On the Complexity of Infinite-Dimensional Quadrature Problems

Thomas Müller-Gronbach, Uni Passau

Infinite-Dimensional Quadrature Problems

Given

- a Borel probability measure μ on a separable Banach space $(\mathfrak{X}, \|\cdot\|)$,
- a class F of functionals $f : \mathfrak{X} \to \mathbb{R}$.

Compute

$$\int_{\mathfrak{X}} f \, d\mu \qquad \text{for } f \in F.$$

Infinite-Dimensional Quadrature Problems

Given

- a Borel probability measure μ on a separable Banach space $(\mathfrak{X}, \|\cdot\|)$,
- a class F of functionals $f : \mathfrak{X} \to \mathbb{R}$.

Compute

$$\int_{\mathfrak{X}} f \, d\mu \qquad \text{for } f \in F.$$

Examples

- μ Wiener measure on $\mathfrak{X} = C([0,1])$ ($\hookleftarrow$ path integrals).

Infinite-Dimensional Quadrature Problems

Given

- a Borel probability measure μ on a separable Banach space $(\mathfrak{X}, \|\cdot\|)$,
- a class F of functionals $f : \mathfrak{X} \longrightarrow \mathbb{R}$.

Compute

$$\int_{\mathfrak{X}} f \, d\mu \qquad \text{for } f \in F.$$

Examples

- μ Gaussian measure on an infinite-dimensional space $\mathfrak{X}$.

Infinite-Dimensional Quadrature Problems

Given

- a Borel probability measure μ on a separable Banach space $(\mathfrak{X}, \|\cdot\|)$,
- a class F of functionals $f : \mathfrak{X} \to \mathbb{R}$.

Compute

$$\int_{\mathfrak{X}} f \, d\mu \qquad \text{for } f \in F.$$

Examples

- μ Gaussian measure on an infinite-dimensional space $\mathfrak{X}$.
- μ distribution of the solution of an
  - SDE on the path space $\mathfrak{X} = C([0,1], \mathbb{R}^k)$ ($\hookleftarrow$ comp. finance),

Infinite-Dimensional Quadrature Problems

Given

- a Borel probability measure μ on a separable Banach space $(\mathfrak{X}, \|\cdot\|)$,
- a class F of functionals $f : \mathfrak{X} \longrightarrow \mathbb{R}$.

Compute

$$\int_{\mathfrak{X}} f \, d\mu \qquad \text{for } f \in F.$$

Examples

- μ Gaussian measure on an infinite-dimensional space $\mathfrak{X}$.
- μ distribution of the solution of an
  - SDE on the path space $\mathfrak{X} = C([0,1], \mathbb{R}^k)$ ($\hookleftarrow$ comp. finance),
  - SPDE on the path space $\mathfrak{X} = C([0,1], L_2(D))$ ($\hookleftarrow$ fluid dyn.).

Outline

- Complexity of Numerical Problems
- Setup for Infinite-Dim. Quadrature: Algorithms, Error, Cost
- Results for $F = \mathrm{Lip}(1)$
  - SDEs
  - The general case
  - Gaussian measures

Complexity of Numerical Problems

(I) **Computational problem**: quadrature, PDE, optimization, . . .

(II) **Computational means**: class $\mathcal{A}$ of algorithms.

(III) **Quality criterion**: error and cost of an algorithm.

(IV) **Minimal error and complexity**:

$$e_N = \inf\{\mathrm{error}(A) : A \in \mathcal{A} \text{ such that } \mathrm{cost}(A) \le N\},$$

$$\mathrm{comp}(\varepsilon) = \inf\{\mathrm{cost}(A) : \ A \in \mathcal{A} \text{ such that } \mathrm{error}(A) \le \varepsilon\}.$$

Complexity of Numerical Problems

(I) **Computational problem**: quadrature, PDE, optimization, ...

(II) **Computational means**: class $\mathcal{A}$ of algorithms.

