Multivariate Verfahren 2

contingency tables - the log-linear model

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March 12th and 13th 2012

the log-linear model

saturated model

The saturated log-linear model is given by

$$\ln m_{ij} = \mu + \mu_{A(i)} + \mu_{B(j)} + \mu_{AB(ij)} \qquad \text{with side conditions}$$

$$\sum_i \mu_{A(i)} = \sum_j \mu_{B(j)} = 0, \quad \sum_i \mu_{AB(ij)} = \sum_j \mu_{AB(ij)} = 0, \quad \forall i, j$$

The above side conditions are to avoid overparametrisation of the model. Additional side conditions are due to the sampling schema:

for the multinomial schema: $x_{++} = \sum_{i,j} \exp(\mu + \mu_{A(i)} + \mu_{B(j)} + \mu_{AB(ij)})$

for the product multinomial schema:
$x_{i+} = \sum_j \exp(\mu + \mu_{A(i)} + \mu_{B(j)} + \mu_{AB(ij)}) \qquad i = 1 \ldots, I$

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Similar to ANOVA the parameters of the saturated log-linear model are

$\mu_{A(i)}, \mu_{B(j)}$ ... main effects

$\mu_{AB(ij)}$ ... components of interaction

Caution! The interpretation of the parameters is not according to ANOVA

In the saturated model the number of free model parameters (taking into account the side conditions) is

- for the Poisson schema: $1 + (I-1) + (J-1) + (I-1)(J-1) = I \cdot J$
- for the multinomial schema: $I \cdot J - 1$
- for the product multinomial schema: $I \cdot (J-1)$

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With the saturated model we may describe every set $\{m_{ij}\}$ of expected cell frequencies. The log-linear independence model is a special case with $\mu_{AB(ij)} = 0$.

The interaction parameter may be computed as follows:

$$\mu_{AB(ij)} = \ln m_{ij} - (\mu + \mu_{A(i)} + \mu_{B(j)})$$

In the log-linear independence model the number of free model parameters (taking into account the side conditions) is

- for the Poisson schema: $1 + (I - 1) + (J - 1) = I + J - 1$
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connection with ANOVA

Model equation of an ANOVA with two factors:

$$\mu_{ij} = \mu + \alpha_i + \beta_j + \alpha\beta_{ij}$$

We observe a strong similarity to the saturated log-linear model **but** there is one basic difference:

In ANOVA we model the **expectation of the dependent variable** as the sum of effects of factors.

In the log-linear model we model **logarithmized cell frequencies** as sum of effects

If we treat the logarithmized cell frequencies as response variable and if we are interested in the influence of the values of the variates on the frequencies, then especially with the Poisson schema we may interpret the parameters similar to ANOVA.

In log-linear models we basically want to know if there is a connection between the variates.

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connection with ANOVA

From this follows the basic difference to ANOVA:

Example

		healing			
		fast	normal	slow	
therapy	a	x_{11}	x_{12}	x_{13}	x_{1+}
	b	x_{21}	x_{22}	x_{23}	x_{2+}
	c	x_{31}	x_{32}	x_{33}	x_{3+}
		x_{+1}	x_{+2}	x_{+3}	x_{++}

With ANOVA we would have healing as dependent variable and therapy as the only factor:

$$\mu_i = \mu + \alpha_i$$

with μ_i the expectation of healing given a factor-level i

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differences to ANOVA

In the log-linear model we have main effects μ_A, μ_B for the factor as well as the response variable.

If the **interaction effects** vanish in our log-linear model we have independent variates A and B.

In contrast in ANOVA missing significance of the factors ($\hat{=}$ independence of the variates) yields vanishing **main effects**.

In ANOVA main effects always aim at the response variable itself.

In log-linear models the main effects just have connection to the cell frequencies and only the interaction effects permit the evaluation of dependencies of variates.

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odds-ratio

The odds-ratio (relative risk) α is defined as follows:

$$\alpha_{i_1 i_2 j_1 j_2} := \frac{m_{i_1 j_1} m_{i_2 j_2}}{m_{i_1 j_2} m_{i_2 j_1}} = \frac{P(B_{j_1}|A_{i_1})/P(B_{j_2}|A_{i_1})}{P(B_{j_1}|A_{i_2})/P(B_{j_2}|A_{i_2})}$$

The odds-ratio expresses the connection between the values A_{i_1} and A_{i_2} of variate A and the values B_{j_1} and B_{j_2} of variate B.

