}_1^{[001,002]}\left(\mathrm{d}y^1_{[001]}-y^1_{[002]}\mathrm{d}Y^3\right)\wedge\partial_3\rfloor\mathrm{d}Y^2\wedge\mathrm{d}Y^3\\
&\quad+\mathfrak{p}_1^{[010,011]}\left(\mathrm{d}y^1_{[010]}-y^1_{[011]}\mathrm{d}Y^3\right)\wedge\partial_3\rfloor\mathrm{d}Y^2\wedge\mathrm{d}Y^3\\
&\quad+\mathfrak{p}_1^{[000,010]}\left(\mathrm{d}y^1_{[000]}-y^1_{[010]}\mathrm{d}Y^2\right)\wedge\partial_2\rfloor\mathrm{d}Y^2\wedge\mathrm{d}Y^3\\
&\quad+\mathfrak{p}_1^{[000,001]}\left(\mathrm{d}y^1_{[000]}-y^1_{[001]}\mathrm{d}Y^3\right)\wedge\partial_3\rfloor\mathrm{d}Y^2\wedge\mathrm{d}Y^3\\
&\quad+\mathrm{d}\left(\mathfrak{p}_1^{[00,01;0]}\left(\mathrm{d}y^1_{[000]}-y^1_{[010]}\mathrm{d}Y^2\right)\wedge\partial_2\rfloor\mathrm{d}Y^2\right)\Bigg)
\end{aligned}$$

whose time derivative is of interest.

9.3.5 Domain impact

Thus we consider

$$\begin{aligned}
\mathrm{L}_{\partial_t}\int_{\mathcal{D}_S}\left(j^3\sigma\right)^*\breve{c} &= \int_{\mathcal{D}_S}\left(j^4\sigma\right)^*\mathrm{L}_{d_t}\left(\breve{c}\right)\\
&= \int_{\mathcal{D}_S}\left(j^4\sigma\right)^*d_t\rfloor\mathrm{d}\breve{c}+\int_{\partial\mathcal{D}_S}\left(j^4\bar{\sigma}\right)^*d_t\rfloor\left(\left(\Psi^3\right)^*\breve{c}\right)\ .
\end{aligned}$$

This leads to

$$\begin{aligned}
& \int_{\mathcal{D}_S} \left(j^4\sigma\right)^* d_t \rfloor \left((D_t \rfloor \mathrm{d}\check{c}) \wedge \mathrm{d}Y^2 \wedge \mathrm{d}Y^3 \right) \\
= & \int_{\mathcal{D}_S} \left(j^4\sigma\right)^* d_t \rfloor \left(\left(\frac{\partial h}{\partial y^1_{[000]}} - d_2 \rfloor \mathrm{d}\mathfrak{p}_1^{[000,010]} - d_3 \rfloor \mathrm{d}\mathfrak{p}_1^{[000,001]} \right) \mathrm{d}y^1_{[000]} \right. \\
& \left. + \frac{\partial h}{\partial p_1^{[000,100]}} \mathrm{d}p_1^{[000,100]} \right) \mathrm{d}Y^2 \wedge \mathrm{d}Y^3 \\
= & \int_{\mathcal{D}_S} \left(j^4\sigma\right)^* \left(\left(\frac{\partial h}{\partial y^1_{[000]}} - d_2 \rfloor \mathrm{d}\mathfrak{p}_1^{[000,010]} - d_3 \rfloor \mathrm{d}\mathfrak{p}_1^{[000,001]} \right) d_t \rfloor \mathrm{d}y^1_{[000]} \right. \\
& \left. + \frac{\partial h}{\partial p_1^{[000,100]}} d_t \rfloor \mathrm{d}p_1^{[000,100]} \right) \mathrm{d}Y^2 \wedge \mathrm{d}Y^3 \\
= & \int_{\mathcal{D}_S} \left(j^4\sigma\right)^* \left(\left(+\varsigma \left(y^1_{[040]} + \nu y^1_{[022]} \right) + 2\varsigma \left(1-\nu\right) y^1_{[022]} + \varsigma \left(y^1_{[004]} + \nu y^1_{[022]} \right) \right) y^1_{[100]} \right. \\
& \left. + y^1_{[100]} \left(-\varsigma y^1_{[040]} - \varsigma y^1_{[004]} - 2\varsigma y^1_{[022]} - R y^1_{[100]} \right) \right) \mathrm{d}Y^2 \wedge \mathrm{d}Y^3 \\
= & \int_{\mathcal{D}_S} \left(j^4\sigma\right)^* \left(-R \left(y^1_{[100]} \right)^2 \mathrm{d}Y^2 \wedge \mathrm{d}Y^3 \right)
\end{aligned}$$

on the domain.

9.3.6 Boundary impact

The boundary conditions supply

$$\int_{\partial\mathcal{D}_S} \left(j^4\bar{\sigma}\right)^* d_t \rfloor \left(\left(\Psi^3\right)^* \check{c} \right) = \int_{\partial\mathcal{D}_{S1}} \left(j^4\bar{\sigma}_1\right)^* d_t \rfloor \left(d_2 \rfloor \left(\left(\Psi^3\right)^* \check{c} \right) \wedge \mathrm{d}X^2 \right)$$

where

$$\begin{aligned}
\left(\left(\Psi^3\right)^* \check{c} \right) = & \; \mathfrak{p}_1^{[001,002]} \mathrm{d}y^1_{[00;1]} \wedge \mathrm{d}Y^2 + \mathfrak{p}_1^{[010,011]} \mathrm{d}y^1_{[01;0]} \wedge \mathrm{d}Y^2 \\
& + \mathfrak{p}_1^{[000,001]} \mathrm{d}y^1_{[00;0]} \wedge \mathrm{d}Y^2 + \mathrm{d}\left(\mathfrak{p}_1^{[00,01;0]} \left(\mathrm{d}y^1_{[00;0]} - y^1_{[01;0]} \mathrm{d}Y^2 \right) \right)
\end{aligned}$$

and consequently

$$\begin{aligned}
\left(d_2 \rfloor \left(\left(\iota_S\right)^* \bar{c} \right) \wedge \mathrm{d}X^2 \right) = & \; \mathfrak{p}_1^{[001,002]} \mathrm{d}y^1_{[00;1]} \wedge \mathrm{d}Y^2 + \mathfrak{p}_1^{[010,011]} \mathrm{d}y^1_{[01;0]} \wedge \mathrm{d}Y^2 \\
& + \mathfrak{p}_1^{[000,001]} \mathrm{d}y^1_{[00;0]} \wedge \mathrm{d}Y^2 - d_2 \rfloor \mathrm{d}\mathfrak{p}_1^{[00,01;1]} \wedge \mathrm{d}y^1_{[00;0]} \wedge \mathrm{d}Y^2 \\
& - \mathfrak{p}_1^{[00,01;1]} \mathrm{d}y^1_{[01;0]} \wedge \mathrm{d}Y^2 \\
= & \; \mathfrak{p}_1^{[001,002]} \mathrm{d}y^1_{[00;1]} \wedge \mathrm{d}Y^2 + \left(\mathfrak{p}_1^{[000,001]} - d_2 \rfloor \mathrm{d}\mathfrak{p}_1^{[00,01;0]} \right) \mathrm{d}y^1_{[00;0]} \wedge \mathrm{d}Y^2 \,.
\end{aligned}$$

Thus

$$\begin{aligned}
\int_{\partial\mathcal{D}_{S1}} \left(j^4\bar{\sigma}_1\right)^* d_t \rfloor \left(d_2 \rfloor \left(\left(\Psi^3\right)^* \check{c} \right) \wedge \mathrm{d}X^2 \right) = & \int_{\partial\mathcal{D}_{S1}} \left(j^4\bar{\sigma}_1\right)^* \left(\mathfrak{p}_1^{[001,002]} d_t \rfloor \mathrm{d}y^1_{[00;1]} \wedge \mathrm{d}Y^2 \right. \\
& \left. + \left(\mathfrak{p}_1^{[000,001]} - d_2 \rfloor \mathrm{d}\mathfrak{p}_1^{[00,01;1]} \right) d_t \rfloor \mathrm{d}y^1_{[00;0]} \wedge \mathrm{d}Y^2 \right)
\end{aligned}$$

as $\mathrm{d}y^1_{[00;0]} \wedge \mathrm{d}Y^2 = 0$ at the boundary this simplifies to

$$\int_{\partial\mathcal{D}_{S1}} \left(j^4\bar{\sigma}_1\right)^* \left(\mathfrak{p}_1^{[001,002]} d_t \rfloor \mathrm{d}y^1_{[00;1]} \wedge \mathrm{d}Y^2\right) = \int_{\partial\mathcal{D}_{S1}} \left(j^4\bar{\sigma}_1\right)^* \left(-R_\partial \left(y^1_{[10;1]}\right)^2 \wedge \mathrm{d}Y^2\right) .$$

9.4 Time-invariant Hamiltonian representation

The next part of this thesis is dedicated to the so called time-invariant port Hamiltonian systems. Their description will be carried out on a manifold $\mathcal{M}$ in the finite- and on a bundle $(\mathcal{H}, \eta, \mathcal{D}_S)$ in the infinite-dimensional case. The "time"-coordinate t will appear in this context as simple curve parameter on $\mathcal{H}$.

In order to derive this bundle structure from $(\mathcal{E}, \pi, \mathcal{D})$ we introduce the map

$$\begin{array}{rccc} \iota_t : & \mathcal{D}_S & \rightarrow & \mathcal{D} \\ & \left(Y^2, \ldots, Y^r\right) & \rightarrow & \left(Y^1 = \text{const.}, Y^2, \ldots, Y^r\right) . \end{array}$$

This special inclusion map determines a new bundle structure by deriving the pull-back of the domain jet framework along the contact bundle morphism (Ψ_t^n, ι_t). Similarly to the considerations on the determination of the boundary conditions, we obtain a jet bundle structure with dependent coordinates $y^\alpha_{[j_1;0\ldots0]}$ on the total manifold $\mathcal{H}$. Additionally, we confine ourselves to Lagrangians that depend only on jet coordinates $y^\alpha_{[j_1\ J_t]}$ with $j_1 \in \{0,1\}$. Furthermore we assume that the functions $p_\alpha^{[0,1_1]} \in C^\infty(\mathcal{H})$ allow a change of coordinates such that the bundle $(\mathcal{H}, \eta, \mathcal{D}_S)$ is equipped with the local coordinates $\left(Y^i, y^\alpha, p_\alpha^{[0,1_1]}\right)$, $i = 2, \ldots, r$, $\alpha = 1, \ldots, s$. Under these restrictions we are able to identify the derived conditions on the total time derivatives $d_1 \rfloor y^\alpha$ and $d_1 \rfloor p_\alpha^{[0,1_1]}$ with the coordinates of a vertical vector field

$$v_h = \dot{y}^\alpha \partial_\alpha + \dot{p}^{\tilde{\alpha}} \partial_{\tilde{\alpha}}$$

with $p^{\tilde{\alpha}} = \delta^{\tilde{\alpha}\tilde{\beta}} p_{\tilde{\beta}}^{[0,1_1]}$ and

$$\begin{array}{rcl} \dot{y}^\alpha & = & d_1 \rfloor y^\alpha \\ \dot{p}^{\tilde{\alpha}} & = & \delta^{\tilde{\alpha}\tilde{\beta}} d_1 \rfloor p_{\tilde{\beta}}^{[0,1_1]} . \end{array}$$

Here we have used the Kronecker symbol $\delta^{\tilde{\alpha}\tilde{\beta}}$.The geometrical properties of such a Hamiltonian operator v_h are discussed in part III.

9.4.1 Application - the Kirchhoff plate

The map

$$\delta^{1\tilde{1}} p_1^{[000,100]} = \rho\Lambda y^1_{[100]} = p^{\tilde{1}}$$

enables us to pull-back the form $h\mathrm{d}t \wedge \mathrm{d}Y^2 \wedge \mathrm{d}Y^3$ and we obtain

$$\begin{array}{rcl} h\mathrm{d}Y^2 \wedge \mathrm{d}Y^3 & = & \left(\frac{1}{2}\varsigma\left(\left(y^1_{[0;20]}\right)^2 + \left(y^1_{[0;02]}\right)^2 + 2\nu y^1_{[0;02]} y^1_{[0;20]} + 2(1-\nu)\left(y^1_{[0;11]}\right)^2\right) + \right. \\ & & \left. \frac{1}{2\rho\Lambda}\left(p^{\tilde{1}}\right)^2\right) \mathrm{d}Y^2 \wedge \mathrm{d}Y^3 . \end{array}$$

Having this form and equation (9.4) at ones disposal, we are able to determine the Hamilton operator by

$$\begin{aligned}
\dot{y}^1 &= \frac{1}{\rho\Lambda} p^{\tilde{1}} \\
\dot{p}^{\tilde{1}} &= -\varsigma y^1_{[0;40]} - \varsigma y^1_{[0;04]} - 2\varsigma y^1_{[0;22]} - R\frac{1}{\rho\Lambda} p^{\tilde{1}} \ .
\end{aligned}$$

Remark 9.9 *In this case the Hamilton operator is nothing else than a generalized vector field (see, e.g., [Saunders, 1989]).*

Part III

Hamiltonian Systems

Structure makes life much easier!

Some structures on manifolds as, e.g., the Poisson, symplectic, or Dirac structure, where already treated in the introduction. It is now remarkable that every dynamic system, whose mathematical description represents one of these structures is equipped with a pleasant underlying property. This additional structural information could, e.g., ease the proof of stability. In this part of the thesis we will discuss Poisson structure systems that incorporate dissipation and in addition in- and outputs by means of ports .

Such port Hamiltonian systems with dissipation, or pHd systems [van der Schaft, 2000] for short, have turned out to be a versatile tool for the mathematical modeling in control theory. This class of systems comes along with a mathematical description that separates structural properties, storage elements, and dissipative parts. Thus a network description of such plants, which is very useful for simulation and control, becomes available.

We present in the subsequent investigations an extension of the finite-dimensional pHd description to the infinite-dimensional case. It is shown, which differential geometric objects have to be introduced and how boundary conditions come into play. Additionally, the key property of pHd systems – their behavior with respect to interconnection – is investigated for domain and boundary interconnections.

In the first chapter some well known results for finite-dimensional pHd systems are recalled. The following chapter is dedicated to the introduction of a possible extension of the approach to the infinite-dimensional case. We consider systems with 1^{st} and n^{th} order Hamiltonian density. Special attention is paid on the interconnection of two infinite-dimensional pHd systems via power conserving interconnections. Additionally, it is remarkable that we confine ourselves to the case where no differential operators are used in the system description. Finally the developed representation is applied to the Kirchhoff plate.

Chapter 10

Finite-dimensional port Hamiltonian Systems with Dissipation (F-pHd Systems)

In this section the geometrical structure and some additional properties of finite-dimensional pHd systems are under investigation. The precise definition of the used spaces will serve as a basis for the subsequent analysis of the infinite-dimensional case.

10.1 Geometrical structure of F-pHd systems

Let $\mathcal{M}$ denote the s-dimensional state manifold with coordinates (x^α), $\alpha = 1, \ldots, s$. The canonical product $\mathcal{T}(\mathcal{M}) \times \mathcal{T}^*(\mathcal{M}) \to C^\infty(\mathcal{M})$ is given by the interior product $\dot{x}^\alpha \partial_\alpha \rfloor \dot{x}_\beta \mathrm{d}x^\beta = \dot{x}^\alpha \dot{x}_\alpha$, $\beta = 1, \ldots, s$. Let $\mathcal{U} = \operatorname{span}\{\mathrm{e}_\varsigma\}$ with coordinates (u^ς), $\varsigma = 1, \ldots, m$ denote the input vector space. Consequently we choose the dual vector space $\mathcal{Y} = \mathcal{U}^* = \operatorname{span}\{\mathrm{e}^\varsigma\}$ with coordinates (y_ς) as the output vector space. The structure of a F-pHd system with state (x^α), input (u^ς), output (y_ς) and Hamiltonian $H_0 \in C^\infty(\mathcal{M})$ is given by

$$\dot{x} = (J - R)\rfloor \mathrm{d}H_0 + u \rfloor B \tag{10.1}$$

$$y = B \rfloor \mathrm{d}H_0 \tag{10.2}$$

or visualized as commutative diagram

$$\begin{tikzcd}
\mathcal{U} \arrow[r, "B"] & \mathcal{T}(\mathcal{M}) \arrow[dr, "\tau_{\mathcal{M}}"'] & & \mathcal{T}^*(\mathcal{M}) \arrow[ll, "(J-R)"'] \arrow[r, "B^*"] \arrow[dl, "\tau^*_{\mathcal{M}}"] & \mathcal{Y} \\
& & \mathcal{M} & C^\infty(\mathcal{M}) \arrow[u, "\mathrm{d}"'] &
\end{tikzcd}$$

where $J = J^{\alpha\beta}\partial_\alpha \otimes \partial_\beta$, $J^{\alpha\beta} = -J^{\beta\alpha}$, $R = R^{\alpha\beta}\partial_\alpha \otimes \partial_\beta$, $R^{\alpha\beta} = R^{\beta\alpha}$, $B = B^\alpha_\varsigma e^\varsigma \otimes \partial_\alpha$ is used. Additionally, the matrix $\left[R^{\alpha\beta}\right]$ is positive semidefinite and all coefficients are assumed to meet $J^{\alpha\beta}, R^{\alpha\beta}, B^{\alpha\varsigma} \in C^\infty(\mathcal{M})$. Obviously, the tensors J, R are maps of the form

$$\begin{aligned} J - R : \mathcal{T}^*(\mathcal{M}) &\to \mathcal{T}(\mathcal{M}) \\ \mathrm{d}H_0 &\to (J-R)\rfloor \mathrm{d}H_0 \,. \end{aligned}$$

The tensor B is a map

$$\begin{aligned} B : \mathcal{U} &\rightarrow \mathcal{T}(\mathcal{M}) \\ u &\rightarrow u \rfloor B \end{aligned}$$

and its adjoint map is given by

$$\begin{aligned} B^* : \mathcal{T}^*(\mathcal{M}) &\rightarrow \mathcal{Y} \\ \mathrm{d}H_0 &\rightarrow B \rfloor \mathrm{d}H_0 \,. \end{aligned}$$

The exterior derivative d,

$$\mathrm{d}H_0 = \partial_\alpha H_0 \mathrm{d}x^\alpha$$

serves here as a map $\mathrm{d} : C^\infty(\mathcal{M}) \rightarrow \mathcal{T}^*(\mathcal{M})$. The circumstance that the introduced Hamiltonian system is roughly speaking enveloped by the two linear spaces $\mathcal{U}$ and $\mathcal{Y}$ is visualized is figure 10.1.

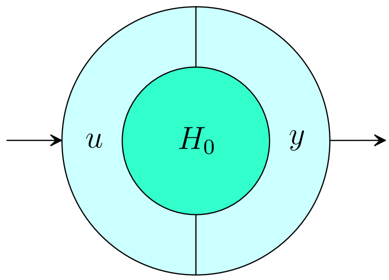

Figure 10.1: A F-pHd system.

Remark 10.1 *The map B is a special two feet tensor, as the first feet is an element of a linear space, i.e. a vector and the second feet is a vector field on the manifold $\mathcal{M}$. This construction guaranties that the pairing of in- and output $u \rfloor y$ takes place at the footing $p \in \mathcal{M}$ of the vector field part of the tensor B. Consequently a port consists of two dual vector spaces and an additional footing information.*

10.2 The Hamilton operator and collocation

Let us introduce the Hamilton operator[1] $v_H = \dot{x}^\alpha \partial_a$ with $\dot{x}^\alpha$ from (10.1) and visualize it on the state manifold in Fig. 10.2.

Remark 10.2 *This introduction of the Hamilton operator is compatible to the classical definition of a Hamilton vector field (see section 1.3.1), if the system is free, no dissipation is considered, i.e. $R = 0$, and the functions $J^{\alpha\beta}$ meet equation (1.4). In this case the tensor J induces a Poisson structure on the manifold $\mathcal{M}$.*

[1]The introduced Hamilton operator is not a vector field on $\mathcal{M}$ because of its dependence on the input u. In fact it is a submanifold of $\mathcal{T}(\mathcal{M})$ parametrized by u.

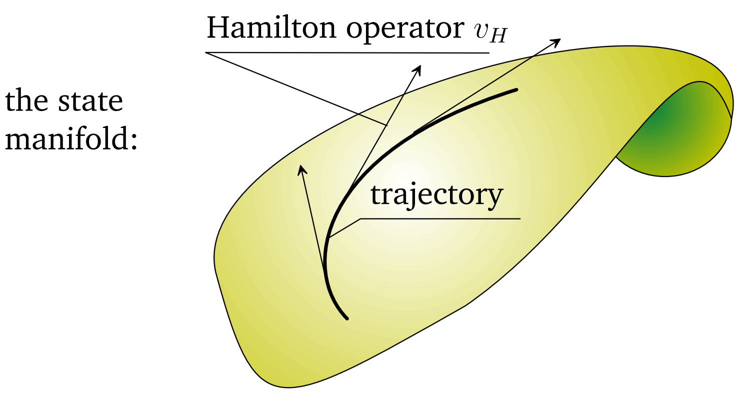

Figure 10.2: The finite dimensional Hamilton operator v_H (for $u = u(x)$).

Taking into account this definition, we easily obtain the well known relation

$$\mathrm{L}_{v_H}(H_0) = v_H \rfloor \mathrm{d}(H_0) = -\left(R \rfloor \mathrm{d}H_0\right) \rfloor \mathrm{d}H_0 + u \rfloor y \ . \tag{10.3}$$

Obviously the product $u \rfloor y$ equals the external impact on the formal time derivative of the Hamiltonian H_0. One can often interpret this product as the power fed into the system. It is common to call such an in- and output configuration collocated in- and outputs.

If the input map is given by $B = -e^{\varsigma} \otimes J \rfloor \mathrm{d}H_\varsigma$ with suitable functions H_ς, then from

$$\mathrm{L}_{v_H}(H_\varsigma) = ((J - R) \rfloor \mathrm{d}H_0 - u^\omega J \rfloor \mathrm{d}H_\omega) \rfloor \mathrm{d}H_\varsigma \ , \quad \omega = 1 \ldots m$$

it follows that $\mathrm{L}_{v_H}(H_\varsigma) = y_\varsigma$ is fulfilled for the case that $(R \rfloor \mathrm{d}H_0) \rfloor \mathrm{d}H_\varsigma = (J \rfloor \mathrm{d}H_\omega) \rfloor \mathrm{d}H_\varsigma = 0$. This often applies in mechanics.

Chapter 11

Infinite-dimensional port Hamiltonian Systems (I-pHd Systems)

In order to extend the pHd approach from the finite- to the infinite-dimensional case, we have to replace the state manifold $\mathcal{M}$, its tangent bundle $\mathcal{T}(\mathcal{M})$, its cotangent bundle $\mathcal{T}^*(\mathcal{M})$ and the smooth functions $C^\infty(\mathcal{M})$ by new spaces. Furthermore, the free Hamiltonian H_0, the maps J, R, B and the exterior derivative d have to be substituted by new functions and operators.

11.1 Geometrical structure of I-pHd systems

First we introduce the bounded base manifold $\mathcal{D}$ with local coordinates (X^i) [1], $i = 1, \ldots, r$. Commonly these coordinates will represent the independent spatial coordinates according to the analyzed plant. Additionally, let $(\mathcal{H}, \eta, \mathcal{D})$ be the "state" bundle with local coordinates (X^i, x^α), $\alpha = 1, \ldots, s$, where x^α represents the dependent coordinates.
From η we derive four important bundles:

- The n^{th} jet bundle $(J^n\eta, \eta^n, \mathcal{D})$ with the adapted coordinates $\left(X^i, x^\alpha, x^\alpha_{[J]}\right)$ according to the total manifold $J^n\eta$;
- the vertical tangent bundle $\left(V\eta, \tau_{\mathcal{H}}|_{V\eta}, \mathcal{H}\right)$ with coordinates $(X^i, x^\alpha, \dot{x}^\alpha)$;
- the exterior bundle $\left(\wedge^0_r\mathcal{T}^*(\mathcal{H}), \tau_{\wedge^0_r\mathcal{T}^*(\mathcal{H})}, \mathcal{H}\right)$ where $\wedge^0_r(\mathcal{T}^*(\mathcal{H})) = \mathrm{span}\{\mathrm{dX}\}$ and adapted local coordinates $(X^i, x^\alpha, \mathfrak{r})$;
- and the exterior bundle $\left(\wedge^1_r\mathcal{T}^*(\mathcal{H}), \tau_{\wedge^1_r(\mathcal{T}^*(\mathcal{H}))}, \mathcal{H}\right)$ where $\wedge^1_r(\mathcal{T}^*(\mathcal{H})) = \mathrm{span}\{\mathrm{d}x^\alpha \wedge \mathrm{dX}\}$ and adapted local coordinates $(X^i, x^\alpha, \dot{\mathfrak{r}}_\alpha)$.

Here the volume form $\mathrm{dX} = \mathrm{d}X^1 \wedge \cdots \wedge \mathrm{d}X^r$ is used.

[1] Here the independent coordinates do not incorporate the time coordinate, as we consider only time-invariant pHd systems. Thus the base domain equals the $r-1$ dimension spatial domain $\mathcal{D}_S$ of the Euler-Lagrange part.

Furthermore we define the interior product

$$\begin{array}{rcl} \rfloor \quad : V\eta \times \wedge_r^1 \left(\mathcal{T}^*\left(\mathcal{H}\right)\right) & \rightarrow & \wedge_r^0 \mathcal{T}^*\left(\mathcal{H}\right) \\ \left(\dot{x}^\alpha \partial_\alpha, \dot{\mathfrak{r}}_\alpha \mathrm{d}x^\alpha \wedge \mathrm{dX}\right) & \rightarrow & \dot{x}^\alpha \partial_\alpha \rfloor \dot{\mathfrak{r}}_\alpha \mathrm{d}x^\alpha \wedge \mathrm{dX} = \dot{x}^\alpha \dot{\mathfrak{r}}_\alpha \mathrm{dX} \end{array}$$

to represent the canonical product of the I-pHd system. An n^{th} order I-pHd system is equipped with a Hamiltonian $h_0 \mathrm{dX}$, $h_0 \in C^\infty\left(J^n \eta\right)$. Considering the previously introduced bundles we see that the Hamiltonian is a section on the pull-back bundle $h_0 \mathrm{dX} \in \Gamma\left(\left(\eta_0^n\right)^* \tau_{\wedge_r^0 \mathcal{T}^*(\mathcal{H})}\right)$.

Now we are able to replace $\mathcal{T}(\mathcal{M})$, $\mathcal{T}^*(\mathcal{M})$, $C^\infty\left(\mathcal{M}\right)$ of chapter 10 by the pull-back bundles $\left(\eta_0^{2n}\right)^* \tau_{\mathcal{H}}|_{V\eta}$, $\left(\eta_0^{2n}\right)^* \tau_{\wedge_r^1(\mathcal{T}^*(\mathcal{H}))}$, $\left(\eta_0^n\right)^* \tau_{\wedge_r^0 \mathcal{T}^*(\mathcal{H})}$. Additionally, we use the variational derivative $\delta : \left(\eta_0^n\right)^* \wedge_r^0 \mathcal{T}^*\left(\mathcal{H}\right) \rightarrow \left(\eta_0^{2n}\right)^* \wedge_r^1 \left(\mathcal{T}^*\left(\mathcal{H}\right)\right)$ instead of the exterior derivative of the chapter 10 and substitute the tensors J, R by suitable maps $\mathfrak{J}, \mathfrak{R} : \left(\eta_0^{2n}\right)^* \wedge_r^1 \left(\mathcal{T}^*\left(\mathcal{H}\right)\right) \rightarrow \left(\eta_0^{2n}\right)^* V\eta$. The map $\mathfrak{J}$ is assumed to be skew symmetric, i.e. $\mathfrak{J}^{\alpha\beta} = -\mathfrak{J}^{\beta\alpha}$, and $\mathfrak{R}$ to be a symmetric positive semidefinite map.

Remark 11.1 *The maps $\mathfrak{J}, \mathfrak{R}$ could also be differential operators (see, e.g., [Olver, 1986]), i.e. maps of the form*

$$\left(\eta_0^{2n}\right)^* \wedge_r^1 \left(\mathcal{T}^*\left(\mathcal{H}\right)\right) \rightarrow \left(\eta_0^{2n+m}\right)^* V\eta\,,$$

where $m > 0$. Here we will confine ourselves to the non-differential-operator case.

For the input space we choose a vector bundle $(\mathcal{U}, \eta_{\mathcal{U}}, \mathcal{D})$ with local coordinates (X^i, u^ς), $\varsigma = 1, \ldots, m$ and basis $\{e_\varsigma\}$. Of course, the output space $\mathcal{Y} = \mathcal{U}^*$ is given by the dual vector bundle, where we use the coordinates (X^i, y_ς) and the basis $\{e^\varsigma \otimes \mathrm{dX}\}$. Furthermore, we conclude that there exists a bilinear map $\mathcal{U} \times_{\mathcal{D}} \mathcal{Y} \rightarrow \wedge_r^0 \left(\mathcal{T}^*\left(\mathcal{H}\right)\right)$ given by $u^\varsigma e_\varsigma \rfloor y_\varsigma e^\varsigma \otimes \mathrm{dX} = u^\varsigma y_\varsigma \mathrm{dX}$. The input map

$$\begin{array}{rrcl} \mathfrak{B} : & \mathcal{U} & \rightarrow & \left(\eta_0^{2n}\right)^* V\eta \\ & \left(u^\varsigma e_\varsigma\right) & \rightarrow & u^\varsigma e_\varsigma \rfloor B_\varsigma^\alpha e^\varsigma \otimes \partial_\alpha \end{array}$$

and its adjoint – the output map $\mathfrak{B}^*$

$$\begin{array}{rcl} \mathfrak{B}^* : \left(\eta_0^{2n}\right)^* \wedge_r^1 \left(\mathcal{T}^*\left(\mathcal{H}\right)\right) & \rightarrow & \mathcal{Y} \\ \delta_\alpha h_0 \mathrm{d}x^\alpha \wedge \mathrm{dX} & \rightarrow & B_\varsigma^\alpha e^\varsigma \otimes \partial_\alpha \rfloor \delta_\alpha h_0 \mathrm{d}x^\alpha \wedge \mathrm{dX} \end{array}$$

are defined by the tensor $B_\varsigma^\alpha e^\varsigma \otimes \partial_\alpha$.

Finally we propose the structure of an infinite-dimensional port Hamiltonian system

$$\dot{x} = \left(\mathfrak{J} - \mathfrak{R}\right)\left(\delta\left(h_0 \mathrm{dX}\right)\right) + \mathfrak{B}\left(u\right) \tag{11.1}$$

$$y = \mathfrak{B}^*\left(\delta\left(h_0 \mathrm{dX}\right)\right)\,, \tag{11.2}$$

or visualized in the following commutative diagram.

$$
\begin{tikzcd}
\mathcal{U} \arrow[r, "\mathfrak{B}"] \arrow[dd, "\eta_{\mathcal{U}}"'] & \left(\eta_0^{2n}\right)^* V\eta \arrow[dr, "\left(\eta_0^{2n}\right)^* \tau_{\mathcal{H}}|_{V\eta}"'] & & & & \left(\eta_0^{2n}\right)^* \wedge_p^1 \mathcal{T}^*\left(\mathcal{H}\right) \arrow[llll, "(\mathfrak{J}-\mathfrak{R})"'] \arrow[r, "\mathfrak{B}^*"] \arrow[dl, "\left(\eta_0^{2n}\right)^* \tau_{\wedge_p^1\mathcal{T}^*(\mathcal{H})}"'] & \mathcal{Y} \arrow[dd, "\eta_{\mathcal{Y}}"] \\
& & J^{2n}\eta \arrow[dr, "\eta_0^{2n}"] & & J^{2n}\eta \arrow[dl, "\eta_0^{2n}"'] & \left(\eta_0^{n}\right)^* \wedge_p^0 \mathcal{T}^*\left(\mathcal{H}\right) \arrow[u, "\delta"'] & \\
\mathcal{D} & V\eta \arrow[rr, "\tau_{\mathcal{H}}|_{V\eta}"'] & & \mathcal{H} \arrow[d, "\eta"'] & & \wedge_p^1\mathcal{T}^*\left(\mathcal{H}\right) \arrow[ll, "\tau_{\wedge_p^1\mathcal{T}^*(\mathcal{H})}"] & \mathcal{D} \\
& & & \mathcal{D}\,. & & &
\end{tikzcd}
$$

The maps $\mathfrak{J}, \mathfrak{R}$ are in local coordinates given by

$$
\begin{aligned}
\mathfrak{J} : \left(\eta_0^{2n}\right)^* \wedge_r^1 \left(\mathcal{T}^*\left(\mathcal{H}\right)\right) &\rightarrow \left(\eta_0^{2n}\right)^* V\eta \\
\dot{\mathfrak{r}}_\alpha \mathrm{d}x^\alpha \wedge \mathrm{dX} &\rightarrow J^{\beta\alpha}\dot{\mathfrak{r}}_\alpha \partial_\beta
\end{aligned}
$$

with $J^{\beta\alpha} = -J^{\alpha\beta}$ and

$$
\begin{aligned}
\mathfrak{R} : \left(\eta_0^{2n}\right)^* \wedge_r^1 \left(\mathcal{T}^*\left(\mathcal{H}\right)\right) &\rightarrow \left(\eta_0^{2n}\right)^* V\eta \\
\dot{\mathfrak{r}}_\alpha \mathrm{d}x^\alpha \wedge \mathrm{dX} &\rightarrow R^{\beta\alpha}\dot{\mathfrak{r}}_\alpha \partial_\beta
\end{aligned}
$$

with $R^{\beta\alpha} = R^{\alpha\beta}$, $R^{\beta\alpha}\dot{\mathfrak{r}}_\alpha\dot{\mathfrak{r}}_\beta \geq 0$.

