3^{rd} practice sheet multivariate methods II

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9. (a) Determine a rotation matrix R_x for $\mathbf{R}^3$ that rotates a point $P = (P_x, P_y, P_z)^T$ counterclockwise through an angle α about the x-axis. I.e. we want to get the rotated point P^* as $P^* = R_x \cdot P$.

   (b) Show that the rotation matrix R_x is orthogonal.

10. (a) Show: The product of orthogonal matrices is again orthogonal.

    (b) Rotation through an angle α about a rotation-axis $a = (a_x, a_y, a_z)^T$: The rotation-axis a is given as a unit vector. Rotation about a can be done by the following consecutive rotations:

    - Rotation of axis a into the x-axis, the corresponding rotation matrix is R_a.
    - Rotation of the object about the x-axis, the corresponding rotation matrix is R_x.
    - Re-rotation of the axis a into its original position, the corresponding rotation matrix is R_a^{-1}.

    I.e. the rotation about a may be described in the following way: $P^* = R_a^{-1} \cdot R_x \cdot R_a \cdot P$.

    Show: R_a has the following form and R_a is orthogonal.

$$R_a = \begin{pmatrix} x & y & -z \\ -\frac{xy}{\sqrt{x^2+z^2}} & \sqrt{x^2+z^2} & \frac{yz}{\sqrt{x^2+z^2}} \\ \frac{z}{\sqrt{x^2+z^2}} & 0 & \frac{x}{\sqrt{x^2+z^2}} \end{pmatrix}$$

11. Let the Wishart-distribution with n degrees of freedom be defined as in our lecture. The pdf of the Wishart-distribution is

$$f_{\mathbf{W}}(w) = \frac{|w|^{(n-p-1)/2} \exp[-\mathrm{tr}(\frac{w}{2} \cdot \Sigma^{-1})]}{2^{np/2} \cdot |\Sigma|^{n/2} \cdot \Gamma_p(\frac{n}{2})}$$

where $\Gamma_p(\frac{n}{2}) = \pi^{p(p-1)/4} \cdot \prod_{j=1}^{p} \Gamma(\frac{n+1-j}{2})$ is the multivariate Gamma-function.

Further we have the following theorem: If S is Wishart-distributed $S \sim \mathbf{W}_p(\Sigma, n)$ and $\mathbf{C}$ is a $(q \times p)$-matrix with rank q, then we have: $\mathbf{C}S\mathbf{C}^T \sim \mathbf{W}_q(\mathbf{C}\Sigma\mathbf{C}^T, n)$.

Using the theorem above show that the marginal distributions of the diagonal elements of S are Chi-square-distributions: $\frac{s_{jj}}{\sigma_j^2} \sim \boldsymbol{\chi}_n^2$

12. We want to test a factor model with k factors assuming normal distributed data, i.e. we test

$$H_0 : \Sigma = L \cdot L^T + V; \quad L \ldots (p \times k)\text{-matrix}; \quad V > 0$$

against $H_1 : \Sigma \ldots (p \times p)$-matrix, arbitrary, positive definite.

Use the conditional equations for $\hat{L}$ and $\hat{V}$ and derive the test statistic U_k for the likelihood ratio test of the above hypotheses:

$$U_k = (n-1) \cdot (\ln|\hat{\Sigma}| - \ln|S|)$$