(III) **Quality criterion**: error and cost of an algorithm.

(IV) **Minimal error and complexity**:

$$e_N = \inf\{\mathrm{error}(A) : A \in \mathcal{A} \text{ such that } \mathrm{cost}(A) \leq N\},$$

$$\mathrm{comp}(\varepsilon) = \inf\{\mathrm{cost}(A) : \ A \in \mathcal{A} \text{ such that } \mathrm{error}(A) \leq \varepsilon\}.$$

Leads to

- benchmarks for existing algorithms,
- definition of optimal algorithms,
- construction of new algorithms (sometimes).

Typical result

$$e_N \asymp N^{-\alpha},$$

consists of

- **upper bound**: existence (construction) of algorithms $A_N \in \mathcal{A}$ with

$$\mathrm{cost}(A_N) \leq N \quad \wedge \quad \mathrm{error}(A_N) = O\left(N^{-\alpha}\right),$$

- **lower bound**: $\exists\, c > 0$ $\forall$ algorithm $A \in \mathcal{A}$:

$$\mathrm{cost}(A) \leq N \quad \Rightarrow \quad \mathrm{error}(A) \geq c \cdot N^{-\alpha}.$$

See

Traub, Wasilkowski, Woźniakowski (1988), *Novak* (1988),
Werschulz (1991), *Plaskota* (1996), *Ritter* (2000), . . .

Algorithms, Error, and Cost

(I) **Problem**: Compute $S(f) = \int_{\mathfrak{X}} f \, d\mu$ for $f \in F$.

(II) **Randomized algorithm** for approximation of $S : F \to \mathbb{R}$

$$\widehat{S} : F \times \Omega \to \mathbb{R}$$

with any probability space $(\Omega, \mathcal{A}, \mathbb{P})$, where

$$\widehat{S}(f, \omega) = \varphi\big(\omega, f(X_1(\omega)), \ldots, f(X_{\nu(\omega)}(\omega))\big)$$

and

$$X_i(\omega) \in \mathfrak{X} \qquad \textbf{(full space sampling)}.$$

Algorithms, Error, and Cost

(I) **Problem**: Compute $S(f) = \int_{\mathfrak{X}} f \, d\mu$ for $f \in F$.

(II) **Randomized algorithm** for approximation of $S : F \longrightarrow \mathbb{R}$

$$\widehat{S} : F \times \Omega \longrightarrow \mathbb{R}$$

with any probability space $(\Omega, \mathcal{A}, \mathbb{P})$, where

$$\widehat{S}(f,\omega) = \varphi\big(\omega, f(X_1(\omega)), \ldots, f(X_{\nu(\omega)}(\omega))\big)$$

and

$$X_i(\omega) \in \mathfrak{X}$$ **(full space sampling)**.

(III) Error and cost

$$\mathrm{error}(\widehat{S}) = \sup_{f \in F} \big(\mathbb{E}|\widehat{S}(f) - S(f)|^2\big)^{1/2}, \qquad \mathrm{cost}(\widehat{S}) = \mathbb{E}(\nu).$$

Remark

1. More generally, we cover adaptive algorithms, too.
2. In the case $\dim(\mathfrak{X}) < \infty$

   numerous results (algorithms, complexity) for explicitly or implicitly given distributions μ.
3. In the case $\dim(\mathfrak{X}) = \infty$

   sampling of f at any point $x \in \mathfrak{X}$ at unit cost is unrealistic in general.