Remark 11.2 *Here we have again introduced a combination of in- and output maps $\mathfrak{B}$ and $\mathfrak{B}^*$ that provides all three ingredients of a port. Of course two components of the port are given by the dual vector bundles $\mathcal{U}$ and $\mathcal{Y}$ representing the space of in- and output variables. The third component is again given by the footing $p \in \mathcal{H}$ of the vertical vector field part of the tensor $B_\varsigma^\alpha e^\varsigma \otimes \partial_\alpha$.*

This general definitions will now be used to investigate the boundary conditions of I-pHd systems with 1^{st} order Hamiltonian. After that, we extend the achieved results to the case of systems with n^{th} order Hamiltonian.

11.2 Boundary Ports of I-pHd Systems

The central object of I-pHd systems is given by the Hamiltonian functional

$$
\mathfrak{H}\left(\sigma\right) = \int_{\mathcal{D}} \left(j^n\sigma\right)^* \left(h_0 \mathrm{dX}\right) \ ,
$$

whereat the n^{th} prolongation of the section $\sigma \in \Gamma(\eta)$ is applied. The evolution of the I-pHd system is determined by the η-vertical operator[2] $v_h = \dot{x}^\alpha \partial_\alpha \in \left(\eta_0^{2n}\right)^* \tau_{\mathcal{H}}|_{V\eta}$ with $\dot{x}^\alpha$ from (11.1) and denoted as the Hamilton operator. Thus we are able to depict v_h in figure 11.1, whereby the fibre preserving property of the corresponding automorphism, i.e. the supposed solution, is stressed.

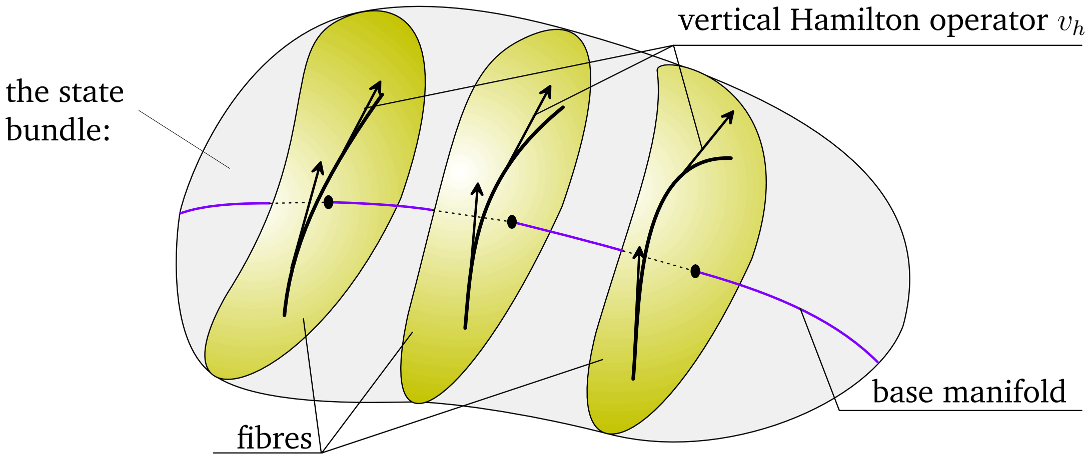

Figure 11.1: The η-vertical Hamilton operator v_h (Here we assume $u = u\left(X^i, x^\alpha_{[J]}\right)$ and consequently v_h becomes a generalized vector field).

In order to visualize the special properties of the Hamilton operator and to derive the impact of the boundary conditions, we have to consider the time derivative of $\mathfrak{H}$ along the solution of the corresponding I-pHd system. We are able to formulate the *formal time derivative*[3] of the Hamiltonian functional as

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathfrak{H} = \mathrm{L}_{j^n v_h}(\mathfrak{H}) = \int_{\mathcal{D}} \left(j^{2n}\sigma\right)^* \mathrm{L}_{j^n v_h}(h_0 \mathrm{dX}) \ . \tag{11.3}$$

It is obvious that this mathematical problem is similar to the tasks treated in the calculus of variations. Consequently we will make heavy use of the cognitions obtained in chapter 8.

First we introduce new local coordinates (Z^i) resp. (Z^i, z^α) for the total resp. the base manifold of the bundle η such that the requirements stated in section 8.1.1[4] are met. Consequently we are able to pull-back the Hamiltonian $h_0 \mathrm{dX}$ and obtain $h'_0 \mathrm{dZ}$. Again we will suppress the superscript and use $h_0 \mathrm{dZ}$ instead of $h'_0 \mathrm{dZ}$. The Hamilton operator is denoted by $v_h = \dot{z}^\alpha \partial_\alpha$ in the local coordinates (Z^j, z^α). The corresponding boundary bundle is defined by $\left(\bar{\mathcal{H}}, \bar{\eta}, \partial\mathcal{D}\right)$, equipped with the coordinates $\left(\bar{Z}^i, z^\alpha_{[0\ldots 0;j_r]}\right)$ and the boundary section $\bar{\sigma} : \partial\mathcal{D} \to \bar{\mathcal{H}}$. Additionally, we introduce the domain and boundary jet framework Π^n resp. Π^n_∂ and the corresponding contact ideals I_n resp. $I_{\partial n}$.

This definitions enable us to introduce the extended Hamiltonian

$$h_{ext} = h_0 \mathrm{dZ} + \omega + \mathrm{d}(\omega_\partial) \ , \quad \omega \in I_n \ , \quad (\Psi^n)^* \omega_\partial \in I_{\partial n} \ ,$$

[2]Again, this operator is not a vector field, but a submanifold of $\left(\eta_0^2\right)^* \tau_{\mathcal{H}}|_{V\eta}$ parametrized in u.

[3]The introduced formal time derivative becomes the time derivative, if the solution exists and is parametrized in the time t.

[4]Here we use the local coordinates (Z^i, z^α) instead of (Y^i, y^α) in order to prevent confusion with the local coordinates of the output bundle $\eta_{\mathcal{Y}}$.

where we made use of boundary contact bundle morphism (Ψ^n, ι).

It is obvious that the additional forms ω, $(\Psi^n)^* \omega_\partial$ as elements of the contact ideals I_n, $I_{\partial n}$ do not modify the Hamiltonian functional, i.e.

$$\mathfrak{H}(\sigma) = \int_{\mathcal{D}} (j^n\sigma)^* (h_0 \mathrm{dZ}) = \int_{\mathcal{D}} (j^n\sigma)^* (h_{ext}) \ .$$

Now it is left to determine the forms ω, ω_∂ in an appropriate fashion, such that we can determine the domain and boundary impact of the system evolution on the Hamiltonian functional. Similarly to the calculus of variations, we will discuss these points for systems with 1^{st} and n^{th} order Hamiltonian separately.

11.3 Systems with 1^{st} order Hamiltonian

In this restrictive case we are able to construct the extended Hamiltonian analogously to the Cartan form, i.e.

$$h_{ext} = h_0 \mathrm{dZ} + p_\alpha^{[1_i]} \left(\mathrm{d}z^\alpha \wedge \partial_i \rfloor \mathrm{dZ} - z^\alpha_{[1_i]} \mathrm{dZ}\right) \ .$$

The unknown functions $p_\alpha^{[1_i]}$ are similarly defined and given by

$$p_\alpha^{[1_i]} = \frac{\partial h_0}{\partial z^\alpha_{[1_i]}} \ .$$

Finally we get

$$\begin{aligned}
\int_{\mathcal{D}} \left(j^2\sigma\right)^* \mathrm{L}_{j^1 v_h} (h_0 \mathrm{dZ}) \ &= \ \int_{\mathcal{D}} \left(j^2\sigma\right)^* \mathrm{L}_{j^1 v_h} \left(h_0 \mathrm{dZ} + \frac{\partial h_0}{\partial z^\alpha_{[1_i]}} \left(\mathrm{d}z^\alpha - z^\alpha_{[1_i]} \mathrm{d}Z^i\right) \wedge \partial_i \rfloor \mathrm{dZ} \right) \\
&= \ \int_{\mathcal{D}} \left(j^2\sigma\right)^* \left(v_h \rfloor \mathrm{d} \left(h_0 \mathrm{dZ} + \frac{\partial h_0}{\partial z^\alpha_{[1_i]}} \left(\mathrm{d}z^\alpha - z^\alpha_{[1_i]} \mathrm{d}Z^i\right) \wedge \partial_i \rfloor \mathrm{dZ} \right) \right. \\
&\quad \left. + \mathrm{d} \left(v_h \rfloor \left(\frac{\partial h_0}{\partial z^\alpha_{[1_i]}} \mathrm{d}z^\alpha \wedge \partial_i \rfloor \mathrm{dZ} \right) \right) \right) \\
&= \ \int_{\mathcal{D}} \left(j^2\sigma\right)^* \left(v_h \rfloor \delta_\alpha h_0 \mathrm{d}z^\alpha \wedge \mathrm{dZ} + \right. \\
&\quad \left. \mathrm{d}_h \left(v_h \rfloor \left(\frac{\partial h_0}{\partial z^\alpha_{[1_i]}} \mathrm{d}z^\alpha \wedge \partial_i \rfloor \mathrm{dZ} \right) \right) + \mathrm{d}_v \left(v_h \rfloor \left(\frac{\partial h_0}{\partial z^\alpha_{[1_i]}} \mathrm{d}z^\alpha \wedge \partial_i \rfloor \mathrm{dZ} \right) \right) \right) \\
&= \ \int_{\mathcal{D}} \left(j^2\sigma\right)^* v_h \rfloor \left(\delta_\alpha h_0 \mathrm{d}z^\alpha \wedge \mathrm{dZ} - \mathrm{d}_h \left(\partial_\alpha^{[1_i]} (h_0) \, \mathrm{d}z^\alpha \wedge \partial_i \rfloor \mathrm{dZ} \right) \right) \ .
\end{aligned}$$

Thus we are able to conclude that

$$\begin{aligned}
\left(j^2\sigma\right)^* \mathrm{L}_{j^1 v_h} (h_0 \mathrm{dZ}) \ &= \ \left(j^2\sigma\right)^* j^1 v_h \rfloor \mathrm{d} (h_0 \mathrm{dZ}) \\
&= \ \left(j^2\sigma\right)^* v_h \rfloor \left(\delta_\alpha h_0 \mathrm{d}z^\alpha \wedge \mathrm{dZ} - \mathrm{d}_h \left(\partial_\alpha^{[1_i]} (h_0) \, \mathrm{d}z^\alpha \wedge \partial_i \rfloor \mathrm{dZ} \right) \right) \ ,
\end{aligned} \tag{11.4}$$

whereby the definition of the variational derivative as a map between the spaces $(\eta_0^1)^* \wedge_r^0 (\mathcal{T}^* (\mathcal{H}))$ and $(\eta_0^2)^* \wedge_r^1 (\mathcal{T}^* (\mathcal{H}))$ is confirmed. This result can be used in (11.3) and consequently it follows from the definition of the Hamilton operator (11.1) and (11.2) that

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathfrak{H} &= \int_{\mathcal{D}} \left(j^2\sigma\right)^* \left(-\mathfrak{R}\left(\delta\left(h_0 \mathrm{dZ}\right)\right) \rfloor \delta\left(h_0 \mathrm{dZ}\right) + \left(u^{\varsigma} y_{\varsigma}\right) \mathrm{dZ}\right) \\
&\quad - \int_{\mathcal{D}} \left(j^2\sigma\right)^* \left(v_h \rfloor \mathrm{d}_h \left(\partial_\alpha^{[1_i]}\left(h_0\right) \mathrm{d}z^\alpha \wedge \partial_i \rfloor \mathrm{dZ}\right)\right) \\
&= \int_{\mathcal{D}} \left(j^2\sigma\right)^* \left(-\mathfrak{R}\left(\delta\left(h_0 \mathrm{dZ}\right)\right) \rfloor \delta\left(h_0 \mathrm{dZ}\right) + \left(u^{\varsigma} y_{\varsigma}\right) \mathrm{dZ}\right) \\
&\quad + \int_{\mathcal{D}} \left(j^2\sigma\right)^* \mathrm{d}_h \left(v_h \rfloor \left(\partial_\alpha^{[1_i]}\left(h_0\right) \mathrm{d}z^\alpha \wedge \partial_i \rfloor \mathrm{dZ}\right)\right) \\
&= \int_{\mathcal{D}} \left(j^2\sigma\right)^* \left(-\mathfrak{R}\left(\delta\left(h_0 \mathrm{dZ}\right)\right) \rfloor \delta\left(h_0 \mathrm{dZ}\right) + \left(u^{\varsigma} y_{\varsigma}\right) \mathrm{dZ}\right) \\
&\quad + \int_{\mathcal{D}} \mathrm{d}\ \left(j^2\sigma\right)^* \left(v_h \rfloor \left(\partial_\alpha^{[1_i]}\left(h_0\right) \mathrm{d}z^\alpha \wedge \partial_i \rfloor \mathrm{dZ}\right)\right) \\
&= \int_{\mathcal{D}} \left(j^2\sigma\right)^* \left(-\mathfrak{R}\left(\delta\left(h_0 \mathrm{dZ}\right)\right) \rfloor \delta\left(h_0 \mathrm{dZ}\right) + \left(u^{\varsigma} y_{\varsigma}\right) \mathrm{dZ}\right) \\
&\quad + \int_{\partial\mathcal{D}} \iota^* \left(\left(j^2\sigma\right)^* \left(v_h \rfloor \left(\partial_\alpha^{[1_i]}\left(h_0\right) \mathrm{d}z^\alpha \wedge \partial_i \rfloor \mathrm{dZ}\right)\right)\right)
\end{aligned}
\tag{11.5}
$$

is met. Here we have taken into account the special properties of the horizontal differential d_h (see definition 4.16).

Equation (11.5) states that the dissipative operator $\mathfrak{R}$, the pairing $u^{\varsigma} y_{\varsigma}$ on the domain, and the boundary term

$$
\begin{aligned}
&\int_{\partial\mathcal{D}} \iota^* \left(\left(j^2\sigma\right)^* \left(v_h \rfloor \partial_\alpha^{[1_i]}\left(h_0\right) \mathrm{d}z^\alpha \wedge \partial_i \rfloor \mathrm{dZ}\right)\right) = \\
&= \int_{\partial\mathcal{D}} \left(j^2\bar{\sigma}\right)^* \ \left(\Psi^2\right)^* \left(v_h \rfloor \partial_\alpha^{[1_i]}\left(h_0\right) \mathrm{d}z^\alpha \wedge \partial_i \rfloor \mathrm{dZ}\right) \\
&= \int_{\partial\mathcal{D}} \left(j^2\bar{\sigma}\right)^* \ \left(v_{h\partial} \rfloor \left(\left(\Psi^2\right)^* \left(\partial_\alpha^{[1_i]}\left(h_0\right) \mathrm{d}z^\alpha \wedge \partial_i \rfloor \mathrm{dZ}\right)\right)\right) \\
&= \int_{\partial\mathcal{D}} \left(j^2\bar{\sigma}\right)^* \left(\underbrace{\left(\dot{z}^\alpha \circ \Psi^2\right) \partial_\alpha \rfloor \left(\left(\partial_\alpha^{[1_r]} h_0 \circ \Psi^2\right)\ \mathrm{d}z^\alpha \wedge \mathrm{d}\bar{\mathrm{Z}}\right)}_{\lambda_\partial}\right)
\end{aligned}
\tag{11.6}
$$

determine the evolution of the Hamiltonian functional $\mathcal{H}$. Here the second prolongation of the boundary section $(j^2\bar{\sigma}) : \partial\mathcal{D} \to J^2\bar{\eta}$, and the boundary volume form $\mathrm{d}\bar{\mathrm{Z}} = \partial_r \rfloor \mathrm{dZ} = (-1)^{r-1} \mathrm{d}\bar{Z}^1 \wedge \ldots \wedge \mathrm{d}\bar{Z}^{r-1}$ are introduced.

Consequently we see that the special construction of the Hamilton operator enables us to determine the domain part of the time derivative of the Hamiltonian functional in a straight forward manner. Additionally, we are able to identify the boundary impact to be given by the form λ_∂. Obviously it is left to introduce the boundary ports for the I-pHd system such that the system purely interacts with the surrounding through ports.

Before we are able to extract this information, we have to define the appropriate spaces, where the vector and form components of the form λ_∂ are living in.

11.3.1 Definition of the boundary spaces

In contrary to the determination procedure of the collocated output y on the domain, as stated in equation (11.2), it will turn out that it is no more possible to give a unique separation of the in- and output variables at the boundary. In order to overcome this problem we introduce two pairs of dual bundles and state that both of them could be used as boundary ports for the I-pHd system.

The first pair is given by the boundary input vector bundle $\left(\bar{\mathcal{U}}, \bar{\eta}_{\bar{\mathcal{U}}}, \partial\mathcal{D}\right)$ with local coordinates $\left(\bar{Z}^j, \bar{u}^\gamma\right), j = 1, \ldots, (r-1), \gamma = 1, \ldots, \bar{m}$ and the basis $\{e_\gamma\}$ and its dual – the boundary output vector bundle $\left(\bar{\mathcal{Y}}, \bar{\eta}_{\bar{\mathcal{Y}}}, \partial\mathcal{D}\right)$ with local coordinates $\left(\bar{Z}^j, \bar{y}_\gamma\right)$ and basis $\left\{\bar{e}^\gamma \otimes \mathrm{d}\bar{\mathrm{Z}}\right\}$. The second pair is given by the boundary input vector bundle $\left(\tilde{\mathcal{U}}, \bar{\eta}_{\tilde{\mathcal{U}}}, \partial\mathcal{D}\right)$ with local coordinates $\left(\bar{Z}^j, \tilde{u}_\gamma\right), j = 1, \ldots, (r-1), \gamma = 1, \ldots, \tilde{m}$ and the basis $\{\tilde{e}^\gamma\}$ and its dual – the boundary output vector bundle $\left(\tilde{\mathcal{Y}}, \bar{\eta}_{\tilde{\mathcal{Y}}}, \partial\mathcal{D}\right)$ with local coordinates $\left(\bar{Z}^j, \tilde{y}^\gamma\right)$ and basis $\left\{\mathrm{d}\bar{\mathrm{Z}} \otimes \tilde{e}_\gamma\right\}$.

Additionally, we introduce similarly to the domain vector bundles four vector bundles according to the boundary bundle $\bar{\eta}$:

- The n^{th} jet bundle $(J^n\bar{\eta}, \bar{\eta}^n, \partial\mathcal{D})$ with the adapted coordinates $\left(\bar{Z}^i, z^\alpha, z^\alpha_{\left[\bar{J};j_r\right]}\right)$ according to the total manifold $J^n\bar{\eta}$;
- the vector bundle $\left(\wedge^0_{r-1}\mathcal{T}^*\left(\bar{\mathcal{H}}\right), \tau_{\wedge^0_{r-1}\mathcal{T}^*\left(\bar{\mathcal{H}}\right)}, \bar{\mathcal{H}}\right)$, where $\wedge^0_{r-1}\mathcal{T}^*\left(\bar{\mathcal{H}}\right) = \mathrm{span}\left\{\mathrm{d}\bar{\mathrm{Z}}\right\}$ and local coordinates $\left(\bar{Z}^i, z^\alpha, \mathfrak{r}_\partial\right)$ and
- the bundle $\left(\wedge^1_{r-1}\mathcal{T}^*\left(\bar{\mathcal{H}}\right), \tau_{\wedge^1_{r-1}\mathcal{T}^*\left(\bar{\mathcal{H}}\right)}, \bar{\mathcal{H}}\right)$, where $\wedge^1_{r-1}\mathcal{T}^*\left(\bar{\mathcal{H}}\right) = \mathrm{span}\left\{\mathrm{d}z^\alpha \wedge \mathrm{d}\bar{\mathrm{Z}}\right\}$and local coordinates $\left(\bar{Z}^i, z^\alpha, \dot{\mathfrak{r}}_{\partial\alpha}\right)$.
- The dual bundle to $\tau_{\wedge^1_{r-1}\mathcal{T}^*\left(\bar{\mathcal{H}}\right)}$ is given by the vertical boundary tangent bundle $\left(V\bar{\eta}, \tau_{\bar{\mathcal{H}}}|_{V\bar{\eta}}, \bar{\mathcal{H}}\right)$.

By construction we obtain the bilinear products

$$
\begin{array}{rcl}
\bar{\mathcal{U}} \times_{\partial\mathcal{D}} \bar{\mathcal{Y}} & \to & \wedge^0_{r-1}\mathcal{T}^*\left(\bar{\mathcal{H}}\right) \\
\left(\bar{u}^\gamma \bar{e}_\gamma, \bar{y}_\varsigma \bar{e}^\varsigma \otimes \mathrm{d}\bar{\mathrm{Z}}\right) & \to & \bar{u}^\gamma \bar{e}_\gamma \rfloor \bar{y}_\varsigma \bar{e}^\varsigma \otimes \mathrm{d}\bar{\mathrm{Z}}
\end{array}
$$

respectively

$$
\begin{array}{rcl}
\tilde{\mathcal{Y}} \times_{\partial\mathcal{D}} \tilde{\mathcal{U}} & \to & \wedge^0_{r-1}\mathcal{T}^*\left(\bar{\mathcal{H}}\right) \\
\left(\tilde{y}^\gamma \mathrm{d}\bar{\mathrm{Z}} \otimes \tilde{e}_\gamma, \tilde{u}_\varsigma \tilde{e}^\varsigma\right) & \to & \tilde{y}^\gamma \mathrm{d}\bar{\mathrm{Z}} \otimes \tilde{e}_\gamma \rfloor \tilde{u}_\varsigma \tilde{e}^\varsigma \, .
\end{array}
$$

The form λ_∂ stated in equation (11.6) meets obviously $\lambda_\partial \in \Gamma\left(\left(\bar{\eta}^2_0\right)^* \tau_{\wedge^0_{r-1}\mathcal{T}^*\left(\bar{\mathcal{H}}\right)}\right)$. It is now assumed that λ_∂ is generated by both bilinear products, i.e.

$$
\begin{array}{rcl}
\lambda_\partial & = & \left(\dot{z}^\alpha \circ \Psi^2\right) \partial_\alpha \rfloor \left(\left(\partial^{[1_r]}_\alpha h_0 \circ \Psi^2\right) \, \mathrm{d}z^\alpha \wedge \mathrm{d}\bar{\mathrm{Z}}\right) \\
 & = & \bar{u}^\gamma \bar{e}_\gamma \rfloor \bar{y}_\varsigma \bar{e}^\varsigma \otimes \mathrm{d}\bar{\mathrm{Z}} \\
 & = & \tilde{y}^\gamma \mathrm{d}\bar{\mathrm{Z}} \otimes \tilde{e}_\gamma \rfloor \tilde{u}_\varsigma \tilde{e}^\varsigma \\
 & = & \bar{u}^\gamma \bar{y}_\gamma \mathrm{d}\bar{\mathrm{Z}} = \tilde{y}^\gamma \tilde{u}_\gamma \mathrm{d}\bar{\mathrm{Z}} \, .
\end{array}
$$

It is worth mentioning that the vector field and the form part of λ_∂ are elements of the introduced boundary vector bundles [5], i.e. $(\dot{z}^\alpha \circ \Psi^2)\,\partial_\alpha \in \Gamma\left(\left(\bar{\eta}_0^2\right)^* \tau_{\bar{\mathcal{H}}|V\bar{\eta}}\right)$, and $\left(\partial_\alpha^{[1_r]} h_0 \circ \Psi^2\right)\;\mathrm{d}z^\alpha \wedge \mathrm{d}\bar{\mathrm{Z}} \in \Gamma\left(\left(\bar{\eta}_0^1\right)^* \tau_{\wedge^1_{r-1}\mathcal{T}^*(\bar{\mathcal{H}})}\right)$.

Now it is left to formulate the relation between the form λ_∂ and the dual in- and output bundles by means of maps $\bar{\mathfrak{B}}$, $\bar{\mathfrak{B}}^*$ resp. $\tilde{\mathfrak{B}}$, $\tilde{\mathfrak{B}}^*$.

11.3.2 Determination of the boundary maps

At first we consider the bundle pairing $\bar{\eta}_{\bar{\mathcal{U}}}$ and $\bar{\eta}_{\bar{\mathcal{Y}}}$ and formulate the boundary input map $\bar{\mathfrak{B}}$ to determine the vector part of λ_∂ by

$$\begin{aligned} \bar{\mathfrak{B}}\left(\bar{u}\right) &= \bar{u}^\gamma \bar{e}_\gamma \rfloor \bar{B}_\zeta^\alpha \bar{e}^\zeta \otimes \partial_\alpha \\ &= \left(\dot{z}^\alpha \circ \Psi^2\right)\partial_\alpha \ . \end{aligned}$$

Obviously we have introduced a map of the form

$$\bar{\mathfrak{B}} : \bar{\mathcal{U}} \to \left(\bar{\eta}_0^2\right)^* V\bar{\eta} \ .$$

Consequently we can reformulate λ_∂ and get

$$\lambda_\partial = \bar{u}^\gamma \bar{B}_\gamma^\alpha \partial_\alpha \rfloor \left(\left(\partial_\alpha^{[1_r]} h_0 \circ \Psi^2\right)\;\mathrm{d}z^\alpha \wedge \mathrm{d}\bar{\mathrm{Z}}\right) \ .$$

This leads directly to the adjoint map

$$\bar{\mathfrak{B}}^* : \left(\bar{\eta}_0^2\right)^* \wedge^1_{r-1} \mathcal{T}^*\left(\bar{\mathcal{H}}\right) \to \bar{\mathcal{Y}}$$

given by

$$\begin{aligned} \bar{\mathfrak{B}}^*\left(\partial_\alpha^{[1_r]} h_0 \circ \Psi^2\right) &= \bar{B}_\zeta^\alpha \bar{e}^\zeta \otimes \partial_\alpha \rfloor \left(\left(\partial_\alpha^{[1_r]} h_0 \circ \Psi^2\right)\;\mathrm{d}z^\alpha \wedge \mathrm{d}\bar{\mathrm{Z}}\right) \\ &= \bar{B}_\zeta^\alpha \left(\partial_\alpha^{[1_r]} h_0 \circ \Psi^2\right) \bar{e}^\zeta \otimes \mathrm{d}\bar{\mathrm{Z}} \\ &= \bar{y}_\zeta \bar{e}^\zeta \otimes \mathrm{d}\bar{\mathrm{Z}} \ . \end{aligned}$$

We see that this port configuration is fully defined by the tensor

$$\bar{B}_\zeta^\alpha \bar{e}^\zeta \otimes \partial_\alpha \ .$$

[5] Here and subsequently we will use vector bundles and their corresponding pull-back bundles synonymously.

This procedure can be visualized in a diagram of the form

$$
\begin{tikzcd}
 & \mathcal{D} & \partial\mathcal{D} \arrow[l, "\iota"'] & & \\
 & & \bar{\mathcal{H}} \arrow[u, "\hat{\eta}"] & & \\
\partial\mathcal{D} & & J^2\bar{\eta} \arrow[u, "\bar{\eta}_0^2"] & & \partial\mathcal{D} \\
\bar{\mathcal{U}} \arrow[u, "\bar{\eta}_{\bar{\mathcal{U}}}"] \arrow[r, "\bar{\mathfrak{B}}"] & \left(\bar{\eta}_0^2\right)^* V\bar{\eta} \arrow[ur, "\left(\bar{\eta}_0^2\right)^* \tau_{\bar{\mathcal{H}}|V\bar{\eta}}"] & & \left(\bar{\eta}_0^2\right)^* \wedge_{r-1}^1 \mathcal{T}^*\left(\bar{\mathcal{H}}\right) \arrow[ul, "\left(\bar{\eta}_0^2\right)^* \tau_{\wedge_{r-1}^1 \mathcal{T}^*\left(\bar{\mathcal{H}}\right)}"'] \arrow[r, "\bar{\mathfrak{B}}^*"] & \bar{\mathcal{Y}}. \arrow[u, "\bar{\eta}_{\bar{\mathcal{Y}}}"']
\end{tikzcd}
$$

Now we consider $\bar{\eta}_{\tilde{\mathcal{U}}}$ and $\bar{\eta}_{\tilde{\mathcal{Y}}}$ and formulate the boundary input $\tilde{\mathfrak{B}}$ map to determine the form part of λ_∂ by

$$
\begin{aligned}
\tilde{\mathfrak{B}}\left(\tilde{u}\right) &= \tilde{B}_\alpha^\gamma \mathrm{d}z^\alpha \wedge \mathrm{d}\bar{\mathrm{Z}} \otimes \tilde{e}_\gamma \rfloor \tilde{u}_\zeta \tilde{e}^\zeta \\
&= \left(\partial_\alpha^{[1_r]} h_0 \circ \Psi^2\right) \mathrm{d}z^\alpha \wedge \mathrm{d}\bar{\mathrm{Z}} \, .
\end{aligned}
$$

Here the input map $\tilde{\mathfrak{B}}$ is given by

$$
\tilde{\mathfrak{B}} : \tilde{\mathcal{U}} \to \left(\bar{\eta}_0^2\right)^* \wedge_{r-1}^1 \mathcal{T}^*\left(\bar{\mathcal{H}}\right) \; .
$$

The definition of the input map results in

$$
\lambda_\partial = \left(\dot{z}^\alpha \circ \Psi^2\right) \partial_\alpha \rfloor \tilde{B}_\alpha^{[0_r]\gamma} \mathrm{d}z^\alpha \wedge \mathrm{d}\bar{\mathrm{Z}} \otimes \tilde{e}_\gamma \rfloor \tilde{u}_\zeta \tilde{e}^\zeta
$$

and consequently the adjoint map

$$
\tilde{\mathfrak{B}}^* : \left(\bar{\eta}_0^2\right)^* V\bar{\eta} \to \tilde{\mathcal{Y}}
$$

is given by

$$
\begin{aligned}
\tilde{\mathfrak{B}}^*\left(\dot{z}^\alpha \circ \iota\right) &= \left(\dot{z}^\alpha \circ \Psi^2\right) \partial_\alpha \rfloor \bar{B}_\alpha^{[0_r]\gamma} \mathrm{d}z^\alpha \wedge \mathrm{d}\bar{\mathrm{Z}} \otimes \tilde{e}_\gamma \\
&= \left(\dot{z}^\alpha \circ \Psi^2\right) \bar{B}_\alpha^{[0_r]\gamma} \mathrm{d}\bar{\mathrm{Z}} \otimes \tilde{e}_\gamma \\
&= \tilde{y}^\gamma \mathrm{d}\bar{\mathrm{Z}} \otimes \tilde{e}_\gamma \, .
\end{aligned}
$$

We see that this port configuration is purely defined by the tensor

$$
\tilde{B}_\alpha^\gamma \mathrm{d}z^\alpha \wedge \mathrm{d}\bar{\mathrm{Z}} \otimes \tilde{e}_\gamma \, .
$$

This procedure can be visualized in a diagram of the form

$$\begin{tikzcd}
 & \mathcal{D} & \partial\mathcal{D} \arrow[l, "\iota"] & & \\
 & & \bar{\mathcal{H}} \arrow[u, "\bar{\eta}"] & & \\
\partial\mathcal{D} & & J^2\bar{\eta} \arrow[u, "\bar{\eta}_0^2"] & & \partial\mathcal{D} \\
\tilde{\mathcal{Y}} \arrow[u, "\eta_{\tilde{\mathcal{Y}}}"] & \left(\bar{\eta}_0^2\right)^* V\bar{\eta} \arrow[l, "\tilde{\mathfrak{B}}^*"'] \arrow[ur, "\left(\bar{\eta}_0^2\right)^* \tau_{\bar{\mathcal{H}}|V\bar{\eta}}"] & & \left(\bar{\eta}_0^2\right)^* \wedge_{r-1}^1 \mathcal{T}^*\left(\bar{\mathcal{H}}\right) \arrow[ul, "\left(\bar{\eta}_0^2\right)^* \tau_{\wedge_{r-1}^1 \mathcal{T}^*(\bar{\mathcal{H}})}"'] & \tilde{\mathcal{U}}. \arrow[u, "\eta_{\tilde{\mathcal{U}}}"'] \arrow[l, "\tilde{\mathfrak{B}}"']
\end{tikzcd}$$

If $\tilde{B}_{\alpha}^{\gamma} \mathrm{d}z^{\alpha} \wedge \mathrm{d}\bar{\mathrm{Z}} \otimes \tilde{e}_{\gamma} = \mathrm{d}\left(H_{\partial}^{\gamma} \mathrm{d}\bar{\mathrm{Z}}\right) \otimes \tilde{e}^{\zeta}$ is met, we are additionally able to identify the collocated output with the time derivative

$$\tilde{y}^{\alpha} = \left(\dot{z}^{\beta} \circ \Psi^2\right) \partial_{\beta} \rfloor \mathrm{d}\left(H_{\partial}^{\gamma}\right)$$

of the functions H_{∂}^{γ} along the solution of the system.

