The GARCH Structural Credit Risk Model

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Motivation

- It was Merton (1974) who first adapted the Black and Scholes (1973) and Merton (1973) option pricing framework to the valuation of corporate securities.
- Structural bond pricing models are now used by institutions around the world to value the risky debt of banks and firms, as well as for applications such as the determination of capital adequacy ratios.
- The equity of the firm is valued as a European call option, with the same strike price. This strike price of the implicit options in the Merton model is commonly referred to as the default barrier. In practice, the time series of asset values of the firm, and the volatility of asset returns, cannot be observed directly, and must be inferred from the time series of observed equity values.

How to estimate asset and parameter values?

There are two primary approaches to solving the problem of inferring the value and volatility of firm assets from information on firm equity.

- Merton finds two equations to solve for the current asset value and the asset return volatility. $\Rightarrow$ implied value of the risky debt, risk measures, such as the spread over the risk-free rate associated with the debt of the firm, and the probability of default, over a given time horizon.
- A second approach is provided by the Duan (1994) maximum likelihood method which views the observed equity time series as a transformed data set with the equity pricing formula.

Relevant litterature

- Ericsson and Reneby (2005), in a simulation study, that the maximum likelihood approach of Duan (1994) to estimating structural bond pricing models is superior to the calibration approach of estimating the Merton (1974) model, in the sense that it provides a less biased and more efficient estimator of asset values, asset volatilities, and spreads.
- In practice, the asset return volatility found using Merton's (1974) calibration method can exhibit significant variation over time for many firms and banks (same case for MLE Duan (1994)). Significant shifts in estimated asset return volatility can induce significant shifts in risk indicators, such as spreads. $\Rightarrow$ Strong reason to believe that the volatility of firms asset returns is stochastic, rather than constant.
- Firms and industries alike go through periods of high levels of uncertainty regarding their future rates of asset growth, as well as periods of relative tranquility.

The Garch structural credit risk model

- We propose a structural credit risk model (SCR) that allows for the stochastic volatility of asset returns. In the work following the seminal paper of Black and Scholes (1973) and Merton (1973), it was recognized that the assumption of constant asset return volatility in option pricing is too restrictive.
- We refer to our model as the GARCH structural credit risk model, in light of the fact that firm asset return volatility (in fact, volatility squared) is assumed to follow a GARCH process, as in HN.
- We derive an EM algorithm to estimate the GARCH SCR model, in a manner equivalent to that used by Duan (1994) and Duan et al. (2004). We choose this approach due to the demonstrable superiority of maximum likelihood methods for estimation, as documented by Ericsson and Reneby (2005) in the case of several previous structural credit risk models.

The Heston Nandi option pricing model

Let S_t is the price of the underlying at time t, define the log return at time t as $r_t \equiv \log\left(\frac{S_t}{S_{t-\Delta}}\right)$, where returns are calculated over a time interval of length Δ. The joint dynamics of the log returns and the return volatility are given as follows Heston and Nandi (2000):

$$r_t = r + \lambda\sigma_t^2 + \sigma_t z_t \quad (1)$$

$$\sigma_t^2 = \omega + \sum_{i=1}^{p} \beta_i \sigma_{t-i\Delta}^2 + \sum_{i=1}^{q} \alpha_i (z_{t-i\Delta} - \gamma_i \sigma_{t-i\Delta})^2 \quad (2)$$

Here r is the risk-free interest rate, σ_t^2 is the conditional variance at time t, z_t is a standard normal disturbance, and $\omega, \beta_i, \alpha_i, \gamma_i$, and λ are the HN model parameters. Posterior results confirm $p = q = 1$ are enough to model the asset returns adequately.

Facts about the Heston Nandi option pricing model

The one period ahead variance of the process, $\sigma^2_{t+\Delta}$, can be directly computed at time t as a function of the current period log return r_t, as follows:

$$\sigma^2_{t+\Delta} = \omega + \beta\sigma^2_t + \alpha\frac{(r_t - r - (\lambda+\gamma)\sigma^2_t)^2}{\sigma^2_t} \tag{3}$$

- Process is stationary with finite mean and variance when $\beta + \alpha\gamma^2 < 1$.
- α determines the kurtosis of the distribution and $\alpha = 0 \Rightarrow$ deterministic time varying variance.