Reasonable restrictions

a) For any finite-dimensional subspace $\mathfrak{X}_0 \subset \mathfrak{X}$

$$X_1(\omega), \ldots, X_{\nu(\omega)}(\omega) \in \mathfrak{X}_0 \qquad \textbf{(fixed subspace sampling)}$$

$$\mathrm{cost}(\widehat{S}) = \mathbb{E}(\nu) \cdot \dim \mathfrak{X}_0.$$

Reasonable restrictions

a) For any finite-dimensional subspace $\mathfrak{X}_0 \subset \mathfrak{X}$

$$X_1(\omega), \ldots, X_{\nu(\omega)}(\omega) \in \mathfrak{X}_0 \qquad \textbf{(fixed subspace sampling)}$$

$$\mathrm{cost}(\widehat{S}) = \mathbb{E}(\nu) \cdot \dim \mathfrak{X}_0.$$

Example: Consider an SDE and the Monte-Carlo Euler scheme

$$\widehat{S}(f) = 1/n \sum_{i=1}^{n} f(X_{i,k})$$

with step-size $1/k$ and piecewise linear interpolation. Then

$$\text{cost}(\widehat{S}) \asymp n \cdot k = \#\text{replications} \cdot \#\text{Euler steps}.$$

Reasonable restrictions

a) For any finite-dimensional subspace $\mathfrak{X}_0 \subset \mathfrak{X}$

$$X_1(\omega), \ldots, X_{\nu(\omega)}(\omega) \in \mathfrak{X}_0 \qquad \textbf{(fixed subspace sampling)}$$

$$\operatorname{cost}(\widehat{S}) = \mathbb{E}(\nu) \cdot \dim \mathfrak{X}_0.$$

b) For any scale of finite-dimensional subspaces $\mathfrak{X}_1 \subset \mathfrak{X}_2 \subset \ldots \subset \mathfrak{X}$

$$X_1(\omega), \ldots, X_{\nu(\omega)}(\omega) \in \bigcup_{m=1}^{\infty} \mathfrak{X}_m \qquad \textbf{(variable subspace sampling)}$$

$$\operatorname{cost}(\widehat{S}) = \mathbb{E}\Big(\sum_{i=1}^{\nu} \inf\{\dim \mathfrak{X}_m : X_i \in \mathfrak{X}_m\}\Big).$$

(IV) Minimal Errors (in all three sampling regimes)

$$e_N = \inf\{\mathrm{error}(\widehat{S}) : \mathrm{cost}(\widehat{S}) \leq N\}.$$

Notation

$$\begin{aligned} e_N^{\mathrm{full}} &: \quad \text{full space sampling} \\ e_N^{\mathrm{fix}} &: \quad \text{fixed subspace sampling} \\ e_N^{\mathrm{var}} &: \quad \text{variable subspace sampling} \end{aligned}$$

(IV) Minimal Errors (in all three sampling regimes)

$$e_N = \inf\{\mathrm{error}(\widehat{S}) : \mathrm{cost}(\widehat{S}) \leq N\}.$$

Notation

e_N^{full} : full space sampling

e_N^{fix} : fixed subspace sampling

e_N^{var} : variable subspace sampling

Questions

- Order of convergence of $e_N^{\mathrm{full}} \leq e_N^{\mathrm{var}} \leq e_N^{\mathrm{fix}}$?
- Construction of algorithms $\widehat{S}_N$ such that

$$\text{cost}(\widehat{S}_N) \leq N \qquad \text{and} \qquad \mathrm{error}(\widehat{S}_N) \asymp e_N?$$

Results

In the sequel we consider the class

$$F = \mathrm{Lip}(1)$$

of functionals $f : \mathfrak{X} \longrightarrow \mathbb{R}$ with

$$|f(x) - f(y)| \le \|x - y\|, \qquad x, y \in \mathfrak{X}.$$

Concerning μ we consider

- the SDE case,
- the general case,
- the case of Gaussian measures.

The SDE Case

Consider

$$dX(t) = a(X(t))\,dt + b(X(t))\,dW(t), \quad t \in [0,1],$$
$$X(0) = u_0 \in \mathbb{R},$$

where

- a Lipschitz,
- $b \in C^2(\mathbb{R})$, b', b'' bounded,
- $b(u_0) \neq 0$.