11.3.3 Representation as I-pHd system

Finally we are able to define the representation of I-pHd systems with 1^{st} order Hamiltonian on η with local coordinates (Z^i, z^{α}) given by

$$\begin{aligned}
\dot{z} &= (\mathfrak{J} - \mathfrak{R})\left(\delta\left(h_0 \mathrm{dZ}\right)\right) + \mathfrak{B}(u) \\
y &= \mathfrak{B}^*\left(\delta\left(h_0 \mathrm{dZ}\right)\right) \\
\dot{z}^{\alpha} \circ \Psi^2 &= \bar{\mathfrak{B}}(\bar{u}) \\
\bar{y} &= \bar{\mathfrak{B}}^*\left(\partial_{\alpha}^{[1_r]} h_0 \circ \Psi^2\right)
\end{aligned}$$

respectively

$$\begin{aligned}
\dot{z} &= (\mathfrak{J} - \mathfrak{R})\left(\delta\left(h_0 \mathrm{dZ}\right)\right) + \mathfrak{B}(u) \\
y &= \mathfrak{B}^*\left(\delta\left(h_0 \mathrm{dZ}\right)\right) \\
\partial_{\alpha}^{[1_r]} h_0 \circ \Psi^2 &= \tilde{\mathfrak{B}}(\tilde{u}) \\
\tilde{y} &= \tilde{\mathfrak{B}}^*\left(\left(\dot{z}^{\alpha} \circ \Psi^2\right) \partial_{\alpha}\right) \ .
\end{aligned}$$

Both representations could be used to introduce a symbol for I-pHd systems – see figure 11.2.

Remark 11.3 *The in- and output maps $\mathfrak{B}$, $\mathfrak{B}^*$, $\bar{\mathfrak{B}}$, $\bar{\mathfrak{B}}^*$, $\tilde{\mathfrak{B}}$, $\tilde{\mathfrak{B}}^*$ must result from a modeling procedure as, e.g., calculus of variations. Here only the necessity of such maps in order to obtain a pHd description was discussed.*

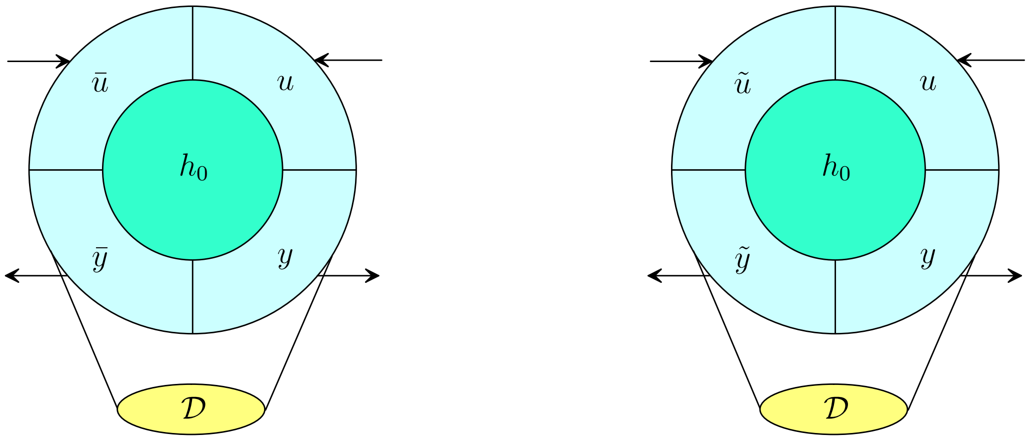

Figure 11.2: The representation I-pHd system with 1st order Hamiltonian.

11.4 Systems with n^{th} order Hamiltonian

We state, following the results of chapter 8, that the extended Hamiltonian is built by

$$h_{ext} = h_0 \mathrm{dZ} + p_\alpha^{[J-I_1,J]} \omega^\alpha_{[J-I_1]} \wedge \partial_{I_1} \rfloor \mathrm{dZ} + \mathrm{d}\left(p_\alpha^{\left[\bar{J}-I_2,\bar{J};j_r\right]} \omega^\alpha_{\partial\left[\bar{J}-I_2;j_r\right]} \wedge \partial_{I_2} \rfloor \mathrm{d}\bar{\mathrm{Z}} \right)$$

where

$$\begin{aligned} \#J &= 1,\ldots,n\,, \quad \#\bar{J} = 1,\ldots,n\,, \quad j_r = 0,\ldots,k_r \\ \#I_1 &= 1\,, \quad I_1 = 1_j\,, \quad j = 1,\ldots,r \\ \#I_2 &= 1\,, \quad I_2 = 1_i\,, \quad i = 1,\ldots,r-1 \end{aligned}$$

is used. Similarly to the construction of the extended Cartan form, we have to determine the functions $p_\alpha^{[J-I_1,J]}$ resp. $p_\alpha^{\left[\bar{J}-I_2,\bar{J};j_r\right]}$ such that the prolongation of the Hamilton operator is suppressed in the domain condition resp. the boundary condition. By construction it is possible to make use of equation (7.9), where we have to replace l by h_0 in order to obtain the condition for the functions $p_\alpha^{[J-I_1,J]}$. Additionally, equation (8.3) supplies the corresponding relations for the functions $p_\alpha^{\left[\bar{J}-I_2,\bar{J};j_r\right]}$.

Having the extended Hamiltonian form at ones disposal, we obtain

$$
\begin{aligned}
&\int_{\mathcal{D}} \left(j^{2n}\sigma\right)^* \mathrm{L}_{j^n v_h}\left(h_{ext}\right) = \qquad (11.7)\\
&\quad = \int_{\mathcal{D}} \left(j^{2n}\sigma\right)^* \left(v_h \rfloor \mathrm{d}\left(h_{ext}\right) + \mathrm{d}\left(j^n v_h \rfloor h_{ext}\right)\right)\\
&\quad = \int_{\mathcal{D}} \left(j^{2n}\sigma\right)^* \left(v_h \rfloor \mathrm{d}\left(h_{ext}\right)\right) + \int_{\partial\mathcal{D}} \iota^*\left(\left(j^{2n}\sigma\right)^* \left(j^n v_h \rfloor h_{ext}\right)\right)\\
&\quad = \int_{\mathcal{D}} \left(j^{2n}\sigma\right)^* \left(v_h \rfloor \mathrm{d}\left(h_{ext}\right)\right) + \int_{\partial\mathcal{D}} \left(j^{2n}\bar{\sigma}\right)^* \left(v_{h\partial} \rfloor \left(\Psi^{2n}\right)^* h_{ext}\right)\\
&\quad = \int_{\mathcal{D}} \left(j^{2n}\sigma\right)^* \left(v_h \rfloor \left(\delta_\alpha h_0 \mathrm{d}z^\alpha \wedge \mathrm{dZ}\right)\right)\\
&\qquad + \int_{\partial\mathcal{D}} \left(j^{2n}\bar{\sigma}\right)^* \left(v_{h\partial} \rfloor \left(\left(\delta_\alpha^{[J_r]}\left(h_0\right) \circ \Psi^{2n}\right) \mathrm{d}z^\alpha_{[\bar{J}_r]} \wedge \mathrm{d}\bar{\mathrm{Z}}\right)\right)\\
&\quad = \int_{\mathcal{D}} \left(j^{2n}\sigma\right)^* \left(v_h \rfloor \left(\delta_\alpha h_0 \mathrm{d}z^\alpha \wedge \mathrm{dY}\right)\right)\\
&\qquad + \int_{\partial\mathcal{D}} \left(j^{2n}\bar{\sigma}\right)^* \left(\underbrace{\left(\mathrm{L}_{d_{[J_r]}}\left(\dot{z}^\alpha\right) \circ \Psi^{2n}\right) \partial_\alpha^{[\bar{J}_r]} \rfloor \left(\left(\delta_\alpha^{[J_r]}\left(h_0\right) \circ \Psi^{2n}\right) \mathrm{d}z^\alpha_{[\bar{J}_r]} \wedge \mathrm{d}\bar{\mathrm{Z}}\right)}_{\lambda_\partial}\right) .
\end{aligned}
$$

Obviously we obtain precisely the same configuration according to a certain multi-index $[J_r] = [0\ldots 0 j_r]$ resp. $\left[\bar{J}_r\right] = [0\ldots 0; j_r]$, as we did in the 1st order case, i.e. $[J_r] = [0\ldots 00]$ resp. $\left[\bar{J}_r\right] = [0\ldots 0; 0]$. Consequently we have to carry out the same procedure as we have used for the 1st order case according to every multi-index $[J_r]$.

Definition of the boundary spaces

The first pair of boundary spaces is given by the boundary input vector bundle $\left(\bar{\mathcal{U}}_{(j_r)}, \bar{\eta}_{\bar{\mathcal{U}}_{(j_r)}}, \partial\mathcal{D}\right)$ with local coordinates $\left(\bar{Z}^j, \bar{u}_{(j_r)}{}^\gamma\right)$, $j = 1, \ldots, (r-1)$, $\gamma = 1, \ldots, \bar{m}_{(j_r)}$ and the basis $\left\{\bar{e}_{(j_r)}{}_\gamma\right\}$ and its dual – the boundary output vector bundle $\left(\bar{\mathcal{Y}}_{(j_r)}, \bar{\eta}_{\bar{\mathcal{Y}}_{(j_r)}}, \partial\mathcal{D}\right)$ with local coordinates $\left(\bar{Z}^j, \bar{y}_{(j_r)}{}_\gamma\right)$ and basis $\left\{\bar{e}_{(j_r)}{}^\gamma \otimes \mathrm{d}\bar{\mathrm{Z}}\right\}$. The second pair is given by the boundary input vector bundle $\left(\tilde{\mathcal{U}}_{(j_r)}, \bar{\eta}_{\tilde{\mathcal{U}}_{(j_r)}}, \partial\mathcal{D}\right)$ with local coordinates $\left(\bar{Z}^j, \tilde{u}_{(j_r)}{}_\gamma\right)$, $j = 1, \ldots, (r-1)$, $\gamma = 1, \ldots, \tilde{m}_{(j_r)}$ and the basis $\left\{\tilde{e}_{(j_r)}{}^\gamma\right\}$ and its dual – the boundary output vector bundle $\left(\tilde{\mathcal{Y}}_{(j_r)}, \bar{\eta}_{\tilde{\mathcal{Y}}_{(j_r)}}, \partial\mathcal{D}\right)$ with local coordinates $\left(\bar{Z}^j, \tilde{y}_{(j_r)}{}^\gamma\right)$ and basis $\left\{\mathrm{d}\bar{\mathrm{Z}} \otimes \tilde{e}_{(j_r)}{}_\gamma\right\}$.

Additionally, we introduce similarly to the 1st order case the vector bundle

$$
\left(\wedge^1_{r-1}\mathcal{T}^*\left(\bar{\mathcal{H}}\right), \tau_{\wedge^1_{r-1}\mathcal{T}^*\left(\bar{\mathcal{H}}\right)}, \bar{\mathcal{H}}\right) ,
$$

where $\wedge^1_{r-1}\mathcal{T}^*\left(\bar{\mathcal{H}}\right) = \mathrm{span}\left\{\mathrm{d}z^\alpha_{[\bar{J}_r]} \wedge \mathrm{d}\bar{\mathrm{Z}}\right\}$ [6] and local coordinates $\left(\bar{Z}^i, z^\alpha, \dot{\mathfrak{r}}^{(j_r)}_\alpha\right)$.

[6]The total manifold $\mathcal{E}_{h\partial}$ has the local coordinates $\left(\bar{X}^i, z^\alpha_{[J_r]}\right)$

By construction we obtain the bilinear products

$$\begin{aligned} \bar{\mathcal{U}}_{(j_r)} \times_{\partial\mathcal{D}} \bar{\mathcal{Y}}_{(j_r)} &\rightarrow \wedge^0_{r-1}\mathcal{T}^*\left(\bar{\mathcal{H}}\right) \\ \left(\bar{u}_{(j_r)}{}^{\gamma}\,\bar{e}_{(j_r)}{}_{\gamma}\,,\bar{y}_{(j_r)}{}_{\zeta}\,\bar{e}_{(j_r)}{}^{\zeta}\otimes \mathrm{d}\bar{\mathrm{Z}}\right) &\rightarrow \bar{u}_{(j_r)}{}^{\gamma}\,\bar{e}_{(j_r)}{}_{\gamma}\rfloor\,\bar{y}_{(j_r)}{}_{\zeta}\,\bar{e}_{(j_r)}{}^{\zeta}\otimes \mathrm{d}\bar{\mathrm{Z}} \end{aligned}$$

respectively

$$\begin{aligned} \tilde{\mathcal{Y}}_{(j_r)} \times_{\partial\mathcal{D}} \tilde{\mathcal{U}}_{(j_r)} &\rightarrow \wedge^0_{r-1}\mathcal{T}^*\left(\bar{\mathcal{H}}\right) \\ \left(\tilde{y}_{(j_r)}{}^{\gamma}\,\mathrm{d}\bar{\mathrm{Z}}\otimes\tilde{e}_{(j_r)}{}_{\gamma}\,,\tilde{u}_{(j_r)}{}_{\zeta}\,\tilde{e}_{(j_r)}{}^{\zeta}\right) &\rightarrow \tilde{y}_{(j_r)}{}^{\gamma}\,\mathrm{d}\bar{\mathrm{Z}}\otimes\tilde{e}_{(j_r)}{}_{\gamma}\rfloor\,\tilde{u}_{(j_r)}{}_{\zeta}\,\tilde{e}_{(j_r)}{}^{\zeta}\ . \end{aligned}$$

The form λ_∂ stated in equation (11.6) meets obviously $\lambda_\partial \in \Gamma\left(\left(\bar{\eta}^{2n}_0\right)^*\tau_{\wedge^0_{r-1}\mathcal{T}^*(\bar{\mathcal{H}})}\right)$ and is built by

$$\lambda_\partial = \sum_{j_r=0}^{k}\lambda^{(j_r)}_\partial\ .$$

Now we assume all forms $\lambda^{(j_r)}_\partial$ to be generated by both bilinear products, i.e.

$$\begin{aligned} \lambda^{(j_r)}_\partial &= \left(\mathrm{L}_{d_{J_r}}\left(\dot{z}^\alpha\right)\circ\Psi^{2n}\right)\partial^{[\bar{J}_r]}_\alpha\rfloor\left(\left(\delta^{[J_r]}_\alpha\left(h_0\right)\circ\Psi^{2n}\right)\mathrm{d}z^\alpha_{[\bar{J}_r]}\wedge\mathrm{d}\bar{\mathrm{Z}}\right) \\ &= \bar{u}_{(j_r)}{}^{\gamma}\,\bar{e}_{(j_r)}{}_{\gamma}\rfloor\,\bar{y}_{(j_r)}{}_{\zeta}\,\bar{e}_{(j_r)}{}^{\zeta}\otimes\mathrm{d}\bar{\mathrm{Z}} \\ &= \tilde{y}_{(j_r)}{}^{\gamma}\,\mathrm{d}\bar{\mathrm{Z}}\otimes\tilde{e}_{(j_r)}{}_{\gamma}\rfloor\,\tilde{u}_{(j_r)}{}_{\zeta}\,\tilde{e}_{\partial}{}^{\zeta} \\ &= \bar{u}_{(j_r)}{}^{\gamma}\,\bar{y}_{(j_r)}{}_{\gamma}\,\mathrm{d}\bar{\mathrm{Z}} = \tilde{y}_{(j_r)}{}^{\gamma}\,\tilde{u}_{(j_r)}{}_{\gamma}\,\mathrm{d}\bar{\mathrm{Z}}\ . \end{aligned}$$

Determination of the boundary maps

At first we consider the bundles $\bar{\eta}_{\bar{\mathcal{U}}_{(j_r)}}$, $\bar{\eta}_{\bar{\mathcal{Y}}_{(j_r)}}$ and formulate the boundary input map such that the vector part of $\lambda^{(j_r)}_\partial$ is determined, i.e.

$$\begin{aligned} \bar{\mathfrak{B}}^{(j_r)}\left(\bar{u}_{(j_r)}\right) &= \bar{u}_{(j_r)}{}^{\gamma}\,\bar{e}_{(j_r)}{}_{\gamma}\rfloor\bar{B}^{(j_r),\alpha}_{\zeta}\,\bar{e}_{(j_r)}{}^{\zeta}\otimes\partial^{[\bar{J}_r]}_\alpha \\ &= \left(\mathrm{L}_{d_{[J_r]}}\left(\dot{z}^\alpha\right)\circ\Psi^{2n}\right)\partial^{[\bar{J}_r]}_\alpha\ . \end{aligned}$$

Obviously we have introduced a map of the form

$$\bar{\mathfrak{B}}^{(j_r)}:\bar{\mathcal{U}}_{(j_r)}\rightarrow\left(\bar{\eta}^{2n}_0\right)^*V\eta\ .$$

Consequently we can reformulate $\lambda^{(j_r)}_\partial$ and get

$$\lambda^{(j_r)}_\partial = \bar{u}_{(j_r)}{}^{\gamma}\,\bar{B}^{(j_r),\alpha}_{\zeta}\partial^{[\bar{J}_r]}_\alpha\rfloor\left(\left(\delta^{[J_r]}_\alpha\left(h_0\right)\circ\Psi^{2n}\right)\mathrm{d}z^\alpha_{[\bar{J}_r]}\wedge\mathrm{d}\bar{\mathrm{Z}}\right)\ .$$

This leads directly to the adjoint map

$$\bar{\mathfrak{B}}^{(j_r)*}:\left(\bar{\eta}^{2n}_0\right)^*\wedge^1_{r-1}\mathcal{T}^*\left(\bar{\mathcal{H}}\right)\rightarrow\bar{\mathcal{Y}}_{(j_r)}$$

given by

$$
\begin{aligned}
\bar{\mathfrak{B}}^{(j_r)*}\left(\delta_\alpha^{[J_r]}\left(h_0\right)\circ\Psi^{2n}\right) &= \bar{B}_\zeta^{(j_r),\alpha}\,\bar{e}_{(j_r)}{}^\zeta\otimes\partial_\alpha^{\left[\bar{J}_r\right]}\rfloor\left(\left(\delta_\alpha^{[J_r]}\left(h_0\right)\circ\Psi^{2n}\right)\mathrm{d}z_{\left[\bar{J}_r\right]}^\alpha\wedge\mathrm{d}\bar{\mathrm{Z}}\right)\\
&= \bar{B}_\zeta^{(j_r),\alpha}\left(\delta_\alpha^{[J_r]}\left(h_0\right)\circ\Psi^{2n}\right)\bar{e}_{(j_r)}{}^\zeta\otimes\mathrm{d}\bar{\mathrm{Z}}\\
&= \bar{y}_{(j_r)\,\zeta}\,\bar{e}_{(j_r)}{}^\zeta\otimes\mathrm{d}\bar{\mathrm{Z}}\,.
\end{aligned}
$$

We see that this port configuration is defined by the tensor

$$
\bar{B}_\zeta^{(j_r),\alpha}\,\bar{e}_{(j_r)}{}^\zeta\otimes\partial_\alpha^{\left[\bar{J}_r\right]}\,,\quad\left[\bar{J}_r\right]=\left[0\ldots0;j_r\right].
$$

This procedure can be visualized in a diagram of the form

$$
\begin{tikzcd}
 & \mathcal{D} & \partial\mathcal{D} \arrow[l, "\iota"] & & \\
 & & \bar{\mathcal{H}} \arrow[u, "\bar{\eta}"] & & \\
\partial\mathcal{D} & & J^{2n}\bar{\eta} \arrow[u, "\bar{\eta}_0^{2n}"] & & \partial\mathcal{D} \\
\bar{\mathcal{U}}_{(j_r)} \arrow[u, "\bar{\eta}_{\bar{\mathcal{U}}_{(j_r)}}"] \arrow[r, "\bar{\mathfrak{B}}^{(j_r)}"] & \left(\bar{\eta}_0^{2n}\right)^* V\bar{\eta} \arrow[ur, "\left(\bar{\eta}_0^{2n}\right)^*\tau_{\bar{\mathcal{H}}|V\bar{\eta}}"] & & \left(\bar{\eta}_0^{2n}\right)^*\wedge_{r-1}^1\mathcal{T}^*\left(\bar{\mathcal{H}}\right) \arrow[ul, "\left(\bar{\eta}_0^{2n}\right)^*\tau_{\wedge_{r-1}^1\mathcal{T}^*\left(\bar{\mathcal{H}}\right)}"'] \arrow[r, "\bar{\mathfrak{B}}^{(j_r)*}"] & \bar{\mathcal{Y}}_{(j_r)}. \arrow[u, "\eta_{\bar{\mathcal{Y}}_{(j_r)}}"']
\end{tikzcd}
$$

Now we consider the bundles $\bar{\eta}_{\tilde{\mathcal{U}}_{(j_r)}}, \bar{\eta}_{\tilde{\mathcal{Y}}_{(j_r)}}$ and formulate the boundary input map to determine the form part of $\lambda_\partial^{(j_r)}$

$$
\begin{aligned}
\tilde{\mathfrak{B}}^{(j_r)}\left(\tilde{u}_{(j_r)}\right) &= \tilde{B}_\alpha^{(j_r),\gamma}\mathrm{d}z_{\left[\bar{J}_r\right]}^\alpha\wedge\mathrm{d}\bar{\mathrm{Z}}\otimes\tilde{e}_{(j_r)\,\gamma}\rfloor\,\tilde{u}_{(j_r)\,\zeta}\,\tilde{e}_{(j_r)}{}^\zeta\\
&= \left(\delta_\alpha^{[J_r]}\left(h_0\right)\circ\Psi^{2n}\right)\mathrm{d}z_{\left[\bar{J}_r\right]}^\alpha\wedge\mathrm{d}\bar{\mathrm{Z}}\,.
\end{aligned}
$$

Here the input map $\tilde{\mathfrak{B}}^{(j_r)}$ is given by

$$
\tilde{\mathfrak{B}}^{(j_r)}:\tilde{\mathcal{U}}_{(j_r)}\rightarrow\left(\bar{\eta}_0^{2n}\right)^*\wedge_{r-1}^1\mathcal{T}^*\left(\bar{\mathcal{H}}\right)\ .
$$

The definition of the input map results in

$$
\lambda_\partial^{(j_r)}=\left(\mathrm{L}_{d_{[J_r]}}\left(\dot{z}^\alpha\right)\circ\Psi^{2n}\right)\partial_\alpha^{\left[\bar{J}_r\right]}\rfloor\tilde{B}_\alpha^{(j_r),\gamma}\mathrm{d}z_{\left[\bar{J}_r\right]}^\alpha\wedge\mathrm{d}\bar{\mathrm{Z}}\otimes\tilde{e}_{(j_r)\,\gamma}\rfloor\,\tilde{u}_{(j_r)\,\zeta}\,\tilde{e}_{(j_r)}{}^\zeta
$$

and consequently the adjoint map

$$
\tilde{\mathfrak{B}}^{(j_r)*}:\left(\bar{\eta}_0^{2n}\right)^*V\bar{\eta}\rightarrow\tilde{\mathcal{Y}}_{(j_r)}
$$

is given by

$$
\begin{aligned}
\tilde{\mathfrak{B}}^{(j_r)*}\left(\left(\mathrm{L}_{d_{[J_r]}}\left(\dot{z}^{\alpha}\right)\circ\Psi^{2n}\right)\right) &= \left(\mathrm{L}_{d_{[J_r]}}\left(\dot{z}^{\alpha}\right)\circ\Psi^{2n}\right)\partial_{\alpha}^{\left[\bar{J}_r\right]}\rfloor\tilde{B}_{\alpha}^{(j_r),\gamma}\mathrm{d}z^{\alpha}_{\left[\bar{J}_r\right]}\wedge\mathrm{d}\bar{\mathrm{Z}}\otimes\bar{e}_{(j_r)}{}_{\gamma} \\
&= \left(\mathrm{L}_{d_{[J_r]}}\left(\dot{z}^{\alpha}\right)\circ\Psi^{2n}\right)\tilde{B}_{\alpha}^{(j_r),\gamma}\mathrm{d}\bar{\mathrm{Z}}\otimes\bar{e}_{(j_r)}{}_{\gamma} \\
&= \tilde{y}_{(j_r)}{}^{\gamma}\,\mathrm{d}\bar{\mathrm{Z}}\otimes\tilde{e}_{(j_r)}{}_{\gamma}\ .
\end{aligned}
$$

We see that this port configuration is purely defined by the tensor

$$
\tilde{B}_{\alpha}^{(j_r),\gamma}\mathrm{d}z^{\alpha}_{\left[\bar{J}_r\right]}\wedge\mathrm{d}\bar{\mathrm{Z}}\otimes\tilde{e}_{(j_r)}{}_{\gamma}\quad,\qquad\left[\bar{J}_r\right]=\left[0\ldots0;j_r\right].
$$

This procedure can be visualized in a diagram of the form

$$
\begin{tikzcd}
 & \mathcal{D} & \partial\mathcal{D} \arrow[l, "\iota"'] & & \\
 & & \bar{\mathcal{H}} \arrow[u, "\bar{\eta}"] & & \\
\partial\mathcal{D} & & J^{2n}\bar{\eta} \arrow[u, "\bar{\eta}_0^{2n}"] & & \partial\mathcal{D} \\
\tilde{\mathcal{Y}}_{(j_r)} \arrow[u, "\bar{\eta}_{\tilde{\mathcal{Y}}_{(j_r)}}"] & \left(\bar{\eta}_0^{2n}\right)^{*}V\bar{\eta} \arrow[l, "\tilde{\mathfrak{B}}^{(j_r)*}"'] \arrow[ur, "\left(\bar{\eta}_0^{2n}\right)^{*}\tau_{\bar{\mathcal{H}}|V\bar{\eta}}"] & & \left(\bar{\eta}_0^{2n}\right)^{*}\wedge_{r-1}^{1}\mathcal{T}^{*}\left(\bar{\mathcal{H}}\right) \arrow[ul, "\left(\bar{\eta}_0^{2n}\right)^{*}\tau_{\wedge_{r-1}^{1}\mathcal{T}^{*}\left(\bar{\mathcal{H}}\right)}"'] & \tilde{\mathcal{U}}_{(j_r)}. \arrow[l, "\tilde{\mathfrak{B}}^{(j_r)}"'] \arrow[u, "\bar{\eta}_{\tilde{\mathcal{U}}_{(j_r)}}"']
\end{tikzcd}
$$

11.4.1 Representation as I-pHd system

Finally we are able to define the representation of I-pHd systems with n^{th} order Hamiltonian on η with local coordinates (Z^i, z^{α}). We will consider here the example of a 3^{rd} order I-pHd system having the following structure

$$
\begin{aligned}
\dot{z} &= \left(\mathfrak{J}-\mathfrak{R}\right)\left(\delta\left(h_0\mathrm{dZ}\right)\right)+\mathfrak{B}\left(u\right) \\
y &= \mathfrak{B}^{*}\left(\delta\left(h_0\mathrm{dZ}\right)\right) \\
\dot{z}^{\alpha}\circ\Psi^{6} &= \bar{\mathfrak{B}}^{(0)}\left(u_{(0)}\right) \\
y_{(0)} &= \bar{\mathfrak{B}}^{(0)*}\left(\delta_{\alpha}^{[0\ldots00]}\left(h_0\right)\circ\Psi^{6}\right) \\
\delta_{\alpha}^{[0\ldots01]}\left(h_0\right)\circ\Psi^{6} &= \tilde{\mathfrak{B}}^{(1)}\left(\tilde{u}_{(1)}\right) \\
\tilde{y}_{(1)} &= \tilde{\mathfrak{B}}^{(1)*}\left(\left(\mathrm{L}_{d_{[1_r]}}\left(\dot{z}^{\alpha}\right)\circ\Psi^{6}\right)\right) \\
\mathrm{L}_{d_{[2_r]}}\left(\dot{z}^{\alpha}\right)\circ\Psi^{6} &= \bar{\mathfrak{B}}^{(2)}\left(\bar{u}_{(2)}\right) \\
\bar{y}_{(2)} &= \bar{\mathfrak{B}}^{(2)*}\left(\delta_{\alpha}^{[0\ldots02]}\left(h_0\right)\circ\Psi^{6}\right)\ .
\end{aligned}
$$

In fact there exist 8 different possible boundary configurations for a I-pHd system with 3^{rd} order Hamiltonian. A symbolic representation of the state 3^{rd} order example is depicted in figure 11.3.

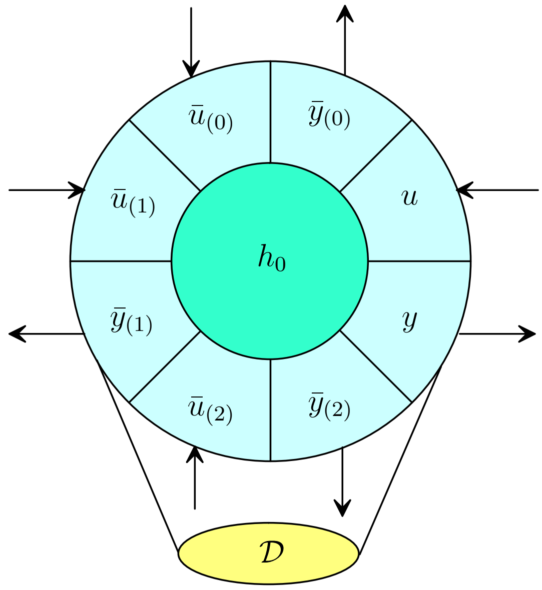

Figure 11.3: The representation an I-pHd system with 3^{rd} order Hamiltonian.

Remark 11.4 *A consequence of the proposed pHd system structure is that F-pHd resp. I-pHd systems cannot be subdivided in several F-pHd resp. I-pHd subsystems in general, because one must be able to introduce subsystems interacting through linear spaces.*

11.4.2 Application - the Kirchhoff plate

Following the investigations of section 9.4.1 we are able to define the Hamiltonian to be given by

$$\begin{aligned} h_0 \mathrm{d}Z^1 \wedge \mathrm{d}Z^2 &= \left(\frac{1}{2}\varsigma\left(\left(z^1_{[20]}\right)^2 + \left(z^1_{[02]}\right)^2 + 2\nu z^1_{[02]} z^1_{[20]} + 2\left(1-\nu\right)\left(z^1_{[11]}\right)^2\right) + \right. \\ &\left. \frac{1}{2\rho\Lambda}\left(z^2\right)^2\right) \mathrm{d}Z^1 \wedge \mathrm{d}Z^2 \,, \end{aligned}$$

where $\varsigma, \nu, \rho, \Lambda \in \mathbb{R}^+$. Here we use the local coordinates (Z^1, Z^2, z^1, z^2) rather then $\left(Y^2, Y^3, y^1, p^{\tilde{1}}\right)$ and suppress the first entry in the multi-index $[0, J_t]$, i.e. we use simply the $[J_t]$ instead.

Hamilton operator

The Hamilton operator $v_h = \dot{z}^1 \frac{\partial}{\partial z^1} + \dot{z}^2 \frac{\partial}{\partial z^2}$ is defined by

$$\begin{aligned} \dot{z}^1 &= \frac{1}{\rho\Lambda} z^2 \\ \dot{z}^2 &= -\varsigma z^1_{[40]} - \varsigma z^1_{[04]} - 2\varsigma z^1_{[22]} - R\frac{1}{\rho\Lambda} z^2 \,, \quad R \in \mathbb{R}^+ \,, \end{aligned}$$

or better structured

$$\begin{bmatrix} \dot{z}^1 \\ \dot{z}^2 \end{bmatrix} = \left(\underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{\mathfrak{J}} - \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & R \end{bmatrix}}_{\mathfrak{R}} \right) \begin{bmatrix} \delta_1 h \\ \delta_2 h \end{bmatrix} .$$

Here we have used a matrix representation to illustrate the appearance of the maps $\mathfrak{J}$ and $\mathfrak{R}$. Obviously it is left to introduce the boundary ports on $\partial\mathcal{D}_1$. The bearing of the Kirchhoff plate leads to the following map

$$\begin{aligned} \delta_\alpha^{[01]}(h_0) \circ \Psi_1^4 &= \varsigma\left(z^1_{[0;2]} + \nu z^1_{[2;0]}\right) \\ &= \tilde{B}_1^{(1),1}\, \tilde{u}_{(1)\,1} = \tilde{u}_{(1)\,1} \\ \tilde{y}_{(1)\,1} &= \tilde{B}_1^{(1),1}\left(\mathrm{L}_{d_{J_r}}\left(\dot{z}^1\right) \circ \Psi_1^4\right) \\ &= \dot{z}^1_{[01]} \circ \Psi_1^4 = \frac{1}{\rho\Lambda} z^2_{[0;1]} \,. \end{aligned}$$

The introduced damping torque M could be easily incorporated in this description by the assignment of $\tilde{B}_1^{(1),1} = 1$ and

$$\tilde{u}_{(1)\,1} = -R_\partial\, \tilde{y}_{(1)\,1} = -R_\partial \frac{1}{\rho\Lambda} z^2_{[0;1]} \,, \quad R_\partial \in \mathbb{R}^+ \,.$$

From the fact that $\delta_2^{[00]}(h_0) = 0$ resp. $\delta_2^{[01]}(h_0) = 0$ we conclude that no tensor entries $\bar{B}_2^{(0),i} = \bar{B}_2^{(1),i} = 0$ exist. Additionally, $\dot{z}^1 \circ \Psi^4 = 0$ on $\partial\mathcal{D}$ due to the restraint support and thus also $\bar{B}_1^{(0),i} = 0$.