The odds-ratios $\alpha_{i_1 i_2 j_1 j_2}$ contain the whole information about the dependencies of variates A and B.

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odds-ratio - properties

- $\alpha_{i_1 i_2 j_1 j_2} = 1$ iff A and B are independent.
- if we know all $\alpha_{i_1 i_2 j_1 j_2}$ and the marginal distributions p_{i+} and p_{+j} we may compute all probabilities $P(A_i \cap B_j)$ and $P(B_j | A_i)$ respectively
- $\alpha_{i_1 i_2 j_1 j_2}$ is a direct function of the interaction parameters μ_{AB}:

$$\alpha_{i_1 i_2 j_1 j_2} = \exp((\mu_{AB(i_1 j_1)} - \mu_{AB(i_1 j_2)}) - (\mu_{AB(i_2 j_1)} - \mu_{AB(i_2 j_2)}))$$

remember double differences!

In the independence model all double differences of the interaction parameters are zero $\Longrightarrow \exp(0) = 1 = \alpha$

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- $\alpha_{i_1 i_2 j_1 j_2}$ is a direct function of the interaction parameters μ_{AB}:

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remember double differences!

In the independence model all double differences of the interaction parameters are zero $\Longrightarrow \exp(0) = 1 = \alpha$

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3-dim models

If we increase the number of variates we get a considerable increase in potential connections between the variates (e.g. all variates are independent; two independent variates depend on a third variate etc.).

So we have to test whether there is a connection between the variates and if yes we have to check the form of the dependency additionally.

$$\begin{array}{llll} \text{3 variates:} & A = 1, \ldots, I & & m_{ijk} = E(x_{ijk}) \\ & B = 1, \ldots, J & & \\ & C = 1, \ldots, K & & p_{ijk} = \frac{m_{ijk}}{N} \end{array}$$

We have 9 different types of models which differ in the form of the dependency between the variates.

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1) 3-dim saturated model (ABC)

$$\ln m_{ijk} = \underbrace{\mu}_{\text{grand mean}} + \underbrace{\mu_{A(i)} + \mu_{B(j)} + \mu_{C(k)}}_{\text{main effects}} +$$

$$+ \underbrace{\mu_{AB(ij)} + \mu_{AC(ik)} + \mu_{BC(jk)}}_{\text{interactions of two effects}} + \underbrace{\mu_{ABC(ijk)}}_{\text{interaction of three effects}}$$

side conditions avoiding overparametrization:

$\sum_i \mu_{A(i)} = \sum_j \mu_{B(j)} = \sum_k \mu_{C(k)} = 0,$

$\sum_i \mu_{AB(ij)} = \sum_j \mu_{AB(ij)} = \sum_i \mu_{AC(ik)} = \sum_k \mu_{AC(ik)} = \sum_j \mu_{BC(jk)} =$
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2) model without three-way interactions (AB/AC/BC)

first simplification of the saturated model:

The connection of two variates remains unaffected by the third variate.

The constraints of the model concerning m_{ijk} and any of the conditional odds-ratios respectively are:

$$\alpha_{i_1 i_2 j_1 j_2}(C_{k_1}) = \frac{m_{i_1 j_1 k_1} m_{i_2 j_2 k_1}}{m_{i_1 j_2 k_1} m_{i_2 j_1 k_1}} = \frac{m_{i_1 j_1 k_2} m_{i_2 j_2 k_2}}{m_{i_1 j_2 k_2} m_{i_2 j_1 k_2}} = \alpha_{i_1 i_2 j_1 j_2}(C_{k_2})$$

i.e. the odds-ratio remains the same, no matter what value C_k is observed for C

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the log-linear model

3) model of conditional independence (e.g. AC/BC)

no three-way interaction, only 2 two-way interactions (here AC and BC).