Facts about the Heston Nandi option pricing model (Contd.)

- γ allows a large negative shock z_t to raise the variance more than a large positive z_t. γ controls the skewness of the distribution of log returns, and the distribution of log returns is symmetric when γ and λ are both equal to zero.
- the correlation between volatility and realized log returns is given by

$$Cov_{t-\Delta}[\sigma^2_{t+\Delta}, r_t] = -2\alpha\gamma\sigma^2_t \tag{4}$$

Positive values for α and γ imply a negative correlation between volatility and spot returns, which is consistent with the leverage effect documented by e.g. Christie (1982).

Facts about the Heston Nandi option pricing model (Contd.)

The risk-neutral version of the model is given by

$$r_t = r + \lambda^\star \sigma_t^2 + \sigma_t z_t^\star \tag{5}$$
$$\sigma_t^2 = \omega + \beta \sigma_{t-\Delta}^2 + \alpha (z_{t-\Delta}^\star - \gamma^\star \sigma_{t-\Delta})^2 \tag{6}$$

where the transformed parameters are

$$\lambda^\star = -\frac{1}{2}$$
$$\gamma^\star = \gamma + \lambda + \frac{1}{2}$$
$$z_t^\star = z_t + \left(\lambda + \frac{1}{2}\right)\sigma_t$$

Facts about the Heston Nandi option pricing model (Contd.)

The formula for the HN call option price is given by:

$$E_t = V_t \mathbb{P}_1 - K e^{-r(T-t)} \mathbb{P}_2 \tag{7}$$

where

$$\mathbb{P}_1 = \frac{1}{2} + \frac{e^{-r(T-t)}}{\pi V_t} \int_0^\infty \mathcal{R}\left[\frac{K^{-i\phi} f^\star(i\phi+1)}{i\phi}\right] d\phi$$

and

$$\mathbb{P}_2 = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \mathcal{R}\left[\frac{K^{-i\phi} f^\star(i\phi)}{i\phi}\right] d\phi$$

are the delta of the call value, and the risk neutral probability of the asset price V_T being greater than the strike price K at maturity, respectively.

Facts about the Heston Nandi option pricing model (Contd.)

The generating function of the return process in the model defined by (1) and (2) is given by

$$f(\phi) = S_t^{\phi} \exp\left(A_t + B_t \sigma_{t+\Delta}^2\right), \tag{8}$$

with coefficient functions defined by

$$A_t = A_{t+\Delta} + \phi r + B_{t+\Delta}\omega - \frac{1}{2}\ln\left(1 - 2\alpha B_{t+\Delta}\right) \tag{9}$$

$$B_t = \phi(\lambda+\gamma) - \frac{1}{2}\gamma^2 + \beta B_{t+\Delta} + \frac{0.5(\phi-\gamma)^2}{1-2\alpha B_{t+\Delta}} \tag{10}$$

The RN $f^{\star}(\phi)$, is obtained by substituting $\lambda^{\star}$ and $\gamma^{\star}$ for λ and γ in the function $f(\phi)$ above. A_t and B_t are solved recursively, given the terminal conditions $A_T = 0$ and $B_T = 0$. Feller (1971) shows how to calculate probabilities by inverting the characteristic function.

The GARCH SCR model and its estimation

1. Set the elements of the HN parameter vector θ^0 equal to 0.001, where $\theta^0 \equiv \{\alpha^0, \beta^0, \gamma^0, \lambda^0, \omega^0\}$. Initialize the elements of the vector for the times series of asset values $\{V_t^0\}_{t=1}^n$ to any value k.
2. Given the parameter vector θ^i computed in iteration i, compute the vector of asset values $\{V_t^{i+1}\}_{t=1}^n$ for the iteration $i+1$ by inverting the call option equation (7), given the time series of equity values $\{E_t\}_{t=1}^n$.
3. Compute the series of log returns using the formula $r_t^{(i+1)} = \log\left(V_t^{(i+1)} / V_{t-1}^{(i+1)}\right)$ from the extracted series of asset values $\{V_t^{(i+1)}\}_{t=1}^n$ which are used estimate the conditional variances $\{\sigma_{t+j}^{2,(i+1)}\}_{j=1}^{n-1}$ together with the parameter vector $\theta^{(i+1)} \equiv \{\alpha^{(i+1)}, \beta^{(i+1)}, \gamma^{(i+1)}, \lambda^{(i+1)}, \omega^{(i+1)}\}$ as in Bollerslev (1986).