Derivative of Hamiltonian functional

Now we are able to derive the time derivative of the Hamiltonian functional, i.e.

$$\begin{aligned} \mathrm{L}_{v_h} \int_{\mathcal{D}} \left(j^2\sigma\right)^* h_0 \mathrm{dZ} &= \int_{\mathcal{D}} \left(j^4\sigma\right)^* \left(v_h \rfloor \left(\delta_1 h_0 \mathrm{d}z^1 \wedge \mathrm{dZ} + \delta_2 h_0 \mathrm{d}z^2 \wedge \mathrm{dZ}\right)\right) \\ &\quad + \int_{\partial\mathcal{D}_1} \left(j^4\bar{\sigma}_1\right)^* \left(\left(\dot{z}^1_{[01]} \circ \Psi_1^4\right) \partial_1^{[0;1]} \rfloor \left(\left(\delta_1^{[01]}(h_0) \circ \Psi_1^4\right) \mathrm{d}z^1_{[0;1]} \wedge \mathrm{d}\bar{\mathrm{Z}}\right)\right) \\ &= \int_{\mathcal{D}} \left(j^4\sigma\right)^* \left(-\delta_2 h_0\, R\, \delta_2 h_0 \mathrm{dZ}\right) \\ &\quad + \int_{\partial\mathcal{D}_1} \left(j^4\bar{\sigma}_1\right)^* \left(-\frac{1}{\rho\Lambda} z^2_{[0;1]}\, R_\partial\, \frac{1}{\rho\Lambda} z^2_{[0;1]} \mathrm{d}\bar{\mathrm{Z}}\right) \\ &\leq 0 \,. \end{aligned}$$

Here we have already used the general results presented in equation (11.7). Additionally, we have restricted the boundary integral to the partition $\partial\mathcal{D}_1$ as the damping torque acts only along this part of the boundary (see figure 8.7).

One of the most important properties of F-pHd systems is their structural invariance with respect to power conserving interconnections. Thus we investigate in the subsequent section the behavior of I-pHd systems with respect to domain and boundary interconnections.

11.5 Interconnection of I-pHd systems

Here we will confine ourselves to the case of systems with 1st order Hamiltonians. It is obvious, that the treatment of this system class is sufficient to introduce the interconnection procedure, as the extension to n^{th} order I-pHd systems equals only the use of a more involved notation.

11.5.1 Introduction of the considered I-pHd systems

In the following the I-pHd systems, which are generated by two interconnected I-pHd systems, are formulated on a product bundle $\left(\mathcal{H}_1 \times \mathcal{H}_2, \eta_{\mathcal{H}_1} \times \eta_{\mathcal{H}_2}, \mathcal{D}_1 \times \mathcal{D}_2\right)$.

We will investigate three different cases of interconnection – domain ⇔ domain, boundary ⇔ boundary, and boundary ⇔ domain. The considered systems are defined by

$$\begin{aligned} \dot{z}_1{}^{\alpha} &= (J_1 - R_1)^{\alpha\beta}\ \delta_\beta h_{01} + u_1{}^{\varsigma}\ B_1{}_{\varsigma}^{\alpha} \\ y_1{}_{\varsigma} &= B_1{}_{\varsigma}^{\beta}\ \delta_\beta h_{01}\ , \end{aligned}$$

with boundary conditions

$$\begin{aligned} \bar{u}_1{}^{\gamma}\ \bar{B}_1{}_{\gamma}^{\alpha} &= \dot{z}_1{}^{\alpha} \circ \Psi_1^2\ , \quad y_1{}_{\gamma} = \bar{B}_1{}_{\gamma}^{\alpha}\left(\partial_\alpha^{[1_r]} h_{01} \circ \Psi_1^2\right) \\ \text{or} \qquad \tilde{B}_1{}_{\alpha}^{\gamma}\ \tilde{u}_1{}_{\gamma} &= \left(\partial_\alpha^{[1_r]} h_{01} \circ \Psi_1^2\right)\ , \quad \tilde{y}_1{}^{\gamma} = \tilde{B}_1{}_{\alpha}^{\gamma}\left(\dot{z}_1{}^{\alpha} \circ \Psi_1^2\right) \end{aligned}$$

and

$$\begin{aligned} \dot{z}_2{}^{\alpha} &= (J_2 - R_2)^{\alpha\beta}\, \delta_\beta h_{02} + u_2{}^{\varsigma}\ B_2{}_{\varsigma}^{\alpha} \\ y_2{}_{\varsigma} &= B_2{}_{\varsigma}^{\beta}\ \delta_\beta h_{02}\ , \end{aligned}$$

with boundary conditions

$$\begin{aligned} \bar{u}_2{}^{\gamma}\ \bar{B}_2{}_{\gamma}^{\alpha} &= \dot{z}_2{}^{\alpha} \circ \Psi_2^2\ , \quad y_2{}_{\gamma} = \bar{B}_2{}_{\gamma}^{\alpha}\left(\partial_\alpha^{[1_r]} h_{02} \circ \Psi_2^2\right) \\ \text{or} \qquad \tilde{B}_2{}_{\alpha}^{\gamma}\ \tilde{u}_2{}_{\gamma} &= \left(\partial_\alpha^{[1_r]} h_{02} \circ \Psi_2^2\right)\ , \quad \tilde{y}_2{}^{\gamma} = \bar{B}_2{}_{\alpha}^{\gamma}\left(\dot{z}_2{}^{\alpha} \circ \Psi_2^2\right)\ . \end{aligned}$$

In all three cases the systems are linked by a I-pH system without dynamics defined by

$$y_{I\,\alpha} = I_{\alpha\beta}\, u_I{}^{\beta}\ , \quad I_{\alpha\beta} = -I_{\beta\alpha}$$

or in matrix representation

$$\begin{bmatrix} y_{I1} \\ y_{I2} \end{bmatrix} = \begin{bmatrix} I_{11} & I_{12} \\ I_{21} & I_{22} \end{bmatrix} \begin{bmatrix} u_{I1} \\ u_{I2} \end{bmatrix}$$

with $I_{11} = -I_{11}^{\intercal}$, $I_{22} = -I_{22}^{\intercal}$, $I_{21} = -I_{12}^{\intercal}$. This system belongs to the class of power-conserving interconnections.

The time derivative of the interconnected Hamiltonian functional

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathfrak{H}_{12} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathcal{D}_1} h_{01}\mathrm{dX}_1 + \int_{\mathcal{D}_2} h_{02}\mathrm{dX}_2\right)$$

will be analyzed for all considered interconnections.

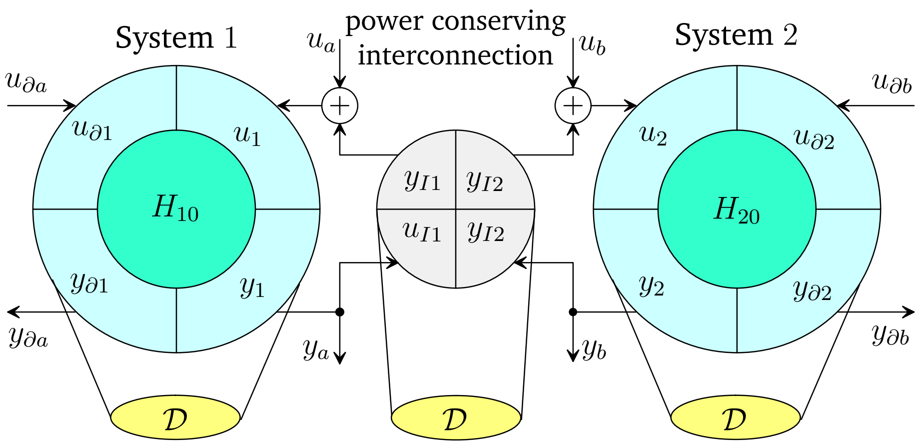

Figure 11.4: The domain⇔domain interconnection.

11.5.2 The domain⇔domain interconnection

This interconnection represents the case, where the domain of two I-pHd systems coincides, see Fig. 11.4. Thus the introduced product bundle reduces to the special case of a fibred product bundle (see Def. 3.8). Consequently we get the following assignments

$$\begin{aligned} {u_{I1}}^{\alpha} &= (K_1)^{\alpha\varsigma}\, {y_1}_{\varsigma} \\ {u_{I2}}^{\alpha} &= (K_2)^{\alpha\varsigma}\, {y_2}_{\varsigma} \\ {u_1}^{\varsigma} &= {u_a}^{\varsigma} + (K_1)^{\alpha\varsigma}\, {y_{I1}}_{\alpha} \\ {u_2}^{\varsigma} &= {u_b}^{\varsigma} + (K_2)^{\alpha\varsigma}\, {y_{I2}}_{\alpha}\ . \end{aligned}$$

Here and in the subsequent analysis we use $(K_i)^{\alpha\varsigma} = (K_i)_{\alpha\varsigma} = (K_i)^{\alpha}_{\varsigma} = \begin{cases} 0 \text{ if } \alpha \neq \varsigma \\ 1 \text{ if } \alpha = \varsigma \end{cases}$ with $i \in \{1,2\}$.

The domain inputs result in

$$\begin{aligned} {u_1}^{\varsigma} &= {u_a}^{\varsigma} + {\bar{I}_{11}}^{\varsigma\varepsilon}\, {B_1}^{\varphi}_{\varepsilon}\, \delta_{\varphi} h_{01} + {\bar{I}_{12}}^{\varsigma\varepsilon}\, {B_2}^{\varphi}_{\varepsilon}\, \delta_{\varphi} h_{02} \\ {u_2}^{\varsigma} &= {u_b}^{\varsigma} + {\bar{I}_{21}}^{\varsigma\varepsilon}\, {B_1}^{\varphi}_{\varepsilon}\, \delta_{\varphi} h_{01} + {\bar{I}_{22}}^{\varsigma\varepsilon}\, {B_2}^{\varphi}_{\varepsilon}\, \delta_{\varphi} h_{02} \end{aligned}$$

with ${\bar{I}_{11}}^{\varsigma\varepsilon} = (K_1)^{\varsigma\beta}\, {I_{11}}_{\beta\psi}\, (K_1)^{\psi\varepsilon}$, ${\bar{I}_{12}}^{\varsigma\varepsilon} = (K_1)^{\varsigma\beta}\, {I_{12}}_{\beta\psi}\, (K_2)^{\psi\varepsilon}$, ${\bar{I}_{21}}^{\varsigma\varepsilon} = (K_2)^{\varsigma\beta}\, {I_{21}}_{\beta\psi}\, (K_1)^{\psi\varepsilon}$, ${\bar{I}_{22}}^{\varsigma\varepsilon} = (K_2)^{\varsigma\beta}\, {I_{22}}_{\beta\psi}\, (K_2)^{\psi\varepsilon}$ and the boundary inputs are denoted by $\bar{u}_1 = \bar{u}_a$, $\bar{u}_2 = \bar{u}_b$. Consequently we end up with an I-pHd system on $\left(\mathcal{H}_1 \times \mathcal{H}_2, \eta_{\mathcal{H}_1 \times_{\mathcal{D}} \mathcal{H}_2}, \mathcal{D}\right)$ defined by

$$\begin{bmatrix} \dot{z}_1^{\alpha} \\ \dot{z}_2^{\alpha} \end{bmatrix} = (J - R) \begin{bmatrix} \delta_{\beta} h_{01} \\ \delta_{\beta} h_{02} \end{bmatrix} + B \begin{bmatrix} {u_a}^{\varsigma} \\ {u_b}^{\varsigma} \end{bmatrix}$$

where

$$\begin{aligned} J &= \begin{bmatrix} J_1^{\alpha\beta} + {B_1}^{\alpha}_{\varsigma}\, {\bar{I}_{11}}^{\varsigma\varepsilon}\, {B_1}^{\beta}_{\varepsilon} & {B_1}^{\alpha}_{\varsigma}\, {\bar{I}_{12}}^{\varsigma\varepsilon}\, {B_2}^{\beta}_{\varepsilon} \\ {B_2}^{\alpha}_{\varsigma}\, {\bar{I}_{21}}^{\varsigma\varepsilon}\, {B_1}^{\beta}_{\varepsilon} & J_2^{\alpha\beta} + {B_2}^{\alpha}_{\varsigma}\, {\bar{I}_{22}}^{\varsigma\varepsilon}\, {B_2}^{\beta}_{\varepsilon} \end{bmatrix} \\ R &= \begin{bmatrix} R_1^{\alpha\beta} & 0 \\ 0 & R_2^{\alpha\beta} \end{bmatrix},\ B = \begin{bmatrix} {B_1}^{\beta}_{\varsigma} & 0 \\ 0 & {B_2}^{\beta}_{\varsigma} \end{bmatrix} \end{aligned}$$

is used. The collocated outputs are given by

$$\begin{aligned} {y_1}_{\varsigma} &= {B_1}^{\beta}_{\varsigma}\, \delta_{\beta} h_{01} = {y_a}_{\varsigma} \\ {y_2}_{\varsigma} &= {B_2}^{\beta}_{\varsigma}\, \delta_{\beta} h_{02} = {y_b}_{\varsigma}\ . \end{aligned}$$

Thus the domain⇔domain interconnection preserves the structure of an I-pHd system. The time derivative of the interconnected Hamiltonian functional leads to similar results as already shown in equation (11.5) for the general case.

11.5.3 The boundary⇔boundary interconnection

This interconnection represents the case, where two I-pHd Systems are interconnected on a common boundary $\partial\mathcal{D}_{12}$ defined by $\partial\mathcal{D}_1 \supset \partial\mathcal{D}_{12} \subset \partial\mathcal{D}_2$. The boundary inputs on $\partial\mathcal{D}_{12}$ are in

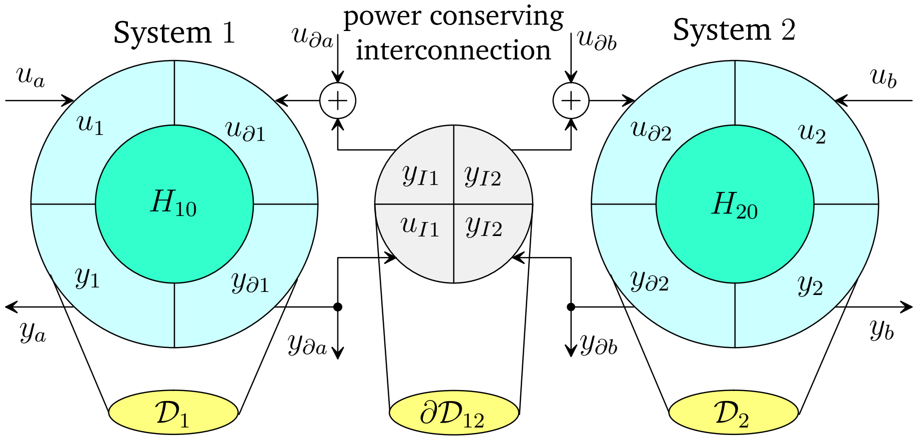

Figure 11.5: The boundary⇔boundary interconnection.

this case given by

$$
\begin{aligned}
\bar{u}_1{}^{\gamma} &= \bar{u}_a{}^{\gamma} + (K_1)^{\gamma\beta}\left(I_{11\,\beta\psi}\,(K_1)^{\psi\varepsilon}\,\bar{y}_{1\,\varepsilon} + I_{12\,\beta\psi}\,(K_2)^{\psi\varepsilon}\,\bar{y}_{2\,\varepsilon}\right)\\
\bar{u}_2{}^{\gamma} &= \bar{u}_b{}^{\gamma} + (K_2)^{\gamma\beta}\left(I_{21\,\beta\psi}\,(K_1)^{\psi\varepsilon}\,\bar{y}_{1\,\varepsilon} + I_{22\,\beta\psi}\,(K_2)^{\psi\varepsilon}\,\bar{y}_{2\,\varepsilon}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\bar{u}_1{}^{\gamma} &= \bar{u}_a{}^{\gamma} + (K_1)^{\gamma\beta}\left(I_{11\,\beta\psi}\,(K_1)^{\psi\varepsilon}\,\bar{y}_{1\,\varepsilon} + I_{12\,\beta\psi}\,(K_2)^{\psi}_{\varepsilon}\,\tilde{y}_2{}^{\varepsilon}\right)\\
\tilde{u}_{2\,\gamma} &= \tilde{u}_{b\,\gamma} + (K_2)^{\beta}_{\gamma}\left(I_{21\,\beta\psi}\,(K_1)^{\psi\varepsilon}\,\bar{y}_{1\,\varepsilon} + I_{22\,\beta\psi}\,(K_2)^{\psi}_{\varepsilon}\,\tilde{y}_2{}^{\varepsilon}\right) .
\end{aligned}
$$

In the analysis of this interconnection, we have to specify the maps $\bar{\mathfrak{B}}$ or equally $\tilde{\mathfrak{B}}$ for both systems. In the following we will introduce only the map $\bar{\mathfrak{B}}_1$ for the first system and the maps $\bar{\mathfrak{B}}_2$ and $\tilde{\mathfrak{B}}_2$ for the second system. The combination $\tilde{\mathfrak{B}}_1$ and $\tilde{\mathfrak{B}}_2$ supplies similar results and is omitted here. The considered inputs are given by

$$
\begin{aligned}
\bar{u}_1{}^{\gamma} &= \bar{u}_a{}^{\gamma} + \hat{I}_{11}{}^{\gamma\varepsilon}\,\bar{B}_1{}^{\alpha}_{\varepsilon}\left(\partial^{[1_r]}_{\alpha} h_{01}\circ\Psi^2_1\right) + \hat{I}_{12}{}^{\gamma\varepsilon}\,\bar{B}_2{}^{\alpha}_{\varepsilon}\left(\partial^{[1_r]}_{\alpha} h_{02}\circ\Psi^2_2\right)\\
\bar{u}_2{}^{\gamma} &= \bar{u}_b{}^{\gamma} + \hat{I}_{21}{}^{\gamma\varepsilon}\,\bar{B}_1{}^{\alpha}_{\varepsilon}\left(\partial^{[1_r]}_{\alpha} h_{01}\circ\Psi^2_1\right) + \hat{I}_{22}{}^{\gamma\varepsilon}\,\bar{B}_2{}^{\alpha}_{\varepsilon}\left(\partial^{[1_r]}_{\alpha} h_{02}\circ\Psi^2_2\right) .
\end{aligned}
$$

with $\hat{I}_{11}{}^{\varsigma\varepsilon} = (K_1)^{\varsigma\beta}\,I_{11\,\beta\psi}\,(K_1)^{\psi\varepsilon}$, $\hat{I}_{12}{}^{\varsigma\varepsilon} = (K_1)^{\varsigma\beta}\,I_{12\,\beta\psi}\,(K_2)^{\psi\varepsilon}$, $\hat{I}_{21}{}^{\varsigma\varepsilon} = (K_2)^{\varsigma\beta}\,I_{21\,\beta\psi}\,(K_1)^{\psi\varepsilon}$, $\hat{I}_{22}{}^{\varsigma\varepsilon} = (K_2)^{\varsigma\beta}\,I_{22\,\beta\psi}\,(K_2)^{\psi\varepsilon}$ and

$$
\begin{aligned}
\bar{u}_1{}^{\gamma} &= \bar{u}_a{}^{\gamma} + \hat{I}_{11}{}^{\gamma\varepsilon}\,\bar{B}_1{}^{\alpha}_{\varepsilon}\left(\partial^{[1_r]}_{\alpha} h_{01}\circ\Psi^2_1\right) + \hat{I}_{12}{}^{\gamma}_{\varepsilon}\,\tilde{B}_2{}^{\varepsilon}_{\alpha}\left(\dot{z}_2{}^{\alpha}\circ\Psi^2_2\right)\\
\tilde{u}_{2\,\gamma} &= \tilde{u}_{b\,\gamma} + \hat{I}_{21}{}^{\varepsilon}_{\gamma}\,\bar{B}_1{}^{\alpha}_{\varepsilon}\left(\partial^{[1_r]}_{\alpha} h_{01}\circ\Psi^2_1\right) + \hat{I}_{22\,\gamma\varepsilon}\,\tilde{B}_2{}^{\varepsilon}_{\alpha}\left(\dot{z}_2{}^{\alpha}\circ\Psi^2_2\right)
\end{aligned}
$$

with $\hat{I}_{11}{}^{\varsigma\varepsilon} = (K_1)^{\varsigma\beta}\, I_{11\,\beta\psi}\,(K_1)^{\psi\varepsilon}$, $\hat{I}_{12}{}^{\gamma}_{\varepsilon} = (K_1)^{\gamma\beta}\, I_{12\,\beta\psi}\,(K_2)^{\psi}_{\varepsilon}$, $\hat{I}_{21}{}^{\varepsilon}_{\gamma} = (K_2)^{\beta}_{\gamma}\, I_{21\,\beta\psi}\,(K_1)^{\psi\varepsilon}$, $\hat{I}_{22\,\gamma\varepsilon} = (K_2)^{\beta}_{\gamma}\, I_{22\,\beta\psi}\,(K_2)^{\psi}_{\varepsilon}$.

The time derivative of the interconnected Hamiltonian $\mathfrak{H}_{12}$ equals the sum of the individual derivatives except the $\partial\mathcal{D}_{12}$-part. Here $\int_{\partial\mathcal{D}_{12}} \left(\bar{y}_{1\,\gamma}\,\bar{u}_1{}^{\gamma} + \bar{y}_{2\,\gamma}\,\bar{u}_2{}^{\gamma}\right) \mathrm{d}\bar{\mathrm{Z}}$ resp. $\int_{\partial\mathcal{D}_{12}} \left(\bar{y}_{1\,\gamma}\,\bar{u}_1{}^{\gamma} + \tilde{y}_{2\,\gamma}\,\tilde{u}_2{}^{\gamma}\right) \mathrm{d}\bar{\mathrm{Z}}$ has to be analyzed. In the first case we get

$$
\begin{aligned}
&\int_{\partial\mathcal{D}_{12}} \left(\partial^{[1_r]}_{\omega} h_{01} \circ \Psi^2_1\right) \bar{B}_1{}^{\omega}_{\gamma} \left(\bar{u}_a{}^{\gamma} + \hat{I}_{11}{}^{\gamma\varepsilon}\, \bar{B}_1{}^{\alpha}_{\varepsilon} \left(\partial^{[1_r]}_{\alpha} h_{01} \circ \Psi^2_1\right) + \hat{I}_{12}{}^{\gamma\varepsilon}\, \bar{B}_2{}^{\alpha}_{\varepsilon} \left(\partial^{[1_r]}_{\alpha} h_{02} \circ \Psi^2_2\right)\right) \mathrm{d}\bar{\mathrm{Z}} \\
&+ \int_{\partial\mathcal{D}_{12}} \left(\partial^{[1_r]}_{\omega} h_{02} \circ \Psi^2_2\right) \bar{B}_2{}^{\omega}_{\gamma} \left(\bar{u}_b{}^{\gamma} + \hat{I}_{21}{}^{\gamma\varepsilon}\, \bar{B}_1{}^{\alpha}_{\varepsilon} \left(\partial^{[1_r]}_{\alpha} h_{01} \circ \Psi^2_1\right) + \hat{I}_{22}{}^{\gamma\varepsilon}\, \bar{B}_2{}^{\alpha}_{\varepsilon} \left(\partial^{[1_r]}_{\alpha} h_{02} \circ \Psi^2_2\right)\right) \mathrm{d}\bar{\mathrm{Z}}
\end{aligned}
$$

and in the second case this results in

$$
\begin{aligned}
&\int_{\partial\mathcal{D}_{12}} \left(\partial^{[1_r]}_{\omega} h_{01} \circ \Psi^2_1\right) \bar{B}_1{}^{\omega}_{\gamma} \left(\bar{u}_a{}^{\gamma} + \hat{I}_{11}{}^{\gamma\varepsilon}\, \bar{B}_1{}^{\alpha}_{\varepsilon} \left(\partial^{[1_r]}_{\alpha} h_{01} \circ \Psi^2_1\right) + \hat{I}_{12}{}^{\gamma}_{\varepsilon}\, \tilde{B}_2{}^{\varepsilon}_{\alpha} \left(\dot{z}_2{}^{\alpha} \circ \Psi^2_2\right)\right) \mathrm{d}\bar{\mathrm{Z}} \\
&+ \int_{\partial\mathcal{D}_{12}} \left(\dot{z}_2{}^{\omega} \circ \Psi^2_2\right) \tilde{B}_2{}^{\gamma}_{\omega} \left(\tilde{u}_{b\,\gamma} + \hat{I}_{21}{}^{\varepsilon}_{\gamma}\, \bar{B}_1{}^{\alpha}_{\varepsilon} \left(\partial^{[1_r]}_{\alpha} h_{01} \circ \Psi^2_1\right) + \hat{I}_{22\,\gamma\varepsilon}\, \tilde{B}_2{}^{\varepsilon}_{\alpha} \left(\dot{z}_2{}^{\alpha} \circ \Psi^2_2\right)\right) \mathrm{d}\bar{\mathrm{Z}}\ .
\end{aligned}
$$

Because of the condition $\hat{I}_{12}{}^{\gamma\alpha} = -\,\hat{I}_{21}{}^{\alpha\gamma}$ the first integrals simplify to

$$
\int_{\partial\mathcal{D}_{12}} \left(\bar{y}_{1\,\gamma}\,\bar{u}_a{}^{\gamma} + \bar{y}_{2\,\gamma}\,\bar{u}_b{}^{\gamma}\right) \mathrm{d}\bar{\mathrm{Z}}\ .
$$

The second integral simplifies similarly to

$$
\int_{\partial\mathcal{D}_{12}} \left(\bar{y}_{1\,\gamma}\,\bar{u}_a{}^{\gamma} + \tilde{y}_2{}^{\gamma}\,\tilde{u}_{b\,\gamma}\right) \mathrm{d}\bar{\mathrm{Z}}
$$

because of $\hat{I}_{12}{}^{\gamma}_{\varepsilon} = -\,\hat{I}_{21}{}^{\varepsilon}_{\gamma}$. Consequently the time derivative of the interconnected Hamiltonian functional caused on $\partial\mathcal{D}_{12}$ is purely determined by the collocation of $\bar{u}_a$ and $\bar{u}_b$ (resp. $\tilde{u}_b$) with $\bar{y}_1$ and $\bar{y}_2$ (resp. $\tilde{y}_2$). It is worth mentioning that this is a simple consequence of the power-conserving interconnection.

11.5.4 The boundary⇔domain interconnection

This interconnection represents the combination of a r-dimensional I-pHd system, i.e. $\dim(\mathcal{D}_1) = r$, with a $(r-1)$-dimension system, i.e. $\dim(\mathcal{D}_2) = r-1$, along $\partial\mathcal{D}_{12}$ defined by $\partial\mathcal{D}_1 \supset \partial\mathcal{D}_{12} \subset \mathcal{D}_2$. Again we have to consider both boundary maps for the first system. Consequently the inputs of the systems are given by

$$
\begin{aligned}
\bar{u}_1{}^{\gamma} &= \bar{u}_a{}^{\gamma} + \hat{I}_{11}{}^{\gamma\varepsilon}\, \bar{B}_1{}^{\alpha}_{\varepsilon} \left(\partial^{[1_r]}_{\alpha} h_{01} \circ \Psi^2_1\right) + \hat{I}_{12}{}^{\gamma\varepsilon}\, B_2{}^{\varphi}_{\varepsilon}\ \delta_{\varphi} h_{02} \\
u_2{}^{\varsigma} &= u_b{}^{\varsigma} + \hat{I}_{21}{}^{\varsigma\varepsilon}\, \bar{B}_1{}^{\alpha}_{\varepsilon} \left(\partial^{[1_r]}_{\alpha} h_{01} \circ \Psi^2_1\right) + \hat{I}_{22}{}^{\varsigma\varepsilon}\, B_2{}^{\varphi}_{\varepsilon}\ \delta_{\varphi} h_{02}
\end{aligned}
$$

with $\hat{I}_{11}{}^{\varsigma\varepsilon} = (K_1)^{\varsigma\beta}\, I_{11\,\beta\psi}\,(K_1)^{\psi\varepsilon}$, $\hat{I}_{12}{}^{\varsigma\varepsilon} = (K_1)^{\varsigma\beta}\, I_{12\,\beta\psi}\,(K_2)^{\psi\varepsilon}$, $\hat{I}_{21}{}^{\varsigma\varepsilon} = (K_2)^{\varsigma\beta}\, I_{21\,\beta\psi}\,(K_1)^{\psi\varepsilon}$, $\hat{I}_{22}{}^{\varsigma\varepsilon} = (K_2)^{\varsigma\beta}\, I_{22\,\beta\psi}\,(K_2)^{\psi\varepsilon}$
or in the second case

$$
\begin{aligned}
\tilde{u}_{1\,\gamma} &= \tilde{u}_{a\,\gamma} + \hat{I}_{11\,\gamma\varepsilon}\, \tilde{B}_1{}^{\varepsilon}_{\alpha} \left(\dot{z}_1{}^{\alpha} \circ \Psi^2_1\right) + \hat{I}_{12}{}^{\varepsilon}_{\gamma}\, B_2{}^{\varphi}_{\varepsilon}\ \delta_{\varphi} h_{02} \\
u_2{}^{\gamma} &= u_b{}^{\gamma} + \hat{I}_{21}{}^{\gamma}_{\varepsilon}\, \tilde{B}_1{}^{\varepsilon}_{\alpha} \left(\dot{z}_1{}^{\alpha} \circ \Psi^2_1\right) + \hat{I}_{22}{}^{\gamma\varepsilon}\, B_2{}^{\varphi}_{\varepsilon}\ \delta_{\varphi} h_{02}
\end{aligned}
$$

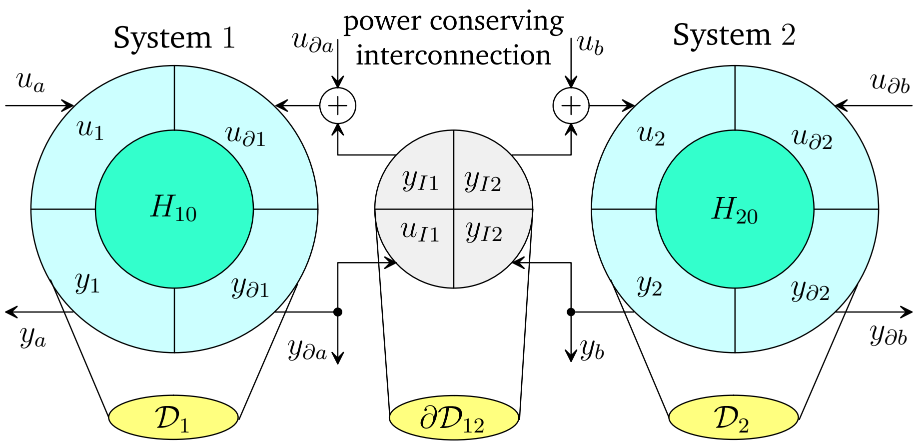

Figure 11.6: The boundary⇔domain interconnection.

with $\hat{I}_{11\,\gamma\varepsilon} = (K_1)^{\beta}_{\gamma}\, I_{11\,\beta\psi}\,(K_1)^{\psi}_{\varepsilon}$, $\hat{I}_{12\,\gamma}^{\;\;\varepsilon} = (K_1)^{\beta}_{\gamma}\, I_{12\,\beta\psi}\,(K_2)^{\psi\varepsilon}$, $\hat{I}_{21\;\varepsilon}^{\;\;\gamma} = (K_2)^{\gamma\beta}\, I_{21\,\beta\psi}\,(K_1)^{\psi}_{\varepsilon}$, $\hat{I}_{22}^{\;\;\gamma\varepsilon} = (K_2)^{\gamma\beta}\, I_{22\,\beta\psi}\,(K_2)^{\psi\varepsilon}$.

