Beispiel 2 Lineare Hülle

a) $\{(1+X^2), (1-X^2)\}$

$$\mathrm{Lin}(1+X^2, 1-X^2) = \{\lambda(1+X^2) + \mu(1-X^2) \mid \lambda, \mu \in \mathbb{R}\}$$
$$= \{(\lambda+\mu) + (\lambda-\mu)X^2 \mid \lambda, \mu \in \mathbb{R}\}$$
$$= \{\gamma + \delta X^2 \mid \gamma, \delta \in \mathbb{R}\} = \mathrm{Lin}[1, X^2]$$

$$\begin{aligned} \gamma &= \lambda+\mu \\ \delta &= \lambda-\mu \end{aligned} \Rightarrow \quad \lambda = \frac{\gamma+\delta}{2}; \quad \mu = \frac{\gamma-\delta}{2}$$

b) $\mathrm{Lin}(1, 1+X, (1+X)^2, (1+X)^3, (1+X)^4)$

$$= \{\lambda_0 \cdot 1 + \lambda_1(1+X) + \lambda_2(1+X)^2 + \lambda_3(1+X)^3 + \lambda_4(1+X)^4 \mid \lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{R}\} =$$

$$= \{(\lambda_0+\lambda_1+\lambda_2+\lambda_3+\lambda_4) + (\lambda_1+2\lambda_2+3\lambda_3+4\lambda_4)X + (\lambda_2+3\lambda_3+6\lambda_4)X^2 + (\lambda_3+4\lambda_4)X^3 + \lambda_4 X^4 \mid \lambda_0, \lambda_1, \lambda_2, \lambda_3 \lambda_4 \in \mathbb{R}\}$$

$$\overset{(*)}{=} \{\mu_0 + \mu_1 X + \mu_2 X^2 + \mu_3 X^3 + \mu_4 X^4 \mid \mu_0, \mu_1, \mu_2, \mu_3, \mu_4 \in \mathbb{R}\}$$

$$= \mathrm{Lin}(1, X, X^2, X^3, X^4) \equiv P_4$$

Beweis von $(*)$

$$\mu_0 = \lambda_0+\lambda_1+\lambda_2+\lambda_3+\lambda_4$$
$$\mu_1 = \lambda_1+2\lambda_1+3\lambda_3+4\lambda_4$$
$$\mu_2 = \lambda_2+3\lambda_3+6\lambda_4$$
$$\mu_3 = \lambda_3+4\lambda_4$$
$$\mu_4 = \lambda_4$$

$$\underbrace{\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 & 6 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}}_{A} \cdot \underbrace{\begin{pmatrix} \lambda_0 \\ \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{pmatrix}}_{\vec{\lambda}} = \underbrace{\begin{pmatrix} \mu_0 \\ \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \end{pmatrix}}_{\vec{\mu}}$$

Für beliebige vorgegebene $\mu_0, \mu_1, \mu_2, \mu_3, \mu_4 \in \mathbb{R}$ hat das Gleichungssystem immer eindeutige Lösung, da

$$\det(A) = 1 \neq 0 \Rightarrow \vec{\lambda} = A^{-1} \cdot \vec{\mu}$$