The GARCH SCR model and its convergence

The convergence of the above algorithm is guaranteed, but as usual in nonlinear problems with potentially multi-modal likelihood functions:

- convergence can be slow,
- and multiple solutions are possible.

Thus, it is useful to experiment with multiple initial seeds of the algorithm in order to verify that the estimates of the parameters are robust to choice of initial values. We made an effort to do this in the current study, in particular in our empirical application to the banks that follows our simulation study below. Some extensions and solutions:

The GARCH SCR model and benchmark

- Benchmark the GARCH SCR model against the calibrated Merton (1974) model, and the Duan (2004) MLE the Merton model, using simulated data.
- We consider in a $2X2$ matrix combination of asset volatility (business risk) and leverage ratio (financial risk) for the firm.
- σ equal to 20% and 40%, and (K/V) equal to 0.5 and 1.0. These ratios are generated by setting the initial asset value equal to 100 and setting the value of K equal to 50 or 100, respectively.
- $\Rightarrow$ Empirical evidence indicates that such a leverage ratio of 1.0 may even be an underestimate of the leverage ratios of some major investment banks during that period.

The GARCH SCR model and benchmark

- Evaluate the models' performances under two different data generated processes (DGPs) for the underlying asset returns and return volatility. The first DGP is the discrete time version of the Black and Scholes (1973) model, with constant volatility, found by setting $\alpha = \beta = 0$ in the HN model with one lag. This case is of interest because it allows us to measure the performance of the GARCH structural credit risk model estimation algorithm against the Merton and Duan et al. methods when the true model is actually that assumed by the latter models.
- The second DGP is the HN model with $\alpha > 0$ and $\beta > 0$, and it is of interest because it allows us to measure the difference in performance between the Merton and Duan et al. methods and our method in a setting of stochastic volatility.

The GARCH SCR model and benchmark: DGP=GBM

- The time horizon is assumed to be one year throughout, and $r = 5\%$.
- DGP of the firm's assets is a geometric Brownian motion, we set the asset value, the default barrier, and the asset volatility according to the desired combination of business and financial risk,
- simulate 52 weeks of asset values, and then simulate 52 weeks of equity values by pricing equity as a call option on the firm's assets using the Black-Scholes formula.

The GARCH SCR model and benchmark: DGP=GBM

- Take that 52 week time series of equity values as an input to our three models under consideration.
- Produce estimates of the asset value, the asset volatility, and the fair spread on the firm's debt, priced according to each of our three models, for the last week in the 52 week sample.
- These estimates can be compared directly against the true asset value, asset volatility, and spread calculated under the Merton model for the last day in the sample, and such comparisons serve as a measure of the out-of-sample accuracy of each model under study.

The GARCH SCR model and benchmark: DGP=GARCH

- DGP of the asset time series is a GARCH process, the above procedure when DGP=GBM is repeated.
- It is necessary to parameterize the GARCH process to produce the asset volatility required by the experiment.
- For each unique experiment, we ran 100 simulations of the 52 week period necessary to log one out-of-sample vector of assets, asset volatility, and spreads per model.
- Using the histogram of these 100 out-of-sample vectors of results, along with the true asset values, volatilities, and spreads, we calculate the sample mean and standard deviation of the difference between the predicted and true asset value, asset volatility, and spread for each model, in each of the eight scenarios described above.
- No changes when running experiment for 1000 values.