The time derivative of $\mathfrak{H}_{12}$ is again given by the sum of the individual derivatives except the $\partial\mathcal{D}_{12}$-part, plus the result of $\int_{\partial\mathcal{D}_{12}} \left(\bar{y}_{1\,\gamma}\,\bar{u}_1^{\;\gamma} + y_{2\,\varsigma}\,u_2^{\;\varsigma}\right) \mathrm{d}\bar{Z}$ resp. $\int_{\partial\mathcal{D}_{12}} \left(\tilde{y}_1^{\;\gamma}\,\tilde{u}_{1\,\gamma} + y_{2\,\varsigma}\,u_2^{\;\varsigma}\right) \mathrm{d}\bar{Z}$. These integrals leads to

$$\int_{\partial\mathcal{D}_{12}} \left(\partial^{[1_r]}_{\omega} h_{01} \circ \Psi_1^2\right) \bar{B}_{1\;\gamma}^{\;\omega} \left(u_{\partial a}^{\;\;\gamma} + \bar{I}_{11}^{\;\;\gamma\varepsilon}\, \bar{B}_{1\;\varepsilon}^{\;\alpha} \left(\partial^{[1_r]}_{\alpha} h_{01} \circ \Psi_1^2\right) + \bar{I}_{12}^{\;\;\gamma\varepsilon}\, B_{2\;\varepsilon}^{\;\varphi}\;\delta_{\varphi} h_{02}\right) \mathrm{d}\bar{Z}$$
$$+ \int_{\partial\mathcal{D}_{12}} \delta_{\omega} h_{02}\; B_{2\;\gamma}^{\;\omega} \left(u_b^{\;\gamma} + \bar{I}_{21}^{\;\;\gamma\varepsilon}\, \bar{B}_{1\;\varepsilon}^{\;\alpha} \left(\partial^{[1_r]}_{\alpha} h_{01} \circ \Psi_1^2\right) + \bar{I}_{22}^{\;\;\gamma\varepsilon}\, B_{2\;\varepsilon}^{\;\varphi}\;\delta_{\varphi} h_{02}\right) \mathrm{d}\bar{Z}$$

or equally

$$\int_{\partial\mathcal{D}_{12}} \left(\dot{z}_1^{\;\omega} \circ \Psi_1^2\right) \tilde{B}_{1\;\gamma}^{\;\omega} \left(u_{\partial a\,\gamma} + \hat{I}_{11\,\gamma\varepsilon}\, \tilde{B}_{1\;\alpha}^{\;\varepsilon} \left(\dot{z}_1^{\;\alpha} \circ \Psi_1^2\right) + \hat{I}_{12\,\gamma}^{\;\;\varepsilon}\, B_{2\;\varepsilon}^{\;\varphi}\;\delta_{\varphi} h_{02}\right) \mathrm{d}\bar{Z}$$
$$+ \int_{\partial\mathcal{D}_{12}} \delta_{\omega} h_{02}\; B_{2\;\gamma}^{\;\omega} \left(u_b^{\;\gamma} + \hat{I}_{21\;\varepsilon}^{\;\;\gamma}\, \tilde{B}_{1\;\alpha}^{\;\varepsilon} \left(\dot{z}_1^{\;\alpha} \circ \Psi_1^2\right) + \hat{I}_{22}^{\;\;\gamma\varepsilon}\, B_{2\;\varepsilon}^{\;\varphi}\;\delta_{\varphi} h_{02}\right) \mathrm{d}\bar{Z}\,.$$

Once again the condition $\hat{I}_{12}^{\;\;\gamma\alpha} = -\,\hat{I}_{21}^{\;\;\alpha\gamma}$ simplifies the first integral to

$$\int_{\partial\mathcal{D}_{12}} \left(\bar{y}_{1\,\gamma}\,\bar{u}_a^{\;\gamma} + y_{2\,\gamma}\,u_b^{\;\gamma}\right) \mathrm{d}\bar{Z}\,, \tag{11.8}$$

and $\hat{I}_{12\,\gamma}^{\;\;\varepsilon} = -\,\hat{I}_{21\;\varepsilon}^{\;\;\gamma}$ simplifies the second integral to

$$\int_{\partial\mathcal{D}_{12}} \left(\tilde{y}_1^{\;\gamma}\,\tilde{u}_{a\,\gamma} + y_{2\,\gamma}\,u_b^{\;\gamma}\right) \mathrm{d}\bar{Z}\,. \tag{11.9}$$

The time evolution of $\mathfrak{H}_{12}$ is consequently determined by the individual damping $\mathfrak{R}_1$, $\mathfrak{R}_2$ on $\mathcal{D}_1$, $\mathcal{D}_2$, the pairings $y_1\rfloor u_1$, $y_2\rfloor u_2$ on $\mathcal{D}_1$, $\mathcal{D}_2 - \partial\mathcal{D}_{12}$, the pairings $\bar{y}_1\rfloor\bar{u}_1$ (resp. $\tilde{y}_1\rfloor\tilde{u}_1$), $\bar{y}_2\rfloor\bar{u}_2$ (resp. $\tilde{y}_2\rfloor\tilde{u}_2$) on $\partial\mathcal{D}_1 - \partial\mathcal{D}_{12}$, $\partial\mathcal{D}_2$ and the quantity in equation (11.8) resp. (11.9).

11.6 Application - Membrane and String

In this section we will investigate a mechanical structure, whose infinite dimensional components can be modelled using the introduced 1st order I-pHd description. The considered construction consists of a rectangular undamped membrane and an attached undamped string. The proposed interconnection of this systems is shown in Fig. 11.7. In the mathematical

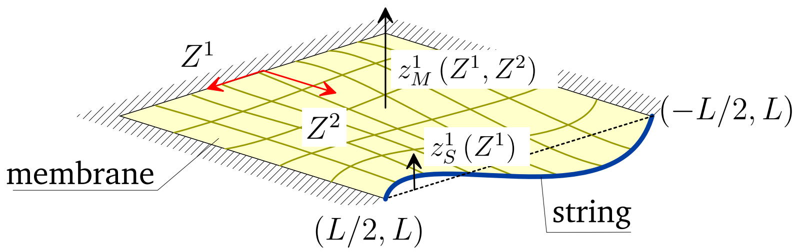

Figure 11.7: The membrane-string interconnection.

modeling, we assume that for both components only small vertical displacements $z_M^1(Z^1, Z^2)$, $z_S^1(Z^1)$ appear. Consequently we are able to formulate the potential energy density of the membrane [Villaggio, 1997]

$$e_{PM} = \frac{S_M}{2}\left(\left(z^1_{M[10]}\right)^2 + \left(z^1_{M[01]}\right)^2\right) \mathrm{d}Z^1 \wedge \mathrm{d}Z^2 \, .$$

Here the constant membrane tension $S_M \in \mathbb{R}^+$ is introduced. Similarly we are able to define the potential energy of the string

$$e_{PS} = \frac{S_S}{2}\left(z^1_{S[1]}\right)^2 \mathrm{d}Z^1$$

with the string tension $S_S \in \mathbb{R}^+$. The kinetic energy is given by

$$e_{KM} = \frac{1}{2\rho_M}\left(z^2_M\right)^2 \mathrm{d}Z^1 \wedge \mathrm{d}Z^2$$

respectively

$$e_{KS} = \frac{1}{2\rho_S}\left(z^2_S\right)^2 \mathrm{d}Z^1 \, ,$$

where the constant mass per unit area $\rho_M \in \mathbb{R}^+$ and mass per unit length $\rho_S \in \mathbb{R}^+$ are used. The Hamiltonian densities

$$\begin{aligned} h_{0M} &= \left(\frac{S_M}{2}\left(z^1_{M[10]}\right)^2 + \frac{S_M}{2}\left(z^1_{M[01]}\right)^2 + \frac{1}{2\rho_M}\left(z^2_M\right)^2\right) \mathrm{d}Z^1 \wedge \mathrm{d}Z^2 \\ h_{0S} &= \left(\frac{S_S}{2}\left(z^1_{S[1]}\right)^2 + \frac{1}{2\rho_S}\left(z^2_S\right)^2\right) \mathrm{d}Z^1 \end{aligned}$$

can now be used to define the corresponding I-pHd representations. The membrane is described by

$$\begin{bmatrix} \dot z^1_M \\ \dot z^2_M \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \delta_{z^1_M} h_{0M} \\ \delta_{z^2_M} h_{0M} \end{bmatrix} = \begin{bmatrix} \frac{1}{\rho_M} z^2_M \\ S_M\left(z^1_{M[20]} + z^1_{M[02]}\right) \end{bmatrix}$$

with the boundary conditions

$$\begin{aligned}\frac{1}{\rho_M} z_M^2 \circ \Psi_M^2 &= \dot{z}_M^1 \circ \Psi_M^2 = \bar{B}_{M\,1}^{\;1}\, \bar{u}_M^{\;1} \\ \bar{y}_{M\,1} &= \left(\partial_\alpha^{[01]} H_{0M} \circ \Psi_M^2\right) \bar{B}_{M\,1}^{\;\alpha} = \left(S_M z_{M[01]}^1 \circ \Psi_M^2\right) \bar{B}_{M\,1}^{\;1} \ ,\end{aligned}$$

where the map

$$\begin{aligned}\Psi_M^2 \quad : \quad & \left(\bar{Z}, z^1_{M[0;0]}, z^2_{M[0;0]}\ , z^1_{M[1;0]}, z^2_{M[1;0]}, z^1_{M[0;1]}, z^2_{M[0;1]}, \right. \\ & \left. z^1_{M[2;0]}, z^2_{M[2;0]}, z^1_{M[1;1]}, z^2_{M[1;1]}, z^1_{M[0;2]}, z^2_{M[0;2]}\right) \\ \rightarrow \quad & \left(Z^1 = \bar{Z},\ Z^2 = L\ , z^1_M = z^1_{M[0;0]}, z^2_M = z^2_{M[0;0]}, \right. \\ & z^1_{M[10]} = z^1_{M[1;0]}, z^2_{M[10]} = z^2_{M[1;0]}, z^1_{M[01]} = z^1_{M[0;1]}, \\ & z^2_{M[01]} = z^2_{M[0;1]}, z^1_{M[20]} = z^1_{M[2;0]}, z^2_{M[20]} = z^2_{M[2;0]}, \\ & z^1_{M[11]} = z^1_{M[1;1]}, z^2_{M[11]} = z^2_{M[1;1]}, z^1_{M[02]} = z^1_{M[0;2]}, \\ & \left. z^2_{M[0;2]} = z^2_{M[0;2]}\right)\end{aligned}$$

is used. The string is described by

$$\begin{bmatrix} \dot{z}_S^1 \\ \dot{z}_S^2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \delta_{z_S^1} h_{0S} \\ \delta_{z_S^2} h_{0S} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_S = \begin{bmatrix} \frac{1}{\rho_S} z_S^2 \\ S_M z_{S[2]}^1 + u_S \end{bmatrix}$$

and the boundary conditions ($\iota_S : \left\{-\frac{L}{2}, \frac{L}{2}\right\} \rightarrow \left\{Z^1 = -\frac{L}{2}, Z^1 = \frac{L}{2}\right\}$)

$$\begin{aligned}\tilde{B}_{S\,1}^{\;1}\, \tilde{u}_{S\,1} &= \partial_1^{[1]} h_{0S} \circ \Psi_S^2 = \left(S_S z_{S[1]}^1\right) \circ \Psi_S^2 \\ \tilde{y}_S^{\;1} &= \tilde{B}_{S\,1}^{\;1}\, (\dot{z}_S^1 \circ \Psi_S^2) = \tilde{B}_{S\,1}^{\;1} \left(\tfrac{1}{\rho_S} z_S^2 \circ \Psi_S^2\right) \ .\end{aligned}$$

The power conserving interconnection is in this case given by

$$y_{I1} = u_{I2}\ ,\ y_{I2} = -u_{I1}\ ,$$

with

$$u_{I1} = \bar{y}_{M\,1}\ \ ,\ u_{I2} = y_{S\,1}\ \ \text{and}\ \ \bar{u}_M^{\;1} = y_{I1}\ ,\ u_S^{\;1} = y_{I2}\ .$$

Thus we are able to derive

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathcal{D}_1} h_{0M} \mathrm{d}X^1 \wedge \mathrm{d}X^2 + \int_{\mathcal{D}_2} h_{0S} \mathrm{d}X^2\right) = \int_{\partial\mathcal{D}_1 - \partial\mathcal{D}_{12}} \bar{u}_M^{\;1}\, \bar{y}_{M\,1}\, \partial_i \rfloor \mathrm{dZ} + \left[\tilde{u}_{S\,1}\, \tilde{y}_S^{\;1}\right]_{-L/2}^{L/2}$$

whereby it is visualized that only the boundary inputs on the membrane and on the string modify the interconnected Hamiltonian functional $\mathfrak{H}_{12}$. If we take the restraint support of the membrane as visualized in figure 11.7 into account we conclude that $\bar{u}_M^{\;1} = \tilde{u}_{S\,1} = 0$ and consequently the interconnected Hamiltonian functional is invariant under the motion of the coupled system.

Part IV

Stability and Control

Stability is the cornerstone of control!

The modification of plants as, e.g., achieved by means of control laws, is often intended to improve a certain behavior of the system. In fact normally all the applied action has to fulfill a special minimum requirement – it has to ensure the stability of the plant. Consequently control engineers are forced to investigate their models with regards to certain stability criteria.

The previous parts II, III of this thesis where dedicated to the modeling of two classes of infinite-dimensional systems. In fact all these considerations gave a deep insight to the natural properties of the analyzed systems. In this final part we want to take advantage of the achieved results in the analysis of the stability of these systems in the sense of Lyapunov.

It is remarkable, that modeling and simulation of infinite-dimensional systems is a well established discipline in engineering. The formulation of partial differential equations (PDEs) for field problems as, e.g., known from mechanics, fluid- and electrodynamics or even from coupled field problems, is well established and numerical methods for simulation purposes like, e.g., the finite element method are available and implemented in sophisticated computer software. In fact stability investigation of infinite-dimensional systems are not that widely used in engineering.

In the subsequent analysis the well known definition of stability according to Lyapunov and some additional results on finite dimensional systems are recalled (following [Hahn, 1967]). After that a possible application of Lyapunov's stability notion to infinite-dimensional systems is discussed. It will be shown, how Sobolev spaces and class K-functions allow the definition of a stability criterion for such systems. In order to illustrate the applicability of the stated criteria we will investigate the stability of the damped Kirchhoff plate as introduced in part II and III.

Chapter 12

Stability of infinite-dimensional systems

Before we investigate the infinite-dimensional case, we want to recall the stability of systems in the sense of Lyapunov following [Hahn, 1967].

12.1 The finite-dimensional case

Here we consider the time-invariant, autonomous system

$$\dot{x}^\alpha = f^\alpha(x), \quad x|_{t=t_0} = x_0, \quad \alpha = 1, \ldots, s \; . \tag{12.1}$$

From a differential geometric point of view, this system equals a vector field $v = \dot{x}^\alpha \partial_\alpha$ formulated in the local coordinates $x^\alpha, \alpha = 1, \ldots, s$ on an s-dimensional manifold $\mathcal{M}$.

The solution of the system is defined as a map

$$\begin{array}{rcl} \phi : \mathbb{R} \times \mathcal{M} & \rightarrow & \mathcal{M} \\ (t - t_0, x_0) & \rightarrow & x = \phi(t - t_0, x_0) \; , \end{array}$$

whereby we have introduced the flow ϕ of the system.

Remark 12.1 *Here we have assumed that the existence and uniqueness of the solution for the system 12.1 is guaranteed.*

Consequently the solution of such a system, defined by $\phi(t - t_0, x_0)$, equals a curve on the manifold $\mathcal{M}$ parametrized by the time t. Additionally it is remarkable, that there is no need to introduce the time t as coordinate on the manifold $\mathcal{M}$. The flow $\phi(t - t_0, x_0)$ (also denoted $\phi_{t-t_0}(x_0)$) qualifies as a one parameter group of transformations (see, e.g., [Olver, 1986]).

We will consider the stability of the origin $x = 0$ under the following definition.

Definition 12.2 *The equilibrium of the differential equation (12.1) is called stable (in the sense of Lyapunov) if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that*

$$\|\phi(t - t_0, x_0)\| < \varepsilon \text{ for } t > t_0$$

is valid whenever

$$\|x_0\| < \delta(\varepsilon) \; .$$

In order to derive a criterion for the proof of stability we introduce

Definition 12.3 *A real-valued function $\varphi(r)$ belongs to class K ($\varphi \in K$) if it is defined, continuous, and strictly increasing on $0 \leq r \leq r_1$, resp. $0 \leq r \leq \infty$, and if it vanishes at $r = 0 : \varphi(0) = 0$.*

and obtain the following criterion.

Definition 12.4 *The equilibrium of the differential equation (12.1) is stable in case there exists a function φ of class K that*

$$\|\phi(t - t_0, x_0)\| \leq \varphi(\|x_0\|), \text{ for } t \geq t_0$$

(see [Hahn, 1967])

Finally this criterion leads to the subsequent theorem.

Theorem 12.5 *If there exist two class K functions φ_1, φ_2 and a function v, which meets*

$$\varphi_1(\|x\|) \leq v(x) \leq \varphi_2(\|x\|)$$

and whose formal time derivative $\dot{v}$ for (12.1) is negative semi-definite, then the equilibrium is stable. (see [Hahn, 1967])

Proof. If we apply the function $v(x)$ on a solution of (12.1), we may formulate the inequality

$$v(\phi(t - t_0, x_0)) \leq v(x_0)$$

because of $\dot{v}(x) \leq 0$. Consequently we are able to state that

$$\varphi_1(\|\phi(t - t_0, x_0)\|) \leq v(\phi(t - t_0, x_0)) \leq v(x_0) \leq \varphi_2(\|x_0\|).$$

The introduction of the class K function $\varphi_3 = \varphi_1{}^{-1} \circ \varphi_2$ enables us finally to define

$$\|\phi(t - t_0, x_0)\| \leq \varphi_3(\|x_0\|)$$

and thus the stability is shown (see Definition 12.4). ■

It is worth mentioning, that theorem 12.5 states pure formal conditions on $v(x)$ and $\dot{v}(x)$. This implies that these conditions can be evaluated without any analytical knowledge of the solution of the system. But in fact the results of the conditions are only meaningful, if the existence and uniqueness of the solution given.

Remark 12.6 *The inequality stated in theorem 12.5 implies that the function $v(x)$ is positive definite.*

The equivalence of all norms in a finite-dimensional vector space is responsible for the fact, that the specification of the used norm is suppressed in the previously stated theorems, proofs, and definitions. Another consequence of this circumstance is that the proof of stability for a certain plant with respect to a chosen norm proves stability with respect to all norms. Unfortunately this wonderful property will not apply anymore for the infinite-dimensional case.

12.2 The infinite-dimesional case

We will assume, that the infinite-dimensional and time-invariant systems under investigation are formulate on the smooth bundle $(\mathcal{E}, \pi, \mathcal{D})$ and its n^{th} order jet framework Π^n. Here the bounded, spatial base manifold $\mathcal{D}$ is equipped with the independent coordinates (Y^i), $i = 1 \ldots r$ and for the total manifold $\mathcal{E}$ we use the independent and dependent coordinates (Y^i, y^α), $\alpha = 1 \ldots s$.

The map $\sigma : \mathcal{D} \to \mathcal{E}$ resp. $j^n\sigma : \mathcal{D} \to J^n\mathcal{E}$ is called section resp. the n^{th} order prolongation of a section and equals the assignment of the dependent coordinates resp. their partial derivatives as functions of the independent coordinates.

12.2.1 Sobolev spaces

The sections $\sigma \in \Gamma(\pi)$ on the bundle π are assumed to be elements of a special infinite-dimensional vector space – the Sobolev space $W_{k,p}$

$$\sigma \in W_{k,p}(\mathcal{D}) \ .$$

This Sobolev space is equipped with the norm

$$\|\sigma\|_{k,p} = \left[\sum_{\#J=0}^{k} \int_{\mathcal{D}} \sum_{\alpha=1}^{s} \left|\partial_{[J]}\sigma^\alpha\right|^p \mathrm{dX} \right]^{1/p} \ , \quad 1 \leq p < \infty.$$

It is now remarkable that for every section $\sigma \in W_{k_2,p}$ the inequality

$$\sum_{\#J=0}^{k_1} \int_{\mathcal{D}} \sum_{\alpha=1}^{s} \left|\partial_{[J]}\sigma^\alpha\right|^p \mathrm{dX} \leq \sum_{\#J=0}^{k_2} \int_{\mathcal{D}} \sum_{\alpha=1}^{s} \left|\partial_{[J]}\sigma^\alpha\right|^p \mathrm{dX}$$

is met, iff $k_1 \leq k_2$. This equals the relation

$$\|\sigma\|_{k_1,p} \leq \|\sigma\|_{k_2,p} \ .$$

This inequality is sufficient for the subsequent analysis, but it is worth mentioning that there exists a generalization of this relation referred to as Sobolev embedding theorem (see Def. 12.11).

12.2.2 System representation

We will confine ourselves to autonomous, infinite-dimensional systems given in the form

$$\dot{x} = F\left(X^i, x^\alpha\right) \ , \quad \sigma^\alpha|_{t=t_0} = \sigma_0^\alpha \ , \quad i = 1, \ldots, r \ , \ \alpha = 1, \ldots, s \tag{12.2}$$

with the differential operator F. This type is also referred to as evolution equation [Olver, 1986]. From a geometrical point of view equation (12.2) represents the definition of a generalized vertical vector field on a certain pull-back bundle of the total manifold, i.e. $(\pi_0^n)^* \mathcal{T}(\mathcal{E})$ in local coordinates. Obviously the previously defined Hamilton operator v_h (with assigned input u) is of this class.

12.2.3 Stability of the equilibrium

Now we assume the existence of a solution of the system[1] and defined it as a map

$$\begin{aligned} \Phi : \mathbb{R} \times \Gamma(\pi) &\rightarrow \Gamma(\pi) \\ (t - t_0, \sigma_0) &\rightarrow \sigma^\alpha = \Phi^\alpha (t - t_0, \sigma_0) \end{aligned}$$

that meets

$$\sigma = \Phi(0, \sigma) \ .$$

Additional we impute, that the map Φ qualifies as a one-parameter C^0-semigroup (see. e.g. [Renardy, R.C. Rogers, 2004]) of transformations with identity $e = \Phi(0, \cdot)$.

Furthermore we let the analyzed equilibrium section σ, which meets

$$0 = F\left(X^i, \sigma^\alpha\right) \ ,$$

be given by the origin, i.e. $\sigma^\alpha(X) = 0$. The equilibrium section represents a point in the Sobolev space $W_{k_2,p}$ resp. $W_{k_1,p}$, whose stability is of interest.

Definition 12.7 *The equilibrium point $\sigma = 0 \in W_{k_2,p} \subseteq W_{k_1,p},\ k_1 \leq k_2$ is called a stable equilibrium point of the evolution equation (12.2) with respect to the Sobolev norm $\|\cdot\|_{k_2,p}$, if for all $\varepsilon > 0$, there exists a $\delta(\varepsilon)$ such that*

$$\|\sigma_0\|_{k_2,p} < \delta(\varepsilon) \ \textit{implies} \implies \|\Phi(t - t_0, \sigma_0)\|_{k_1,p} < \varepsilon \ \forall t \geq t_0$$

where $\Phi(t - t_0, \sigma_0)$ is the solution of 12.2 with the initial conditions σ_0 and parametrization t.

Having the definitions of the finite-dimensional case at ones disposal, it is obvious that the specification of the used norms is an essential additional ingredient used in the infinite-dimensional case.

Furthermore we are able to incorporate the notion of class K functions and get the following definition.

Definition 12.8 *The equilibrium of the differential equation (12.2) is stable with respect to the norm $\|\cdot\|_{k_2,p}$ in case there exists a function φ of class K such that*

$$\|\Phi(t - t_0, \sigma_0)\|_{k_1,p} \leq \varphi\left(\|\sigma_0\|_{k_2,p}\right) \ , \quad k_1 \leq k_2 \ \textit{for} \ t \geq t_0 \ .$$

Following [Hahn, 1967] we are now able to introduce the notion of a Lyapunov functional in the following theorem.

Theorem 12.9 *If there exists a functional $V(\sigma) = \int_{\mathcal{D}} (j^n\sigma)^* (f\mathrm{dX})\ , f \in C^\infty(J^n\mathcal{E})$, which fulfills the condition*

$$\varphi_1\left(\|\sigma\|_{k_1,p}\right) \leq V(\sigma) \leq \varphi_2\left(\|\sigma\|_{k_2,p}\right) \ , \quad k_1 \leq k_2$$

on $D_r = \left\{\sigma | \ \|\sigma\|_{k_2,p} \leq r\right\}$ with the class K functions φ_1 and φ_2 and whose formal time derivative along (12.2) meets

$$\mathrm{L}_{j^n(\dot{x}\partial_\alpha)} V(\sigma) = \int_{\mathcal{D}} (j^n\sigma)^* \left(\mathrm{L}_{j^n(\dot{x}\partial_\alpha)} (f\mathrm{dX})\right) \leq 0$$

then the equilibrium point $\sigma = 0$ is stable in the sense of Lyapunov with respect to the norm $\|\cdot\|_{k_2,p}$.

[1]It is worth mentioning that this assumption is a rather strong assumption.

The formal time derivative coincides with the time derivative, i.e.

$$\mathrm{L}_{j^n(\dot{x}\partial_\alpha)}V(\sigma) = \mathrm{L}_{\partial_t}(V(\Phi(t-t_0,\sigma_0)))$$

if the solution $\Phi(t-t_0,\sigma_0)$ parametrized in t exists. As this is assumed here, we can make use of both formulations.
Proof. The condition $\mathrm{L}_{j^n(\dot{x}\partial_\alpha)}V(\sigma) \leq 0$ implies that

$$V(\Phi(t-t_0,\sigma_0)) \leq V(\sigma_0) \text{ with } t_0 \leq t\,.$$

Consequently we have the relation

$$\begin{array}{rcl} \varphi_1\left(\|\Phi(t-t_0,\sigma_0)\|_{k_1,p}\right) & \leq & V(\Phi(t-t_0,\sigma_0)) \\ & \leq & V(\sigma_0) \\ & \leq & \varphi_2\left(\|\sigma_0\|_{k_2,p}\right) \end{array}$$

and by applying the inverse function of $\varphi_1(\cdot)$

$$\begin{array}{rcl} \|\Phi(t-t_0,\sigma_0)\|_{k_1,p} & \leq & {\varphi_1}^{-1}(V(\Phi(t-t_0,\sigma_0))) \\ & \leq & {\varphi_1}^{-1}(V(\sigma_0)) \\ & \leq & {\varphi_1}^{-1}\left(\varphi_2\left(\|\sigma_0\|_{k_2,p}\right)\right) \end{array}$$

we get the inequality

$$\|\Phi(t-t_0,\sigma_0)\|_{k_1,p} \leq \varphi_3\left(\|\sigma_0\|_{k_2,p}\right)\,,$$

whereby the class K function $\varphi_3 = {\varphi_1}^{-1} \circ \varphi_2$ is introduced. Consequently the stability of $\sigma = 0$ on D_r is shown. ■

Finally we are able to define a criterion for asymptotic stability, where we follow again the considerations presented in [Hahn, 1967].

Theorem 12.10 *If the functional $V(\sigma)$ meets the conditions of theorem 12.9 and in addition*

$$\mathrm{L}_{j^n(\dot{x}\partial_\alpha)}V(\sigma) \leq -\varphi_4\left(\|\sigma\|_{k_2,p}\right)\,, \quad \varphi_4 \in K$$

then the equilibrium point $\sigma = 0$ of (12.2) is asymptotically stable.

Proof. From

$$\|\sigma\|_{k_1,p} \leq \varphi_1^{-1}(V(\sigma))\,, \quad \varphi_2^{-1}(V(\sigma)) \leq \|\sigma\|_{k_2,p}$$

we conclude that

$$\begin{array}{rcl} \mathrm{L}_{\partial_t}(V(\Phi^\alpha(t-t_0,\sigma_0))) & \leq & -\varphi_4\left(\varphi_2^{-1}(V(\Phi^\alpha(t-t_0,\sigma_0)))\right) \\ & = & :-\chi(V(\Phi^\alpha(t-t_0,\sigma_0)))\,, \quad \chi \in K\,. \end{array}$$

Now we consider the auxiliary scalar differential equation

$$\dot{w} = -\chi(w)\,,\ w \geq 0\,.$$

We know that

$$w\left(t\right) \leq q\left(w_0\right)\rho\left(t-t_0\right)\ , \quad q \in K,\ \rho \in L\ ,\ w_0 = w\left(t_0\right)\ .$$

If $V_0 = w_0$ then $V\left(\Phi^\alpha\left(t-t_0,\sigma_0\right)\right) \leq w\left(t\right)$ for all t. Hence

$$V\left(\Phi^\alpha\left(t-t_0,\sigma_0\right)\right) \leq q\left(V_0\right)\rho\left(t-t_0\right)$$

is fulfilled. Finally we end up with

$$\left\|\Phi\left(t-t_0,\sigma_0\right)\right\|_{k_1,p} \leq \varphi_1^{-1}\left(q\left(V_0\right)\rho\left(t-t_0\right)\right) \leq \varphi_1^{-1}\left(q\left(\varphi_2\left(\left\|\sigma_0\right\|_{k_2,p}\right)\right)\rho\left(t-t_0\right)\right)$$

or rewritten in

$$\left\|\Phi\left(t-t_0,\sigma_0\right)\right\|_{k_1,p} \leq \varphi\left(\left\|\sigma_0\right\|_{k_2,p}\right)\rho\left(t-t_0\right)\ , \quad \varphi \in K,\ \sigma \in L\ .$$

■

It is worth mentioning, that the proposed stability criteria make only use of the formal time derivative and consequently the solution of the system is not necessary for the determination of these quantities.

In order to show the applicability of the derived criteria, we investigate the stability of the damped rectangular Kirchhoff plate.

12.3 Applications - Kirchhoff plate

We consider a fully supported plate as shown in figure 12.1. Consequently both $y^1_{[0;0]}$ and $y^1_{[0;1]}$

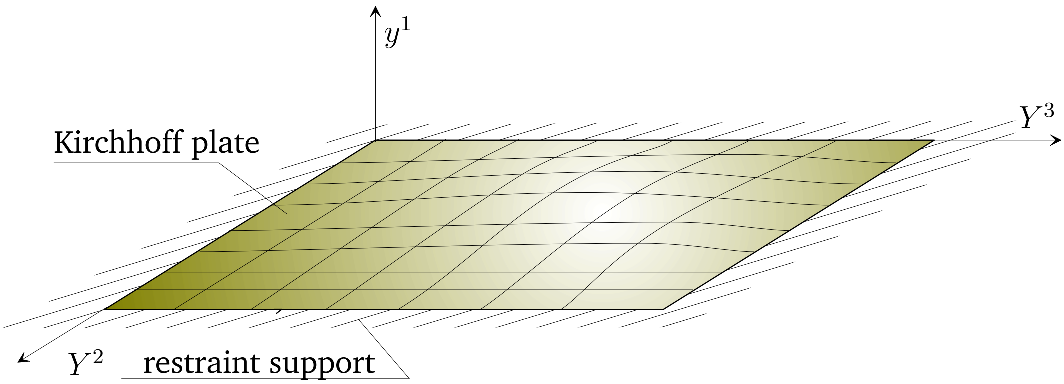

Figure 12.1: The fully supported rectangular Kirchhoff plate.

vanish on the entire boundary $\partial\mathcal{D}$.

The Hamiltonian density for the rectangular Kirchhoff plate is given by

$$h = \frac{1}{2\rho\Lambda}(p)^2 + \frac{1}{2}\varsigma\left(\left(y^1_{[20]}\right)^2 + \left(y^1_{[02]}\right)^2 + 2\nu y^1_{[02]}y^1_{[20]} + 2(1-\nu)\left(y^1_{[11]}\right)^2\right),$$

where $\varsigma, \nu, \rho, \Lambda \in \mathbb{R}^+$ and $\nu < 1$. At first we determine the lower bound $\varphi_1\left(\|\sigma\|_{k_1,p}\right)$ used in theorem 12.9.

12.3.1 Determination of lower bound

Because of the fact that the Hamiltonian density is built up by several additive terms, we are able to focus in a first step on the second part of h given by

$$\begin{aligned}
&\left(y^1_{[20]}\right)^2 + \left(y^1_{[02]}\right)^2 + 2\nu y^1_{[02]}y^1_{[20]} + 2(1-\nu)\left(y^1_{[11]}\right)^2 = \\
&= (1-\nu)\left(y^1_{[20]}\right)^2 + (1-\nu)\left(y^1_{[02]}\right)^2 \\
&\quad +\nu\left(\left(y^1_{[02]}\right)^2 + 2y^1_{[02]}y^1_{[20]} + \left(y^1_{[20]}\right)^2\right) + 2(1-\nu)\left(y^1_{[11]}\right)^2 \\
&= (1-\nu)\left(y^1_{[20]}\right)^2 + (1-\nu)\left(y^1_{[02]}\right)^2 + \nu\left(y^1_{[02]} + y^1_{[20]}\right)^2 + 2(1-\nu)\left(y^1_{[11]}\right)^2 \\
&= (1-\nu)\left(\left(y^1_{[20]}\right)^2 + 2\left(y^1_{[11]}\right)^2 + \left(y^1_{[02]}\right)^2\right) + \nu\left(y^1_{[02]} + y^1_{[20]}\right)^2 \\
&\geq (1-\nu)\left(\left(y^1_{[20]}\right)^2 + 2\left(y^1_{[11]}\right)^2 + \left(y^1_{[02]}\right)^2\right).
\end{aligned}$$

Thus we have already found a lower bound. In order to derive $\varphi_1(\cdot)$ from this result we have to consider Poincare-Friedrich [Zeidler, 1995] inequality

$$\begin{aligned}
C_1\int\left(j^1\sigma\right)^*\left(y^1_{[10]}\right)^2 \mathrm{dY} &\leq \int\left(j^2\sigma\right)^*\left(\left(y^1_{[20]}\right)^2 + \left(y^1_{[11]}\right)^2\right)\mathrm{dY} \\
C_2\int\left(j^1\sigma\right)^*\left(y^1_{[01]}\right)^2 \mathrm{dY} &\leq \int\left(j^2\sigma\right)^*\left(\left(y^1_{[11]}\right)^2 + \left(y^1_{[02]}\right)^2\right)\mathrm{dY},
\end{aligned}$$

which is applicable because of the fact, that both $\left(j^1\sigma\right)^*\left(y^1_{[10]}\right)$ and $\left(j^1\sigma\right)^*\left(y^1_{[01]}\right)$ are elements of $\mathring{W}_{1,2}$ (see [Zeidler, 1995]) due to the used restricted bearing of the plate. This supplies the following inequality

$$(1-\nu)\left(\left(y^1_{[20]}\right)^2+2\left(y^1_{[11]}\right)^2+\left(y^1_{[02]}\right)^2\right)\geq C_3\left(\left(y^1_{[10]}\right)^2+\left(y^1_{[01]}\right)^2\right) \ .$$

Now we are again able to apply the Poincare-Friedrich inequality, i.e.

$$C_4\int\sigma^*\left(y^1_{[00]}\right)^2\mathrm{dY}\leq C_3\int\left(j^1\sigma\right)^*\left(\left(y^1_{[10]}\right)^2+\left(y^1_{[01]}\right)^2\right)\mathrm{dY}$$

and consequently we are able to formulate

$$\varphi_1\left(\|\sigma\left(X\right)\|_{0,2}\right)\leq\int\left(j^2\sigma\right)^*h\ \mathrm{dY}=\mathfrak{H}\left(\sigma\right) \ ,$$

where

$$\begin{array}{rcl}\varphi_1\left(\left(\|\sigma\|_{0,2}\right)^2\right):\Gamma\pi & \rightarrow & \mathbb{R}\\ \sigma & \rightarrow & C_5\int\sigma^*\left(\left((p)^2+\left(y^1_{[00]}\right)^2\right)\mathrm{dY}\right)\end{array}$$

and

$$C_5=\min\left(C_4,\frac{1}{2\rho\Lambda}\right) \ .$$

Now its left to determine the upper bound $\varphi_2\left(\|\sigma\|_{k_2,p}\right)$.