The constraints of the model concerning m_{ijk} are:

$$m_{ijk} = \frac{m_{i+k} m_{+jk}}{m_{++k}}$$

equivalent are the following constraints:

$$p_{ijk} = \frac{p_{i+k} p_{+jk}}{p_{++k}} \iff \frac{p_{ijk}}{p_{++k}} = \frac{p_{i+k}}{p_{++k}} \cdot \frac{p_{+jk}}{p_{++k}} \iff$$

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4) model of independence of one variate (e.g. AC/B)

only main effects and one two-way interaction (here AC).

The constraints of the model concerning m_{ijk} and p_{ijk} respectively are:

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i.e. A and C conjoined are independent of B.

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4) model of independence of one variate (e.g. AC/B)

only main effects and one two-way interaction (here AC).

The constraints of the model concerning m_{ijk} and p_{ijk} respectively are:

$$m_{ijk} = \frac{m_{i+k} m_{+j+}}{m_{+++}} \quad \Longleftrightarrow \quad p_{ijk} = p_{i+k} \cdot p_{+j+} \quad \Longleftrightarrow$$

$$\Longleftrightarrow \quad P(A_i \cap B_j \cap C_k) = P(A_i \cap C_k) \cdot P(B_j)$$

i.e. A and C conjoined are independent of B.

5) model of total independence (A/B/C)

no interactions: $P(A_i \cap B_j \cap C_k) = P(A_i) \cdot P(B_j) \cdot P(C_k)$

The following models are mentioned only for the sake of completeness:

the log-linear model

6) saturated model for two variates (e.g. AC): $\ln m_{ijk} = \mu + \mu_{A(i)} + \mu_{C(k)} + \mu_{AC(ik)}$

7) model with only two main effects (e.g. A/C): $\ln m_{ijk} = \mu + \mu_{A(i)} + \mu_{C(k)}$

8) model with only one main effect (e.g. A): $\ln m_{ijk} = \mu + \mu_{A(i)}$

9) null model: $\ln m_{ijk} = \mu$, $P(A_i \cap B_j \cap C_k) = \frac{1}{I \cdot J \cdot K}$

These models are degenerated for three variates in a certain sense.

Models 3) to 9) are called **multiplicative** because their joint density function (pdf) may be factorized by marginal density functions.

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sampling schemata in 3-dim models

Multinomial and Poisson schemata are analogous to bivariate models.

In product-multinomial stratified schemata we have to distinguish whether the sample sizes of one or of two variates are fixed. Accordingly we have product-multinomial schemata in one or two variates.

If we stratify according to variate A, i.e. the marginal sums x_{i++} are fixed, we have

$$P(x_{111} \dots x_{IJK}) = \prod_i \frac{x_{i++}!}{\prod_{j,k} x_{ijk}!} \prod_{j,k} p_{ijk}^{x_{ijk}}$$

Contingency tables with product-multinomial sampling schemata reduce the set of possible models (1) to 9)) and need additional side conditions.

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<u>example:</u> the marginal sums x_{i++} are fixed. Then the log-linear model has to satisfy the side conditions

$$x_{i++} = \sum_{j,k} \exp(\mu + \mu_{A(i)} + \mu_{B(j)} + \mu_{C(k)} + \mu_{AB(ij)} + \dots) \qquad i = 1, \dots, I$$

These conditions can only be satisfied if we have the main effect μ_A in our model.

<u>example:</u> the marginal sums for two variates are fixed, e.g. $x_{ij+} \quad i = 1, \dots, I \quad j = 1, \dots, J$ hence the log-linear model has to satisfy the side conditions

$$x_{ij+} = \sum_{k} \exp(\mu + \mu_{A(i)} + \dots) \qquad i = 1, \dots, I \quad j = 1, \dots, J$$

i.e. all admissible models must include the interaction effect $\mu_{AB(ij)}$

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admissible models

General limitation for admissible models: If the marginal sums x_M for a given combination of variates M are fixed then the log-linear model must include the parameter μ_M. If M corresponds e.g. to the combination AB, i.e. x_{ij+} are fixed then only models including μ_{AB} are admissible.

If we confine ourself to admissible models estimators and test statistics are the same for all three schamata

hierarchy of models - model fitting cf. ANOVA

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