Let μ denote the distribution of X on $\mathfrak{X} = C([0,1])$.

Theorem 1 (*Creutzig, Dereich, M-G, Ritter 09*)

$$N^{-1/4} \cdot (\ln N)^{-3/4} \preceq e_N^{\mathrm{fix}} \preceq N^{-1/4}$$ (**fixed subspace**)

$$N^{-1/2} \preceq e_N^{\mathrm{var}} \preceq N^{-1/2} \cdot \ln N$$ (**variable subspace**)

$$N^{-1/2} \cdot (\ln N)^{-3/2} \preceq e_N^{\mathrm{full}} \preceq N^{-1/2} \cdot (\ln N)^{-1/2}$$ (**full space**)

Theorem 1 (*Creutzig, Dereich, M-G, Ritter 09*)

$$N^{-1/4} \cdot (\ln N)^{-3/4} \preceq e_N^{\mathrm{fix}} \preceq N^{-1/4} \qquad \textbf{(fixed subspace)}$$

$$N^{-1/2} \preceq e_N^{\mathrm{var}} \preceq N^{-1/2} \cdot \ln N \qquad \textbf{(variable subspace)}$$

$$N^{-1/2} \cdot (\ln N)^{-3/2} \preceq e_N^{\mathrm{full}} \preceq N^{-1/2} \cdot (\ln N)^{-1/2} \qquad \textbf{(full space)}$$

Remark Upper bounds (up to powers of $\ln N$)

- for fixed subspace sampling: Monte-Carlo Euler,
- for variable subspace sampling: multilevel Monte-Carlo Euler.

Theorem 1 (*Creutzig, Dereich, M-G, Ritter 09*)

$$N^{-1/4} \cdot (\ln N)^{-3/4} \preceq e_N^{\mathrm{fix}} \preceq N^{-1/4}$$ (**fixed subspace**)

$$N^{-1/2} \preceq e_N^{\mathrm{var}} \preceq N^{-1/2} \cdot \ln N$$ (**variable subspace**)

$$N^{-1/2} \cdot (\ln N)^{-3/2} \preceq e_N^{\mathrm{full}} \preceq N^{-1/2} \cdot (\ln N)^{-1/2}$$ (**full space**)

Remark Upper bounds (up to powers of $\ln N$)

- for fixed subspace sampling: Monte-Carlo Euler,
- for variable subspace sampling: multilevel Monte-Carlo Euler.
  The basic idea

$$\mathbb{E}(f(X_L)) = \sum_{\ell=2}^{L} \mathbb{E}(f(X_\ell) - f(X_{\ell-1})) + \mathbb{E}(f(X_1)).$$

Theorem 1 (*Creutzig, Dereich, M-G, Ritter 09*)

$$N^{-1/4} \cdot (\ln N)^{-3/4} \preceq e_N^{\text{fix}} \preceq N^{-1/4} \qquad \textbf{(fixed subspace)}$$

$$N^{-1/2} \preceq e_N^{\text{var}} \preceq N^{-1/2} \cdot \ln N \qquad \textbf{(variable subspace)}$$

$$N^{-1/2} \cdot (\ln N)^{-3/2} \preceq e_N^{\text{full}} \preceq N^{-1/2} \cdot (\ln N)^{-1/2} \qquad \textbf{(full space)}$$

Remark Upper bounds (up to powers of $\ln N$)

- for fixed subspace sampling: Monte-Carlo Euler,
- for variable subspace sampling: multilevel Monte-Carlo Euler.

  Math Finance: *Giles* (2008, ...), *Giles, Higham, Mao* (2009), *Avikainen* (2009), *Giles, Watherhouse* (2009).

  Levy-driven, fractional driven SDEs: *Dereich* (2009), *Dereich, Heidenreich* (2009), *Kloeden, Neuenkirch, Pavani* (2009),

  Initiated by: *Heinrich* (1998), *Heinrich, Sindambiwe* (1999) for integral equations, parametric integration.

