# Unbiasedness of multivariate tests

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**Abstract**

Many tests are surprisingly not unbiased under a finite number observations. This is more serious in multivariate models. We shall illustrate this phenomenon on the multivariate two-sample testing problem and show that the two-sample tests are safely finite-sample unbiased against alternatives which are one-sided in some (geometric) scalar entities, while they are unbiased against two-sided alternatives only under some special conditions [e.g. when the parent distribution is symmetric]. Consider the situation that we have two data clouds of $p$-dimensional observations with generally unknown distributions $F$ and $G$, and wish to verify the hypothesis $\mathbf{H}: F \equiv G$, against some alternatives, $F$ continuous but unknown. If possible, the test of $\mathbf{H}$ should be:

(i) distribution free under $\mathbf{H}$,
(ii) affine invariant with respect to changes of coordinate system,
(iii) consistent against distant alternatives, and consistent against any fixed alternative as $n \to \infty$,
(iv) finite-sample unbiased against a broad class of alternatives of interest.

Unfortunately, a test satisfying all these conditions does not exist in the multivariate setup.

The tests proposed in the literature are either concentrated on some geometric entities of the data clouds, or are of permutation type. They are typically consistent against distant alternatives, but the alternatives are often only vaguely specified. The asymptotic null distribution is given for many of the tests, for some also the local asymptotic power. Some tests are affine-invariant. However, generally, the tests are not unbiased under finite number of observations. They are unbiased against one-sided alternatives; but then we should be careful and consider how ordering the geometric entities (e.g. depths) is related to ordering the observations. The two-sided alternatives are problematic even in the univariate model; the Wilcoxon test, biased for a skew distribution, was illustrated already by Sugiura (1965).

We shall describe several tests, based on the ranks of geometric distances of data; the ordering of distances is natural. These rank tests are finite-sample unbiased against a broad class of location/scale alternatives, one-sided with respect to the stochastic ordering. The tests are finite-sample distribution-free under the hypothesis and some alternatives. Each of them is locally most powerful against a specific alternative. The question of affine invariance of two-sample multivariate tests will be also disscussed.

**References**

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