12.3.2 Determination of upper bound

Again we have to focus on the second part of the Hamiltonian density. We are able to find an upper bound by

$$\begin{array}{rl}(1-\nu)\left(y^1_{[20]}\right)^2+(1-\nu)\left(y^1_{[02]}\right)^2+\nu\left(y^1_{[02]}+y^1_{[20]}\right)^2+2\left(1-\nu\right)\left(y^1_{[11]}\right)^2 & =\\ \leq\ (1-\nu)\left(y^1_{[20]}\right)^2+(1-\nu)\left(y^1_{[02]}\right)^2+\nu\left(y^1_{[02]}+y^1_{[20]}\right)^2 & \\ +\nu\left(y^1_{[02]}-y^1_{[020]}\right)^2+2\left(1-\nu\right)\left(y^1_{[11]}\right)^2 & \\ =\ (1+\nu)\left(y^1_{[20]}\right)^2+(1+\nu)\left(y^1_{[02]}\right)^2+2\left(1-\nu\right)\left(y^1_{[11]}\right)^2 \ . & \end{array}$$

Finally we are able to state

$$\mathfrak{H}\left(\sigma\right)=\int\left(j^2\sigma\right)^*h\ \mathrm{dY}\leq\varphi_2\left(\left(\|\sigma\|_{2,2}\right)^2\right) \ ,$$

where

$$\begin{array}{rcl}\varphi_2\left(\|\sigma\left(X\right)\|_{2,2}\right) & : & \Gamma\left(\pi\right)\rightarrow\mathbb{R}\\ \sigma & \rightarrow & C_6\int\left((p)^2+\left(y^1_{[00]}\right)^2+\left(y^1_{[10]}\right)^2+\left(y^1_{[01]}\right)^2\right.\\ & & \left.+\left(y^1_{[20]}\right)^2+\left(y^1_{[11]}\right)^2+\left(y^1_{[02]}\right)^2\right)\mathrm{dY}\end{array}$$

and

$$C_6 = \max\left(\frac{1}{2\rho\Lambda}, \frac{1}{2}\varsigma\left(1+\nu\right), \varsigma\left(1-\nu\right)\right) \ .$$

In the previous part (see section 11.4.2) we have already shown, that the formal time derivative $\mathrm{L}_{j^n(\dot{x}\partial_\alpha)}\mathfrak{H}\left(\sigma\right) \leq 0$. Consequently the damped rectangular Kirchhoff plate is stable in the sense of Lyapunov for the norm $\|\cdot\|_{2,2}$, i.e. if $\|\sigma_0^\alpha\left(X^i\right)\|_{2,2}$ is bounded, then the norm of the solution $\|\Phi^\alpha\left(t-t_0,\sigma_0\right)\|_{0,2}$ will also be bounded for $t \geq t_0$.

Unfortunately it is not possible to determine the $-\varphi_4\left(\|\sigma\|_{k_2,p}\right)$ class K-function as the time derivative does not depend on $y^1_{[00]}$. Thus we are not able to show the asymptotic stability.

12.4 Remarks on further extensions

The relations between different Sobolev norms are defined by the so called Sobolev embedding theorem.

Definition 12.11 (Sobolev embedding theorem) *The space $W_{k,p}\left(\mathcal{D}\right)$ is contained in the space $W_{j,q}\left(\mathcal{D}\right)$, i.e.*

$$W_{k,p}\left(D\right) \subseteq W_{j,q}\left(D\right) \ , \ 0 \leq j < k \ , \ 1 \leq p \ , \ q < \infty$$

on the r-dimensional domain $\mathcal{D}$ with piecewise smooth boundary $\partial\mathcal{D}$. The embedding

$$W_{k,p}\left(D\right) \rightarrow W_{j,q}\left(D\right)$$

is continuous and compact for $d < 1/q$ where d is given by

$$d = \frac{1}{p} - \frac{k-j}{r} \ .$$

(see [Zeidler, 1990])

These link between different Sobolev spaces, can be used to increase the amount of Sobolev norms that are bounded during the motion of the infinite-dimensional system, if the initial condition meets certain boundaries.

Chapter 13

Remarks on the Design of Infinite-Dimensional Control Systems

This final chapter is intended to summarize the cognitions gained from the different parts of this thesis and to give an outlook of what can be achieved by the presented methods. Additionally several remarks on the design of infinite-dimensional control systems are stated.

13.1 Summary

The modeling of infinite-dimensional physical systems using Hamilton's principle, the introduction of a Hamiltonian representation, and the definition of a stability criterion is contained in this thesis. In fact the main part is dedicated to the mathematical modeling by means of the calculus of variations. From this analysis a rather general algorithm for the determination of the equations of motion for infinite-dimensional Euler-Lagrange systems is derived. The introduction of jet theory and boundary contact bundle morphisms enables the definition of the so called boundary bundle, which is indispensable for the extraction of boundary conditions of n^{th} order Euler-Lagrange systems. After the incorporation of external inputs in the presented framework, the time evolution of such systems is under investigation and supplies some information about invariant quantities along the solution of Euler-Lagrange systems. These investigations enlighten a certain structure of EL systems such that they qualify as port Hamiltonian system. This class is subsequently analyzed in the finite- and infinite-dimensional case. The cognitions gained from the boundary conditions derivation of EL systems is used to introduce boundary ports for n^{th} order I-pHd systems. Finally the interconnection via power conserving interconnections is discussed. After this part on the structured representation of infinite-dimensional systems, a stability criterion for infinite-dimensional systems using Sobolev spaces is presented.

In fact the treatment of physical systems, from a control theoretical point of view, is always linked to a modification of the system behavior by means of control action. Unfortunately this demand is not treated in this thesis, but in the next section some general remarks on that task are formulated.

13.2 Controller design

Having dynamic systems at ones disposal, the design of a controller is a methodology that determines an assignment of system inputs by means of system outputs. In fact this assignment is in many cases realized by means of a dynamic systems. Thus the mathematical representation of the controller becomes identical to the mathematical representation of the controlled plant.

In the case of finite-dimensional systems, the mathematical representation is given by ordinary differential equations and all in- and output signals can be seen as simple functions of time. In fact the controllers can be realized by means of, e.g., electric circuits. These simple considerations lead to the first question concerning the controller design for infinite-dimensional systems.

The mathematical representation of a controller for infinite-dimensional systems is given by partial-differential equations in general. There are several application, where one is able to overcome this intrinsic problem by the use of boundary control of spatially one dimensional problems. In all this case it is possible to define the controller by means of ordinary differential equations. It is obvious that this procedure does not solve the general problem.

Commonly this question is solved by the introduction of a discretization of the problem i.e. an approximation of the problem by means of ordinary differential equations. Thus the mathematical model according to the system is modified and the proof of stability of a control loop based on the approximation model becomes rather questionable. Effects like spill-over illustrate this problem. Consequently it would be of interest to design controllers on an approximative model using ordinary differential equations and to test the stability of the closed loop taking into account the infinite-dimensional model.

Another solution of bringing the ideas of automatic control to infinite-dimensional systems is given by a system design that directly incorporates control laws into the system. This could be achieved by the use of, e.g., smart materials in the construction of the plant. Obviously this approach requires the control engineer to be involved in the design of the plant. Thus the engineer could prevent the necessity of external inputs to the system by an appropriate design.

Part V

Appendix

Appendix A

Definitions

A.1 Algebra

A.1.1 Sets and functions

The used notion of sets is defined in [Michel, C.J. Herget, 1997]

The terms mapping, map, operator, transformation, and function are used interchangeably.

Definition A.1 (function) *Let X and Y be non-empty sets. A function f from X into Y is a subset of $X \times Y$ such that for every $x \in X$ there is one and only one $y \in Y$ (i.e., there is unique $y \in Y$) such that $(x, y) \in f$. The set X is called the domain of f (or the domain of definition of f), and we say that f is defined on X. The set $\{y \in Y : (x, y) \in f$ for some $x \in X\}$ is called the range of f and is denoted by $\mathrm{range}\,(f)$. For each $(x, y) \in f$, we call y the value of f at x and denote ist by f. We sometimes write $f : X \to Y$ to denote the function f from X into Y. (see [Michel, C.J. Herget, 1997])*

Definition A.2 (one-to-one, onto) *Let f be a function from X into Y. If $\mathrm{range}\,(f) = Y$ the f is said to be surjective or a surjection, and we say that f maps X onto Y. If f is a function such that for every $x_1, x_2 \in X$, $f\,(x_1) = f\,(x_2)$ implies that $x_1 = x_2$, then f is said to be injective or a one-to-one mapping, or an injection. If f is both injective and surjective, we say that f is bijective or one-to-one and onto, or a bijection. (see [Michel, C.J. Herget, 1997])*

Definition A.3 (inverse function) *Let f be an injective mapping onf X into Y. Then we say that f has an inverse, and we call the mapping $\{(y, x) \in \mathrm{range}\,(f) \times X : (x, y) \in f\}$ the inverse of f. Hereafter, we will denote the inverse of f by f^{-1}. (see [Michel, C.J. Herget, 1997])*

Definition A.4 (composite function) *Let $X, Y,$ and Z be non-empty sets. Suppose that $f : X \to Y$ and $g : Y \to Z$. For each $x \in X$, we have $f\,(x) \in Y$ and $g\,(f\,(x)) \in Z$. Since f and g are mappings from X into Y and from Y into Z, respectively, it follows that for each $x \in X$ there is one and only one element $g\,(f\,(x)) \in Z$. Hence the set*

$$\{(x, z) \in X \times Z : z = g\,(f\,(x))\,, x \in X\}$$

is a function from X into Z. We call this function the composite function of g and f and denote it by $g \circ f$. The value of $g \circ f$ at x is given by

$$(g \circ f)\,(x) = g \circ f\,(x) \triangleq g\,(f\,(x))\ .$$

(see [Michel, C.J. Herget, 1997])

Definition A.5 (identity function) *Let X be a non-empty set. Let* $\mathrm{id} : X \to X$ *be defined by* $\mathrm{id}(x) = x$ *for all* $x \in X$*. We call* id *the identity function on* X*. (see [Michel, C.J. Herget, 1997])*

Definition A.6 (image of a function) *Let f be a function from a set X into a set Y. Let $A \subset X$, and let $B \subset Y$. We define the image of A under f, denoted by $f(A)$, to be the set*

$$f(A) = \{y \in Y : y = f(x), x \in A\} \ .$$

We define the inverse image of B under f, denoted by $f^{-1}(B)$ to be the set

$$f^{-1}(B) = \{x \in X : f(x) \in B\} \ .$$

(see [Michel, C.J. Herget, 1997])

A.1.2 Algebraic Structures

Definition A.7 (semigroup) *Let X be a non-empty set with operation α denoted by $\{X; \alpha\}$. We call $\{X; \alpha\}$ a semigroup if α is an associative operation on X. (see [Michel, C.J. Herget, 1997])*

Definition A.8 (group, abelian group) *A group is a semigroup, $\{X; \alpha\}$, with identity e in which every element is invertible. If in addition the operation α is commutative, the group is referred to as commutative or abelian group. (see [Michel, C.J. Herget, 1997])*

Definition A.9 (ring) *Let X be a non-empty set, and let α and β be operations on X. The set X together with the operations α and β on X, denoted by $\{X; \alpha, \beta\}$, is called a ring if*

- *$\{X; \alpha\}$ is an abelian group;*
- *$\{X; \beta\}$ is a semigroup;*
- *and β is distributive over α.*

(see [Michel, C.J. Herget, 1997])

Definition A.10 (division ring) *Let $\{X; +, \cdot\}$ be a non-trivial ring, and let $X^{\#} = X - \{0\}$. The ring X is called a division ring if $\{X^{\#}; \cdot\}$ is a subgroup of $\{X; \cdot\}$. (see [Michel, C.J. Herget, 1997])*

Definition A.11 (field) *Let $\{X; +, \cdot\}$ be a division ring. Then X is called a field if the operation $\cdot$ is commutative. (see [Michel, C.J. Herget, 1997])*

Definition A.12 (module) *Let $\{R; +, \cdot\}$ be a ring with identity, e, and let $\{X; +\}$ be an abelian group. Let $\mu : R \times X \to X$ be any function satisfying the following four conditions for all $r_1, r_2 \in R$ and for all $x_1, x_2 \in X$:*

- $\mu(r_1 + r_2, x_1) = \mu(r_1, x_1) + \mu(r_2, x_1)$
- $\mu(r_1, x_2 + x_2) = \mu(r_1, x_1) + \mu(r_1, x_2)$
- $\mu(r_1, \mu(r_2, x_1)) = \mu(r_1 \cdot r_2, x_1)$*, and*

- $\mu(e, x_1) = x_1$.

Then the composite system $\{R, X, \mu\}$ is called a module (or R-module). (see [Michel, C.J. Herget, 1997])

Definition A.13 (vector space) *Let $\{F; +, \cdot\}$ be a field, and let $\{X; +\}$ be an abelian group. If X is an F-module, then X is called a vector space over F. (see [Michel, C.J. Herget, 1997])*

Definition A.14 (homo-, iso-, endo-, automorphism) *Let $\{X; \alpha\}$ and $\{Y; \beta\}$ be two semigroups (not necessarily distinct). A mapping ρ of set X into set Y is called a homomorphism of the semigroup $\{X; \alpha\}$ into the semigroup $\{Y; \beta\}$ if*

$$\rho(x \; \alpha \; y) = \rho(x) \;\; \beta \; \rho(y)$$

for every $x, y \in X$.

- *If ρ is a mapping of X onto Y, we say that X and Y are homomorphic semigroups, and we refer to X as being homomorphic to Y*
- *If ρ is a one-to-one mapping of X into Y, then ρ is called an isomorphism of X into Y.*
- *If ρ is a mapping which is onto and one-to-one, we say that semigroup X is isomorphic to semigroup Y.*
- *If $X = Y$ (i.e., ρ is a homomorphism of semigroup X into itself) then ρ is called an endomorphism.*
- *If $X = Y$ and if ρ is an isomorphism (i.e., ρ is an isomorphism of semigroup X into itself), then ρ is called an automorphism of X.*

(see [Michel, C.J. Herget, 1997])

Definition A.15 (kernel) *Let ρ be a homomorphism of a semigroup X into a semigroup Y. If $\rho(X)$ has identity element, e', then the subset of X, K_ρ, defined by*

$$K_\rho = \{x \in X : \rho(x) = e'\}$$

is called the kernel of the homomorphism ρ. (see [Michel, C.J. Herget, 1997])

A.1.3 Topology

Definition A.16 (topology, topological space) *A system T of subsets of a set X defines a topology on X if U contains*

- *the empty set $\{\} \in T$ and the set $X \in T$ itself,*
- *the intersection of every one of its finite subsystems ($U_1, \ldots, U_n \in T$, $n \in \mathbb{N}$, then $\bigcap\limits_{j=1}^{n} U_j \in T$),*

- *the union of very one of its subsystems ($U_\alpha \in T \Longrightarrow \bigcup_{\alpha \in A} U_\alpha \in T$, for every $\alpha \in$ arbitrary index set).*

The sets in U are called the open sets of the topological space $\{X, U\}$ often abbreviated to X. (see [Zeidler, 1995] or [Choquet-Bruhat, Cecile DeWitt-Morette, 1982])

Definition A.17 (neighborhood) *A neighborhood of a point x (or a set A) in X is a set $N(x)$ (a set $N(A)$) containing an open set which contains the point x (the set A). (see [Choquet-Bruhat, Cecile DeWitt-Morette, 1982])*

Definition A.18 (limit point) *A point $x \in X$ is a limit point of $A \subseteq X$ if every neighborhood $N(x)$ of x contains at least one point $a \in A$ different from $x : (N(x) - \{x\}) \cap A \neq 0, \forall N(x)$. (see [Choquet-Bruhat, Cecile DeWitt-Morette, 1982])*

Definition A.19 (closed) *A set $A \subset X$ is closed if is $X \backslash A$ open. (see [Choquet-Bruhat, Cecile DeWitt-Morette, 1982])*

Theorem A.20 (closed) *A set A closed iff it contains all its limit points. (see [Choquet-Bruhat, Cecile DeWitt-Morette, 1982])*

Definition A.21 (closure) *The closure $\bar{A}$ of A in X is the union of A and all its limit points; it is the smallest closed set containing A. (see [Choquet-Bruhat, Cecile DeWitt-Morette, 1982])*

Definition A.22 (support) *The support of a function f, denoted by $\mathrm{supp}(f)$, is the smallest closed set outside which f vanishes identically. (see [Choquet-Bruhat, Cecile DeWitt-Morette, 1982])*

Definition A.23 (interior,dense) *The interior of a set A is the largest open set $\mathring{A}$ contained in A. The set A is dense in X if $\bar{A} = X$. (see [Choquet-Bruhat, Cecile DeWitt-Morette, 1982])*

Definition A.24 (covering) *A system $\{U_i\}$ of open subsets of X is an open covering if each element in X belongs to at least one U_i (i.e. $\cup U_i = X$). If the system $\{U_i\}$ has a finite number of elements, the covering is said to be finite. (see [Choquet-Bruhat, Cecile DeWitt-Morette, 1982])*

Definition A.25 (Hausdorff) *A topological space is Hausdorff (separated) if any two distinct points posses disjoint neighborhoods. (see [Choquet-Bruhat, Cecile DeWitt-Morette, 1982])*

Definition A.26 (compact) *A subset $A \subset X$ is compact if it is Hausdorff and if every covering of A has a finite subcovering. (see [Choquet-Bruhat, Cecile DeWitt-Morette, 1982])*

Theorem A.27 (compact) *A compact subspace of a Hausdorff space is necessarily closed. (see [Choquet-Bruhat, Cecile DeWitt-Morette, 1982])*

Definition A.28 (disconnected) *A topological space X is called disconnected, if there exist two disjoint non empty subsets A_1 and A_2 both open in X and such that $A_1 \cup A_2 = X$. Since A_2 is the complement of the open set A_1, it is closed as well as open. Similarly A_1 is closed as well as open. (see [Choquet-Bruhat, Cecile DeWitt-Morette, 1982])*

Theorem A.29 (connected) *A topological space is connnected iff the only subsets which are both open and closed are the void set and the space X itself. (see [Choquet-Bruhat, Cecile DeWitt-Morette, 1982])*

Definition A.30 (continuous function) *A mapping f from a topological space X to a topological space Y is continuous at $x \in X$ if given any neighborhood $N \subset Y$ of f there exists a neighbourhood M of $x \in X$ such that $f(M) \subset N$. f is continuous on X if it is continuous at all points x of X. (see [Choquet-Bruhat, Cecile DeWitt-Morette, 1982])*

Definition A.31 (homeomorphism) *A homeomorphism is a bijection f which is bicontinuous (f and f^{-1} are continuous). (see [Choquet-Bruhat, Cecile DeWitt-Morette, 1982])*

Definition A.32 (topological group) *A topological space X with the group operation α (i.e. $\{X, \alpha\}$) and a topology T builds a topological group, if the group operation is continuous. (see [Choquet-Bruhat, Cecile DeWitt-Morette, 1982])*

Definition A.33 (topological vector space) *A topological space X which is also a vector space on $\mathbb{K}$ is a topological vector space if the operations of addition and scalar multiplication are continuous. (see [Choquet-Bruhat, Cecile DeWitt-Morette, 1982])*

Definition A.34 (metric space) *A metric space is a set X together with a map $d : X \times X \to \mathbb{R}$ such that*

$$\begin{aligned} d(x,y) &\geq 0 \\ d(x,y) &= 0 \textit{ iff } x = y \\ d(x,y) &= d(y,x) \\ d(x,z) &\leq d(x,y) + d(y,z) \ . \end{aligned}$$

$d(x,y)$ is called the distance between x and y. (see [Choquet-Bruhat, Cecile DeWitt-Morette, 1982])

Definition A.35 (complete metric space) *A metric space is called complete if every Cauchy sequence $d(x_n, x_m)$ in the space is convergent*

$$\lim_{n,m\to\infty} d(x_n, x_m) = 0 \ .$$

Consequently there exists N such that, for $n, m > N$, $d(x_n, x_m) < \varepsilon$ for every preassigned $\varepsilon > 0$. (see [Choquet-Bruhat, Cecile DeWitt-Morette, 1982])

Definition A.36 (norm) *The mapping $x \to \|x\|$ of a vector space X on $\mathbb{K}$ into $\mathbb{R}$ is a norm if for $x \in X$ and $\lambda \in \mathbb{K}$*

$$\begin{aligned} \|x+y\| &\leq \|x\| + \|y\| \\ \|\lambda x\| &\leq |\lambda| \, \|x\| \\ \|x\| &= 0 \textit{ iff } x = 0 \ . \end{aligned}$$

(see [Choquet-Bruhat, Cecile DeWitt-Morette, 1982])

Definition A.37 (Banach space) *A complete normed vector space is a Banach space. (see [Choquet-Bruhat, Cecile DeWitt-Morette, 1982])*

A.2 Manifolds

Definition A.38 (manifold) *A n-dimensional (topological) manifold $\mathcal{M}$ is a Hausdorff topological space such that every point has a neighbourhood homeomorphic to the euclidean space $\mathbb{R}^n$. (see [Choquet-Bruhat, Cecile DeWitt-Morette, 1982])*

Definition A.39 (smooth manifold) *Smooth manifolds are manifolds with atlases of class C^∞. (see [Choquet-Bruhat, Cecile DeWitt-Morette, 1982])*

Definition A.40 (immersion, submersion) *A smooth function $f : \mathcal{M} \to \mathcal{N}$ between smooth manifolds is said to be an immersion (submersion) if $\mathsf{rank}\,(f) = m = \dim(\mathcal{M})$ ($\mathsf{rank}\,(f) = n = \dim(\mathcal{N})$) everywhere. (see [Boothby, 1986])*

Definition A.41 (embedding) *A injective immersion is an embedding. (see [Choquet-Bruhat, Cecile DeWitt-Morette, 1982])*

A.3 Bundles

Definition A.42 ((global) trivialization) *If $(\mathcal{E}, \pi, \mathcal{B})$ is a fibred manifold then a (global) trivialization of π is a pair $(\mathcal{F}, t)$ where $\mathcal{F}$ is a manifold (called typical fibre of π) and $t : \mathcal{E} \to \mathcal{B} \times \mathcal{F}$ is a diffeomorphism satisfying the condition*

$$pr_1 \circ t = \pi \,.$$

A fibred manifold which has at least one trivialization is called trivial. (see [Saunders, 1989])

A.3.1 Tangent- and Cotangent bundle

The complementary entity to the vertical bundle is called the transverse bundle.

Definition A.43 (transverse bundle) *The transverse bundle to π is the pull-back vector bundle $(\pi^*(\mathcal{T}(\mathcal{B})), \pi^*(\tau_{\mathcal{B}}), \mathcal{E})$. (see [Saunders, 1989])*

A.3.2 Tensors bundles

Definition A.44 (tensor) *A tensor Φ on a vector space V (e.g., $\mathcal{T}_p(\mathcal{M})$) is by definition a multilinear map*

$$T_g^r : \underbrace{V \times \cdots \times V}_{r} \times \underbrace{V^* \times \cdots \times V^*}_{g} \to \mathbb{R}\,,$$

where V^* (e.g., $\mathcal{T}_p^*(\mathcal{M})$) denotes the dual space to V, $r \geq 0$ its covariant order, and $g \geq 0$ its contravariant order.(see [Boothby, 1986])
A tensor on $p \in \mathcal{M}$ will be denoted

$$T_{g,p}^r : \bigotimes_g^r \mathcal{M}|_p = \underbrace{\mathcal{T}_p(\mathcal{M}) \times \cdots \times \mathcal{T}_p(\mathcal{M})}_{r} \times \underbrace{\mathcal{T}_p^*(\mathcal{M}) \times \cdots \times \mathcal{T}_p^*(\mathcal{M})}_{g} \to \mathbb{R}\,.$$

The space $\bigotimes_g^r \mathcal{M}|_p$ represents a vector space of dimension $r + g$.

Definition A.45 (tensor bundle) *The tensor bundle* $\left(\bigotimes_g^r \mathcal{M}, \tau_{\otimes_g^r \mathcal{M}}, \mathcal{M}\right)$ *of a manifold* $\mathcal{M}$ *consists of the total manifold*

$$\bigotimes_g^r \mathcal{M} = \bigcup_{p\in\mathcal{M}} \bigotimes_g^r \mathcal{M}|_p$$

and the natural projection

$$\begin{array}{rccl} \tau_{\otimes_g^r \mathcal{M}}: & \bigotimes_g^r \mathcal{M} & \to & \mathcal{M} \\ & \bigotimes_g^r \mathcal{M}|_p & \to & p\,. \end{array}$$

Definition A.46 (tensor field) *A* C^∞ *tensor field on a* C^∞ *manifold* $\mathcal{M}$ *is a function* T_g^r *which assigns to each* $p \in \mathcal{M}$ *an element* $T_{g,p}^r$ *and which has the additional property that, given any* $v_1, \ldots, v_r$, $\omega_1, \ldots, \omega_g$, C^∞ *vector fields on an open subset* U *of* $\mathcal{M}$, *then* $T_g^r\left(v_1, \ldots, v_r, \omega_1, \ldots, \omega_g\right)$ *is a* C^∞ *function on* U, *defined by* $T_g^r\left(v_1, \ldots, v_r, \omega_1, \ldots, \omega_g\right)(p) = T_{g,p}^r\left(v_1(p), \ldots, v_r(p),\right.$ $\left.\omega_1(p), \ldots, \omega_g(p)\right)$. *(see [Boothby, 1986])*
Thus a tensor field T_g^r *represents a section on the bundle* $\left(\bigotimes_g^r \mathcal{M}, \tau_{\otimes_g^r \mathcal{M}}, \mathcal{M}\right)$.

Definition A.47 (symmetric tensor field) *A covariant tensor field* T_0^r *is called symmetric iff for each* $1 \le i, j \le r$, *we have*

$$\Phi\left(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_r\right) = \Phi\left(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_r\right)\ .$$

Similarly a contravariant tensor field is called symmetric iff for each $1 \le i, j \le r$, *we have*

$$\Phi\left(\omega_1, \ldots, \omega_i, \ldots, \omega_j, \ldots, \omega_r\right) = \Phi\left(\omega_1, \ldots, \omega_j, \ldots, \omega_i, \ldots, \omega_r\right)\ .$$

(see [Boothby, 1986])

Definition A.48 (skew symmetric tensor field) *A covariant tensor field* T_0^r *is called skew symmetric (or alternating) iff for each* $1 \le i, j \le r$, *we have*

$$\Phi\left(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_r\right) = -\Phi\left(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_r\right)\ .$$

Similarly a contravariant tensor field is called skew symmetric (or alternating) iff for each $1 \le i, j \le r$, *we have*

$$\Phi\left(\omega_1, \ldots, \omega_i, \ldots, \omega_j, \ldots, \omega_r\right) = -\Phi\left(\omega_1, \ldots, \omega_j, \ldots, \omega_i, \ldots, \omega_r\right)\ .$$

(see [Boothby, 1986])

Definition A.49 (tensor product) *The product of the* r^{th}*-order covariant tensor* φ^r *and* s^{th}*-order covariant tensor* ψ^s *is a tensor of order* $r+s$ *defined by*

$$\varphi_0^r \otimes \psi_0^s\left(v_1, \ldots, v_r, \ldots, v_{r+1}, \ldots, v_{r+s}\right) = \varphi_0^r\left(v_1, \ldots, v_r\right)\psi_0^s\left(v_{r+1}, \ldots, v_{r+s}\right)\ .$$

Definition A.50 (symmetrizing and alternating mapping) *The map*

$$\begin{array}{rcl} \mathrm{sym}: \bigotimes_0^r \mathcal{M} & \to & \bigotimes_0^r \mathcal{M} \\ T_0^r\left(v_1, \ldots, v_r\right) & \to & \dfrac{1}{r!}\sum_\sigma T_0^r\left(v_{\sigma(1)}, \ldots, v_{\sigma(r)}\right) \end{array}$$

is referred to as symmetrizing mapping and

$$\begin{aligned} \text{asym} : \bigotimes\nolimits_0^r \mathcal{M} &\to \bigwedge\nolimits^r (\mathcal{M}) \subseteq \bigotimes\nolimits_0^r \mathcal{M} \\ T_0^r (v_1, \ldots, v_r) &\to \frac{1}{r!} \sum_\sigma \operatorname{sgn}(\sigma)\ T_0^r \left(v_{\sigma(1)}, \ldots, v_{\sigma(r)}\right) \end{aligned}$$

is referred to as alternating mapping. Here σ denotes a permutation of $(1, \ldots, r)$ with $(1, \ldots, r) \to (\sigma(1), \ldots, \sigma(r))$ and $\operatorname{sgn}(\sigma)$ the corresponding sign. (see [Boothby, 1986])

Definition A.51 (Lie derivative) *The Lie derivative of a tensor field T_g^r with respect to a vector field $v \in \Gamma(\tau_{\mathcal{M}})$ is a derivation on the algebra of differentiable tensor fields $\bigotimes \mathcal{M}$, i.e.*

$$\begin{aligned} \mathrm{L}_v (u + w) &= \mathrm{L}_v (u) + \mathrm{L}_v (w) \\ \mathrm{L}_v (u \otimes w) &= \mathrm{L}_v (u) \otimes w + u \otimes \mathrm{L}_v (w)\ , \quad w, u \in \Gamma\left(\bigotimes \mathcal{M}\right)\ . \end{aligned}$$

(see [Choquet-Bruhat, Cecile DeWitt-Morette, 1982])

Remark A.52 *The Lie bracket $[v, w]$ represents the Lie derivative $\mathrm{L}_v (w)$ of a contravariant tensor field $w \in \Gamma(\tau_{0\mathcal{M}}^1) = \Gamma(\tau_{\mathcal{M}})$ with respect to the vector $v \in \Gamma(\tau_{\mathcal{M}})$.*

Definition A.53 (Lie bracket) *The Lie bracket of two vector fields $v, w \in \Gamma(\tau_{\mathcal{M}})$ is defined by*

$$[v, w](f) = v(w(f)) - w(v(f))\ , \quad f \in C^\infty \mathcal{M}$$

and is again a vector field $[v, w] \in \Gamma(\tau_{\mathcal{M}})$. (see [Choquet-Bruhat, Cecile DeWitt-Morette, 1982])

A.3.3 Exterior Algebra

Definition A.54 (exterior form) *Exterior r-forms are alternating tensors fields T_0^r of order r and form a subspace $\bigwedge^r \mathcal{M} \subseteq \bigotimes_0^r \mathcal{M}$. (see [Boothby, 1986])*

Definition A.55 (exterior bundle) *The tensor bundle $(\bigwedge^r \mathcal{M}, \tau_{\wedge^r \mathcal{M}}, \mathcal{M})$ of a manifold $\mathcal{M}$ consists of the total manifold $\bigwedge^r \mathcal{M}$ and the natural projection*

$$\begin{aligned} \tau_{\wedge^r \mathcal{M}} : \quad \bigwedge\nolimits^r \mathcal{M} &\to \mathcal{M} \\ \bigwedge\nolimits^r \mathcal{M}|_p &\to p\ . \end{aligned}$$

In particular the bundle $\tau_{\wedge^1 \mathcal{M}}$ equals $\bar{\tau}_{\mathcal{M}}$.

Definition A.56 (exterior product) *The mapping from $\bigwedge^r \mathcal{M} \times \bigwedge^s \mathcal{M} = \bigwedge^{r+s} \mathcal{M}$, defined by*

$$(\varphi, \psi) \to \frac{(r+s)!}{r!\ s!} \text{asym}(\varphi \otimes \psi)$$

is called the exterior product (or wedge product) of φ and ψ and is denoted $\varphi \wedge \psi$. (see [Boothby, 1986])

Definition A.57 (exterior algebra) *The space*

$$\bigwedge \mathcal{M} = \bigwedge^0 \mathcal{M} \oplus \bigwedge^1 \mathcal{M} \oplus \bigwedge^2 \mathcal{M} \oplus \cdots \oplus \bigwedge^n \mathcal{M}$$

on a n-dimensional with the exterior product $\wedge$ forms an algebra of $\mathbb{R}$. (see [Boothby, 1986])

Definition A.58 (exterior ideal) *An ideal of an exterior algebra $\bigwedge \mathcal{M}$ on a manifold $\mathcal{M}$ is a subspace $I \subset \bigwedge \mathcal{M}$ which has the property that whenever $\varphi \in I$ and $\theta \in \bigwedge \mathcal{M}$, then $\theta \wedge \varphi \in I$. (see [Boothby, 1986])*

Definition A.59 (degree of a linear operator) *Let L be a linear operator on $\bigwedge (\mathcal{M})$*

$$L(\lambda\omega + \mu\theta) = \lambda L(\omega) + \mu L(\theta) \ , \quad \lambda, \mu \in \mathbb{R} \ , \quad \omega, \theta \in \bigwedge (\mathcal{M}) \ .$$

Then the degree s of the operator is determined by

$$L : \bigwedge^r (\mathcal{M}) \to \bigwedge^{r+s} (\mathcal{M}) \ .$$

Definition A.60 (derivation) *A linear operator L is a derivation on $\bigwedge (\mathcal{M})$ if its degree is even and if it obeys the Leibnitz rule*

$$L(\omega \wedge \theta) = L(\omega) \wedge \theta + \omega \wedge L(\theta) \ , \quad \omega, \theta \in \bigwedge (\mathcal{M}) \ .$$

Definition A.61 (antiderivation) *A linear operator L is an antiderivation on $\bigwedge (\mathcal{M})$ if its degree is odd and if it obeys the "antiLeibnitz" rule*

$$L(\omega \wedge \theta) = L(\omega) \wedge \theta + (-1)^{\deg(\omega)} \omega \wedge L(\theta) \ , \quad \omega, \theta \in \bigwedge (\mathcal{M}) \ .$$

Definition A.62 (exterior derivative) *Let $\mathcal{M}$ be any C^∞ manifold and let $\bigwedge (\mathcal{M})$ be the algebra of exterior differential forms on $\mathcal{M}$. Then there exists a unique $\mathbb{R}$-linear map $\mathrm{d} : \bigwedge (\mathcal{M}) \to \bigwedge (\mathcal{M})$ such that*

- *if $f \in \bigwedge^0 (\mathcal{M}) = C^\infty (\mathcal{M})$, then $\mathrm{d}f$ is the differential of f;*
- *if $\alpha \in \bigwedge^r (\mathcal{M})$ and $\beta \in \bigwedge^s (\mathcal{M})$, then $\mathrm{d}(\alpha \wedge \beta) = \mathrm{d}\alpha \wedge \beta + (-1)^r \alpha \wedge \mathrm{d}\beta$;*
- $\mathrm{d}(\mathrm{d}(\cdot)) = 0$.