Theorem 1 (*Creutzig, Dereich, M-G, Ritter 09*)

$$N^{-1/4} \cdot (\ln N)^{-3/4} \preceq e_N^{\text{fix}} \preceq N^{-1/4} \qquad \textbf{(fixed subspace)}$$

$$N^{-1/2} \preceq e_N^{\text{var}} \preceq N^{-1/2} \cdot \ln N \qquad \textbf{(variable subspace)}$$

$$N^{-1/2} \cdot (\ln N)^{-3/2} \preceq e_N^{\text{full}} \preceq N^{-1/2} \cdot (\ln N)^{-1/2} \qquad \textbf{(full space)}$$

Remark Upper bounds (up to powers of $\ln N$)

- for fixed subspace sampling: Monte-Carlo Euler,
- for variable subspace sampling: multilevel Monte-Carlo Euler.

Note Strong optimality property of the multilevel algorithm.

The General Case

Basis for analysis and results: approximation of distributions.

Wasserstein metric of order $r \geq 1$

$$\Delta^{(r)}(\mu, \widetilde{\mu}) = \inf_{\rho} \left(\int_{\mathfrak{X} \times \mathfrak{X}} \|x - y\|^r \, d\rho(x, y) \right)^{1/r}$$

with $\inf$ over all Borel prob. measures ρ on $\mathfrak{X} \times \mathfrak{X}$ with marginals $\mu, \widetilde{\mu}$.

The General Case

Basis for analysis and results: approximation of distributions.

Wasserstein metric of order $r \geq 1$

$$\Delta^{(r)}(\mu, \widetilde{\mu}) = \inf_{\rho} \left(\int_{\mathfrak{X} \times \mathfrak{X}} \|x - y\|^r \, d\rho(x, y) \right)^{1/r}$$

with $\inf$ over all Borel prob. measures ρ on $\mathfrak{X} \times \mathfrak{X}$ with marginals $\mu, \widetilde{\mu}$.

Connection to quadrature problem on $F = \mathrm{Lip}(1)$:

Kantorovich-Rubinstein Theorem

$$\Delta^{(1)}(\mu, \widetilde{\mu}) = \sup_{f \in F} \left| \int_{\mathfrak{X}} f \, d\mu - \int_{\mathfrak{X}} f \, d\widetilde{\mu} \right|.$$

N**-th quantization number** of order r

$$q_N^{(r)} = \inf\{\Delta^{(r)}(\mu, \widetilde{\mu}) : |\operatorname{supp}(\widetilde{\mu})| \leq N\}.$$

N**-th quantization number** of order r

$$q_N^{(r)} = \inf\{\Delta^{(r)}(\mu, \widetilde{\mu}) : |\operatorname{supp}(\widetilde{\mu})| \leq N\}.$$

N**-th average Kolmogorov width** of order r

$$d_N^{(r)} = \inf\{\Delta^{(r)}(\mu, \widetilde{\mu}) : \dim(\operatorname{span}(\operatorname{supp}(\widetilde{\mu}))) \leq N\}.$$

N**-th quantization number** of order r

$$q_N^{(r)} = \inf\{\Delta^{(r)}(\mu, \widetilde{\mu}) : |\operatorname{supp}(\widetilde{\mu})| \le N\}.$$

N**-th average Kolmogorov width** of order r

$$d_N^{(r)} = \inf\{\Delta^{(r)}(\mu, \widetilde{\mu}) : \dim(\operatorname{span}(\operatorname{supp}(\widetilde{\mu}))) \le N\}.$$

Theorem 2 (*Creutzig, Dereich, M-G, Ritter 09*)

$$N^{1/2} \cdot \sup_{n \ge 4N} (q_{n-1}^{(1)} - q_n^{(1)}) \preceq e_N^{\text{full}} \preceq N^{-1/2} \cdot q_N^{(2)}$$

$$\max(e_N^{\text{full}}, d_{2N}^{(1)}) \preceq e_N^{\text{var}}$$

$$\inf_{k \cdot n \le N} \max(e_n^{\text{full}}, d_k^{(1)}) \preceq e_N^{\text{fix}} \preceq \inf_{k \cdot n \le N} (n^{-1/2} + d_k^{(2)})$$

If $d_N^{(2)}$ is regular varying: $e_N^{\text{var}} \preceq \max(N^{-1/2}, d_N^{(2)}) \cdot \ln N$.