Simulation results: DGP=GBM

- Increases in business and financial are associated with larger average out-of-sample estimation errors for the asset, the volatility, and the spread for each of the three models, as well as larger error standard deviations.
- Duan and GARCH models outperform the calibrated Merton model in all cases, in terms of achieving a lower average error in estimation of assets, asset volatility, and the spread, except for the low business risk, low financial risk case, in which the performance of the three models is similar in their estimation of the asset level and volatility.

Simulation results: DGP=GBM (contd)

- Results are in general consistent with the comparison of the Merton and Duan methods in Ericsson and Reneby (2005).
- The GARCH and Duan models still outperform the calibrated Merton model in their estimation of spreads in that case, however...
- We surmise that the slight under-performance of the GARCH model versus the Duan is that the extra complexity of the GARCH model produces a slight cost in terms of estimation precision when asset volatility is constant.

Simulation results: DGP=GARCH

- In all of the four cases studied, the GARCH model outperforms both the Duan and the calibrated Merton model in the out-of-sample estimation of asset values, asset volatility, and spreads, in terms of having lower average sample errors for all three variables of interest.
- The standard deviation of the estimation error under the GARCH model increases with higher business risk, and with higher financial risk.
- The sample standard deviations of the estimation errors for all three variables are lower in the high financial risk, high business risk case for the Merton and Duan models than for the low business risk, high financial risk case, but otherwise are increasing in business and financial risk.

Simulation results

Data Generating Process: Constant Volatility

Scenario	Business Risk	Low		Low		High		High	
	Financial Risk	Low		High		Low		High	
	Model	MAE	St. Dev.	MAE	St. Dev.	MAE	St. Dev.	MAE	St. Dev.
Asset	Merton	0.01	0.04	5.51	7.10	0.67	1.40	9.24	8.85
	Duan	0.01	0.02	2.09	4.15	0.27	0.51	3.89	3.91
	GARCH	0.01	0.02	2.08	4.25	0.29	0.51	3.93	3.91
Asset Volatility	Merton	0.02	0.02	0.11	0.08	0.07	0.06	0.17	0.12
	Duan	0.02	0.01	0.04	0.04	0.04	0.03	0.07	0.05
	GARCH	0.02	0.02	0.06	0.06	0.05	0.04	0.10	0.08
Spread	Merton	1.65	8.96	666.63	985.67	150.97	339.31	1208.60	1286.50
	Duan	1.01	4.64	275.35	700.97	61.45	123.56	634.76	602.86
	GARCH	1.04	4.97	274.44	706.97	69.64	138.09	652.23	701.13

Data Generating Process: Stochastic Volatility (GARCH)

Scenario	Business Risk	Low		Low		High		High	
	Financial Risk	Low		High		Low		High	
	Model	MAE	St. Dev.	MAE	St. Dev.	MAE	St. Dev.	MAE	St. Dev.
Asset	Merton	0.03	0.19	20.04	25.43	0.96	2.56	22.66	21.80
	Duan	0.08	0.65	15.56	25.29	0.89	2.16	18.84	21.44
	GARCH	0.02	0.14	7.13	17.84	0.70	1.76	13.34	15.54
Asset Volatility	Merton	0.05	0.05	0.53	0.95	0.11	0.12	0.53	0.71
	Duan	0.04	0.03	0.45	1.10	0.06	0.07	0.42	0.69
	GARCH	0.03	0.03	0.12	0.09	0.03	0.07	0.13	0.11
Spread	Merton	6.94	42.84	5272.00	13173.00	232.78	702.59	5265.60	9256.70
	Duan	18.58	165.16	5836.00	18039.00	216.09	566.61	5736.20	11038.00
	GARCH	5.19	36.51	1120.00	1765.00	177.15	310.12	1231.00	8379.00

Table: Results from our forecasting exercise.

Conclusions

- The GARCH model achieves the lowest out-of-sample RMSE and MAD of the three models considered in six out of the ten cases considered, and
- Achieves lowest RMSE and MAD in five out of the six cases in which Fannie Mae and Freddie Mac were excluded from the sample.
- New Structural Credit Risk Method/Model (GSCR) using MLE (EM) method that has shown superior results than existing others