The exterior derivative maps $\bigwedge^r (\mathcal{M})$ into $\bigwedge^{r+1} (\mathcal{M})$ and is additionally an antiderivation.

Definition A.63 (interior product) *The contracted multiplication or interior product of a form $\omega \in \bigwedge (\mathcal{M})$ and a vector $v \in \Gamma(\tau_{\mathcal{M}})$ denoted by $v \rfloor \omega$ (or $i_v(\omega)$) is an antiderivation*

$$v\rfloor (\omega \wedge \theta) = v\rfloor (\omega) \wedge \theta + (-1)^{\deg(\omega)} \omega \wedge v\rfloor (\theta) \quad \theta \in \bigwedge (\mathcal{M})$$

and is defined by

- *if $\omega \in \bigwedge^0 (\mathcal{M}) = C^\infty (\mathcal{M})$, then $v\rfloor\omega = 0$;*
- *if $\omega \in \bigwedge^1 (\mathcal{M}) = \mathcal{T}^* (\mathcal{M})$, then $v\rfloor\omega = \mathrm{L}_v(\omega)$ equals the Lie derivative;*
- *if $\omega \in \bigwedge^r (\mathcal{M})$, then $v\rfloor (\omega(w_2, \ldots, w_r)) = \omega(v, w_2, \ldots, w_r)$ with $w_2, \ldots, w_r \in \Gamma(\tau_{\mathcal{M}})$;*
- $\mathrm{d}(\mathrm{d}(\cdot)) = 0$.

The exterior derivative maps $\bigwedge^r (\mathcal{M})$ into $\bigwedge^{r+1} (\mathcal{M})$.

Definition A.64 (Lie derivative of an exterior form) *If the Lie derivative $\mathrm{L}_v(\cdot)$ acts on alternating covariant tensors ω, i.e. exterior forms, one is able use H. Cartan's formula*

$$\mathrm{L}_v(\omega) = v\rfloor \mathrm{d}(\omega) + \mathrm{d}(v\rfloor\omega) \ .$$

A.3.4 Special bundle morphisms

Definition A.65 (push-forward, differential) *A smooth mapping $f : \mathcal{M} \to \mathcal{N}$ between the n-dimensional manifolds $\mathcal{M}$ and $\mathcal{N}$ induces a bundle morphism (f_*, f) between the corresponding tangent bundles $\tau_{\mathcal{M}}$ and $\tau_{\mathcal{N}}$.*

$$\begin{tikzcd}
\mathcal{T}(\mathcal{M}) \arrow[r, "(f_*, f)"] \arrow[d, "\tau_{\mathcal{M}}"'] & \mathcal{T}(\mathcal{N}) \arrow[d, "\tau_{\mathcal{N}}"] \\
\mathcal{M} \arrow[r, "f"'] & \mathcal{N}
\end{tikzcd}$$

In local coordinates $x^1, \ldots, x^n$ according to $\mathcal{M}$ and $z^1, \ldots, z^n$ according to $\mathcal{N}$ this morphism has the form

$$\begin{aligned} z^i &= f^i\left(x^k\right), \quad i, k = 1, \ldots, n \\ \dot{z}^i &= f_*^i\left(x^k, \dot{x}^k\right) = \frac{\partial f^i}{\partial x^k}\dot{x}^k . \end{aligned}$$

The corresponding transformation of a section $v = v^k \partial_k \in \Gamma(\tau_{\mathcal{M}})$ (see Def. 3.6), i.e. a vector field, is given by

$$w = w^i \partial_i = f_*(v) = \left(\left(\frac{\partial f^i}{\partial x^k} v^k\right) \circ f^{-1}\right) \partial_i \in \Gamma(\tau_{\mathcal{N}})$$

and denoted push-forward or differential.

Definition A.66 *A smooth mapping $f : \mathcal{N} \to \mathcal{M}$ between the n-dimensional manifolds $\mathcal{M}$ and $\mathcal{N}$ induces a bundle morphism (f^*, f^{-1}) between the corresponding cotangent bundles $\bar{\tau}_{\mathcal{M}}$ and $\bar{\tau}_{\mathcal{N}}$.*

$$\begin{tikzcd}
\mathcal{T}^*(\mathcal{M}) \arrow[r, "(f^*, f^{-1})"] \arrow[d, "\bar{\tau}_{\mathcal{M}}"'] & \mathcal{T}^*(\mathcal{N}) \arrow[d, "\bar{\tau}_{\mathcal{N}}"] \\
\mathcal{M} \arrow[r, "f^{-1}"'] & \mathcal{N}
\end{tikzcd}$$

In local coordinates $x^1, \ldots, x^n$ according to $\mathcal{M}$ and $z^1, \ldots, z^n$ according to $\mathcal{N}$ this morphism has the form

$$\begin{aligned} z^i &= \left(f^{-1}\right)^i\left(x^k\right), \quad i, k = 1, \ldots, n \\ \dot{z}_i &= f^{*i}\left(x^k, \dot{x}_k\right) = \left(\frac{\partial f^k}{\partial z^i}\dot{x}_k\right) \circ f^{-1} . \end{aligned}$$

It is now remarkable, that the corresponding transformation of a section $\omega = \omega_k \mathrm{d}x^k \in \Gamma(\bar{\tau}_{\mathcal{M}})$, i.e. a covector field, does not require any inverse function. The transformation is given by

$$\lambda = \lambda_i \mathrm{d}z^i = f^*\left(\omega_k \mathrm{d}x^k\right) = \left(\left(\frac{\partial f^k}{\partial z^i}\omega_k\right) \circ f\right) \mathrm{d}z^i \in \Gamma(\bar{\tau}_{\mathcal{N}})$$

and denoted the pull-back of a 1-form.

Definition A.67 (pull-back of r-forms) *A smooth mapping $f : \mathcal{N} \to \mathcal{M}$ between the n-dimensional manifolds $\mathcal{M}$ and $\mathcal{N}$ induces a bundle morphism (f^*, f^{-1}) between the corresponding cotangent bundles $(\bigwedge^r \mathcal{M}, \tau_{\wedge^r\mathcal{M}}, \mathcal{M})$ and $(\bigwedge^r \mathcal{N}, \tau_{\wedge^r\mathcal{N}}, \mathcal{N})$, $0 \leq r \leq n$. In local coordinates $x^1, \ldots, x^n, \dot{x}_{i_1,\ldots,i_r}$ according to $\bigwedge^r \mathcal{M}$ and $z^1, \ldots, z^n, \dot{z}_{j_1,\ldots,j_r}$ according to $\bigwedge^r \mathcal{N}$ this morphism has the form*

$$\begin{aligned} z^i &= \left(f^{-1}\right)^i \left(x^k\right), \quad i, k = 1, \ldots, n \\ \dot{z}_{j_1,\ldots,j_r} &= f^*\left(x^k, \dot{x}_{i_1,\ldots,i_r}\right) = \left(\frac{\partial f^{i_1}}{\partial z^{j_1}} \cdots \frac{\partial f^{i_r}}{\partial z^{j_r}} \dot{x}_{i_1,\ldots,i_r}\right) \circ f^{-1}, \quad \text{with } \begin{array}{c} 1 \leq i_1 \leq \cdots \leq i_r \leq n \\ 1 \leq j_1 \leq \cdots \leq j_r \leq n, \end{array} . \end{aligned}$$

It is worth mentioning, that the fibre of such vector bundles $\tau_{\wedge^r\mathcal{M}}$ resp. $\tau_{\wedge^r\mathcal{N}}$ is of dimension $\binom{n}{r}$. The transformation of an r-form $\omega = \omega_{i_1,\ldots,i_r} \mathrm{d}x^{i_1} \wedge \ldots \wedge \mathrm{d}x^{i_r} \in \Gamma\left(\tau_{\wedge^r\mathcal{M}}\right)$ under this bundle morphism is given by

$$\alpha_{j_1,\ldots,j_r} \mathrm{d}z^{j_1} \wedge \ldots \wedge \mathrm{d}z^{j_r} = f^*(\omega) = \left(\left(\frac{\partial f^{i_1}}{\partial z^{j_1}} \cdots \frac{\partial f^{i_r}}{\partial z^{j_r}} \omega_{i_1,\ldots,i_r}\right) \circ f\right) \mathrm{d}z^{j_1} \wedge \ldots \wedge \mathrm{d}z^{j_r}.$$

Thus, there is no need to confine oneselve in the application of the pull-back to invertible functions f.

Definition A.68 (tensor bundle morphism) *A smooth mapping $f : \mathcal{M} \to \mathcal{N}$ between the n-dimensional manifolds $\mathcal{M}$ and $\mathcal{N}$ induces a bundle morphism (F, f) between the corresponding tensor bundles $\tau_{\otimes^r_g\mathcal{M}}$ and $\tau_{\otimes^r_g\mathcal{N}}$ denoted tensor bundle morphism.*

$$\begin{tikzcd} \bigotimes^r_g \mathcal{M} \arrow[r, "{(F, f)}"] \arrow[d, "\tau_{\otimes^r_g\mathcal{M}}"'] & \bigotimes^r_g \mathcal{N} \arrow[d, "\tau_{\otimes^r_g\mathcal{N}}"] \\ \mathcal{M} \arrow[r, "f"'] & \mathcal{N} \end{tikzcd}$$

In local coordinates $x^1, \ldots, x^n$, $\dot{x}^{\,c\ldots d}_{\,r\ldots s}$ according to $\otimes^r_g\mathcal{M}$ and $z^1, \ldots, z^n$, $\dot{z}^{\,i\ldots j}_{\,k\ldots l}$ according to $\otimes^r_g\mathcal{N}$ this morphism has the form

$$\begin{aligned} z^i &= (f)^i \left(x^k\right), \quad i, k, j, l = 1, \ldots, n \\ \dot{z}^{\,i\ldots j}_{\,k\ldots l} &= F^{i\ldots j}_{k\ldots l}\left(\dot{x}^{\,c\ldots d}_{\,r\ldots s}\right) = \left(\underbrace{\frac{\partial f^i}{\partial x^c} \cdots \frac{\partial f^j}{\partial x^d}}_{r} \underbrace{\frac{\partial \left(f^{-1}\right)^r}{\partial z^k} \cdots \frac{\partial \left(f^{-1}\right)^s}{\partial z^l}}_{g} \dot{x}^{\,c\ldots d}_{\,r\ldots s}\right) \circ f . \end{aligned}$$

A.4 Integration on manifolds

Definition A.69 (integrable functions, integrable forms) *A function f on $\mathcal{M}$ is integrable if it is bounded, has compact support (vanishes outside a compact set), and is almost continuous (that is, continuous except possibly on a set of content (measure) zero). An n-form ω on $\mathcal{M}$, in the very general sense of a function assigning to each $p \in \mathcal{M}$ an element ω_p of $\bigwedge^n \mathcal{M}|_p$, is said to be integrable if $\omega = f\ \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n$, where f is an integrable function. (see [Boothby, 1986])*

Theorem A.70 (Stokes's theorem) *Let $\mathcal{M}$ be an oriented compact manifold of dimension n with coherently oriented boundary $\partial\mathcal{M}$. Let ω be a differential form of degree $(n-1)$ of compact support. Then:*

$$\int_{\mathcal{M}} \mathrm{d}\omega = \int_{\partial\mathcal{M}} \iota^*\omega$$

is met with the inclusion mapping $i : \partial\mathcal{M} \to \mathcal{M}$. When $\partial\mathcal{M} = 0$, the integral over $\mathcal{M}$ vanishes. (see [Abraham, J.E. Marsden, T. Ratiu, 1988])

Appendix B

The Kirchhoff Plate

The derivation of the equations of motion, the determination of the boundary conditions, the formulation of the port Hamiltonian representation, and finally the stability analysis of the so called Kirchhoff plate is used through out this thesis to illustrate the applicability of the presented methods. Indeed this problem corresponds to the class of 2nd order Euler-Lagrange systems and consequently the theory of chapter 8 has to be used, if one wants to apply the method of Cartan forms.

In fact one is also able to apply the integration by parts method to derive the domain conditions and as it was done by, e.g., Ritz [Ritz, 1909] also the boundary conditions. Here we will discuss this approach using the classical notation as introduced by, e.g., [Gelfand, S.V. Fomin, 2000]. The considered domain of the rectangular Kirchhoff plate is shown in figure B.1.

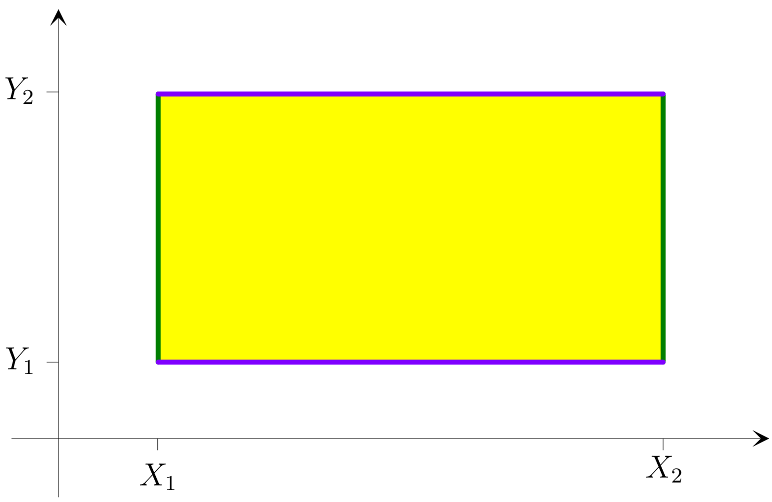

Figure B.1: The rectangular Kirchhoff plate.

The potential energy is given by

$$V = \varsigma\frac{1}{2}\int_{Y_1}^{Y_2}\int_{X_1}^{X_2}\left[\left(\frac{\partial^2 w}{\partial X^2}\right)^2 + \left(\frac{\partial^2 w}{\partial Y^2}\right)^2 + 2\nu\left(\frac{\partial^2 w}{\partial X^2}\right)\left(\frac{\partial^2 w}{\partial Y^2}\right) + 2\left(1-\nu\right)\left(\frac{\partial^2 w}{\partial X \partial Y}\right)^2\right]\mathrm{d}X\,\mathrm{d}Y$$

and the kinetic energy could be formulated by

$$T = \frac{1}{2}\int_{Y_1}^{Y_2}\int_{X_1}^{X_2}\left[\left(\rho\Lambda\frac{\partial w}{\partial t}\right)^2\right]\mathrm{d}X\,\mathrm{d}Y$$

using $\varsigma, \nu, \rho, \Lambda \in \mathbb{R}^+$ (see [Ritz, 1909] or [Bremer, F. Pfeiffer, 1992]). The deviation of the plate is denoted by $w\,(X,Y)$.Taking into account Hamilton's principle, we are able to define the Lagrangian functional

$$\begin{aligned}\mathfrak{L} \;=\; & \int_{t_1}^{t_2}\int_{Y_1}^{Y_2}\int_{X_1}^{X_2} l\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}t = \frac{1}{2}\int_{t_1}^{t_2}\int_{Y_1}^{Y_2}\int_{X_1}^{X_2}\left[\left(\frac{\partial w}{\partial t}\right)^2 - \left(\frac{\partial^2 w}{\partial X^2}\right)^2 - \left(\frac{\partial^2 w}{\partial Y^2}\right)^2\right. \\ & \left. -2\nu\left(\frac{\partial^2 w}{\partial X^2}\right)\left(\frac{\partial^2 w}{\partial Y^2}\right) - 2\,(1-\nu)\left(\frac{\partial^2 w}{\partial X\partial Y}\right)^2\right]\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}t\;,\end{aligned} \tag{B.1}$$

where $\varsigma = \rho = \Lambda = 1$ is used to simplify the upcoming expressions. The variation of the Lagrangian functional $\delta\mathfrak{L}$ leads to

$$\begin{aligned}\delta l \;=\; & \frac{\partial w}{\partial t}\delta\frac{\partial w}{\partial t} - \left(\frac{\partial^2 w}{\partial X^2} + \nu\frac{\partial^2 w}{\partial Y^2}\right)\delta\frac{\partial^2 w}{\partial X^2} \\ & - \left(\frac{\partial^2 w}{\partial Y^2} + \nu\frac{\partial^2 w}{\partial X^2}\right)\delta\frac{\partial^2 w}{\partial Y^2} - 2\,(1-\nu)\,\frac{\partial^2 w}{\partial X\partial Y}\delta\frac{\partial^2 w}{\partial X\partial Y}\;.\end{aligned}$$

In order to obtain the domain conditions one is able to apply integration by parts. Subsequently this procedure is shown for every single part of δl.

For the first part we obtain

$$\int_{t_1}^{t_2}\int_{Y_1}^{Y_2}\int_{X_1}^{X_2}\frac{\partial w}{\partial t}\delta\frac{\partial w}{\partial t}\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}t = \int_{Y_1}^{Y_2}\int_{X_1}^{X_2}\left(\left[\frac{\partial w}{\partial t}\delta w\right]_{t_1}^{t_2} - \int_{t_1}^{t_2}\frac{\partial^2 w}{\partial t^2}\delta w\mathrm{d}t\right)\mathrm{d}X\,\mathrm{d}Y\;.$$

The second part results in

$$\begin{aligned}& \int_{t_1}^{t_2}\int_{Y_1}^{Y_2}\int_{X_1}^{X_2} - \left(\frac{\partial^2 w}{\partial X^2} + \nu\frac{\partial^2 w}{\partial Y^2}\right)\delta\frac{\partial^2 w}{\partial X^2}\mathrm{d}X\,\mathrm{d}Y\mathrm{d}t \;= \\ & = \int_{t_1}^{t_2}\int_{Y_1}^{Y_2}\left(-\left[\underbrace{\left(\frac{\partial^2 w}{\partial X^2} + \nu\frac{\partial^2 w}{\partial Y^2}\right)}_{\text{boundary condition}}\delta\frac{\partial w}{\partial X}\right]_{X_1}^{X_2} + \int_{X_1}^{X_2}\left(\frac{\partial^3 w}{\partial X^3} + \nu\frac{\partial^3 w}{\partial X\partial Y^2}\right)\delta\frac{\partial w}{\partial X}\,\mathrm{d}X\right)\mathrm{d}Y\,\mathrm{d}t\;.\end{aligned}$$

Obviously one is able to apply an additional integration by parts on last entry, i.e. we get

$$\begin{aligned}& \int_{t_1}^{t_2}\int_{Y_1}^{Y_2}\int_{X_1}^{X_2}\left(\frac{\partial^3 w}{\partial X^3} + \nu\frac{\partial^3 w}{\partial X\partial Y^2}\right)\delta\frac{\partial w}{\partial X}\mathrm{d}X\mathrm{d}Y\mathrm{d}t \;= \\ & = \int_{t_1}^{t_2}\int_{Y_1}^{Y_2}\left(\left[\underbrace{\left(\frac{\partial^3 w}{\partial X^3} + \nu\frac{\partial^3 w}{\partial X\partial Y^2}\right)}_{\text{boundary condition}}\delta w\right]_{X_1}^{X_2} - \int_{X_1}^{X_2}\left(\frac{\partial^4 w}{\partial X^4} + \nu\frac{\partial^4 w}{\partial X^2\partial Y^2}\right)\delta w\mathrm{d}X\right)\mathrm{d}Y\mathrm{d}t\;.\end{aligned}$$

The third part has to be treated similarly and we obtain

$$\int_{t_1}^{t_2}\int_{X_1}^{X_2}\int_{Y_1}^{Y_2} -\left(\frac{\partial^2 w}{\partial Y^2}+\nu\frac{\partial^2 w}{\partial X^2}\right)\delta\frac{\partial^2 w}{\partial Y^2}\mathrm{d}Y\,\mathrm{d}X\,\mathrm{d}t =$$

$$= \int_{t1}^{t2}\int_{X1}^{X2}\left(-\left[\underbrace{\left(\frac{\partial^2 w}{\partial Y^2}+\nu\frac{\partial^2 w}{\partial X^2}\right)}_{\text{boundary condition}}\delta\frac{\partial w}{\partial Y}\right]_{Y1}^{Y2} + \int_{Y_1}^{Y_2}\left(\frac{\partial^3 w}{\partial Y^3}+\nu\frac{\partial^3 w}{\partial X^2\partial Y}\right)\delta\frac{\partial w}{\partial Y}\mathrm{d}Y\right)\mathrm{d}X\,\mathrm{d}t$$

and

$$\int_{t_1}^{t_2}\int_{X_1}^{X_2}\int_{Y_1}^{Y_2}\left(\frac{\partial^3 w}{\partial Y^3}+\nu\frac{\partial^3 w}{\partial X^2\partial Y}\right)\delta\frac{\partial w}{\partial Y}\mathrm{d}Y\,\mathrm{d}X\,\mathrm{d}t =$$

$$= \int_{t_1}^{t_2}\int_{X_1}^{X_2}\left(\left[\underbrace{\left(\frac{\partial^3 w}{\partial Y^3}+\nu\frac{\partial^3 w}{\partial X^2\partial Y}\right)}_{\text{boundary condition}}\delta w\right]_{Y_1}^{Y_2} - \int_{Y_1}^{Y_2}\left(\frac{\partial^4 w}{\partial Y^4}+\nu\frac{\partial^4 w}{\partial X^2\partial Y^2}\right)\delta w\mathrm{d}Y\right)\mathrm{d}X\,\mathrm{d}t\ .$$

The fourth part of δl

$$\int_{t_1}^{t_2}\int_{Y_1}^{Y_2}\int_{X_1}^{X_2} -2\left(1-\nu\right)\frac{\partial^2 w}{\partial X\partial Y}\delta\frac{\partial^2 w}{\partial X\partial Y}\mathrm{d}Y\,\mathrm{d}X\,\mathrm{d}t$$

enlightens finally the general problem of n^{th} order Lagrangians, as we obtain two different possibilities for the application of integration by parts.

Remark B.1 *This non-uniqueness of the integration by parts order is equivalent to the non-uniqueness of the used contact form in the construction of the Cartan form.*

We will discuss both possibilities and obtain in the first case, where we apply integration by parts on the X-coordinate

$$\int_{t_1}^{t_2}\int_{Y_1}^{Y_2}\left(-\left[2\left(1-\nu\right)\frac{\partial^2 w}{\partial X\partial Y}\delta\frac{\partial w}{\partial Y}\right]_{X_1}^{X_2} + \int_{X_1}^{X_2}\left(2\left(1-\nu\right)\frac{\partial^3 w}{\partial X^2\partial Y}\right)\delta\frac{\partial w}{\partial Y}\mathrm{d}X\right)\mathrm{d}Y\,\mathrm{d}t\ .$$

Obviously an additional integration by parts on the second part has to be applied, i.e.

$$\int_{t_1}^{t_2}\int_{Y_1}^{Y_2}\int_{X_1}^{X_2}\left(2\left(1-\nu\right)\frac{\partial^3 w}{\partial X^2\partial Y}\right)\delta\frac{\partial w}{\partial Y}\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}t$$

$$= \int_{t_1}^{t_2}\int_{X_1}^{X_2}\left(\left[\underbrace{\left(2\left(1-\nu\right)\frac{\partial^3 w}{\partial X^2\partial Y}\right)}_{\text{boundary condition}}\delta w\right]_{Y_1}^{Y_2} - \int_{Y_1}^{Y_2}\left(2\left(1-\nu\right)\frac{\partial^4 w}{\partial X^2\partial Y^2}\right)\delta w\mathrm{d}Y\right)\mathrm{d}X\,\mathrm{d}t\ .$$

In fact on the first part – on the boundary – integration by parts is also necessary. This leads to

$$\int_{t_1}^{t_2}\int_{Y_1}^{Y_2} - \left[2\left(1-\nu\right)\frac{\partial^2 w}{\partial X \partial Y}\delta\frac{\partial w}{\partial Y}\right]_{X_1}^{X_2} \mathrm{d}Y\ \mathrm{d}t$$

$$= \int_{t_1}^{t_2}\left(\left[-\left[2\left(1-\nu\right)\underbrace{\frac{\partial^2 w}{\partial X \partial Y}}_{\text{edge condition}}\delta w\right]_{X_1}^{X_2}\right]_{Y_1}^{Y_2} + \left[\int_{Y_1}^{Y_2}\underbrace{\left(2\left(1-\nu\right)\frac{\partial^3 w}{\partial X \partial Y^2}\right)}_{\text{boundary condition}}\delta w\ \mathrm{d}Y\right]\right)_{X1}^{X2} \mathrm{d}t\ .$$

Consequently we obtain a condition of the edges of domain of the Kirchhoff plate, as it was stated by Ritz or Lamb [Ritz, 1909].

The second possibility – integration by parts on the Y-coordinate – lead similarly to

$$\int_{t_1}^{t_2}\int_{X_1}^{X_2}\left(-\left[2\left(1-\nu\right)\frac{\partial^2 w}{\partial X \partial Y}\delta\frac{\partial w}{\partial X}\right]_{Y_1}^{Y_2} + \int_{Y_1}^{Y_2}\left(2\left(1-\nu\right)\frac{\partial^3 w}{\partial X \partial Y^2}\right)\delta\frac{\partial w}{\partial X}\mathrm{d}Y\right)\mathrm{d}X\ \mathrm{d}t$$

and

$$\int_{t_1}^{t_2}\int_{Y_1}^{Y_2}\int_{X_1}^{X_2}\left(2\left(1-\nu\right)\frac{\partial^3 w}{\partial X \partial Y^2}\right)\delta\frac{\partial w}{\partial X}\mathrm{d}X\ \mathrm{d}Y\ \mathrm{d}t$$

$$= \int_{t_1}^{t_2}\int_{Y_1}^{Y_2}\left(\left[\underbrace{\left(2\left(1-\nu\right)\frac{\partial^3 w}{\partial X \partial Y^2}\right)}_{\text{boundary condition}}\delta w\right]_{X_1}^{X_2} - \int_{X_1}^{X_2}\left(2\left(1-\nu\right)\frac{\partial^4 w}{\partial X^2 \partial Y^2}\right)\delta w \mathrm{d}X\right)\mathrm{d}Y \mathrm{d}t\ .$$

From

$$\int_{t_1}^{t_2}\int_{X_1}^{X_2} - \left[2\left(1-\nu\right)\frac{\partial^2 w}{\partial X \partial Y}\delta\frac{\partial w}{\partial X}\right]_{Y_1}^{Y_2} \mathrm{d}Y \mathrm{d}t$$

$$= \int_{t_1}^{t_2}\left(\left[-\left[2\left(1-\nu\right)\underbrace{\frac{\partial^2 w}{\partial X \partial Y}}_{\text{edge condition}}\delta w\right]_{Y_1}^{Y_2}\right]_{X_1}^{X_2} + \left[\int_{X_1}^{X_2}\underbrace{\left(2\left(1-\nu\right)\frac{\partial^3 w}{\partial X^2 \partial Y}\right)}_{\text{boundary condition}}\delta w \mathrm{d}X\right]_{Y_1}^{Y_2}\right)\mathrm{d}t$$

we obtain also in this case a condition on the edge of the plate.

Finally the domain condition results in

$$\left(\frac{\partial^2 w}{\partial t^2} + \frac{\partial^4 w}{\partial X^4} + \frac{\partial^4 w}{\partial Y^4} + 2\frac{\partial^4 w}{\partial X^2 \partial Y^2}\right)\delta w\ \mathrm{d}X\ \mathrm{d}Y\ \mathrm{d}t = 0\ ,$$

the boundary condition on $[\quad]_{X_1}^{X_2}$ results in

$$-\left[\left(\frac{\partial^2 w}{\partial X^2} + \nu\frac{\partial^2 w}{\partial Y^2}\right)\delta\frac{\partial w}{\partial X}\right]_{X_1}^{X_2} = 0$$

$$\left[\left(\frac{\partial^3 w}{\partial X^3} + \left(2-\nu\right)\frac{\partial^3 w}{\partial X \partial Y^2}\right)\delta w\right]_{X_1}^{X_2} = 0$$

and on $[\quad]_{Y_1}^{Y_2}$ we obtain

$$\left[\left(\frac{\partial^3 w}{\partial Y^3} + (2-\nu)\,\frac{\partial^3 w}{\partial X^2 \partial Y}\right)\delta w\right]_{Y_1}^{Y_2} = 0$$
$$-\left[\left(\frac{\partial^2 w}{\partial Y^2} + \nu \frac{\partial^2 w}{\partial X^2}\right)\delta \frac{\partial w}{\partial Y}\right]_{Y1}^{Y2} = 0\;.$$

It it worth mentioning that these results coincide with the results of section 8.6, despite the obtained condition on the edges

$$\left[\left[\frac{\partial^2 w}{\partial X \partial Y}\delta w\right]_{Y_1}^{Y_2}\right]_{X_1}^{X_2} = 0\;.$$

As already mentioned in section 5.1, the whole method has a significant drawback - integration by parts is not the appropriate tool for higher dimensional domains (in contrary to Stokes's theorem) and cannot be used without caution. In fact the integration order was changed without any modification of the integrand. In order to incorporate this, we introduce the notion of forms to B.1 and split part 4 in two integrals

$$\int_{t_1}^{t_2}\int_{Y_1}^{Y_2}\int_{X_1}^{X_2} -2\,(1-\nu)\,\frac{\partial^2 w}{\partial X \partial Y}\delta\frac{\partial^2 w}{\partial X \partial Y}\mathrm{d}Y\wedge\mathrm{d}X\wedge\mathrm{d}t$$
$$= \underbrace{\int_{t_1}^{t_2}\int_{Y_1}^{Y_2}\int_{X_1}^{X_2} (1-\nu)\,\frac{\partial^2 w}{\partial X \partial Y}\delta\frac{\partial^2 w}{\partial X \partial Y}\mathrm{d}X\wedge\mathrm{d}Y\wedge\mathrm{d}t}_{\text{use integration by parts – possibility 1}}$$
$$+\underbrace{\int_{t_1}^{t_2}\int_{X_1}^{X_2}\int_{Y_1}^{Y_2} -(1-\nu)\,\frac{\partial^2 w}{\partial X \partial Y}\delta\frac{\partial^2 w}{\partial X \partial Y}\mathrm{d}Y\wedge\mathrm{d}X\wedge\mathrm{d}t}_{\text{use integration by parts – possibility 2}}\;.$$

In the second case we have changed the integration order and consequently also the sign of the integrand. For the first part, where integration by parts along X is applied we get

$$\int_{t_1}^{t_2}\left(\left[\left[(1-\nu)\;\underbrace{\frac{\partial^2 w}{\partial X \partial Y}}_{\text{edge condition}}\;\delta w\right]_{X1}^{X2}\right]_{Y1}^{Y2} - \left[\int_{Y_1}^{Y_2}\underbrace{\left((1-\nu)\,\frac{\partial^3 w}{\partial X \partial Y^2}\right)}_{\text{boundary condition}}\delta w\mathrm{d}Y\right]_{X1}^{X2}\right)\mathrm{d}t$$

and the second part results in

$$\int_{t_1}^{t_2}\left(\left[\left[-(1-\nu)\;\underbrace{\frac{\partial^2 w}{\partial X \partial Y}}_{\text{edge condition}}\;\delta w\right]_{Y1}^{Y2}\right]_{X1}^{X2} + \left[\int_{X_1}^{X_2}\underbrace{\left((1-\nu)\,\frac{\partial^3 w}{\partial X^2 \partial Y}\right)}_{\text{boundary condition}}\delta w\mathrm{d}X\right]_{Y1}^{Y2}\right)\mathrm{d}t\;.$$

Finally one has to sum up all conditions on the domain, on all parts of the boundary, and on the edges. It is obvious that the edge condition vanishes completely. The domain and boundary conditions are not modified and coincide again with the results of section 8.6.