N**-th quantization number** of order r

$$q_N^{(r)} = \inf\{\Delta^{(r)}(\mu, \widetilde{\mu}) : |\mathrm{supp}(\widetilde{\mu})| \le N\}.$$

N**-th average Kolmogorov width** of order r

$$d_N^{(r)} = \inf\{\Delta^{(r)}(\mu, \widetilde{\mu}) : \dim(\mathrm{span}(\mathrm{supp}(\widetilde{\mu}))) \le N\}.$$

Theorem 2 (*Creutzig, Dereich, M-G, Ritter 09*)

$$N^{1/2} \cdot \sup_{n \ge 4N} (q_{n-1}^{(1)} - q_n^{(1)}) \preceq e_N^{\mathrm{full}} \preceq N^{-1/2} \cdot q_N^{(2)}$$

$$\max(e_N^{\mathrm{full}}, d_{2N}^{(1)}) \preceq e_N^{\mathrm{var}}$$

$$\inf_{k \cdot n \le N} \max(e_n^{\mathrm{full}}, d_k^{(1)}) \preceq e_N^{\mathrm{fix}} \preceq \inf_{k \cdot n \le N} (n^{-1/2} + d_k^{(2)})$$

If $d_N^{(2)}$ is regular varying: $e_N^{\mathrm{var}} \preceq \max(N^{-1/2}, d_N^{(2)}) \cdot \ln N$.

For SDEs $q_N^{(r)} \asymp (\ln N)^{-1/2}$, $d_N^{(r)} \asymp N^{-1/2}$.

Quantization numbers

Finite-dimensional spaces: studied since around 1950

Graf, Luschgy (2000)

Infinite-dimensional spaces: studied since around 2000

Aurzada, Creutzig, Dereich, Fehringer, Luschgy, M-G, Matoussi,

Pagès, Printems, Ritter, Scheutzow,...

Applications to computational finance:

Pagés, Pham, Printems (2004), *Pagés, Printems* (2005)

Quantization numbers

Average Kolmogorov widths

Gaussian measures:

Mathé (1990), *Maiorov* (1992,...), ..., *Creutzig* (2002)

Diffusion processes:

Creutzig, Dereich, M-G, Ritter (2009)

Quantization numbers

Average Kolmogorov widths

Optimal algorithms for quadrature problems

Finite dimensional spaces:

Nikolskij (1950), *Kiefer* (1957), *Bakhvalov* (1959), . . .

Infinite dimensional spaces:

Wasilkowski, Woźniakowski (1996),

Creutzig, Dereich, M-G, Ritter (2009),

Hickernell, Niu (2009)

Hichernell, M-G, Niu, Ritter (2009)

The Gaussian Case

Small ball probabilities for centered Gaussian measure μ on $\mathfrak{X}$

$$\varphi(\varepsilon) = -\ln \mu(\{x \in \mathfrak{X} : \ \|x\| \le \varepsilon\}).$$

The Gaussian Case

Small ball probabilities for centered Gaussian measure μ on $\mathfrak{X}$

$$\varphi(\varepsilon) = -\ln \mu(\{x \in \mathfrak{X} : \|x\| \leq \varepsilon\}).$$

Typically, if $\dim \mathfrak{X} = \infty$,

$$\exists\, \alpha > 0, \beta \in \mathbb{R} : \qquad \varphi(\varepsilon) \asymp \varepsilon^{-\alpha} \cdot (\ln \varepsilon^{-1})^{\beta} \qquad \text{as } \varepsilon \to 0.$$

See ... *Li, Shao* (2001), *Belinski, Linde* (2002), *Fill, Torcaso* (2004), ...

The Gaussian Case

Small ball probabilities for centered Gaussian measure μ on $\mathfrak{X}$

$$\varphi(\varepsilon) = -\ln \mu(\{x \in \mathfrak{X} : \|x\| \le \varepsilon\}).$$

Typically, if $\dim \mathfrak{X} = \infty$,

$$\exists\, \alpha > 0, \beta \in \mathbb{R} : \qquad \varphi(\varepsilon) \asymp \varepsilon^{-\alpha} \cdot (\ln \varepsilon^{-1})^{\beta} \qquad \text{as } \varepsilon \to 0.$$

See . . . *Li, Shao* (2001), *Belinski, Linde* (2002), *Fill, Torcaso* (2004), . . .

Example: Fractional Brownian motion with Hurst index $H \in]0, 1[$ on $\mathfrak{X} = C[0, 1]$. Here

$$\alpha = 1/H, \qquad \beta = 0.$$

The Gaussian Case

Small ball probabilities for centered Gaussian measure μ on $\mathfrak{X}$

$$\varphi(\varepsilon) = -\ln \mu(\{x \in \mathfrak{X} : \|x\| \le \varepsilon\}).$$

Typically, if $\dim \mathfrak{X} = \infty$,

$$\exists\, \alpha > 0, \beta \in \mathbb{R} : \qquad \varphi(\varepsilon) \asymp \varepsilon^{-\alpha} \cdot (\ln \varepsilon^{-1})^{\beta} \qquad \text{as } \varepsilon \to 0.$$

See ... *Li, Shao* (2001), *Belinski, Linde* (2002), *Fill, Torcaso* (2004), ...

Proposition *Creutzig* (2002), *Dereich* (2003)

$$q_N^{(r)} \asymp (\ln N)^{-1/\alpha} \cdot (\ln \ln N)^{\beta/\alpha}, \qquad d_N^{(r)} \asymp N^{-1/\alpha} \cdot (\ln N)^{\beta/\alpha}.$$

Assumption $-\ln \mu(\{x \in \mathfrak{X} : \|x\| \leq \varepsilon\}) \asymp \varepsilon^{-\alpha} \cdot (\ln \varepsilon^{-1})^{\beta}$