The change of the sign due to the introduction of the language of forms could also be illustrated by considering the directions of the used integrations on the boundary, which are a simple result is the used orientation of the domain and the induced orientation of the boundary (see Fig. B.2). Finally we are able to state that the original results of Kirchhoff, which did not

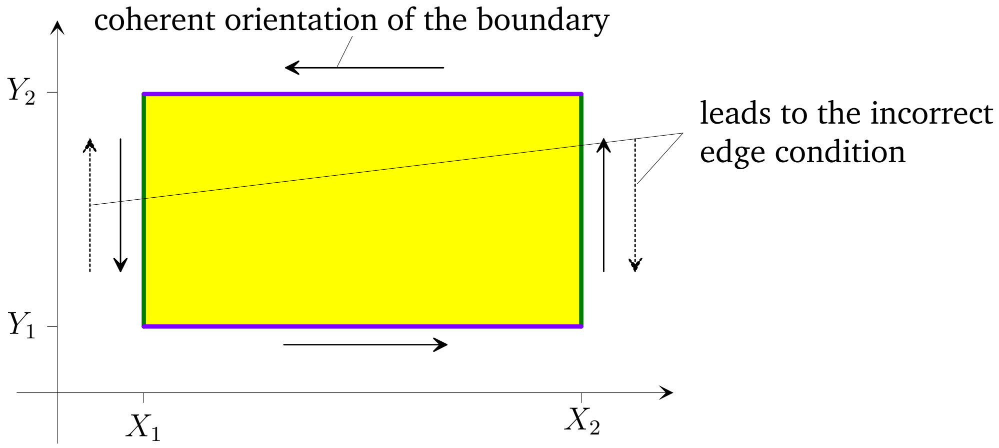

Figure B.2: Integration on the boundary.

contain an edge condition are correct.

Appendix C

The Maple©[Maplesoft, 2004] Package

The application of the calculus of variations on physical problems, which are more complex than simple textbook examples, is an expensive task. Whatever method one applies, the management of huge expressions demands the introduction of computer algebra tools. It is worth mentioning that the theory presented in part II stems from such a development.

In this chapter we present a computer algebra package that implements the determination of the domain and boundary conditions using the Cartan form solution. The actual status of the package makes use of the index order (see Def. 4.2) in the construction procedure of the Cartan form. Additionally it is assumed that the analysed system is formulated in (Y^j, y^α) coordinates, i.e. the last independent coordinate is constant on the boundary. Consequently the determination of the domain and boundary conditions is not limited with respect to system order or dimension.

C.1 General Aims

The MAPLE9.5 [Maplesoft, 2004] Package *JetVariationalCalculus* is intended to provide routines for the management of local coordinates of jet manifolds, contact forms, and total derivatives on a rather general level. Additionally these procedures are used to implement algorithms for the derivation of domain and boundary conditions of Euler-Lagrange systems.

In order to guarantee the reusability of the developed code, its structure should be modular, object oriented and accessible through small, but flexible interfaces. Unfortunately MAPLE9.5 does not provide object oriented programming as known from Java or C++. The generation of independent instances of a module is realized by the implementation of a procedure (constructor), which returns a new module. In the following all modules will provide such a constructor.

C.2 Modules and Interfaces

The *JetVariationalCalculus* package is a framework of three modules that incorporate each other. For basic operations on exterior forms we will use the module "MyLieSymm". Based on this module we introduce the "Jets" module that provides all routines for the management of local jet coordinates and related objects. Finally the module "JetVariationalCalculus" incorporates all the developed methods in order to derive the domain and boundary conditions. In figure C.1 this structure is depicted, where the corresponding constructors are also indicated.

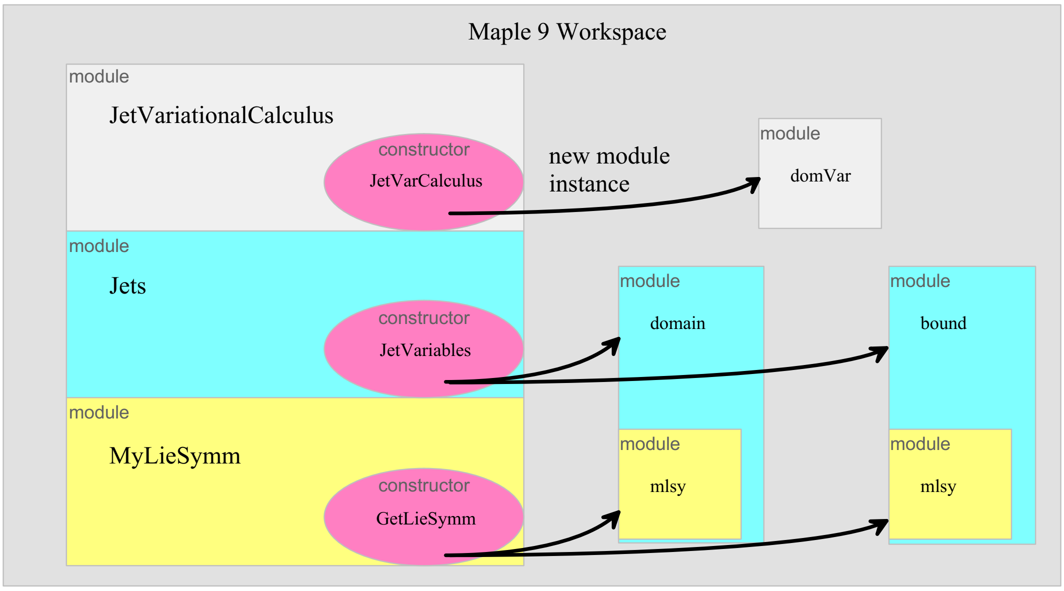

Figure C.1: The JetVariationalCalculus package structure.

In the subsequent sections we will discuss the introduced modules separately

C.2.1 The "MyLieSymm" Module

MAPLE9.5 provides packages for the handling of objects from differential geometry (vectors, forms etc.). In our case the rather old package *liesymm* is used. Unfortunately this package does not provide the necessary object oriented behavior. To overcome this problem a new module called *MyLieSymm* is implemented, which envelops *liesymm* and administers the corresponding data. This module can be added to the MAPLE workspace by evaluating > and provides the constructor . The methods of a module instance can be evaluated by applying the scoping operator " ".

The implemented methods are

- **SetLieSymmVariables, GetLieSymmVariables:**
  Set and get the local coordinates;
- **myd:**
  Implements the exterior derivative $\mathrm{d}(\cdot)$ (see Maple help [liesymm,d]);
- **my&^:**
  Implements the exterior product $\wedge$ (see Maple help [liesymm,wedge]);
- **myhook:**
  Implements the interior product – the hook operator $\rfloor$ (see Maple help [liesymm,hook]),
- **mygetcoeff:**
  Extract the coefficient part of a basis wedge product (see Maple help [liesymm,getcoeff]);

- **mygetform:**
  Extract the basis element of a single wedge product (see Maple help [liesymm,getform]);
- **myLie:**
  Implements the Lie derivative (see Maple help [liesymm,Lie]);
- **mywcollect:**
  Regroup the terms as a sum of wedge products (see Maple help [liesymm,wcollect]);
- **mywdegree:**
  Compute the wedge degree of a form (see Maple help [liesymm,wcollect]);

In figure C.2 the application of the *MyLieSymm* module is depicted
This module is used by the “Jets” module.

C.2.2 The “Jets” Module

The mathematical analysis of the calculus of variations, where manifolds, n^{th} order jet bundles and prolongations are used, illustrates the necessity of managing the corresponding adapted coordinates in a single *Jets* module. This module can be added to the MAPLE workspace by evaluating $>$ and provides the constructor .

The following methods are implemented:

- **SetIndepVars, GetIndepVars:**
  Set and get the independent variables X^i of the variational problem;
- **SetDepVars, GetDepVars:**
  Set and get the dependent variables x^α of the variational problem;
- **GetIndepOfDepVars:**
  Get the independent variables related to a given dependent variable;
- **CalcJetVars:**
  Despite the fact that some objects of our theoretical investigations are defined on the infinite-dimensional jet manifold $J^\infty\mathcal{E}$, the maximal jet order $\#J$ must be restricted to a finite number for practical calculation. This method allows to define the maximal jet order and to calculate the corresponding jet variables $x^\alpha_{[J]}$. The following notation

  $$\begin{array}{rcl} \text{mathematical formula} & \Longleftrightarrow & \text{maple sheet} \\ x^\alpha_{[101]} & \Longleftrightarrow & \text{‘xalpha;101‘} \end{array}$$

  is used for jet coordinates;
- **GetJetVars:**
  Returns the calculated jet variables $x^\alpha_{[J]}$;
- **GetVars:**
  Returns all adapted coordinates $\left(X^i, x^\alpha, x^\alpha_{[J]}\right)$ to the n^{th} jet manifold $J^n\pi$;

- **CalcContactForms:**
  Calculate the contact forms $\omega^{\alpha}_{[J]}$ (see Def. 4.12);
- **GetContactForms:**
  Returns the generated contact forms $\omega^{\alpha}_{[J]}$;
- **GetTotalDiff:**
  Returns the automatically generated total derivative vector fields d_i (see Def. 4.11);
- **GetVarIndex, SetVarIndex:**
  This methods are necessary to extract and modify the multi-index J corresponding to a given jet variable $x^{\alpha}_{[J]}$;
- **GetVector:**
  This method returns a unit vector $v \in \Gamma\left(\mathcal{T}\left(J^n\pi\right)\right)$ corresponding to the given coordinate.
- **GetVolForm:**
  Returns the volume form dY corresponding to the domain $\mathcal{D}$.
- **GetMLSyObj:**
  Returns the underlying *MyLieSymm* module.
- **ProlongVectField:**
  Derives the prolongation of a given vector field.

The application of these methods is depicted in figure C.3 and C.4. The "Jets" module provides all basic manipulation tools, which are needed for the implementation of the derived formulas as computer algebra algorithms.

```
> with(MyLieSymm);# loading the module
                "InitMyLieSymmPackage module by regpro - VERSION 3.0 "
"EU-Proj.: GEOPLEX IST-2001-34166 Copyright (c) 2005 - Institut of Automatic Control and Control Systems Technology \
    "
                "Johannes Kepler University Linz - Austria, All rights reserved. "
Warning, the protected name close has been redefined and unprotected
                                [GetLieSymm]
> mlsy:=GetLieSymm();mlsy2:=GetLieSymm(): # the constructor returns two independent
  instances
                "MyLieSymm - by regpro JKU Linz-Austria"
Warning, the `with' command does not work inside procedures or modules
mlsy := module()
export SetLieSymmVariables, GetLieSymmVariables, myd, `my&^`, myhook, mygetcoeff, mygetform, myLie, mywcollect,
mywdegree, myLieBracket;
    ...
end module
                "MyLieSymm - by regpro JKU Linz-Austria"
Warning, the `with' command does not work inside procedures or modules
> mlsy:-SetLieSymmVariables([x1,x2]):mlsy2:-SetLieSymmVariables([z1,z2,z3]): #
  definition of the local coordinates
> mlsy:-GetLieSymmVariables();mlsy2:-GetLieSymmVariables();
                                [x1, x2]
                                [z1, z2, z3]
> mlsy:-myd(x1);mlsy:-myd(z1);mlsy2:-myd(x3);mlsy2:-myd(z3); # application of the
  exterior derivative
                                d(x1)
                                  0
                                  0
                                d(z3)
> temp2:=mlsy2:-`my&^`(z3,mlsy2:-myd(z1),mlsy2:-myd(z2)); # application of the wedge
  product
                        temp2 := z3 (d(z1) &^ d(z2))
> mlsy2:-myhook(temp2,[v1,v2,v3]);# appication of the hook operator
                        z3 (v1 d(z2) - d(z1) v2)
> mlsy2:-mygetcoeff(temp2);mlsy2:-mygetform(temp2);
                                  z3
                            d(z1) &^ d(z2)
> mlsy2:-mywcollect(temp2);mlsy2:-mywdegree(temp2);
                          z3 (d(z1) &^ d(z2))
                                  2
```

Figure C.2: Example application of the *MyLieSymm* module.

```
> with(Jets);
```

"Jets module by regpro - VERSION 3.0 "

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"

"Johannes Kepler University Linz - Austria, All rights reserved. "

$[JetVariables]$

```
> domain:=JetVariables();
```

"JetVariables - by regpro JKU Linz-Austria"

"MyLieSymm - by regpro JKU Linz-Austria"

```
Warning, the `with' command does not work inside procedures or modules
```

domain := **module**()

export *SetIndepVars, GetIndepVars, SetDepVars, GetDepVars, GetIndepOfDepVars, GetJetOrder, CalcJetVars, SetJetVars, GetJetVars, GetVarIndex, SetVarIndex, GetVars, GetVarPos, CalcContactForms, SetContactForms, GetContactForms, GetTotalDiff, GetVector, GetVolForm, GetMLSyObj, ProlongVectField*;

...

end module

```
> domain:-SetIndepVars(X1,X2):domain:-GetIndepVars(X1,X2);
```

$[X1, X2]$

```
> domain:-SetDepVars([x1,X1],x2):domain:-GetDepVars([x1,X1],x2);
```

$\mathrm{x1}(X1), \mathrm{x2}(X1, X2)$

$[x1, x2]$

```
> domain:-GetIndepOfDepVars(x1);
```

$[X1]$

```
> domain:-CalcJetVars(2): domain:-GetJetVars();
JetVariables successfully generated!
```

$[x1;10, x2;10, x2;01, x1;20, x2;20, x2;11, x2;02]$

```
> domain:-GetVars();
```

$[X1, X2, x1, x2, x1;10, x2;10, x2;01, x1;20, x2;20, x2;11, x2;02]$

```
> domain:-CalcContactForms():domain:-GetContactForms();
ContactForms derived
```

$[\mathrm{d}(x1) - x1;10\ \mathrm{d}(X1), \mathrm{d}(x2) - x2;10\ \mathrm{d}(X1) - x2;01\ \mathrm{d}(X2)],$
$[[\mathrm{d}(x1;10) - x1;20\ \mathrm{d}(X1)], [\mathrm{d}(x2;10) - x2;20\ \mathrm{d}(X1) - x2;11\ \mathrm{d}(X2), \mathrm{d}(x2;01) - x2;11\ \mathrm{d}(X1) - x2;02\ \mathrm{d}(X2)]]$

Figure C.3: Example application of the *Jets* module.

```
> domain:-GetTotalDiff();domain:-GetTotalDiff(1);domain:-GetTotalDiff(2);
```

$$[[1, 0, x1;10, x2;10, x1;20, x2;20, x2;11, 0, 0, 0, 0], [0, 1, 0, x2;01, 0, x2;11, x2;02, 0, 0, 0, 0]]$$

$$[1, 0, x1;10, x2;10, x1;20, x2;20, x2;11, 0, 0, 0, 0]$$

$$[0, 1, 0, x2;01, 0, x2;11, x2;02, 0, 0, 0, 0]$$

```
> domain:-GetVector(X1);domain:-GetVector(`x1;20`);
```

$$[1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]$$

$$[0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0]$$

```
> domain:-GetVarIndex(`x2;02`);domain:-SetVarIndex(x2,[0,1]);
```

$$[x2, [0, 2]]$$

$$x2;01$$

```
> domain:-GetVolForm();
```

$$\mathrm{d}(X1)\ \&\hat{}\ \mathrm{d}(X2)$$

```
>
> mlsy:=domain:-GetMLSyObj();
```

mlsy := **module**()

export *SetLieSymmVariables, GetLieSymmVariables, myd, `my&^`, myhook, mygetcoeff, mygetform, myLie, mywcollect, mywdegree, myLieBracket*;

...

end module

```
> mlsy:-myLie(f(x1,x2),domain:-GetTotalDiff(1));
```

$$x1;10\left(\frac{\partial}{\partial x1}\mathrm{f}(x1, x2)\right) + x2;10\left(\frac{\partial}{\partial x2}\mathrm{f}(x1, x2)\right)$$

Figure C.4: Example application of the *Jets* module.

C.2.3 The "JetVariationalCalculus" Module

This package can be added to the MAPLE workspace by evaluating >
and provides the constructor . All the methods stated below correspond to the presented theoretical investigation and make essential use of the "Jets" module.

The implemented methods are:

- **EulerOp:**
  This method allows to apply the Euler-Lagrange Operator $\delta_\alpha(\cdot)$ of equation (7.10), to a given Lagrangian density $l\left(X^i, x^\alpha, x^\alpha_{[J]}\right)$. The resulting partial differential equation corresponding to the dependent coordinate x^α is returned, whereupon the "$=0$"-statement is neglected. This is the common notation used in MAPLE.

- **CalcCartanForm, GetCartanForm:**
  These two methods determine and return the Cartan form c as defined in equation (7.12) to the MAPLE workspace. Due to the fact that there exists no unique representation of the Cartan form for higher order Lagrangians, this method additionally makes use of the introduced multi-index order (see Def. 4.2).

- **GetDomainConditions:**
  This method evaluates the presented condition (7.14) and returns the domain conditions in terms of PDEs. The equations are returned as a list, whereupon the PDEs are given in combination with the corresponding dependent variable.

- **CalcBoundSystem, GetBoundSystem:**
  The method CalcBoundSystem() determines automatically the boundary jet bundle. By evaluating GetBoundSystem the user gets access to this jet bundle. The local coordinates of the boundary jet bundle are represented in the form

mathematical formula	$\Longleftrightarrow$	maple sheet
$x^\alpha_{[10;1]}$	$\Longleftrightarrow$	‘xalpha_001;10‘

- **CalcBoundECForm, GetBoundECForm:**
  The CalcBoundECForm determines the extended Cartan form on the boundary, i.e. pulled-back on the boundary jet bundle. The evaluation of GetBoundECForm returns the resulting form.

- **GetBoundaryConditions:**
  The presented boundary conditions (7.15) are evaluated by this method. The equations are returned similarly to the domain conditions as a list, whereupon the PDEs are given in combination with the corresponding dependent boundary variable.

The application of these methods is depicted in figure C.5.
The whole package, in combination with some examples, is available on the homepage of the *Institute of Automatic Control and Control Systems Technology*
(.

```
> with(JetVariationalCalculus);
          "JetVariationalCalculus module by regpro - VERSION 3.0 "
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    "
          "Johannes Kepler University Linz, All rights reserved. "
                    [JetVarCalculus]
> domvar:=JetVarCalculus(domain);
          "JetVarCalculus - by regpro JKU Linz-Austria!"
domvar := module()
export GetJetVariable, EulerOp, CalcCartanForm, SetCartanForm, GetCartanForm, GetDomainConditions,
CalcBoundSystem, GetBoundSystem, CalcBoundECForm, GetBoundECForm, GetBoundaryConditions,
GetMomentaFunctions, GetHamiltonianFunction, GetLegendreTransform, GetPCHSystem;
    ...
end module
> l:=`x1`^2+`x2;01`^2;
                    l := x1^2 + x2;01^2
> domvar:-EulerOp(x1,l);domvar:-EulerOp(x2,l);
                    2 x1
                    -2 x2;02
>
> domvar:-CalcCartanForm(l*domain:-GetVolForm()):domvar:-GetCartanForm();
Deriving the cartan form
   (x1^2 + x2;01^2) (d(X1) &^ d(X2)) - 2 x2;01 (d(x2) &^ d(X1)) - 2 x2;01^2 (d(X1) &^ d(X2))
> domvar:-GetDomainConditions();
          [[d(x1) &^ d(X2), -2 x1], [d(x2), -2 x2;02]]
> domvar:-CalcBoundSystem();
   "This method assumes the last independent coordinateto be constant on the boundary!! i.e.:", X2, "=const"
          "JetVariables - by regpro JKU Linz-Austria"
          "MyLieSymm - by regpro JKU Linz-Austria"
Warning, the `with' command does not work inside procedures or modules

Setting independent coordinates of boundary bundle!
Setting dependent coordinates of boundary bundle!
   x1_00(X1), x2_00(X1), x1_01(X1), x2_01(X1), x1_02(X1), x2_02(X1)
Deriving boundary jet structure!
JetVariables successfully generated!
ContactForms derived
> boundary:=domvar:-GetBoundSystem();
boundary := module()
export SetIndepVars, GetIndepVars, SetDepVars, GetDepVars, GetIndepOfDepVars, GetJetOrder, CalcJetVars, SetJetVars,
GetJetVars, GetVarIndex, SetVarIndex, GetVars, GetVarPos, CalcContactForms, SetContactForms, GetContactForms,
GetTotalDiff, GetVector, GetVolForm, GetMLSyObj, ProlongVectField;
    ...
end module
> domvar:-CalcBoundECForm();
          "The extended Cartan form on the boundary is determined!"
> domvar:-GetBoundECForm();
                    -2 x2_01 (d(x2_00) &^ d(X1))
> domvar:-GetBoundaryConditions(bound);
                    [[d(x2_00), -2 x2_01]]
```

Figure C.5: Example application of the *JetVariationalCalculus* module.

C.3 Application

In the following a distributed parameter system – the Timoshenko beam, e.g., [Meirovitch, 1967, Meirovitch] – will be used for a short demonstration of the *JetVariationalCalculus* package. From now on, the time is denoted by t and X^i, $i = 1, 2, 3$ are the canonical coordinates of the 3-dimensional Euclidian space. Following the standard assumptions of linear elasticity, we introduce the displacements x^α, $\alpha = 1, 2, 3$ and the according bundle $(\mathcal{E}, \pi, \mathcal{D})$, $\dim(\mathcal{E}) = 7$, $\dim(\mathcal{D}) = 4$ with adapted coordinates (t, X^i), $t = X^0$ for $\mathcal{D}$ and (t, X^i, x^α) for $\mathcal{E}$. The kinetic energy E_{kin} and the stored energy E_{pot} result in

$$\begin{aligned} E_{\text{kin}} &= \frac{1}{2}\int_{\mathcal{D}_S} x^i_{[1000]}\delta_{ij}x^j_{[1000]}\rho \mathrm{dX}\,, \quad \mathrm{dX} = \mathrm{d}X^1 \wedge \mathrm{d}X^2 \wedge \mathrm{d}X^3 \text{ and} \\ E_{\text{pot}} &= \int_{\mathcal{D}} \sigma^{ij}\mathrm{d}\varepsilon_{ij}(X) \wedge \mathrm{dX} = \int_{\mathcal{D}_S} e_{\text{pot}}\mathrm{dX}\,. \end{aligned} \tag{C.1}$$

Here $\sigma = \sigma^{\alpha\beta}\partial_\alpha \otimes \partial_\beta$, $\sigma^{\alpha\beta} = \sigma^{\beta\alpha}$, $\beta = 1, 2, 3$ and $\varepsilon_{ij} = \varepsilon_{ij}\mathrm{d}X^i \otimes \mathrm{d}X^j$, $2\varepsilon_{ij} = x^j_{[1_i]} + x^i_{[1_j]}$, $j = 1, 2, 3$ denote the stress and strain tensor. The quantity ρ represents the mass density. The integral is taken over the body in the reference configuration. Stress and strain are connected by Hooke's law $\sigma^{\alpha\beta} = C^{\alpha\beta ij}\varepsilon_{ij}$, $C^{\alpha\beta ij} = C^{\beta\alpha ij} = C^{\alpha\beta ji} = C^{ji\alpha\beta}$, where a linear material behavior, characterized by the stiffness tensor $C^{\alpha\beta ij}$, is assumed.

This general assumptions will serve as a basis for the analysis of the application example. Additional simplifications, which are motivated by the spatial shape of the investigated problems, will enable to reduce the amount of independent coordinates.

C.3.1 The Timoshenko Beam

According to Timoshenko, we state that

$$\begin{array}{rcl} x^1 &=& w^1 - \Psi^3 X^2 + \Psi^2 X^3 \\ x^2 &=& w^2 - \Psi^1 X^3 \\ x^3 &=& w^3 + \Psi^1 X^2 \end{array} \text{, which implies } \begin{array}{rcl} \varepsilon_{11} &=& w^1_{[01]} - \Psi^3_{[01]}X^2 + \Psi^2_{[01]}X^3 \\ \varepsilon_{12} &=& -\frac{1}{2}\Psi^3 + \frac{1}{2}w^2_{[01]} - \frac{1}{2}\Psi^1_{[01]}X^3 \\ \varepsilon_{13} &=& \frac{1}{2}\Psi^2 + \frac{1}{2}w^3_{[01]} + \frac{1}{2}\Psi^1_{[01]}X^2 \\ \varepsilon_{33} &=& \varepsilon_{22} = \varepsilon_{23} = 0\,. \end{array} . \tag{C.2}$$

These assumptions are formulated in the coordinates according to Figure (C.6) and incorpo-

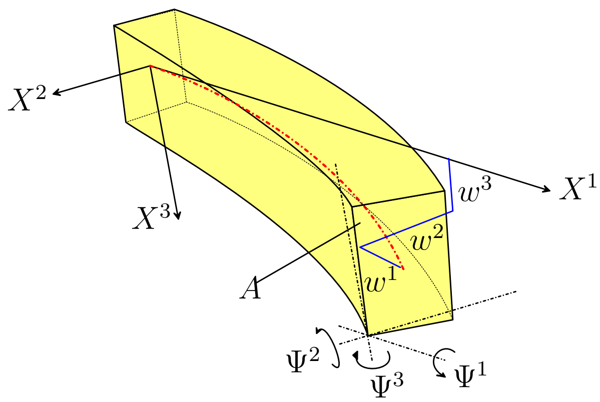

Figure C.6: The Timoshenko Beam.

rate the special spatial distribution of a beam. Now, the independent and dependent variables are (t, X^1) and (w^α, Ψ^α). Thus, we introduce the new bundle $(\bar{\mathcal{E}}, \bar{\pi}, \bar{\mathcal{D}})$, $\dim(\bar{\mathcal{E}}) = 8$, $\dim(\bar{\mathcal{D}}) = 2$ with adapted coordinates (t, X^1), $t = X^0$, $(t, X^1, w^\alpha, \Psi^\alpha)$, $\alpha = 1, 2, 3$ for $\bar{\mathcal{D}}$ and $\bar{\mathcal{E}}$. Therefore, we use the bundle $\bar{\mathcal{E}}$ to describe the Timoshenko beam. Furthermore it is straightforward to see that all the required functions and densities for an energy based modeling of the beam can be expresses as functions of the variables of the first jet manifold $J^1\pi$ of π only.

C.3.2 Program Code

These investigations can be done in MAPLE with the instruction lines as shown in Figure C.7. To keep the relations readable, we restricted the beam deformation to the (X^1, X^3)-plane and assume no beam elongation. Due to these assumptions we have to set $w^1 = 0$, $w^2 = 0$, $\Psi^1 = 0$, $\Psi^3 = 0$. For this case, only two figures of Hooke's law – Young's modulus of elasticity E and the shear modulus G – are necessary to describe the material behavior. For the energy calculation we introduce the area moment of inertia $I_y = \int_A (X^3)^2 \, \mathrm{d}X^1 \wedge \mathrm{d}X^2$ and assume that the integral $\int_A X^3 \, \mathrm{d}X^1 \wedge \mathrm{d}X^2$ vanishes in accordance to the chosen coordinate system.

```
> domain:=JetVariables():
                         "JetVariables - by regpro JKU Linz-Austria"
> domain:-SetIndepVars(t,X);
                                          [t, X]
> domain:-SetDepVars(w3,psi2);
                                     w3(t, X), ψ2(t, X)
                                         [w3, ψ2]
> domain:-CalcJetVars(2): domain:-GetJetVars(); domain:-CalcContactForms():
JetVariables successfully generated!
        [w3;10, w3;01, psi2;10, psi2;01, w3;20, w3;11, w3;02, psi2;20, psi2;11, psi2;02]
ContactForms derived
```

Figure C.7: Timoshenko Beam - loading the package and initializing the variables.

Additionally the cross sectional area A is assumed to stay constant along the x^1-coordinate. After the preliminary work, which is shown in figure C.7, it is possible to calculate all 2 PDEs and corresponding boundary conditions within some instruction lines. Unfortunately the full solution (6 PDEs) would need some terminal pages and cannot be shown here. The kinetic and potential energy density is given by

$$E_{\text{kin}} = \frac{\rho}{2}\left(I_y \left(\psi^2_{[10]}\right)^2 + A\left(w^3_{[10]}\right)^2\right) \;, \text{ and } \; E_{\text{pot}} = \frac{I_y E}{2}\left(\psi^2_{[01]}\right)^2 + \frac{GA}{2}\left(\frac{\psi^2}{2} + \frac{w^3_{[01]}}{2}\right)^2$$

and allow to formulate the Lagrangian $L = E_{\text{kin}} - E_{\text{pot}}$.

The following code lines (see figure C.8) show the calculation of the PDEs based on the "*Cartan form*" approach. The -method provides

$$\frac{GA}{4}\left(w^3_{[02]} + \psi^2_{[01]}\right) - \rho A w^3_{[20]} = 0$$

according to the coordinate w^3 and

$$-\frac{GA}{4}\left(w^3_{[01]} + \psi^2\right) + I_y\left(-\rho\psi^2_{[20]} + E\psi^2_{[02]}\right) = 0$$

```
> domvar:=JetVarCalculus(domain):
                    "JetVarCalculus - by regpro JKU Linz-Austria!"
> Ekin:=1/2*(`psi2;10`^2*rho*Iy+rho*`w3;10`^2*A);
  Epot:=1/2*(`psi2;01`^2*Iy*E+(1/2*psi2+1/2*`w3;01`)^2*G*A);
```

$$Ekin := \frac{psi2;10^2\, \rho\, Iy}{2} + \frac{\rho\, w3;10^2\, A}{2}$$

$$Epot := \frac{psi2;01^2\, Iy\, E}{2} + \frac{\left(\frac{\psi 2}{2} + \frac{w3;01}{2}\right)^2 G\, A}{2}$$

```
> L:=Ekin-Epot:
> domvar:-CalcCartanForm(L*domain:-GetVolForm());
Deriving the cartan form
> domvar:-GetDomainConditions();
```

$$\left[\left[\mathrm{d}(w3), -\rho\, A\, w3;20 + \frac{G\, A\, w3;02}{4} + \frac{G\, A\, psi2;01}{4}\right], \left[\mathrm{d}(\psi 2), -\frac{G\, A\, w3;01}{4} - \rho\, Iy\, psi2;20 - \frac{G\, A\, \psi 2}{4} + Iy\, E\, psi2;02\right]\right]$$

Figure C.8: Timoshenko Beam - definition of the Lagrangian and derivation of the PDEs.

according to the coordinate ψ^2.

Finally we present the extraction of the boundary conditions (see figure C.9), where the variables of the *boundary jet bundle* and the *extended Cartan form on the boundary* are shown. The boundary conditions are given by

$$-\frac{GA}{4}\left(w^3_{[0;1]} + \psi^2_{[0;0]}\right) = 0$$

for $w^3_{[0;0]}$ and

$$I_y\; E\; \psi^2_{[0;1]} = 0$$

for $\psi^2_{[0;0]}$. The presented results coincide with the calculations shown in, e.g., [Meirovitch, 1967, Meirovitch] and illustrate the reduction of work.

The appendix fragment is used only once. Subsequent appendices can be created using the Chapter Section/Body Tag.

```
> domvar:-GetDomainConditions();
```

$$\left[\left[\mathrm{d}(w3), -\rho\, A\; w3;20 + \frac{G\,A\; w3;02}{4} + \frac{G\,A\; psi2;01}{4}\right], \left[\mathrm{d}(\psi 2), -\frac{G\,A\; w3;01}{4} - \rho\; Iy\; psi2;20 - \frac{G\,A\,\psi 2}{4} + Iy\; E\; psi2;02\right]\right]$$

```
> domvar:-CalcBoundSystem();
```

"This method assumes the last independent coordinateto be constant on the boundary!! i.e.:", *X*, "=const"

```
> boundary:=domvar:-GetBoundSystem();
```

boundary := **module**()
export *SetIndepVars, GetIndepVars, SetDepVars, GetDepVars, GetIndepOfDepVars, GetJetOrder, CalcJetVars, SetJetVars, GetJetVars, GetVarIndex, SetVarIndex, GetVars, GetVarPos, CalcContactForms, SetContactForms, GetContactForms, GetTotalDiff, GetVector, GetVolForm, GetMLSyObj, ProlongVectField*;
...
end module

```
> boundary:-GetVars();
```

[*t, w3_00, psi2_00, w3_01, psi2_01, w3_02, psi2_02, w3_00;1, psi2_00;1, w3_01;1, psi2_01;1, w3_02;1, psi2_02;1, w3_00;2, psi2_00;2, w3_01;2, psi2_01;2, w3_02;2, psi2_02;2*]

```
> domvar:-CalcBoundECForm();domvar:-GetBoundECForm();
```

"The extended Cartan form on the boundary is determined!"

$$-psi2\_01\; Iy\; E\; (\mathrm{d}(t)\ \&\wedge\ \mathrm{d}(psi2\_00)) - \left(\frac{G\,A\; psi2\_00}{4} + \frac{G\,A\; w3\_01}{4}\right)(\mathrm{d}(t)\ \&\wedge\ \mathrm{d}(w3\_00))$$

```
> domvar:-GetBoundaryConditions();
```

$$\left[[\mathrm{d}(psi2\_00), psi2\_01\; Iy\; E], \left[\mathrm{d}(w3\_00), \frac{G\,A\; psi2\_00}{4} + \frac{G\,A\; w3\_01}{4}\right]\right]$$

Figure C.9: Timoshenko Beam - derivation of the boundary condition.

Appendix D

Afterword

This work has been done in the context of the European sponsored project **GeoPlex** with reference code IST-2001-34166. Further information is available at http://www.geoplex.cc.

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Linz, 14. Februar 2006

Eidesstattliche Erklärung

Ich versichere, dass ich die vorliegende Dissertation selbstständig verfasst, andere als die angegebenen Quellen und Hilfsmittel nicht benutzt und mich auch sonst keiner unerlaubten Hilfe bedient habe.

Linz, Februar 2006