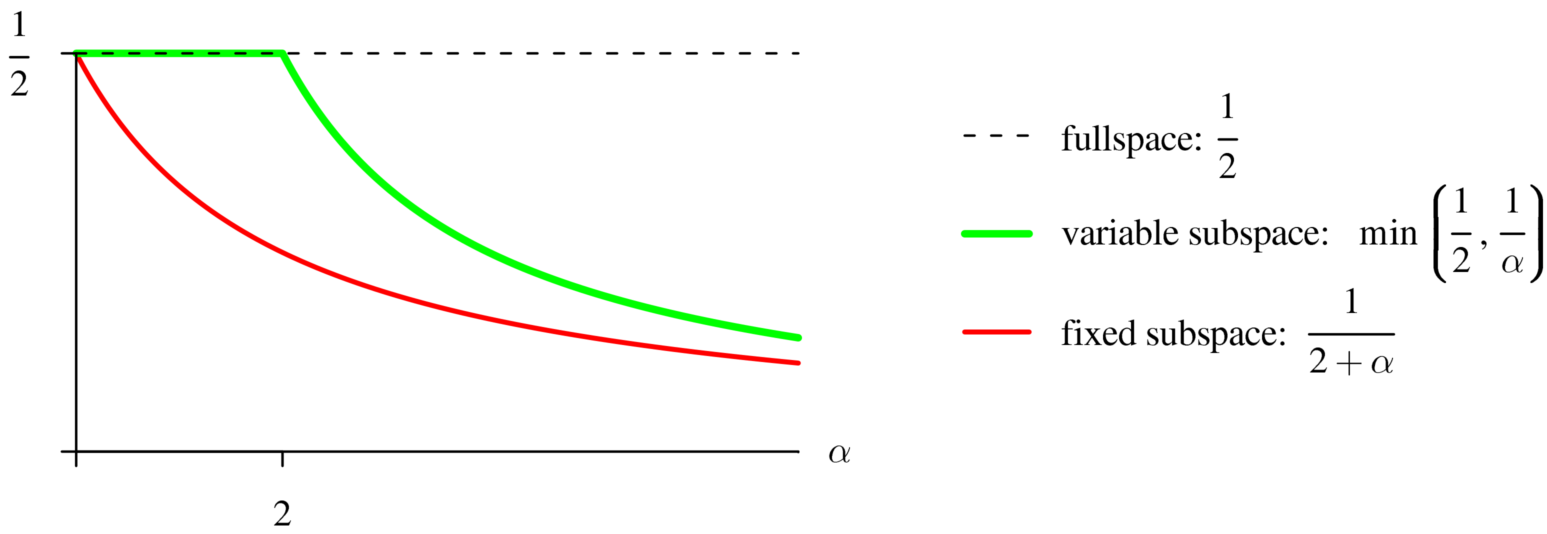
Theorem 3 (*Creutzig, Dereich, M-G, Ritter 09*) Up to powers of $\ln N$

$$0 < \limsup_{N \to \infty} N^{1/2} \cdot e_N^{\text{full}} < \infty$$

$$0 < \limsup_{N \to \infty} N^{1/2} \cdot e_N^{\text{var}} < \infty \qquad \text{for } \alpha < 2$$

$$e_N^{\text{var}} \asymp N^{-1/\alpha} \qquad \text{for } \alpha \geq 2$$

$$0 < \limsup_{N \to \infty} N^{1/(2+\alpha)} \cdot e_N^{\text{fix}} < \infty$$



Summary

- Introduced: a reasonable cost model for the analysis of infinite-dimensional quadrature problems.

Summary

- Introduced: a reasonable cost model for the analysis of infinite-dimensional quadrature problems.
- Established: a close connection between quadrature, approximation of distributions, and strong approximation.
- Hereby: upper and lower bounds for minimal errors on $F = \mathrm{Lip}(1)$.

Summary

- Introduced: a reasonable cost model for the analysis of infinite-dimensional quadrature problems.
- Established: a close connection between quadrature, approximation of distributions, and strong approximation.
- Hereby: upper and lower bounds for minimal errors on $F = \mathrm{Lip}(1)$.
- In particular, for SDEs,

  var. subspace: $N^{-1/2}$, **fixed subspace:** $N^{-1/4}$,

  and for Gaussian measures with 'smoothness' $1/\alpha \in]0, \infty[$

  var. subspace: $N^{-1/\max(2,\alpha)}$, **fixed subspace:** $N^{-1/(2+\alpha)}$

  (up to powers of $\ln N$).

Summary

- Introduced: a reasonable cost model for the analysis of infinite-dimensional quadrature problems.
- Established: a close connection between quadrature, approximation of distributions, and strong approximation.
- Hereby: upper and lower bounds for minimal errors on $F = \mathrm{Lip}(1)$.
- In particular, for SDEs,

  var. subspace: $N^{-1/2}$, **fixed subspace:** $N^{-1/4}$,

  and for Gaussian measures with 'smoothness' $1/\alpha \in \,]0, \infty[$

  var. subspace: $N^{-1/\max(2,\alpha)}$, **fixed subspace:** $N^{-1/(2+\alpha)}$

  (up to powers of $\ln N$).
- Optimality of multilevel Monte-Carlo methods